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► **To cite this version:**

Matthieu Bonnard. The Drag in a Navier-Stokes Flow with Friction-Driven Boundary Conditions. 2011. hal-00565411

**HAL Id: hal-00565411**

**<https://hal.science/hal-00565411>**

Preprint submitted on 13 Feb 2011

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# The Drag in a Navier-Stokes Flow with Friction-Driven Boundary Conditions

Matthieu Bonnivard

## Abstract

We consider the drag of an obstacle in a Navier-Stokes flow, associated to the friction-driven boundary conditions introduced by Bucur, Feireisl and Nečasová [6]. These boundary conditions account for the asymptotic effect of rough boundaries on the solutions to the Navier-Stokes equations. Consequently, they appear as a natural control variable in order to reduce the drag using the effect of micro-rugosities. In this paper, we prove the existence of a drag associated to friction-driven boundary conditions, we give a uniqueness criterion and, in the particular case of Navier's friction law, we prove that the drag is differentiable with respect to the friction coefficient, and compute its gradient.

## 1 Introduction

In this paper, we are interested in the definition and the mathematical properties of the drag of an obstacle in a Navier-Stokes flow, associated to the so-called *friction-driven* boundary conditions on the solid-fluid interface [6].

Let  $N \in \{2, 3\}$  be the space dimension. We define the obstacle  $\mathcal{S}$  as a compact subset of  $\mathbb{R}^N$ , with a Lipschitz boundary. The friction-driven boundary conditions are determined by a triplet  $\{\mu, A, \mathcal{V}\}$ , where:

- $\mu$  is a *capacitary measure* concentrated on  $\partial\mathcal{S}$ ,
- $A$  is a *positive symmetric matrix function*,
- $\mathcal{V} = \{\mathcal{V}(x)\}_{x \in \partial\mathcal{S}}$  is a collection of vector spaces, where for q.e.  $x \in \partial\mathcal{S}$ ,  $\mathcal{V}(x)$  is a *subspace* of the *tangent hyperplane* to  $\mathcal{S}$  at  $x$ .

Let  $\mathbf{u}_\infty \in \mathbb{R}^N$  be the velocity of the fluid at infinity. To define the drag of the solid  $\mathcal{S}$  in the direction  $\mathbf{u}_\infty$ , we (artificially) introduce an open subset  $\Omega \subset \mathbb{R}^N$  such that  $\mathcal{S} \subset \Omega$ , and define the fluid domain

$$\mathcal{F} = \Omega \setminus \mathcal{S}.$$

We note  $L_0^2(\mathcal{F})$  the subspace of  $L^2(\mathcal{F})$  consisting of functions with mean value equal to zero. We consider the following problem: find  $(\mathbf{u}, p) \in H^1(\mathcal{F}, \mathbb{R}^N) \times L_0^2(\mathcal{F})$  such that

$$-\operatorname{div}(\sigma(\mathbf{u}, p)) + (\mathbf{u} \cdot \nabla)\mathbf{u} = 0 \text{ in } \mathcal{F}, \quad (1)$$

$$\operatorname{div} \mathbf{u} = 0 \text{ in } \mathcal{F}, \quad (2)$$

$$\mathbf{u} = \mathbf{u}_\infty \text{ on } \partial\Omega, \quad (3)$$

$$\mathbf{u}(x) \in \mathcal{V}(x) \text{ for q.e. } x \in \partial\mathcal{S}, \quad (4)$$

$$[2\nu\mathbf{D}(\mathbf{u})\mathbf{n} + \mu\mathbf{A}\mathbf{u}] \cdot \phi = 0 \text{ for any } \phi \in \mathcal{V}(x), x \in \partial\mathcal{S}, \quad (5)$$

where  $\sigma(\mathbf{u}, p)$  is the stress tensor defined by

$$\sigma(\mathbf{u}, p) = 2\nu\mathbf{D}(\mathbf{u}) - p\operatorname{Id},$$

$\nu > 0$  being the viscosity of the fluid and  $\mathbf{D}(\mathbf{u})$  the symmetric part of  $\nabla\mathbf{u}$ , defined by

$$\mathbf{D}(\mathbf{u}) = \frac{1}{2} ((\nabla\mathbf{u}) + (\nabla\mathbf{u})^T).$$

In condition (5),  $\mathbf{n}$  stands for the normal vector field to  $\partial\mathcal{S}$  oriented towards the solid.

Condition (4) expresses that the flow is driven by the family of linear spaces  $\{\mathcal{V}(x)\}_{x \in \partial\mathcal{S}}$ . Above,  $[2\nu\mathbf{D}(\mathbf{u})\mathbf{n} + \mu\mathbf{A}\mathbf{u}] \cdot \phi = 0$  for  $\phi \in \mathcal{V}(x)$  is a formal pointwise relation, which has to be understood globally on  $\partial\mathcal{S}$ . Precisely, condition (5) holds provided that

$$2\nu \int_{\partial\mathcal{S}} \mathbf{D}(\mathbf{u})\mathbf{n} \cdot \phi \, dx + \int_{\partial\mathcal{S}} \mathbf{A}\mathbf{u} \cdot \phi \, d\mu = 0$$

for every  $\phi \in H^1(\mathcal{F}, \mathbb{R}^N)$  such that  $\phi(x) \in \mathcal{V}(x)$  for q.e.  $x \in \partial\mathcal{S}$ . In particular, condition (5) is a friction condition, characterized by the measure  $\mu$  and the matrix  $A$ .

Friction-driven boundary conditions have been obtained by Bucur, Feireisl and Nečasová in [6], as the general expression of the boundary conditions resulting from the rugosity effect on perfectly slippery walls. The rugosity effect has been investigated in [11, 10, 8, 2] and in the more recent papers [7, 6, 5]. It consists in approximating a given domain  $U$  by a sequence of geometric perturbations  $(U_\varepsilon)_{\varepsilon>0}$ , and studying the asymptotic behaviour of a family of vector fields  $\mathbf{u}_\varepsilon \in H^1(U_\varepsilon, \mathbb{R}^N)$  in the neighborhood of the boundary of the limit domain  $U$ . In [6], the authors have considered a sequence  $(\mathbf{u}_\varepsilon)_{\varepsilon>0}$  of weak solutions to the Navier-Stokes equations with perfect slip boundary conditions in equi-Lipschitz domains  $(U_\varepsilon)_{\varepsilon>0}$ . They have proved the existence of a triplet  $\{\mu, A, \mathcal{V}\}$ , such that, up to a subsequence,  $(\mathbf{u}_\varepsilon)_{\varepsilon>0}$  converges weakly to a function  $\mathbf{u}$  solution to the Navier-Stokes equations with friction-driven boundary conditions.

