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To cite this version:

Matthieu Bonnivard. The Drag in a Navier-Stokes Flow with Friction-Driven Boundary Conditions. 2011. hal-00565411

HAL Id: hal-00565411

https://hal.science/hal-00565411v1

Preprint submitted on 13 Feb 2011

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The Drag in a Navier-Stokes Flow with Friction-Driven Boundary Conditions

Matthieu Bonnivard

Abstract

We consider the drag of an obstacle in a Navier-Stokes flow, associated to the friction-driven boundary conditions introduced by Bucur, Feireisl and Nečasová [6]. These boundary conditions account for the asymptotic effect of rough boundaries on the solutions to the Navier-Stokes equations. Consequently, they appear as a natural control variable in order to reduce the drag using the effect of micro-rugosities. In this paper, we prove the existence of a drag associated to friction-driven boundary conditions, we give a uniqueness criterion and, in the particular case of Navier's friction law, we prove that the drag is differentiable with respect to the friction coefficient, and compute its gradient.

1 Introduction

In this paper, we are interested in the definition and the mathematical properties of the drag of an obstacle in a Navier-Stokes flow, associated to the so-called *friction-driven* boundary conditions on the solid-fluid interface [6].

Let $N \in \{2, 3\}$ be the space dimension. We define the obstacle $\mathcal{S}$ as a compact subset of $\mathbb{R}^N$, with a Lipschitz boundary. The friction-driven boundary conditions are determined by a triplet $\{\mu, A, \mathcal{V}\}$, where:

- μ is a *capacitary measure* concentrated on $\partial\mathcal{S}$,
- A is a *positive symmetric matrix function*,
- $\mathcal{V} = \{\mathcal{V}(x)\}_{x\in\partial\mathcal{S}}$ is a collection of vector spaces, where for q.e. $x \in \partial\mathcal{S}$, $\mathcal{V}(x)$ is a *subspace* of the *tangent hyperplane to* $\mathcal{S}$ at x.

Let $\mathbf{u}_\infty \in \mathbb{R}^N$ be the velocity of the fluid at infinity. To define the drag of the solid $\mathcal{S}$ in the direction $\mathbf{u}_\infty$, we (artificially) introduce an open subset $\Omega \subset \mathbb{R}^N$ such that $\mathcal{S} \subset \Omega$, and define the fluid domain

$$\mathcal{F} = \Omega \setminus \mathcal{S}.$$

We note $L^2_0(\mathcal{F})$ the subspace of $L^2(\mathcal{F})$ consisting of functions with mean value equal to zero. We consider the following problem: find $(\mathbf{u}, p) \in H^1(\mathcal{F}, \mathbb{R}^N) \times L^2_0(\mathcal{F})$ such that

$$-\mathrm{div}(\sigma(\mathbf{u}, p)) + (\mathbf{u} \cdot \nabla)\mathbf{u} = 0 \text{ in } \mathcal{F}, \tag{1}$$

$$\mathrm{div}\ \mathbf{u} = 0 \text{ in } \mathcal{F}, \tag{2}$$

$$\mathbf{u} = \mathbf{u}_\infty \text{ on } \partial\Omega, \tag{3}$$

$$\mathbf{u}(x) \in \mathcal{V}(x) \text{ for q.e. } x \in \partial\mathcal{S}, \tag{4}$$

$$[2\nu \mathbf{D}(\mathbf{u})\mathbf{n} + \mu A\mathbf{u}] \cdot \phi = 0 \text{ for any } \phi \in \mathcal{V}(x),\ x \in \partial\mathcal{S}, \tag{5}$$

where $\sigma(\mathbf{u}, p)$ is the stress tensor defined by

$$\sigma(\mathbf{u}, p) = 2\nu \mathbf{D}(\mathbf{u}) - p\mathrm{Id},$$

$\nu > 0$ being the viscosity of the fluid and $\mathbf{D}(\mathbf{u})$ the symmetric part of $\nabla\mathbf{u}$, defined by

$$\mathbf{D}(\mathbf{u}) = \frac{1}{2}\left((\nabla\mathbf{u}) + (\nabla\mathbf{u})^T\right).$$

In condition (5), $\mathbf{n}$ stands for the normal vector field to $\partial\mathcal{S}$ oriented towards the solid.

Condition (4) expresses that the flow is driven by the family of linear spaces $\{\mathcal{V}(x)\}_{x\in\partial\mathcal{S}}$. Above, $[2\nu\mathbf{D}(\mathbf{u})\mathbf{n} + \mu A\mathbf{u}] \cdot \phi = 0$ for $\phi \in \mathcal{V}(x)$ is a formal pointwise relation, which has to be understood globally on $\partial\mathcal{S}$. Precisely, condition (5) holds provided that

$$2\nu \int_{\partial\mathcal{S}} \mathbf{D}(\mathbf{u})\mathbf{n} \cdot \phi \ \mathrm{d}x + \int_{\partial\mathcal{S}} A\mathbf{u} \cdot \phi \ \mathrm{d}\mu = 0$$

for every $\phi \in H^1(\mathcal{F}, \mathbb{R}^N)$ such that $\phi(x) \in \mathcal{V}(x)$ for q.e. $x \in \partial S$. In particular, condition (5) is a friction condition, characterized by the measure μ and the matrix A.

Friction-driven boundary conditions have been obtained by Bucur, Feireisl and Nečasová in [6], as the general expression of the boundary conditions resulting from the rugosity effect on perfectly slippery walls. The rugosity effect has been investigated in [11, 10, 8, 2] and in the more recent papers [7, 6, 5]. It consists in approximating a given domain U by a sequence of geometric perturbations $(U_\varepsilon)_{\varepsilon>0}$, and studying the asymptotic behaviour of a family of vector fields $\mathbf{u}_\varepsilon \in H^1(U_\varepsilon, \mathbb{R}^N)$ in the neighborhood of the boundary of the limit domain U. In [6], the authors have considered a sequence $(\mathbf{u}_\varepsilon)_{\varepsilon>0}$ of weak solutions to the Navier-Stokes equations with perfect slip boundary conditions in equi-Lipschitz domains $(U_\varepsilon)_{\varepsilon>0}$. They have proved the existence of a triplet $\{\mu, A, \mathcal{V}\}$, such that, up to a subsequence, $(\mathbf{u}_\varepsilon)_{\varepsilon>0}$ converges weakly to a function $\mathbf{u}$ solution to the Navier-Stokes equations with friction-driven boundary conditions.

This asymptotic process is an indirect proof of the existence of a solution to the Navier-Stokes equations with friction-driven boundary conditions, for *admissible* triplets $\{\mu, A, \mathcal{V}\}$, that is, triplets that can be effectively obtained by a sequence of geometric perturbations of the domain. However, since this result relies on abstract $\Gamma-$convergence tools, the class of admissible triplets cannot be explicitely described, at least for very general perturbations. The purpose of this paper is to prove that, in fact, problem (1)-(5) admits at least one weak solution (in a sense that will be precised below), for every $\nu > 0$, $\mathbf{u}_\infty \in \mathbb{R}^N$ and every triplet

$\{\mu, A, \mathcal{V}\}$. This result is stated in Theorem 1. Moreover, we prove that for a suitable choice of ν, $\mathbf{u}_\infty$ and $\mathcal{V}$, which does not depend on $\{\mu, A\}$, every weak solution to System (1)-(5) is unique (see Theorem 2).