This asymptotic process is an indirect proof of the existence of a solution to the Navier-Stokes equations with friction-driven boundary conditions, for *admissible* triplets  $\{\mu, A, \mathcal{V}\}$ , that is, triplets that can be effectively obtained by a sequence of geometric perturbations of the domain. However, since this result relies on abstract  $\Gamma$ -convergence tools, the class of admissible triplets cannot be explicitly described, at least for very general perturbations. The purpose of this paper is to prove that, in fact, problem (1)-(5) admits at least one weak solution (in a sense that will be precised below), for every  $\nu > 0$ ,  $\mathbf{u}_\infty \in \mathbb{R}^N$  and every triplet

$\{\mu, A, \mathcal{V}\}$ . This result is stated in Theorem 1. Moreover, we prove that for a suitable choice of  $\nu$ ,  $\mathbf{u}_\infty$  and  $\mathcal{V}$ , which does not depend on  $\{\mu, A\}$ , every weak solution to System (1)-(5) is unique (see Theorem 2).

In the second part of the paper, we study the differentiability of the drag function with respect to the friction part of the boundary condition. Let  $\nu > 0$  and  $\mathbf{u}_\infty \in \mathbb{R}$ . We can associate to every triplet  $\{\mu, A, \mathcal{V}\}$  and every weak solution  $\mathbf{u}$  to System (1)-(5), a drag  $\mathcal{T}(\mu, A, \mathcal{V}, \mathbf{u})$  defined by

$$\mathcal{T}(\mu, A, \mathcal{V}, \mathbf{u}) = 2\nu \int_{\mathcal{F}} |\mathbf{D}(\mathbf{u})|^2 dx + \int_{\partial\mathcal{S}} A\mathbf{u} \cdot \mathbf{u} d\mu.$$

In the case where  $\mathbf{u}$  is unique, the drag depends only on the choice of the triplet  $\{\mu, A, \mathcal{V}\}$ , which appears as a natural control variable in drag minimization problems. As a first step in the study of the regularity of the drag with respect to  $\{\mu, A, \mathcal{V}\}$ , we consider Navier's boundary conditions on  $\partial\mathcal{S}$ , which are given by the non penetration condition

$$\mathbf{u} \cdot \mathbf{n} = 0 \text{ on } \partial\mathcal{S} \tag{6}$$

and by the friction law

$$2\nu [\mathbf{D}(\mathbf{u})\mathbf{n}]_{tan} + \beta\mathbf{u} = 0 \text{ on } \partial\mathcal{S}. \tag{7}$$

Here,  $\mathbf{n}$  is the normal vector field to  $\partial\mathcal{S}$  oriented towards the solid and  $\beta \in L^2(\partial\mathcal{S})$  is a friction coefficient. For any vector field  $\mathbf{v}$ , defined on  $\partial\mathcal{S}$ , we note  $[\mathbf{v}]_{tan}$  its tangential part, defined by

$$[\mathbf{v}]_{tan} = \mathbf{v} - (\mathbf{v} \cdot \mathbf{n})\mathbf{n}.$$

Boundary conditions (6)-(7) correspond to the case where  $\mathcal{V}(x)$  is equal to the tangent hyperplane to  $\partial\mathcal{S}$  on q.e.  $x$  on the boundary,  $A \equiv \text{Id}$  and  $\mu$  is absolutely continuous with respect to the surface measure on  $\partial\mathcal{S}$  with a density  $\beta \in L^2(\partial\mathcal{S})$ . We prove that the drag is differentiable with respect to  $\beta$  and compute the gradient of the drag in the  $L^2(\partial\mathcal{S})$  topology (see Theorem 3 and Corollary 4).

The structure of the paper is the following. In Section 2, we introduce the functional framework associated with System (1)-(3) and boundary conditions (4)-(5) and (6)-(7), and state our main results. In Section 3, we recall some abstract material from [9] which we use in the proof of Theorems 1 and 2. Section 4 is devoted to the proof of Theorem 3 and Corollary 4.

## 2 Statement of the Main Results

We note

$$\mathbf{H}^1(\mathcal{F}) = H^1(\mathcal{F}, \mathbb{R}^N).$$

To define the weak solutions to System (1)-(5), we introduce the following subsets of  $\mathbf{H}^1(\mathcal{F})$ :

$$K = \{\mathbf{v} \in \mathbf{H}^1(\mathcal{F}) \mid \text{div } \mathbf{v} = 0 \text{ in } L^2(\mathcal{F}), \mathbf{v}(x) \in \mathcal{V}(x) \text{ for q.e. } x \in \partial\mathcal{S}, \mathbf{v} = \mathbf{u}_\infty \text{ on } \partial\Omega\}$$

and

$$V = \{\phi \in \mathbf{H}^1(\mathcal{F}) \mid \text{div } \phi = 0 \text{ in } L^2(\mathcal{F}), \mathbf{v}(x) \in \mathcal{V}(x) \text{ for q.e. } x \in \partial\mathcal{S}, \phi = 0 \text{ on } \partial\Omega\}.$$

Let us fix the family of linear spaces  $\mathcal{V}$  such that  $V$  is non empty. In particular,  $V$  is a separable Hilbert space for the  $\mathbf{H}^1$  scalar product. We note  $V'$  its dual space for the  $\mathbf{H}^1$  norm. Let  $\mu$  be a capacitary measure concentrated on  $\partial\mathcal{S}$  and  $A$  a positive symmetric matrix function.

We say that  $\mathbf{u} \in \mathbf{H}^1(\mathcal{F})$  is a weak solution to system (1)-(5) if  $\mathbf{u} \in K$  and for every  $\phi \in V$ ,

$$2\nu \int_{\mathcal{F}} \mathbf{D}(\mathbf{u}) : \mathbf{D}(\phi) \, dx + \int_{\mathcal{F}} [(\mathbf{u} \cdot \nabla)\mathbf{u}] \cdot \phi \, dx + \int_{\partial\mathcal{S}} A\mathbf{u} \cdot \phi \, d\mu = 0. \quad (8)$$

The following results hold:

**Theorem 1** *Given  $\nu > 0$  and  $\mathbf{u}_\infty \in \mathbb{R}^N$ , there exists at least one weak solution to System (1)-(5). Moreover, there exists a constant  $M > 0$  depending only on  $\nu$ ,  $\mathbf{u}_\infty$ ,  $\Omega$  and  $\mathcal{S}$ , such that every weak solution  $\mathbf{u} \in K$  satisfies the following estimate:*

$$\|\mathbf{u}\|_{\mathbf{H}^1(\mathcal{F})} \leq M. \quad (9)$$

**Theorem 2** *There exists a constant  $C = C(\Omega, \mathcal{S}) > 0$  such that, for all  $\mathbf{u}_\infty \in \mathbb{R}^N$ , there exists  $\nu_0 > 0$  satisfying*

$$\nu_0 \leq C(|\mathbf{u}_\infty| + |\mathbf{u}_\infty|^2)^{1/2} \quad (10)$$

*and such that for all  $\nu > \nu_0$ , System (1)-(5) has a unique weak solution, which satisfies the following estimate:*

$$\|\mathbf{u}\|_{\mathbf{H}^1(\mathcal{F})} \leq C'(\nu + |\mathbf{u}_\infty|), \quad (11)$$

*where  $C'$  is a positive constant depending only on  $\Omega, \mathcal{S}$ .*

Now we consider System (1)-(3) with Navier's boundary conditions (6)-(7) on  $\partial\mathcal{S}$ . To define weak solutions to this system, we introduce

$$\mathbf{H}_\tau = \{\mathbf{v} \in \mathbf{H}^1(\mathcal{F}) \mid \operatorname{div} \mathbf{v} = 0 \text{ in } L^2(\mathcal{F}), \mathbf{v} \cdot \mathbf{n} = 0 \text{ on } \partial\mathcal{S}\},$$

and define, as above,

$$K_\tau = \{\mathbf{v} \in \mathbf{H}_\tau \mid \mathbf{v} = \mathbf{u}_\infty \text{ on } \partial\Omega\}$$

and

$$V_\tau = \{\phi \in \mathbf{H}_\tau \mid \phi = 0 \text{ on } \partial\Omega\}$$

We say that  $\mathbf{u} \in \mathbf{H}^1(\mathcal{F})$  is a weak solution to system (1)-(3) with boundary conditions (6)-(7) if  $\mathbf{u} \in K_\tau$  and for every  $\phi \in V_\tau$ ,

$$2\nu \int_{\mathcal{F}} \mathbf{D}(\mathbf{u}) : \mathbf{D}(\phi) \, dx + \int_{\mathcal{F}} [(\mathbf{u} \cdot \nabla)\mathbf{u}] \cdot \phi \, dx + \int_{\partial\mathcal{S}} \beta \mathbf{u} \cdot \phi \, d\mathcal{H}^{N-1} = 0. \quad (12)$$

Let  $\mathbf{u}_\infty \in \mathbb{R}^N$  and  $\nu_0 > 0$  as in Theorem 2. We fix  $\nu > \nu_0$ , so that for every  $\beta \in L^2(\partial\mathcal{S})$ , any weak solution to system (1)-(3), (6)-(7), is unique. To each friction coefficient  $\beta \in L^2(\partial\mathcal{S})$ , we associate a drag  $T(\beta)$  defined by

$$T(\beta) = 2\nu \int_{\mathcal{F}} |\mathbf{D}(\mathbf{u})|^2 \, dx + \int_{\partial\mathcal{S}} \beta |\mathbf{u}|^2 \, d\mathcal{H}^{N-1},$$

where  $\mathbf{u} \in K_\tau$  satisfies (12).

To address the differentiability of the drag with respect to  $\beta$ , we introduce the following open subset of  $L^2(\partial\mathcal{S})$

$$\mathcal{O} = \{\beta \in L^2(\partial\mathcal{S}) \mid \beta > 0 \text{ q.e. on } \partial\mathcal{S}\},$$

and define the mapping

$$\begin{aligned} R: \mathcal{O} &\rightarrow \mathbf{H}_\tau \\ \beta &\mapsto \mathbf{u}, \end{aligned}$$

where  $\mathbf{u} \in K_\tau$  satisfies (12). The following result holds:

**Theorem 3** *The mapping*

$$\beta \in \mathcal{O} \rightarrow R(\beta) \in \mathbf{H}_\tau$$

*is differentiable, and for every  $\beta \in \mathcal{O}$ , its derivative  $DR(\beta) \in \mathcal{L}(L^2(\partial\mathcal{S}), \mathbf{H}_\tau)$  at point  $\beta$  is defined by*

$$DR(\beta)h = \mathbf{w} \quad \forall h \in L^2(\partial\mathcal{S}),$$

*where  $\mathbf{w}$  is the unique solution to the system:*

$$\left\{ \begin{array}{ll} -\operatorname{div}(\sigma(\mathbf{w}, p)) + (\mathbf{w} \cdot \nabla)\mathbf{u} + (\mathbf{u} \cdot \nabla)\mathbf{w} = 0 & \text{in } \mathcal{F}, \\ \operatorname{div} \mathbf{w} = 0 & \text{in } \mathcal{F}, \\ \mathbf{w} = 0 & \text{on } \partial\Omega, \\ \mathbf{w} \cdot \mathbf{n} = 0 & \text{on } \partial\mathcal{S}, \\ 2\nu [\mathbf{D}(\mathbf{w})\mathbf{n}]_{tan} + \beta\mathbf{w} = -h\mathbf{u} & \text{on } \partial\mathcal{S}, \end{array} \right. \quad (13)$$

*in the following sense:  $\mathbf{w} \in V_\tau$  and for every  $\phi \in V_\tau$ ,*

$$2\nu \int_{\mathcal{F}} \mathbf{D}(\mathbf{w}) : \mathbf{D}(\phi) \, dx + \int_{\mathcal{F}} [(\mathbf{w} \cdot \nabla)\mathbf{u} + (\mathbf{u} \cdot \nabla)\mathbf{w}] \cdot \phi \, dx + \int_{\partial\mathcal{S}} \beta\mathbf{w} \cdot \phi \, d\mathcal{H}^{N-1} = - \int_{\partial\mathcal{S}} h\mathbf{u} \cdot \phi \, d\mathcal{H}^{N-1}. \quad (14)$$

The following corollary has interesting applications to the drag optimization problem.

**Corollary 4** *The mapping*

$$\beta \in \mathcal{O} \rightarrow T(\beta) \in \mathbb{R}$$

*is differentiable, and for every  $\beta \in \mathcal{O}$ , its derivative  $DT(\beta) \in \mathcal{L}(L^2(\partial\mathcal{S}), \mathbb{R})$  is defined by*

$$DT(\beta)h = 2 \left( 2\nu \int_{\mathcal{F}} D(\mathbf{u}) : D(\mathbf{w}) \, dx + \int_{\partial\mathcal{S}} \beta\mathbf{u} \cdot \mathbf{w} \right) + \int_{\partial\mathcal{S}} h|\mathbf{u}|^2 \, d\mathcal{H}^{N-1} \quad \forall h \in L^2(\partial\mathcal{S})$$

*where  $\mathbf{u} = R(\beta)$  and  $\mathbf{w} = DR(\beta)h$  is the solution to system (13). For every  $\beta$ , the gradient of  $T$  at  $\beta$  with respect to the  $L^2(\partial\mathcal{S})$  topology is given by the following formula:*

$$\nabla T(\beta) = (\mathbf{u} + \psi) \cdot \mathbf{u} \quad \text{q.e. on } \partial\mathcal{S}, \quad (15)$$

where  $\mathbf{u} = R(\beta)$  and  $\psi$  is the unique solution to

$$\begin{cases} -\operatorname{div}(\sigma(\psi, p)) + (\nabla \mathbf{u})^T \psi - (\mathbf{u} \cdot \nabla) \psi = 2(\mathbf{u} \cdot \nabla) \mathbf{u} \text{ in } \mathcal{F}, \\ \operatorname{div} \psi = 0 \text{ in } \mathcal{F}, \\ \psi = 0 \text{ on } \partial\Omega, \\ \psi \cdot \mathbf{n} = 0 \text{ on } \partial\mathcal{S}, \\ 2\nu [\mathbf{D}(\psi) \mathbf{n}]_{\tan} + \beta \psi = 0 \text{ on } \partial\mathcal{S}, \end{cases} \quad (16)$$

in the sense that  $\psi \in V_\tau$  and for every  $\phi \in V_\tau$ ,

$$2\nu \int_{\mathcal{F}} \mathbf{D}(\psi) : \mathbf{D}(\phi) \, dx + \int_{\mathcal{F}} [(\nabla \mathbf{u})^T \psi - (\mathbf{u} \cdot \nabla) \psi] \cdot \phi \, dx + \int_{\partial\mathcal{S}} \beta \psi \cdot \phi \, d\mathcal{H}^{N-1} = 2 \int_{\mathcal{F}} [(\mathbf{u} \cdot \nabla) \mathbf{u}] \cdot \phi \, dx. \quad (17)$$

### 3 Proof of Theorems 1 and 2

In order to prove Theorems 1 and 2, we follow the standard steps from the theory of stationary Navier-Stokes equations, which we adapt to the framework of friction-driven boundary conditions. These arguments are developed in [9], for instance.