In the second part of the paper, we study the differentiability of the drag function with respect to the friction part of the boundary condition. Let $\nu > 0$ and $\mathbf{u}_\infty \in \mathbb{R}$. We can associate to every triplet $\{\mu, A, \mathcal{V}\}$ and every weak solution $\mathbf{u}$ to System (1)-(5), a drag $\mathcal{T}(\mu, A, \mathcal{V}, \mathbf{u})$ defined by

$$\mathcal{T}(\mu, A, \mathcal{V}, \mathbf{u}) = 2\nu \int_{\mathcal{F}} |\mathbf{D}(\mathbf{u})|^2 \ \mathrm{d}x + \int_{\partial\mathcal{S}} A\mathbf{u} \cdot \mathbf{u} \ \mathrm{d}\mu.$$

In the case where $\mathbf{u}$ is unique, the drag depends only on the choice of the triplet $\{\mu, A, \mathcal{V}\}$, which appears as a natural control variable in drag minimization problems. As a first step in the study of the regularity of the drag with respect to $\{\mu, A, \mathcal{V}\}$, we consider Navier's boundary conditions on $\partial\mathcal{S}$, which are given by the non penetration condition

$$\mathbf{u} \cdot \mathbf{n} = 0 \text{ on } \partial\mathcal{S} \tag{6}$$

and by the friction law

$$2\nu \left[\mathbf{D}(\mathbf{u})\mathbf{n}\right]_{tan} + \beta\mathbf{u} = 0 \text{ on } \partial\mathcal{S}. \tag{7}$$

Here, $\mathbf{n}$ is the normal vector field to $\partial\mathcal{S}$ oriented towards the solid and $\beta \in L^2(\partial\mathcal{S})$ is a friction coefficient. For any vector field $\mathbf{v}$, defined on $\partial\mathcal{S}$, we note $[\mathbf{v}]_{tan}$ its tangential part, defined by

$$[\mathbf{v}]_{tan} = \mathbf{v} - (\mathbf{v} \cdot \mathbf{n})\mathbf{n}.$$

Boundary conditions (6)-(7) correspond to the case where $\mathcal{V}(x)$ is equal to the tangent hyperplane to $\partial\mathcal{S}$ on q.e. x on the boundary, $A \equiv \mathrm{Id}$ and μ is absolutely continuous with respect to the surface measure on $\partial\mathcal{S}$ with a density $\beta \in L^2(\partial\mathcal{S})$. We prove that the drag is differentiable with respect to β and compute the gradient of the drag in the $L^2(\partial\mathcal{S})$ topology (see Theorem 3 and Corollary 4).

The structure of the paper is the following. In Section 2, we introduce the functional framework associated with System (1)-(3) and boundary conditions (4)-(5) and (6)-(7), and state our main results. In Section 3, we recall some abstract material from [9] which we use in the proof of Theorems 1 and 2. Section 4 is devoted to the proof of Theorem 3 and Corollary 4.

2 Statement of the Main Results

We note

$$\mathbf{H}^1(\mathcal{F}) = H^1(\mathcal{F}, \mathbb{R}^N).$$

To define the weak solutions to System (1)-(5), we introduce the following subsets of $\mathbf{H}^1(\mathcal{F})$:

$$K = \{\mathbf{v} \in \mathbf{H}^1(\mathcal{F}) \mid \text{div } \mathbf{v} = 0 \text{ in } L^2(\mathcal{F}),\ \mathbf{v}(x) \in \mathcal{V}(x) \text{ for q.e. } x \in \partial\mathcal{S},\ \mathbf{v} = \mathbf{u}_\infty \text{ on } \partial\Omega\}$$

and

$$V = \{\phi \in \mathbf{H}^1(\mathcal{F}) \mid \text{div } \phi = 0 \text{ in } L^2(\mathcal{F}),\ \mathbf{v}(x) \in \mathcal{V}(x) \text{ for q.e. } x \in \partial\mathcal{S},\ \phi = 0 \text{ on } \partial\Omega\}.$$

Let us fix the family of linear spaces $\mathcal{V}$ such that V is non empty. In particular, V is a separable Hilbert space for the $\mathbf{H}^1$ scalar product. We note V' its dual space for the $\mathbf{H}^1$ norm. Let μ be a capacitary measure concentrated on $\partial\mathcal{S}$ and A a positive symmetric matrix function.

We say that $\mathbf{u} \in \mathbf{H}^1(\mathcal{F})$ is a weak solution to system (1)-(5) if $\mathbf{u} \in K$ and for every $\phi \in V$,

$$2\nu \int_{\mathcal{F}} \mathbf{D}(\mathbf{u}) : \mathbf{D}(\phi)\ \mathrm{d}x + \int_{\mathcal{F}} [(\mathbf{u}\cdot\nabla)\mathbf{u}]\cdot\phi\ \mathrm{d}x + \int_{\partial\mathcal{S}} A\mathbf{u}\cdot\phi\ \mathrm{d}\mu = 0. \tag{8}$$

The following results hold:

Theorem 1 *Given $\nu > 0$ and $\mathbf{u}_\infty \in \mathbb{R}^N$, there exists at least one weak solution to System (1)-(5). Moreover, there exists a constant $M > 0$ depending only on ν, $\mathbf{u}_\infty$, Ω and $\mathcal{S}$, such that every weak solution $\mathbf{u} \in K$ satisfies the following estimate:*

$$\|\mathbf{u}\|_{\mathbf{H}^1(\mathcal{F})} \leqslant M. \tag{9}$$

Theorem 2 *There exists a constant $C = C(\Omega, \mathcal{S}) > 0$ such that, for all $\mathbf{u}_\infty \in \mathbb{R}^N$, there exists $\nu_0 > 0$ satisfying*

$$\nu_0 \leqslant C(|\mathbf{u}_\infty| + |\mathbf{u}_\infty|^2)^{1/2} \tag{10}$$

and such that for all $\nu > \nu_0$, System (1)-(5) has a unique weak solution, which satisfies the following estimate:

$$\|\mathbf{u}\|_{\mathbf{H}^1(\mathcal{F})} \leqslant C'(\nu + |\mathbf{u}_\infty|), \tag{11}$$

where C' is a positive constant depending only on $\Omega, \mathcal{S}$.

Now we consider System (1)-(3) with Navier's boundary conditions (6)-(7) on $\partial\mathcal{S}$. To define weak solutions to this system, we introduce

$$\mathbf{H}_\tau = \{\mathbf{v} \in \mathbf{H}^1(\mathcal{F}) \mid \operatorname{div} \mathbf{v} = 0 \text{ in } L^2(\mathcal{F}),\ \mathbf{v}\cdot\mathbf{n} = 0 \text{ on } \partial\mathcal{S}\},$$

and define, as above,

$$K_\tau = \{\mathbf{v} \in \mathbf{H}_\tau \mid \mathbf{v} = \mathbf{u}_\infty \text{ on } \partial\Omega\}$$

and

$$V_\tau = \{\phi \in \mathbf{H}_\tau \mid \phi = 0 \text{ on } \partial\Omega\}$$

We say that $\mathbf{u} \in \mathbf{H}^1(\mathcal{F})$ is a weak solution to system (1)-(3) with boundary conditions (6)-(7) if $\mathbf{u} \in K_\tau$ and for every $\phi \in V_\tau$,

$$2\nu \int_{\mathcal{F}} \mathbf{D}(\mathbf{u}) : \mathbf{D}(\phi)\ \mathrm{d}x + \int_{\mathcal{F}} [(\mathbf{u}\cdot\nabla)\mathbf{u}]\cdot\phi\ \mathrm{d}x + \int_{\partial\mathcal{S}} \beta\mathbf{u}\cdot\phi\ \mathrm{d}\mathcal{H}^{N-1} = 0. \tag{12}$$

Let $\mathbf{u}_\infty \in \mathbb{R}^N$ and $\nu_0 > 0$ as in Theorem 2. We fix $\nu > \nu_0$, so that for every $\beta \in L^2(\partial\mathcal{S})$, any weak solution to system (1)-(3), (6)-(7), is unique. To each friction coefficient $\beta \in L^2(\partial\mathcal{S})$, we associate a drag $T(\beta)$ defined by

$$T(\beta) = 2\nu \int_{\mathcal{F}} |\mathbf{D}(\mathbf{u})|^2\ \mathrm{d}x + \int_{\partial\mathcal{S}} \beta|\mathbf{u}|^2\ \mathrm{d}\mathcal{H}^{N-1},$$

where $\mathbf{u} \in K_\tau$ satisfies (12).

To address the differentiability of the drag with respect to β, we introduce the following open subset of $L^2(\partial\mathcal{S})$

$$\mathcal{O} = \{\beta \in L^2(\partial\mathcal{S}) \mid \beta > 0 \text{ q.e. on } \partial\mathcal{S}\},$$

and define the mapping

$$\begin{array}{rccc} R: & \mathcal{O} & \to & \mathbf{H}_\tau \\ & \beta & \mapsto & \mathbf{u}, \end{array}$$

where $\mathbf{u} \in K_\tau$ satisfies (12). The following result holds:

Theorem 3 *The mapping*

$$\beta \in \mathcal{O} \to R(\beta) \in \mathbf{H}_\tau$$

is differentiable, and for every $\beta \in \mathcal{O}$, its derivative $DR(\beta) \in \mathcal{L}(L^2(\partial\mathcal{S}), \mathbf{H}_\tau)$ at point β is defined by

$$DR(\beta)h = \mathbf{w} \qquad \forall h \in L^2(\partial\mathcal{S}),$$

where $\mathbf{w}$ is the unique solution to the system:

$$\left\{ \begin{array}{rl} -\mathrm{div}(\sigma(\mathbf{w}, p)) + (\mathbf{w}\cdot\nabla)\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{w} = 0 & \textit{in } \mathcal{F}, \\ \mathrm{div}\ \mathbf{w} = 0 & \textit{in } \mathcal{F}, \\ \mathbf{w} = 0 & \textit{on } \partial\Omega, \\ \mathbf{w}\cdot\mathbf{n} = 0 & \textit{on } \partial\mathcal{S}, \\ 2\nu\,[\mathbf{D}(\mathbf{w})\mathbf{n}]_{tan} + \beta\mathbf{w} = -h\mathbf{u} & \textit{on } \partial\mathcal{S}, \end{array} \right. \tag{13}$$

in the following sense: $\mathbf{w} \in V_\tau$ and for every $\phi \in V_\tau$,

$$2\nu \int_{\mathcal{F}} \mathbf{D}(\mathbf{w}) : \mathbf{D}(\phi)\ \mathrm{d}x + \int_{\mathcal{F}} [(\mathbf{w}\cdot\nabla)\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{w}]\cdot\phi\ \mathrm{d}x + \int_{\partial\mathcal{S}} \beta\mathbf{w}\cdot\phi\ \mathrm{d}\mathcal{H}^{N-1} = -\int_{\partial\mathcal{S}} h\mathbf{u}\cdot\phi\ \mathrm{d}\mathcal{H}^{N-1}. \tag{14}$$

The following corollary has interesting applications to the drag optimization problem.

Corollary 4 *The mapping*

$$\beta \in \mathcal{O} \to T(\beta) \in \mathbb{R}$$

is differentiable, and for every $\beta \in \mathcal{O}$, its derivative $DT(\beta) \in \mathcal{L}(L^2(\partial\mathcal{S}), \mathbb{R})$ is defined by

$$DT(\beta)h = 2\left(2\nu \int_{\mathcal{F}} D(\mathbf{u}) : D(\mathbf{w})\ \mathrm{d}x + \int_{\partial\mathcal{S}} \beta\mathbf{u}\cdot\mathbf{w}\right) + \int_{\partial\mathcal{S}} h|\mathbf{u}|^2\ \mathrm{d}\mathcal{H}^{N-1} \qquad \forall h \in L^2(\partial\mathcal{S})$$

where $\mathbf{u} = R(\beta)$ and $\mathbf{w} = DR(\beta)h$ is the solution to system (13). For every β, the gradient of T at β with respect to the $L^2(\partial\mathcal{S})$ topology is given by the following formula:

$$\nabla T(\beta) = (\mathbf{u} + \psi)\cdot\mathbf{u} \quad \textit{q.e. on } \partial\mathcal{S}, \tag{15}$$

where $\mathbf{u} = R(\beta)$ *and* ψ *is the unique solution to*

$$\begin{cases} -\mathrm{div}(\sigma(\psi,p)) + (\nabla\mathbf{u})^T\psi - (\mathbf{u}\cdot\nabla)\psi = & 2(\mathbf{u}\cdot\nabla)\mathbf{u} \ \textit{in}\ \mathcal{F}, \\ \mathrm{div}\ \psi = & 0 \ \textit{in}\ \mathcal{F}, \\ \psi = & 0 \ \textit{on}\ \partial\Omega, \\ \psi\cdot\mathbf{n} = & 0 \ \textit{on}\ \partial\mathcal{S}, \\ 2\nu\,[\mathbf{D}(\psi)\mathbf{n}]_{tan} + \beta\psi = & 0 \ \textit{on}\ \partial\mathcal{S}, \end{cases} \tag{16}$$

in the sense that $\psi \in V_\tau$ *and for every* $\phi \in V_\tau$,

$$2\nu\int_{\mathcal{F}} \mathbf{D}(\psi) : \mathbf{D}(\phi)\ \mathrm{d}x + \int_{\mathcal{F}} \big[(\nabla\mathbf{u})^T\psi - (\mathbf{u}\cdot\nabla)\psi\big]\cdot\phi\ \mathrm{d}x + \int_{\partial\mathcal{S}} \beta\psi\cdot\phi\ \mathrm{d}\mathcal{H}^{N-1} = 2\int_{\mathcal{F}} [(\mathbf{u}\cdot\nabla)\mathbf{u}]\cdot\phi\ \mathrm{d}x. \tag{17}$$

3 Proof of Theorems 1 and 2

In order to prove Theorems 1 and 2, we follow the standard steps from the theory of stationary Navier-Stokes equations, which we adapt to the framework of friction-driven boundary conditions. These arguments are developed in [9], for instance.