We define the bilinear form  $a_0 : \mathbf{H}^1(\mathcal{F}) \times \mathbf{H}^1(\mathcal{F}) \rightarrow \mathbb{R}$  by

$$a_0(\mathbf{v}_1, \mathbf{v}_2) = 2\nu \int_{\mathcal{F}} \mathbf{D}(\mathbf{v}_1) : \mathbf{D}(\mathbf{v}_2) \, dx + \int_{\partial\mathcal{S}} A \mathbf{v}_1 \cdot \mathbf{v}_2 \, d\mu \quad \forall (\mathbf{v}_1, \mathbf{v}_2) \in \mathbf{H}^1(\mathcal{F}) \times \mathbf{H}^1(\mathcal{F}), \quad (18)$$

and the trilinear form  $a_1 : \mathbf{H}^1(\mathcal{F}) \times \mathbf{H}^1(\mathcal{F}) \times \mathbf{H}^1(\mathcal{F}) \rightarrow \mathbb{R}$  by

$$a_1(\mathbf{v}, \mathbf{v}_1, \mathbf{v}_2) = \int_{\mathcal{F}} [(\mathbf{v} \cdot \nabla) \mathbf{v}_1] \cdot \mathbf{v}_2 \, dx \quad \forall (\mathbf{v}, \mathbf{v}_1, \mathbf{v}_2) \in \mathbf{H}^1(\mathcal{F}) \times \mathbf{H}^1(\mathcal{F}) \times \mathbf{H}^1(\mathcal{F}). \quad (19)$$

Let  $\mathbf{u}_0 \in K$  such that

$$\|\mathbf{u}_0\|_{\mathbf{H}^1(\mathcal{F})} \leq C_0 |\mathbf{u}_\infty| \quad (20)$$

where  $C_0 > 0$  depends only on  $\mathcal{F}$ . The existence of  $C_0$  and  $\mathbf{u}_0 \in K$  satisfying (20) was proved by Bogovskiĭ [3]. Setting  $\tilde{\mathbf{u}} = \mathbf{u} - \mathbf{u}_0$ , we get that a function  $\mathbf{u} \in K$  satisfies (12) if and only if  $\tilde{\mathbf{u}} \in V$  and satisfies the following identity:

$$\begin{aligned} \forall \phi \in V \quad a_0(\tilde{\mathbf{u}}, \phi) + a_1(\mathbf{u}_0, \tilde{\mathbf{u}}, \phi) + a_1(\tilde{\mathbf{u}}, \mathbf{u}_0, \phi) + a_1(\tilde{\mathbf{u}}, \tilde{\mathbf{u}}, \phi) \\ = -a_0(\mathbf{u}_0, \phi) - a_1(\mathbf{u}_0, \mathbf{u}_0, \phi). \end{aligned} \quad (21)$$

We define the mapping  $a : \mathbf{H}^1(\mathcal{F}) \times \mathbf{H}^1(\mathcal{F}) \times \mathbf{H}^1(\mathcal{F}) \rightarrow \mathbb{R}$  by

$$\begin{aligned} a(\mathbf{v}, \mathbf{v}_1, \mathbf{v}_2) = a_0(\mathbf{v}_1, \mathbf{v}_2) + a_1(\mathbf{u}_0, \mathbf{v}_1, \mathbf{v}_2) + a_1(\mathbf{v}_1, \mathbf{u}_0, \mathbf{v}_2) + a_1(\mathbf{v}, \mathbf{v}_1, \mathbf{v}_2) \\ \forall (\mathbf{v}, \mathbf{v}_1, \mathbf{v}_2) \in \mathbf{H}^1(\mathcal{F}) \times \mathbf{H}^1(\mathcal{F}) \times \mathbf{H}^1(\mathcal{F}), \end{aligned} \quad (22)$$

and the linear form  $\mathbf{f} = \mathbf{f}(\mathbf{u}_0) \in V'$  by

$$\langle \mathbf{f}, \phi \rangle_{V' \times V} = -a_0(\mathbf{u}_0, \phi) - a_1(\mathbf{u}_0, \mathbf{u}_0, \phi) \quad \forall \phi \in V.$$

The problem we consider is equivalent to the following : find  $\mathbf{u}_0 \in K$  and  $\tilde{\mathbf{u}} \in V$  such that

$$a(\tilde{\mathbf{u}}, \tilde{\mathbf{u}}, \phi) = \langle \mathbf{f}(\mathbf{u}_0), \phi \rangle_{V' \times V} \quad \forall \phi \in V. \quad (23)$$

We recall the following results from [9].

**Theorem 5** *Assume that the following hypotheses hold:*

(i) *for all  $\mathbf{v} \in \mathbf{H}^1(\mathcal{F})$ , the mapping  $(\mathbf{v}_1, \mathbf{v}_2) \rightarrow a(\mathbf{v}, \mathbf{v}_1, \mathbf{v}_2)$  is continuous on  $\mathbf{H}^1(\mathcal{F}) \times \mathbf{H}^1(\mathcal{F})$ ;*

(ii) *there exists a constant  $\alpha > 0$  such that*

$$a(\phi, \phi, \phi) \geq \alpha \|\phi\|_{\mathbf{H}^1(\mathcal{F})}^2 \quad \forall \phi \in V; \quad (24)$$

(iii) *the space  $V$  is separable and, for all  $\phi \in V$ , the mapping*

$$\mathbf{v} \rightarrow a(\mathbf{v}, \mathbf{v}, \phi)$$

*is sequentially weakly continuous on  $V$ , i.e.,*

$$\text{weak } \lim_{m \rightarrow \infty} \mathbf{v}_m = \mathbf{v} \text{ in } V \text{ implies } \lim_{m \rightarrow \infty} a(\mathbf{v}_m, \mathbf{v}_m, \phi) = a(\mathbf{v}, \mathbf{v}, \phi) \quad \forall \phi \in V. \quad (25)$$

*Then, problem (23) admits at least one solution.*

**Theorem 6** *Moreover, assume that*

(iv) *the bilinear form  $a(\mathbf{v}, \cdot, \cdot)$  is uniformly  $V$ -elliptic with respect to  $\mathbf{v}$ , i.e., there exists a constant  $\alpha > 0$  such that*

$$a(\mathbf{v}, \phi, \phi) \geq \alpha \|\phi\|_{\mathbf{H}^1(\mathcal{F})}^2 \quad \forall (\mathbf{v}, \phi) \in V \times V; \quad (26)$$

(v) *there exists a continuous and monotonically increasing function  $L : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  such that for all  $\eta > 0$*

$$\begin{aligned} |a(\mathbf{v}_1, \mathbf{v}_1, \mathbf{v}_2) - a(\mathbf{v}_2, \mathbf{v}_1, \mathbf{v}_2)| &\leq L(\eta) \|\mathbf{v}_1\|_{\mathbf{H}^1(\mathcal{F})} \|\mathbf{v}_2\|_{\mathbf{H}^1(\mathcal{F})} \|\mathbf{v}_1 - \mathbf{v}_2\|_{\mathbf{H}^1(\mathcal{F})} \\ \forall \mathbf{v}_1, \mathbf{v}_2 \in V, \quad \forall \mathbf{v}_1, \mathbf{v}_2 \in S_\eta &= \{\mathbf{v} \in V \mid \|\mathbf{v}\|_{\mathbf{H}^1(\mathcal{F})} \leq \eta\}. \end{aligned} \quad (27)$$

*Then, under the condition*

$$[\|\mathbf{f}\|_{V'} / (\alpha^2)] L(\|\mathbf{f}\|_{V'} / \alpha) < 1, \quad (28)$$

*the solution  $\tilde{\mathbf{u}} \in V$  to problem (23) is unique, and satisfies the following estimate:*

$$\|\tilde{\mathbf{u}}\|_{\mathbf{H}^1(\mathcal{F})} \leq \|\mathbf{f}\|_{V'} / \alpha. \quad (29)$$



**Proof of Theorem 1.** Let us fix  $\mathbf{u}_\infty \in \mathbb{R}^N$  and  $\nu > 0$ . We need to prove that the mapping  $a$  defined by (22) satisfies the assumptions of Theorem 5.

By Sobolev imbedding theorem (see the standard monograph by Adams [1]),  $\mathbf{H}^1(\mathcal{F})$  is continuously imbedded in  $L^4(\mathcal{F}, \mathbb{R}^N)$  for  $N = 2, 3$ . Therefore, by Hölder inequality, the mapping  $a_1$  defined by (19) is continuous. Since  $a_0$  defined by (18) is continuous, we obtain the continuity of  $a$ , and consequently, condition (i) from Theorem 5 holds.

To prove (ii), we prove the stronger condition (iv). We use the following Korn inequality, which is proved for instance in [4] (see the proof of Lemma VII.3.5, p. 329):

**Lemma 7** *There exists  $C_1 > 0$  such that for all  $\phi \in V$ ,*

$$\|\nabla\phi\|_{L^2(\mathcal{F}, \mathbb{R}^{N \times N})} \leq C_1 \|D(\phi)\|_{L^2(\mathcal{F}, \mathbb{R}^{N \times N})}.$$

Consequently, by Poincaré inequality, there exists  $C_2 > 0$  such that for all  $\phi \in V$ ,

$$\frac{1}{C_2} \|\phi\|_{\mathbf{H}^1(\mathcal{F})}^2 \leq 2 \|D(\phi)\|_{L^2(\mathcal{F}, \mathbb{R}^{N \times N})}^2. \quad (30)$$

Let  $\mathbf{v}, \phi \in V$ . We observe that

$$a(\mathbf{v}, \phi, \phi) = a_0(\phi, \phi) + a_1(\phi, \mathbf{u}_0, \phi).$$

Indeed,  $a_1(\mathbf{u}_0, \phi, \phi) = 0$  and  $a_1(\mathbf{v}, \phi, \phi) = 0$ , which comes from the following computation:

$$\begin{aligned} a_1(\mathbf{u}_0, \phi, \phi) &= \frac{1}{2} \int_{\mathcal{F}} (\mathbf{u}_0 \cdot \nabla)(|\phi|^2) \, dx \\ &= \frac{1}{2} \left( \int_{\partial\mathcal{S}} |\phi|^2 \mathbf{u}_0 \cdot \mathbf{n} \, d\mathcal{H}^{N-1} + \int_{\partial\Omega} |\phi|^2 \mathbf{u}_0 \cdot \mathbf{n} \, d\mathcal{H}^{N-1} \right). \end{aligned} \quad (31)$$

Since  $\phi \in V$ ,  $\phi = 0$  on  $\partial\Omega$ , and since  $\mathbf{u}_0 \in K$ ,  $\mathbf{u}_0 \cdot \mathbf{n} = 0$  on  $\partial\mathcal{S}$ . Therefore, last boundary integrals vanish. Replacing  $\mathbf{u}_0$  by  $\mathbf{v}$  we obtain the desired conclusion.

For every  $\mathbf{u}_0 \in K$  satisfying (20), we note  $\rho(\mathbf{u}_0)$  the best positive constant such that

$$|a_1(\phi, \mathbf{u}_0, \phi)| \leq \rho(\mathbf{u}_0) \|\phi\|_{\mathbf{H}^1(\mathcal{F})}^2 \quad \forall \phi \in V, \quad (32)$$

We prove the following result:

**Lemma 8** *For every  $\delta > 0$ , there exists  $\mathbf{u}_0 \in K$  satisfying (20) and such that  $\rho(\mathbf{u}_0) \leq \delta$ .*

**Proof.** We consider a bounded extension operator

$$P : \mathbf{H}^1(\mathcal{F}) \rightarrow H_0^1(\Omega, \mathbb{R}^N).$$

Let  $\delta > 0$ . By the proof of Lemma 2.3 in [9], chapter IV, there exists  $\tilde{\mathbf{u}}_0 \in H^1(\Omega, \mathbb{R}^N)$  such that

$$\begin{cases} \operatorname{div} \tilde{\mathbf{u}}_0 = 0 & \text{in } \Omega, \\ \tilde{\mathbf{u}}_0 = \mathbf{u}_\infty & \text{on } \partial\Omega, \\ \tilde{\mathbf{u}}_0 = 0 & \text{in a neighborhood of } \mathcal{S}, \end{cases}$$

and such that for every  $\varphi \in H_0^1(\Omega, \mathbb{R}^N)$ ,

$$\left| \int_{\Omega} [(\varphi \cdot \nabla) \tilde{\mathbf{u}}_0] \cdot \varphi \, dx \right| \leq \frac{\delta}{\|P\|} \|\varphi\|_{H^1(\Omega, \mathbb{R}^N)}. \quad (33)$$

Let  $\phi \in V$ . By inequality (33), we obtain

$$\begin{aligned} \int_{\Omega} [((P\phi) \cdot \nabla) \tilde{\mathbf{u}}_0] \cdot (P\phi) \, dx &\leq \frac{\delta}{\|P\|} \|P\phi\|_{H^1(\Omega, \mathbb{R}^N)} \\ &\leq \delta \|\phi\|_{\mathbf{H}^1(\mathcal{F})}. \end{aligned}$$