We define the bilinear form $a_0 : \mathbf{H}^1(\mathcal{F}) \times \mathbf{H}^1(\mathcal{F}) \to \mathbb{R}$ by

$$a_0(\mathbf{v}_1,\mathbf{v}_2) = 2\nu\int_{\mathcal{F}} \mathbf{D}(\mathbf{v}_1) : \mathbf{D}(\mathbf{v}_2)\ \mathrm{d}x + \int_{\partial\mathcal{S}} A\mathbf{v}_1\cdot\mathbf{v}_2\ \mathrm{d}\mu \quad \forall(\mathbf{v}_1,\mathbf{v}_2) \in \mathbf{H}^1(\mathcal{F}) \times \mathbf{H}^1(\mathcal{F}), \tag{18}$$

and the trilinear form $a_1 : \mathbf{H}^1(\mathcal{F}) \times \mathbf{H}^1(\mathcal{F}) \times \mathbf{H}^1(\mathcal{F}) \to \mathbb{R}$ by

$$a_1(\mathbf{v},\mathbf{v}_1,\mathbf{v}_2) = \int_{\mathcal{F}} [(\mathbf{v}\cdot\nabla)\mathbf{v}_1]\cdot\mathbf{v}_2\ \mathrm{d}x \qquad \forall(\mathbf{v},\mathbf{v}_1,\mathbf{v}_2) \in \mathbf{H}^1(\mathcal{F}) \times \mathbf{H}^1(\mathcal{F}) \times \mathbf{H}^1(\mathcal{F}). \tag{19}$$

Let $\mathbf{u}_0 \in K$ such that

$$\|\mathbf{u}_0\|_{\mathbf{H}^1(\mathcal{F})} \leqslant C_0|\mathbf{u}_\infty| \tag{20}$$

where $C_0 > 0$ depends only on $\mathcal{F}$. The existence of C_0 and $\mathbf{u}_0 \in K$ satisfying (20) was proved by Bogovskiĭ [3]. Setting $\widetilde{\mathbf{u}} = \mathbf{u} - \mathbf{u}_0$, we get that a function $\mathbf{u} \in K$ satisfies (12) if and only if $\widetilde{\mathbf{u}} \in V$ and satisfies the following identity:

$$\begin{aligned} \forall\phi \in V \qquad a_0(\widetilde{\mathbf{u}},\phi) + a_1(\mathbf{u}_0,\widetilde{\mathbf{u}},\phi) + a_1(\widetilde{\mathbf{u}},\mathbf{u}_0,\phi) + a_1(\widetilde{\mathbf{u}},\widetilde{\mathbf{u}},\phi) \\ = -a_0(\mathbf{u}_0,\phi) - a_1(\mathbf{u}_0,\mathbf{u}_0,\phi). \end{aligned} \tag{21}$$

We define the mapping $a : \mathbf{H}^1(\mathcal{F}) \times \mathbf{H}^1(\mathcal{F}) \times \mathbf{H}^1(\mathcal{F}) \to \mathbb{R}$ by

$$\begin{aligned} a(\mathbf{v},\mathbf{v}_1,\mathbf{v}_2) = a_0(\mathbf{v}_1,\mathbf{v}_2) + a_1(\mathbf{u}_0,\mathbf{v}_1,\mathbf{v}_2) + a_1(\mathbf{v}_1,\mathbf{u}_0,\mathbf{v}_2) + a_1(\mathbf{v},\mathbf{v}_1,\mathbf{v}_2) \\ \forall(\mathbf{v},\mathbf{v}_1,\mathbf{v}_2) \in \mathbf{H}^1(\mathcal{F}) \times \mathbf{H}^1(\mathcal{F}) \times \mathbf{H}^1(\mathcal{F}), \end{aligned} \tag{22}$$

and the linear form $\mathbf{f} = \mathbf{f}(\mathbf{u}_0) \in V'$ by

$$\langle\mathbf{f},\phi\rangle_{V'\times V} = -a_0(\mathbf{u}_0,\phi) - a_1(\mathbf{u}_0,\mathbf{u}_0,\phi) \qquad \forall\phi \in V.$$

The problem we consider is equivalent to the following : find $\mathbf{u}_0 \in K$ and $\widetilde{\mathbf{u}} \in V$ such that

$$a(\widetilde{\mathbf{u}}, \widetilde{\mathbf{u}}, \phi) = \langle \mathbf{f}(\mathbf{u}_0), \phi \rangle_{V' \times V} \qquad \forall \phi \in V. \tag{23}$$

We recall the following results from [9].

Theorem 5 *Assume that the following hypotheses hold:*

(i) for all $\mathbf{v} \in \mathbf{H}^1(\mathcal{F})$, the mapping $(\mathbf{v}_1, \mathbf{v}_2) \to a(\mathbf{v}, \mathbf{v}_1, \mathbf{v}_2)$ is continuous on $\mathbf{H}^1(\mathcal{F}) \times \mathbf{H}^1(\mathcal{F})$;

(ii) there exists a constant $\alpha > 0$ such that

$$a(\phi, \phi, \phi) \geqslant \alpha \|\phi\|^2_{\mathbf{H}^1(\mathcal{F})} \quad \forall \phi \in V; \tag{24}$$

(iii) the space V is separable and, for all $\phi \in V$, the mapping

$$\mathbf{v} \to a(\mathbf{v}, \mathbf{v}, \phi)$$

is sequentially weakly continuous on V, i.e.,

$$\text{weak} \lim_{m \to \infty} \mathbf{v}_m = \mathbf{v} \text{ in } V \text{ implies } \lim_{m \to \infty} a(\mathbf{v}_m, \mathbf{v}_m, \phi) = a(\mathbf{v}, \mathbf{v}, \phi) \quad \forall \phi \in V. \tag{25}$$

Then, problem (23) admits at least one solution.

Theorem 6 *Moreover, assume that*

(iv) the bilinear form $a(\mathbf{v}, \cdot, \cdot)$ is uniformly V-elliptic with respect to $\mathbf{v}$, i.e., there exists a constant $\alpha > 0$ such that

$$a(\mathbf{v}, \phi, \phi) \geqslant \alpha \|\phi\|^2_{\mathbf{H}^1(\mathcal{F})} \quad \forall (\mathbf{v}, \phi) \in V \times V; \tag{26}$$

(v) there exists a continuous and monotonically increasing function $L : \mathbb{R}_+ \to \mathbb{R}_+$ such that for all $\eta > 0$

$$\begin{gathered} |a(\mathbf{v}_1, \mathbf{v}_1, \mathbf{v}_2) - a(\mathbf{v}_2, \mathbf{v}_1, \mathbf{v}_2)| \leqslant L(\eta) \|\mathbf{v}_1\|_{\mathbf{H}^1(\mathcal{F})} \|\mathbf{v}_2\|_{\mathbf{H}^1(\mathcal{F})} \|\mathbf{v}_1 - \mathbf{v}_2\|_{\mathbf{H}^1(\mathcal{F})} \\ \forall \mathbf{v}_1, \mathbf{v}_2 \in V, \qquad \forall \mathbf{v}_1, \mathbf{v}_2 \in S_\eta = \{\mathbf{v} \in V \mid \|\mathbf{v}\|_{\mathbf{H}^1(\mathcal{F})} \leqslant \eta\}. \end{gathered} \tag{27}$$

Then, under the condition

$$[\|\mathbf{f}\|_{V'}/(\alpha^2)] L(\|\mathbf{f}\|_{V'}/\alpha) < 1, \tag{28}$$

the solution $\widetilde{\mathbf{u}} \in V$ to problem (23) is unique, and satisfies the following estimate:

$$\|\widetilde{\mathbf{u}}\|_{\mathbf{H}^1(\mathcal{F})} \leqslant \|\mathbf{f}\|_{V'}/\alpha. \tag{29}$$

Proof of Theorem 1. Let us fix $\mathbf{u}_\infty \in \mathbb{R}^N$ and $\nu > 0$. We need to prove that the mapping a defined by (22) satisfies the assumptions of Theorem 5.

By Sobolev imbedding theorem (see the standard monograph by Adams [1]), $\mathbf{H}^1(\mathcal{F})$ is continuously imbedded in $L^4(\mathcal{F}, \mathbb{R}^N)$ for $N = 2, 3$. Therefore, by Hölder inequality, the mapping a_1 defined by (19) is continuous. Since a_0 defined by (18) is continuous, we obtain the continuity of a, and consequently, condition (i) from Theorem 5 holds.

To prove (ii), we prove the stronger condition (iv). We use the following Korn inequality, which is proved for instance in [4] (see the proof of Lemma VII.3.5, p. 329):

Lemma 7 *There exists $C_1 > 0$ such that for all $\phi \in V$,*

$$\|\nabla \phi\|_{L^2(\mathcal{F},\mathbb{R}^{N\times N})} \leqslant C_1 \|D(\phi)\|_{L^2(\mathcal{F},\mathbb{R}^{N\times N})}.$$

Consequently, by Poincaré inequality, there exists $C_2 > 0$ such that for all $\phi \in V$,

$$\frac{1}{C_2}\|\phi\|^2_{\mathbf{H}^1(\mathcal{F})} \leqslant 2\|D(\phi)\|^2_{L^2(\mathcal{F},\mathbb{R}^{N\times N})}. \tag{30}$$

Let $\mathbf{v}, \phi \in V$. We observe that

$$a(\mathbf{v}, \phi, \phi) = a_0(\phi, \phi) + a_1(\phi, \mathbf{u}_0, \phi).$$

Indeed, $a_1(\mathbf{u}_0, \phi, \phi) = 0$ and $a_1(\mathbf{v}, \phi, \phi) = 0$, which comes from the following computation:

$$\begin{aligned} a_1(\mathbf{u}_0, \phi, \phi) &= \frac{1}{2}\int_{\mathcal{F}} (\mathbf{u}_0 \cdot \nabla)(|\phi|^2)\ \mathrm{d}x \\ &= \frac{1}{2}\left(\int_{\partial\mathcal{S}} |\phi|^2\ \mathbf{u}_0 \cdot \mathbf{n}\ \mathrm{d}\mathcal{H}^{N-1} + \int_{\partial\Omega} |\phi|^2\ \mathbf{u}_0 \cdot \mathbf{n}\ \mathrm{d}\mathcal{H}^{N-1}\right). \end{aligned} \tag{31}$$

Since $\phi \in V$, $\phi = 0$ on $\partial\Omega$, and since $\mathbf{u}_0 \in K$, $\mathbf{u}_0 \cdot \mathbf{n} = 0$ on $\partial\mathcal{S}$. Therefore, last boundary integrals vanish. Replacing $\mathbf{u}_0$ by $\mathbf{v}$ we obtain the desired conclusion.

For every $\mathbf{u}_0 \in K$ satisfying (20), we note $\rho(\mathbf{u}_0)$ the best positive constant such that

$$|a_1(\phi, \mathbf{u}_0, \phi)| \leqslant \rho(\mathbf{u}_0)\|\phi\|^2_{\mathbf{H}^1(\mathcal{F})} \qquad \forall \phi \in V, \tag{32}$$

We prove the following result:

Lemma 8 *For every $\delta > 0$, there exists $\mathbf{u}_0 \in K$ satisfying* (20) *and such that $\rho(\mathbf{u}_0) \leqslant \delta$.*

Proof. We consider a bounded extension operator

$$P : \mathbf{H}^1(\mathcal{F}) \to H^1_0(\Omega, \mathbb{R}^N).$$

Let $\delta > 0$. By the proof of Lemma 2.3 in [9], chapter IV, there exists $\widetilde{\mathbf{u}}_0 \in H^1(\Omega, \mathbb{R}^N)$ such that

$$\left\{ \begin{array}{rcll} \operatorname{div} \widetilde{\mathbf{u}}_0 &=& 0 & \text{in } \Omega, \\ \widetilde{\mathbf{u}}_0 &=& \mathbf{u}_\infty & \text{on } \partial\Omega, \\ \widetilde{\mathbf{u}}_0 &=& 0 & \text{in a neighborhood of } \mathcal{S}, \end{array} \right.$$

and such that for every $\varphi \in H_0^1(\Omega, \mathbb{R}^N)$,

$$\left| \int_\Omega [(\varphi \cdot \nabla)\widetilde{\mathbf{u}}_0] \cdot \varphi \, \mathrm{d}x \right| \leqslant \frac{\delta}{\|P\|} \|\varphi\|_{H^1(\Omega,\mathbb{R}^N)}. \tag{33}$$

Let $\phi \in V$. By inequality (33), we obtain

$$\begin{aligned} \int_\Omega [((P\phi) \cdot \nabla)\widetilde{\mathbf{u}}_0] \cdot (P\phi) \, \mathrm{d}x &\leqslant \frac{\delta}{\|P\|} \|P\phi\|_{H^1(\Omega,\mathbb{R}^N)} \\ &\leqslant \delta \|\phi\|_{\mathbf{H}^1(\mathcal{F})}. \end{aligned}$$

Since $\widetilde{\mathbf{u}}_0$ vanishes on $\mathcal{S}$, observing that

$$\int_\Omega [((P\phi) \cdot \nabla)\widetilde{\mathbf{u}}_0] \cdot (P\phi) \, \mathrm{d}x = \int_\mathcal{F} [(\phi \cdot \nabla)\widetilde{\mathbf{u}}_0] \cdot \phi \, \mathrm{d}x,$$

we obtain the desired result with $\mathbf{u}_0 = (\widetilde{\mathbf{u}}_0)_{|\mathcal{F}}$. □

Let $\mathbf{u}_0 \in K$ such that $\rho(\mathbf{u}_0) < \nu/C_2$, where C_2 is defined by (30). By conditions (30) and (32), for every $\mathbf{v}, \phi \in V$,

$$a(\mathbf{v}, \phi, \phi) \geqslant \left(\frac{\nu}{C_2} - \rho(\mathbf{u}_0) \right) \|\phi\|^2_{\mathbf{H}^1(\mathcal{F})}.$$

Consequently, condition (iv) holds with $\alpha = \nu/C_2 - \rho(\mathbf{u}_0)$.