Since  $\tilde{\mathbf{u}}_0$  vanishes on  $\mathcal{S}$ , observing that

$$\int_{\Omega} [((P\phi) \cdot \nabla) \tilde{\mathbf{u}}_0] \cdot (P\phi) \, dx = \int_{\mathcal{F}} [(\phi \cdot \nabla) \tilde{\mathbf{u}}_0] \cdot \phi \, dx,$$

we obtain the desired result with  $\mathbf{u}_0 = (\tilde{\mathbf{u}}_0)|_{\mathcal{F}}$ .  $\square$

Let  $\mathbf{u}_0 \in K$  such that  $\rho(\mathbf{u}_0) < \nu/C_2$ , where  $C_2$  is defined by (30). By conditions (30) and (32), for every  $\mathbf{v}, \phi \in V$ ,

$$a(\mathbf{v}, \phi, \phi) \geq \left( \frac{\nu}{C_2} - \rho(\mathbf{u}_0) \right) \|\phi\|_{\mathbf{H}^1(\mathcal{F})}^2.$$

Consequently, condition (iv) holds with  $\alpha = \nu/C_2 - \rho(\mathbf{u}_0)$ .

To prove condition (25), we only need to consider the nonlinear term  $a_1(\mathbf{v}_m, \mathbf{v}_m, \phi)$ . Integrating by part and using the incompressibility condition, we obtain

$$a_1(\mathbf{v}_m, \mathbf{v}_m, \phi) = - \int_{\mathcal{F}} [(\mathbf{v}_m \cdot \nabla) \phi] \cdot \mathbf{v} \, dx + \int_{\partial \mathcal{S} \cup \partial \Omega} (\mathbf{v}_m \cdot \phi)(\mathbf{v}_m \cdot \mathbf{n}) \, d\mathcal{H}^{N-1}.$$

Since  $\phi, \mathbf{v}_m \in V$ , we conclude as above that

$$a_1(\mathbf{v}_m, \mathbf{v}_m, \phi) = - \int_{\mathcal{F}} [(\mathbf{v}_m \cdot \nabla) \phi] \cdot \mathbf{v}_m \, dx.$$

Since  $\mathbf{H}^1(\mathcal{F})$  is continuously imbedded in  $L^4(\mathcal{F}, \mathbb{R}^N)$ , we obtain

$$\lim_{m \rightarrow \infty} a_1(\mathbf{v}_m, \mathbf{v}_m, \phi) = a_1(\mathbf{v}, \mathbf{v}, \phi).$$

This proves condition (iii).

Now let  $\mathbf{u} \in K$  be a weak solution to System (1)-(5). We define  $\tilde{\mathbf{u}} = \mathbf{u} - \mathbf{u}_0$ . Applying relation (23) to  $\phi = \tilde{\mathbf{u}}$  and using inequality (26), we obtain

$$\|\tilde{\mathbf{u}}\|_{\mathbf{H}^1(\mathcal{F})} \leq \frac{1}{\alpha} \|\mathbf{f}(\mathbf{u}_0)\|_{V'}.$$

By the triangular inequality, estimate (9) holds with  $M = \|\mathbf{u}_0\|_{\mathbf{H}^1(\mathcal{F})} + 1/\alpha \|\mathbf{f}(\mathbf{u}_0)\|_{V'}$ . This concludes the proof of Theorem 1.  $\square$

**Proof of Theorem 2.** Let us fix  $\mathbf{u}_\infty \in \mathbb{R}^N$ .

To prove (v), we observe that for  $\mathbf{v}, \tilde{\mathbf{v}}, \mathbf{v}_1, \mathbf{v}_2 \in V$ ,

$$a(\mathbf{v}, \mathbf{v}_1, \mathbf{v}_2) - a(\tilde{\mathbf{v}}, \mathbf{v}_1, \mathbf{v}_2) = a_1(\mathbf{v} - \tilde{\mathbf{v}}, \mathbf{v}_1, \mathbf{v}_2).$$

Introducing

$$\mathcal{N} = \sup_{\mathbf{v}, \mathbf{v}_1, \mathbf{v}_2 \in V \setminus \{0\}} \frac{a_1(\mathbf{v}, \mathbf{v}_1, \mathbf{v}_2)}{\|\mathbf{v}\|_{\mathbf{H}^1(\mathcal{F})} \|\mathbf{v}_1\|_{\mathbf{H}^1(\mathcal{F})} \|\mathbf{v}_2\|_{\mathbf{H}^1(\mathcal{F})}}, \quad (34)$$

we obtain that (27) holds with  $L(\eta) = \mathcal{N}$  for all  $\eta > 0$ .

We define

$$\nu_0 = \inf \{C_2 [\rho(\mathbf{u}_0) + (\mathcal{N}\|\mathbf{f}(\mathbf{u}_0)\|_{V'})^{1/2}], \mathbf{u}_0 \in K \text{ satisfying (20)}\}.$$

Estimate (10) results from the existence of a constant  $C_3 > 0$ , such that for every  $\phi \in V$ ,

$$|\langle \mathbf{f}(\mathbf{u}_0), \phi \rangle_{V' \times V}| \leq C_3 \|\mathbf{u}_0\|_{\mathbf{H}^1(\mathcal{F})} (1 + \|\mathbf{u}_0\|_{\mathbf{H}^1(\mathcal{F})}) \|\phi\|_{\mathbf{H}^1(\mathcal{F})}.$$

As a result, by (20), there exists  $C_4 > 0$  such that

$$\|\mathbf{f}(\mathbf{u}_0)\|_{V'} \leq C_4 |\mathbf{u}_\infty| (1 + |\mathbf{u}_\infty|),$$

and consequently, inequality (35) implies that there exists a constant  $C_5 > 0$  such that

$$\nu_0^2 \leq C_5 (|\mathbf{u}_\infty| + |\mathbf{u}_\infty|^2).$$

This proves (10).

Let  $\nu > \nu_0$ . By definition of the infimum, there exists  $\mathbf{u}_0 \in K$  satisfying (20) and such that

$$\nu_0 \leq C_2 [\rho(\mathbf{u}_0) + (\mathcal{N}\|\mathbf{f}(\mathbf{u}_0)\|_{V'})^{1/2}] < \nu. \quad (35)$$

We fix  $\alpha = \nu/C_2 - \rho(\mathbf{u}_0)$ . By (35),  $\alpha > 0$  and by the same argument as above,  $\alpha$  satisfies (26). Now let  $\mathbf{u} \in K$  be a weak solution to System (1)-(5) and let  $\tilde{\mathbf{u}} = \mathbf{u} - \mathbf{u}_0$ . Condition (28) reads

$$\mathcal{N}\|\mathbf{f}(\mathbf{u}_0)\|_{V'} < \alpha^2$$

which is equivalent to

$$(\mathcal{N}\|\mathbf{f}(\mathbf{u}_0)\|_{V'})^{1/2} < \nu/C_2 - \rho(\mathbf{u}_0). \quad (36)$$

By (35), last estimate holds. By Theorem 6,  $\tilde{\mathbf{u}} \in V$  is the unique solution to (23). Using (10) and (36), we obtain the following estimate:

$$\|\tilde{\mathbf{u}}\|_{\mathbf{H}^1(\mathcal{F})} < \frac{1}{\mathcal{N}} \left( \frac{\nu}{C_2} - \rho(\mathbf{u}_0) \right). \quad (37)$$

Consequently, by (20) and the triangular inequality,

$$\|\mathbf{u}\|_{\mathbf{H}^1(\mathcal{F})} \leq C_0 |\mathbf{u}_\infty| + \frac{1}{\mathcal{N}C_2} \nu.$$

This proves (11).