To prove condition (25), we only need to consider the nonlinear term $a_1(\mathbf{v}_m, \mathbf{v}_m, \phi)$. Integrating by part and using the incompressibility condition, we obtain

$$a_1(\mathbf{v}_m, \mathbf{v}_m, \phi) = -\int_\mathcal{F} [(\mathbf{v}_m \cdot \nabla)\phi] \cdot \mathbf{v} \, \mathrm{d}x + \int_{\partial\mathcal{S}\cup\partial\Omega} (\mathbf{v}_m \cdot \phi)(\mathbf{v}_m \cdot \mathbf{n}) \, \mathrm{d}\mathcal{H}^{N-1}.$$

Since $\phi, \mathbf{v}_m \in V$, we conclude as above that

$$a_1(\mathbf{v}_m, \mathbf{v}_m, \phi) = -\int_\mathcal{F} [(\mathbf{v}_m \cdot \nabla)\phi] \cdot \mathbf{v}_m \, \mathrm{d}x.$$

Since $\mathbf{H}^1(\mathcal{F})$ is continuously imbedded in $L^4(\mathcal{F}, \mathbb{R}^N)$, we obtain

$$\lim_{m\to\infty} a_1(\mathbf{v}_m, \mathbf{v}_m, \phi) = a_1(\mathbf{v}, \mathbf{v}, \phi).$$

This proves condition (iii).

Now let $\mathbf{u} \in K$ be a weak solution to System (1)-(5). We define $\widetilde{\mathbf{u}} = \mathbf{u} - \mathbf{u}_0$. Applying relation (23) to $\phi = \widetilde{\mathbf{u}}$ and using inequality (26), we obtain

$$\|\widetilde{\mathbf{u}}\|_{\mathbf{H}^1(\mathcal{F})} \leqslant \frac{1}{\alpha} \|\mathbf{f}(\mathbf{u}_0)\|_{V'}.$$

By the triangular inequality, estimate (9) holds with $M = \|\mathbf{u}_0\|_{\mathbf{H}^1(\mathcal{F})} + 1/\alpha \|\mathbf{f}(\mathbf{u}_0)\|_{V'}$. This concludes the proof of Theorem 1. □

Proof of Theorem 2. Let us fix $\mathbf{u}_\infty \in \mathbb{R}^N$.

To prove (v), we observe that for $\mathbf{v}, \widetilde{\mathbf{v}}, \mathbf{v}_1, \mathbf{v}_2 \in V$,

$$a(\mathbf{v}, \mathbf{v}_1, \mathbf{v}_2) - a(\widetilde{\mathbf{v}}, \mathbf{v}_1, \mathbf{v}_2) = a_1(\mathbf{v} - \widetilde{\mathbf{v}}, \mathbf{v}_1, \mathbf{v}_2).$$

Introducing

$$\mathcal{N} = \sup_{\mathbf{v}, \mathbf{v}_1, \mathbf{v}_2 \in V \setminus \{0\}} \frac{a_1(\mathbf{v}, \mathbf{v}_1, \mathbf{v}_2)}{\|\mathbf{v}\|_{\mathbf{H}^1(\mathcal{F})} \|\mathbf{v}_1\|_{\mathbf{H}^1(\mathcal{F})} \|\mathbf{v}_2\|_{\mathbf{H}^1(\mathcal{F})}}, \tag{34}$$

we obtain that (27) holds with $L(\eta) = \mathcal{N}$ for all $\eta > 0$.

We define

$$\nu_0 = \inf \left\{ C_2 \left[\rho(\mathbf{u}_0) + (\mathcal{N} \|\mathbf{f}(\mathbf{u}_0)\|_{V'})^{1/2} \right], \ \mathbf{u}_0 \in K \text{ satisfying (20)} \right\}.$$

Estimate (10) results from the existence of a constant $C_3 > 0$, such that for every $\phi \in V$,

$$|\langle \mathbf{f}(\mathbf{u}_0), \phi \rangle_{V' \times V}| \leqslant C_3 \|\mathbf{u}_0\|_{\mathbf{H}^1(\mathcal{F})} (1 + \|\mathbf{u}_0\|_{\mathbf{H}^1(\mathcal{F})}) \|\phi\|_{\mathbf{H}^1(\mathcal{F})}.$$

As a result, by (20), there exists $C_4 > 0$ such that

$$\|\mathbf{f}(\mathbf{u}_0)\|_{V'} \leqslant C_4 |\mathbf{u}_\infty| (1 + |\mathbf{u}_\infty|),$$

and consequently, inequality (35) implies that there exists a constant $C_5 > 0$ such that

$$\nu_0^2 \leqslant C_5 (|\mathbf{u}_\infty| + |\mathbf{u}_\infty|^2).$$

This proves (10).

Let $\nu > \nu_0$. By definition of the infimum, there exists $\mathbf{u}_0 \in K$ satisfying (20) and such that

$$\nu_0 \leqslant C_2 \left[\rho(\mathbf{u}_0) + (\mathcal{N} \|\mathbf{f}(\mathbf{u}_0)\|_{V'})^{1/2} \right] < \nu. \tag{35}$$

We fix $\alpha = \nu / C_2 - \rho(\mathbf{u}_0)$. By (35), $\alpha > 0$ and by the same argument as above, α satisfies (26). Now let $\mathbf{u} \in K$ be a weak solution to System (1)-(5) and let $\widetilde{\mathbf{u}} = \mathbf{u} - \mathbf{u}_0$. Condition (28) reads

$$\mathcal{N} \|\mathbf{f}(\mathbf{u}_0)\|_{V'} < \alpha^2$$

which is equivalent to

$$(\mathcal{N} \|\mathbf{f}(\mathbf{u}_0)\|_{V'})^{1/2} < \nu / C_2 - \rho(\mathbf{u}_0). \tag{36}$$

By (35), last estimate holds. By Theorem 6, $\widetilde{\mathbf{u}} \in V$ is the unique solution to (23). Using (10) and (36), we obtain the following estimate:

$$\|\widetilde{\mathbf{u}}\|_{\mathbf{H}^1(\mathcal{F})} < \frac{1}{\mathcal{N}} \left(\frac{\nu}{C_2} - \rho(\mathbf{u}_0) \right). \tag{37}$$

Consequently, be (20) and the triangular inequality,

$$\|\mathbf{u}\|_{\mathbf{H}^1(\mathcal{F})} \leqslant C_0 |\mathbf{u}_\infty| + \frac{1}{\mathcal{N} C_2} \nu.$$

This proves (11).

By taking the infimum over all admissible functions $\mathbf{u}_0$, for $\nu > \nu_0$, we obtain the uniqueness of the function $\widetilde{\mathbf{u}} \in K$ satisfying (21). Therefore, System (1)-(5) admits one unique weak solution $\mathbf{u}_0 + \widetilde{\mathbf{u}} \in \mathbf{H}^1(\mathcal{F})$. □

4 Proof of Theorem 3

In this section, we assume that $\mathcal{V}(x)$ is equal to the tangent hyperplane to $\partial\mathcal{S}$ on q.e. x on the boundary, $A \equiv \mathrm{Id}$ and μ is absolutely continuous with respect to the surface measure on $\partial\mathcal{S}$ with a density $\beta \in L^2(\partial\mathcal{S})$.

Let $\mathbf{u}_\infty \in \mathbb{R}^N$. Let $\nu_0 > 0$ as in Theorem 2, and $\nu > \nu_0$. Let $\mathbf{u}_0 \in K$ satisfying (20) and such that

$$\nu_0 \leqslant C_2 \left[\rho(\mathbf{u}_0) + (\mathcal{N}\|\mathbf{f}(\mathbf{u}_0)\|_{V_\tau'})^{1/2}\right] < \nu. \tag{38}$$

We use the mappings introduced in Section 3. We define $\mathbf{g} = \mathbf{g}(\mathbf{u}_0) \in V_\tau'$ by

$$\langle \mathbf{g}, \phi\rangle_{V_\tau' \times V_\tau} = -a_0(\mathbf{u}_0, \phi) - a_1(\mathbf{u}_0, \mathbf{u}_0, \phi) \qquad \forall \phi \in V_\tau.$$

The problem we consider is equivalent to the following: find $\mathbf{u}_0 \in K_\tau$ and $\widetilde{\mathbf{u}} \in V_\tau$ such that

$$a(\widetilde{\mathbf{u}}, \widetilde{\mathbf{u}}, \phi) = \langle \mathbf{g}(\mathbf{u}_0), \phi\rangle_{V_\tau' \times V_\tau} \qquad \forall \phi \in V_\tau. \tag{39}$$

Proof of Theorem 3. The proof is based on the implicit function theorem. We will prove that the mapping

$$\beta \in \mathcal{O} \to \widetilde{\mathbf{u}} \in V_\tau$$

where $\widetilde{\mathbf{u}}$ satisfies (21), is differentiable. We introduce the open subset of V_τ:

$$V_0 = \left\{ \xi \in V_\tau \mid \|\xi\|_{\mathbf{H}^1(\mathcal{F})} < \frac{1}{\mathcal{N}} \left(\frac{\nu}{C_2} - \rho(\mathbf{u}_0) \right) \right\}$$

and define the mapping

$$F : \mathcal{O} \times V_0 \to V_\tau'$$

by

$$\begin{aligned} \forall \phi \in V_\tau \qquad \langle F(\beta, \mathbf{v}), \phi\rangle_{V_\tau' \times V_\tau} &= a_0(\mathbf{v}, \phi) + a_1(\mathbf{u}_0, \mathbf{v}, \phi) + a_1(\mathbf{v}, \mathbf{u}_0, \phi) + a_1(\mathbf{v}, \mathbf{v}, \phi) \\ &\quad + a_0(\mathbf{u}_0, \phi) + a_1(\mathbf{u}_0, \mathbf{u}_0, \phi). \end{aligned} \tag{40}$$

The mapping F is well defined and F is of class $\mathcal{C}^\infty$. By the proof of Theorems 1 and 2, for every $\beta \in \mathcal{O}$ there exists one unique $\widetilde{\mathbf{u}} \in V_0$ satisfying (21). We need to prove that for every $(\beta, \mathbf{v}) \in \mathcal{O} \times V_0$, $D_{\mathbf{v}}F(\beta, \mathbf{v}) \in \mathcal{L}(V_\tau, V_\tau')$ is invertible with continuous inverse.

For every $(\beta, \mathbf{v}) \in \mathcal{O} \times V_0$, $D_{\mathbf{v}}F(\beta, \mathbf{v}) \in \mathcal{L}(V_\tau, V_\tau')$ is defined by

$$\begin{aligned} \forall \xi, \phi \in V_\tau \qquad \langle D_{\mathbf{v}}F(\beta, \mathbf{v})\xi, \phi\rangle_{V_\tau' \times V_\tau} &= a_0(\xi, \phi) + a_1(\mathbf{u}_0, \xi, \phi) + a_1(\xi, \mathbf{u}_0, \phi) \\ &\quad + a_1(\xi, \mathbf{v}, \phi) + a_1(\mathbf{v}, \xi, \phi). \end{aligned}$$

We need to prove that for every $(\beta, \mathbf{v}) \in \mathcal{O} \times V_0$, there exists a constant $C > 0$ such that for every $\mathbf{h} \in V_\tau'$, there exists one unique $\xi \in V_\tau$ such that

$$D_{\mathbf{v}}F(\beta, \mathbf{v})\xi = \mathbf{h}, \tag{41}$$

with the estimate

$$\|\xi\|_{\mathbf{H}^1(\mathcal{F})} \leqslant C\|\mathbf{h}\|_{V_\tau'}. \tag{42}$$

Fix $(\beta, \mathbf{v}) \in \mathcal{O} \times V_0$. We will prove that the bilinear form associated to $D_{\mathbf{v}}F(\beta, \mathbf{v})$ satisfies the assumptions of Lax-Milgram theorem. The continuity results from the continuity of a_0 and a_1, and the coercivity is a consequence of the definition of V_0. Indeed, by computation (31), for all $\phi \in V_\tau$,

$$\langle D_{\mathbf{v}}F(\beta, \mathbf{v})\phi, \phi\rangle_{V_\tau' \times V_\tau} = a_0(\phi, \phi) + a_1(\phi, \mathbf{u}_0, \phi) + a_1(\phi, \mathbf{v}, \phi).$$

Consequently, using property (30) and definitions (32) and (34), we obtain for all $\phi \in V_\tau$,

$$\langle D_{\mathbf{v}}F(\beta, \mathbf{v})\phi, \phi\rangle_{V_\tau' \times V_\tau} \geqslant \left(\frac{\nu}{C_2} - \rho(\mathbf{u}_0) - \mathcal{N}\|v\|_{\mathbf{H}^1(\mathcal{F})}\right) \|\phi\|^2_{\mathbf{H}^1(\mathcal{F})}.$$

Since $\mathbf{v} \in V_0$,

$$\frac{\nu}{C_2} - \rho(\mathbf{u}_0) - \mathcal{N}\|v\|_{\mathbf{H}^1(\mathcal{F})} > 0.$$

This proves that the mapping

$$\beta \in \mathcal{O} \to \widetilde{\mathbf{u}} \in V_\tau$$

where $\widetilde{\mathbf{u}}$ satisfies (39), is differentiable. Consequently, since for all $\beta \in \mathcal{O}$, $R(\beta) = \mathbf{u}_0 + \widetilde{\mathbf{u}}$, R is differentiable on $\mathcal{O}$. Moreover, for every $\beta \in \mathcal{O}$, $DR(\beta) \in \mathcal{L}(L^2(\partial\mathcal{S}), \mathbf{H})$ is defined by

$$DR(\beta) = -\left[D_v F(\beta, R(\beta) - \mathbf{u}_0)\right]^{-1} D_\beta F(\beta, R(\beta) - \mathbf{u}_0).$$