By taking the infimum over all admissible functions  $\mathbf{u}_0$ , for  $\nu > \nu_0$ , we obtain the uniqueness of the function  $\tilde{\mathbf{u}} \in K$  satisfying (21). Therefore, System (1)-(5) admits one unique weak solution  $\mathbf{u}_0 + \tilde{\mathbf{u}} \in \mathbf{H}^1(\mathcal{F})$ .  $\square$

## 4 Proof of Theorem 3

In this section, we assume that  $\mathcal{V}(x)$  is equal to the tangent hyperplane to  $\partial\mathcal{S}$  on q.e.  $x$  on the boundary,  $A \equiv \text{Id}$  and  $\mu$  is absolutely continuous with respect to the surface measure on  $\partial\mathcal{S}$  with a density  $\beta \in L^2(\partial\mathcal{S})$ .

Let  $\mathbf{u}_\infty \in \mathbb{R}^N$ . Let  $\nu_0 > 0$  as in Theorem 2, and  $\nu > \nu_0$ . Let  $\mathbf{u}_0 \in K$  satisfying (20) and such that

$$\nu_0 \leq C_2 [\rho(\mathbf{u}_0) + (\mathcal{N}\|\mathbf{f}(\mathbf{u}_0)\|_{V'_\tau})^{1/2}] < \nu. \quad (38)$$

We use the mappings introduced in Section 3. We define  $\mathbf{g} = \mathbf{g}(\mathbf{u}_0) \in V'_\tau$  by

$$\langle \mathbf{g}, \phi \rangle_{V'_\tau \times V_\tau} = -a_0(\mathbf{u}_0, \phi) - a_1(\mathbf{u}_0, \mathbf{u}_0, \phi) \quad \forall \phi \in V_\tau.$$

The problem we consider is equivalent to the following: find  $\mathbf{u}_0 \in K_\tau$  and  $\tilde{\mathbf{u}} \in V_\tau$  such that

$$a(\tilde{\mathbf{u}}, \tilde{\mathbf{u}}, \phi) = \langle \mathbf{g}(\mathbf{u}_0), \phi \rangle_{V'_\tau \times V_\tau} \quad \forall \phi \in V_\tau. \quad (39)$$

**Proof of Theorem 3.** The proof is based on the implicit function theorem. We will prove that the mapping

$$\beta \in \mathcal{O} \rightarrow \tilde{\mathbf{u}} \in V_\tau$$

where  $\tilde{\mathbf{u}}$  satisfies (21), is differentiable. We introduce the open subset of  $V_\tau$ :

$$V_0 = \left\{ \xi \in V_\tau \mid \|\xi\|_{\mathbf{H}^1(\mathcal{F})} < \frac{1}{\mathcal{N}} \left( \frac{\nu}{C_2} - \rho(\mathbf{u}_0) \right) \right\}$$

and define the mapping

$$F : \mathcal{O} \times V_0 \rightarrow V'_\tau$$

by

$$\begin{aligned} \forall \phi \in V_\tau \quad \langle F(\beta, \mathbf{v}), \phi \rangle_{V'_\tau \times V_\tau} &= a_0(\mathbf{v}, \phi) + a_1(\mathbf{u}_0, \mathbf{v}, \phi) + a_1(\mathbf{v}, \mathbf{u}_0, \phi) + a_1(\mathbf{v}, \mathbf{v}, \phi) \\ &\quad + a_0(\mathbf{u}_0, \phi) + a_1(\mathbf{u}_0, \mathbf{u}_0, \phi). \end{aligned} \quad (40)$$

The mapping  $F$  is well defined and  $F$  is of class  $\mathcal{C}^\infty$ . By the proof of Theorems 1 and 2, for every  $\beta \in \mathcal{O}$  there exists one unique  $\tilde{\mathbf{u}} \in V_0$  satisfying (21). We need to prove that for every  $(\beta, \mathbf{v}) \in \mathcal{O} \times V_0$ ,  $D_{\mathbf{v}}F(\beta, \mathbf{v}) \in \mathcal{L}(V_\tau, V'_\tau)$  is invertible with continuous inverse.

For every  $(\beta, \mathbf{v}) \in \mathcal{O} \times V_0$ ,  $D_{\mathbf{v}}F(\beta, \mathbf{v}) \in \mathcal{L}(V_\tau, V'_\tau)$  is defined by

$$\begin{aligned} \forall \xi, \phi \in V_\tau \quad \langle D_{\mathbf{v}}F(\beta, \mathbf{v})\xi, \phi \rangle_{V'_\tau \times V_\tau} &= a_0(\xi, \phi) + a_1(\mathbf{u}_0, \xi, \phi) + a_1(\xi, \mathbf{u}_0, \phi) \\ &\quad + a_1(\xi, \mathbf{v}, \phi) + a_1(\mathbf{v}, \xi, \phi). \end{aligned}$$

We need to prove that for every  $(\beta, \mathbf{v}) \in \mathcal{O} \times V_0$ , there exists a constant  $C > 0$  such that for every  $\mathbf{h} \in V'_\tau$ , there exists one unique  $\xi \in V_\tau$  such that

$$D_{\mathbf{v}}F(\beta, \mathbf{v})\xi = \mathbf{h}, \quad (41)$$

with the estimate

$$\|\xi\|_{\mathbf{H}^1(\mathcal{F})} \leq C\|\mathbf{h}\|_{V'_\tau}. \quad (42)$$

Fix  $(\beta, \mathbf{v}) \in \mathcal{O} \times V_0$ . We will prove that the bilinear form associated to  $D_{\mathbf{v}}F(\beta, \mathbf{v})$  satisfies the assumptions of Lax-Milgram theorem. The continuity results from the continuity of  $a_0$  and  $a_1$ , and the coercivity is a consequence of the definition of  $V_0$ . Indeed, by computation (31), for all  $\phi \in V_\tau$ ,

$$\langle D_{\mathbf{v}}F(\beta, \mathbf{v})\phi, \phi \rangle_{V'_\tau \times V_\tau} = a_0(\phi, \phi) + a_1(\phi, \mathbf{u}_0, \phi) + a_1(\phi, \mathbf{v}, \phi).$$

Consequently, using property (30) and definitions (32) and (34), we obtain for all  $\phi \in V_\tau$ ,

$$\langle D_{\mathbf{v}}F(\beta, \mathbf{v})\phi, \phi \rangle_{V'_\tau \times V_\tau} \geq \left( \frac{\nu}{C_2} - \rho(\mathbf{u}_0) - \mathcal{N}\|v\|_{\mathbf{H}^1(\mathcal{F})} \right) \|\phi\|_{\mathbf{H}^1(\mathcal{F})}^2.$$

Since  $\mathbf{v} \in V_0$ ,

$$\frac{\nu}{C_2} - \rho(\mathbf{u}_0) - \mathcal{N}\|v\|_{\mathbf{H}^1(\mathcal{F})} > 0.$$

This proves that the mapping

$$\beta \in \mathcal{O} \rightarrow \tilde{\mathbf{u}} \in V_\tau$$

where  $\tilde{\mathbf{u}}$  satisfies (39), is differentiable. Consequently, since for all  $\beta \in \mathcal{O}$ ,  $R(\beta) = \mathbf{u}_0 + \tilde{\mathbf{u}}$ ,  $R$  is differentiable on  $\mathcal{O}$ . Moreover, for every  $\beta \in \mathcal{O}$ ,  $DR(\beta) \in \mathcal{L}(L^2(\partial\mathcal{S}), \mathbf{H})$  is defined by

$$DR(\beta) = -[D_v F(\beta, R(\beta) - \mathbf{u}_0)]^{-1} D_\beta F(\beta, R(\beta) - \mathbf{u}_0).$$

Since for every  $(\beta, \mathbf{v}) \in \mathcal{O} \times V_0$ ,  $D_\beta F(\beta, \mathbf{v}) \in \mathcal{L}(L^2(\partial\mathcal{S}), V'_\tau)$  is defined by

$$\forall h \in L^2(\partial\mathcal{S}), \forall \phi \in V_\tau \quad \langle D_\beta F(\beta, \mathbf{v}), h \rangle = \int_{\partial\mathcal{S}} h(\mathbf{v} + \mathbf{u}_0) \cdot \phi \, d\mathcal{H}^{N-1},$$

defining  $\mathbf{w} = DR(\beta)h$  for  $h \in L^2(\partial\mathcal{S})$ , we obtain that  $\mathbf{w} \in V_\tau$  satisfies the following identity:

$$\begin{aligned} \forall \phi \in V_\tau \quad & a_0(\mathbf{w}, \phi) + a_1(\mathbf{u}_0, \mathbf{w}, \phi) + a_1(\mathbf{w}, \mathbf{u}_0, \phi) + a_1(\mathbf{w}, R(\beta) - \mathbf{u}_0, \phi) \\ & + a_1(R(\beta) - \mathbf{u}_0, \mathbf{w}, \phi) = - \int_{\partial\mathcal{S}} hR(\beta) \cdot \phi \, d\mathcal{H}^{N-1}. \end{aligned}$$

Using the trilinearity of  $a_1$ , we obtain (14). □

**Proof of Corollary 4.** Noticing that for every  $\phi \in V_\tau$ ,

$$[(\nabla \mathbf{u})^T \psi] \cdot \phi = [(\phi \cdot \nabla) \mathbf{u}] \cdot \psi$$

and

$$- \int_{\mathcal{F}} [(\mathbf{u} \cdot \nabla) \psi] \cdot \phi \, dx = \int_{\mathcal{F}} [(\mathbf{u} \cdot \nabla) \phi] \cdot \psi \, dx,$$

(17) can be written in the following form:

$$\begin{aligned} \forall \phi \in V_\tau \quad & 2\nu \int_{\mathcal{F}} \mathbf{D}(\psi) : \mathbf{D}(\phi) \, dx + \int_{\mathcal{F}} \psi \cdot [(\phi \cdot \nabla) \mathbf{u} + (\mathbf{u} \cdot \nabla) \phi] \, dx + \int_{\partial\mathcal{S}} \beta \psi \cdot \phi \, d\mathcal{H}^{N-1} \\ & = 2 \int_{\mathcal{F}} [(\mathbf{u} \cdot \nabla) \mathbf{u}] \cdot \phi \, dx. \end{aligned} \quad (43)$$

Let us fix  $\beta \in \mathcal{O}$  and  $h \in L^2(\partial\mathcal{S})$ . We note  $\mathbf{u} = R(\beta)$  and  $\mathbf{w} = DR(\beta)h$ . Taking  $\phi = \mathbf{w}$  in (12), we express  $DT(\beta)h$  as follows:

$$DT(\beta)h = -2 \int_{\mathcal{F}} [(\mathbf{u} \cdot \nabla)\mathbf{u}] \cdot \mathbf{w} \, dx + \int_{\partial\mathcal{S}} h|\mathbf{u}|^2 \, d\mathcal{H}^{N-1}. \quad (44)$$

Besides, taking  $\phi = \psi$  in (14) and  $\phi = \mathbf{w}$  in (43), we obtain

$$\begin{aligned} & 2\nu \int_{\mathcal{F}} \mathbf{D}(\mathbf{w}) : \mathbf{D}(\psi) \, dx + \int_{\mathcal{F}} [(\mathbf{w} \cdot \nabla)\mathbf{u} + (\mathbf{u} \cdot \nabla)\mathbf{w}] \cdot \psi \, dx + \int_{\partial\mathcal{S}} \beta \mathbf{w} \cdot \psi \, d\mathcal{H}^{N-1} \\ &= - \int_{\partial\mathcal{S}} h \mathbf{u} \cdot \psi \, d\mathcal{H}^{N-1}, \\ & 2\nu \int_{\mathcal{F}} \mathbf{D}(\psi) : \mathbf{D}(\mathbf{w}) \, dx + \int_{\mathcal{F}} \psi \cdot [(\mathbf{w} \cdot \nabla)\mathbf{u} + (\mathbf{u} \cdot \nabla)\mathbf{w}] \, dx + \int_{\partial\mathcal{S}} \beta \psi \cdot \mathbf{w} \, d\mathcal{H}^{N-1} \\ &= 2 \int_{\mathcal{F}} [(\mathbf{u} \cdot \nabla)\mathbf{u}] \cdot \mathbf{w} \, dx. \end{aligned}$$

Subtracting these equalities, we obtain

$$-2 \int_{\mathcal{F}} [(\mathbf{u} \cdot \nabla)\mathbf{u}] \cdot \mathbf{w} \, dx = \int_{\partial\mathcal{S}} h \mathbf{u} \cdot \psi \, d\mathcal{H}^{N-1}$$

and finally, by (44),

$$DT(\beta)h = \int_{\partial\mathcal{S}} [(\mathbf{u} + \psi) \cdot \mathbf{u}] h \, d\mathcal{H}^{N-1}.$$

This proves (15). □

**Remark 9** The coercivity of the bilinear form associated to the weak formulation (43) results from the following estimate (see the proof of Theorems 1 and 2):

$$\begin{aligned} \rho(\mathbf{u}) &\leq \rho(\mathbf{u}_0) + \rho(\tilde{\mathbf{u}}) \\ &\leq \rho(\mathbf{u}_0) + \mathcal{N} \|\tilde{\mathbf{u}}\|_{\mathbf{H}^1(\mathcal{F})} \\ &< \frac{\nu}{C_2}. \end{aligned}$$

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