Since for every $(\beta, \mathbf{v}) \in \mathcal{O} \times V_0$, $D_\beta F(\beta, \mathbf{v}) \in \mathcal{L}(L^2(\partial\mathcal{S}), V_\tau')$ is defined by

$$\forall h \in L^2(\partial\mathcal{S}), \forall \phi \in V_\tau \qquad \langle D_\beta F(\beta, \mathbf{v}), h\rangle = \int_{\partial\mathcal{S}} h(\mathbf{v} + \mathbf{u}_0) \cdot \phi \ \mathrm{d}\mathcal{H}^{N-1},$$

defining $\mathbf{w} = DR(\beta)h$ for $h \in L^2(\partial\mathcal{S})$, we obtain that $\mathbf{w} \in V_\tau$ satisfies the following identity:

$$\begin{aligned}\forall \phi \in V_\tau \qquad & a_0(\mathbf{w}, \phi) + a_1(\mathbf{u}_0, \mathbf{w}, \phi) + a_1(\mathbf{w}, \mathbf{u}_0, \phi) + a_1(\mathbf{w}, R(\beta) - \mathbf{u}_0, \phi) \\ & + a_1(R(\beta) - \mathbf{u}_0, \mathbf{w}, \phi) = -\int_{\partial\mathcal{S}} hR(\beta) \cdot \phi \ \mathrm{d}\mathcal{H}^{N-1}.\end{aligned}$$

Using the trilinearity of a_1, we obtain (14). □

Proof of Corollary 4. Noticing that for every $\phi \in V_\tau$,

$$[(\nabla\mathbf{u})^T\psi] \cdot \phi = [(\phi \cdot \nabla)\mathbf{u}] \cdot \psi$$

and

$$-\int_{\mathcal{F}} [(\mathbf{u} \cdot \nabla)\psi] \cdot \phi \ \mathrm{d}x = \int_{\mathcal{F}} [(\mathbf{u} \cdot \nabla)\phi] \cdot \psi \ \mathrm{d}x,$$

(17) can be written in the following form:

$$\begin{aligned}\forall \phi \in V_\tau \quad & 2\nu \int_{\mathcal{F}} \mathbf{D}(\psi) : \mathbf{D}(\phi) \ \mathrm{d}x + \int_{\mathcal{F}} \psi \cdot [(\phi \cdot \nabla)\mathbf{u} + (\mathbf{u} \cdot \nabla)\phi] \ \mathrm{d}x + \int_{\partial\mathcal{S}} \beta\psi \cdot \phi \ \mathrm{d}\mathcal{H}^{N-1} \\ & = 2\int_{\mathcal{F}} [(\mathbf{u} \cdot \nabla)\mathbf{u}] \cdot \phi \ \mathrm{d}x. \end{aligned} \tag{43}$$

Let us fix $\beta \in \mathcal{O}$ and $h \in L^2(\partial\mathcal{S})$. We note $\mathbf{u} = R(\beta)$ and $\mathbf{w} = DR(\beta)h$. Taking $\phi = \mathbf{w}$ in (12), we express $DT(\beta)h$ as follows:

$$DT(\beta)h = -2\int_{\mathcal{F}} [(\mathbf{u}\cdot\nabla)\mathbf{u}]\cdot\mathbf{w}\ \mathrm{d}x + \int_{\partial\mathcal{S}} h|\mathbf{u}|^2\ \mathrm{d}\mathcal{H}^{N-1}. \tag{44}$$

Besides, taking $\phi = \psi$ in (14) and $\phi = \mathbf{w}$ in (43), we obtain

$$\begin{aligned}
&2\nu\int_{\mathcal{F}} \mathbf{D}(\mathbf{w}):\mathbf{D}(\psi)\ \mathrm{d}x + \int_{\mathcal{F}} [(\mathbf{w}\cdot\nabla)\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{w}]\cdot\psi\ \mathrm{d}x + \int_{\partial\mathcal{S}} \beta\mathbf{w}\cdot\psi\ \mathrm{d}\mathcal{H}^{N-1}\\
&\quad = -\int_{\partial\mathcal{S}} h\mathbf{u}\cdot\psi\ \mathrm{d}\mathcal{H}^{N-1},\\
&2\nu\int_{\mathcal{F}} \mathbf{D}(\psi):\mathbf{D}(\mathbf{w})\ \mathrm{d}x + \int_{\mathcal{F}} \psi\cdot[(\mathbf{w}\cdot\nabla)\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{w}]\ \mathrm{d}x + \int_{\partial\mathcal{S}} \beta\psi\cdot\mathbf{w}\ \mathrm{d}\mathcal{H}^{N-1}\\
&\quad = 2\int_{\mathcal{F}} [(\mathbf{u}\cdot\nabla)\mathbf{u}]\cdot\mathbf{w}\ \mathrm{d}x.
\end{aligned}$$

Substracting these equalities, we obtain

$$-2\int_{\mathcal{F}} [(\mathbf{u}\cdot\nabla)\mathbf{u}]\cdot\mathbf{w}\ \mathrm{d}x = \int_{\partial\mathcal{S}} h\mathbf{u}\cdot\psi\ \mathrm{d}\mathcal{H}^{N-1}$$

and finally, by (44),

$$DT(\beta)h = \int_{\partial\mathcal{S}} [(\mathbf{u}+\psi)\cdot\mathbf{u}]h\ \mathrm{d}\mathcal{H}^{N-1}.$$

This proves (15). □

Remark 9 The coercivity of the bilinear form associated to the weak formulation (43) results from the following estimate (see the proof of Theorems 1 and 2):

$$\begin{aligned}
\rho(\mathbf{u}) &\leqslant \rho(\mathbf{u}_0) + \rho(\widetilde{\mathbf{u}})\\
&\leqslant \rho(\mathbf{u}_0) + \mathcal{N}\|\widetilde{\mathbf{u}}\|_{\mathbf{H}^1(\mathcal{F})}\\
&< \frac{\nu}{C_2}.
\end{aligned}$$

References

[1] Robert A. Adams. *Sobolev spaces.* Academic Press [A subsidiary of Harcourt Brace Jovanovich, Publishers], New York-London, 1975. Pure and Applied Mathematics, Vol. 65.

[2] Youcef Amirat, Didier Bresch, Jérôme Lemoine, and Jacques Simon. Effect of rugosity on a flow governed by stationary Navier-Stokes equations. *Quart. Appl. Math.*, 59(4):769–785, 2001.

[3] M. E. Bogovskiĭ. Solution of the first boundary value problem for an equation of continuity of an incompressible medium. *Dokl. Akad. Nauk SSSR*, 248(5):1037–1040, 1979.

[4] Franck Boyer and Pierre Fabrie. *Éléments d'analyse pour l'étude de quelques modèles d'écoulements de fluides visqueux incompressibles*, volume 52 of *Mathématiques & Applications (Berlin) [Mathematics & Applications]*. Springer-Verlag, Berlin, 2006.

[5] J. Březina. Asymptotic properties of solutions to the equations of incompressible fluid mechanics. Preprint, Necas Center for Math. Model.

[6] Dorin Bucur, Eduard Feireisl, and Šárka Nečasová. Boundary behavior of viscous fluids: influence of wall roughness and friction-driven boundary conditions. *Arch. Ration. Mech. Anal.*, 197(1):117–138, 2010.

[7] Dorin Bucur, Eduard Feireisl, Šárka Nečasová, and Joerg Wolf. On the asymptotic limit of the Navier-Stokes system on domains with rough boundaries. *J. Differential Equations*, 244(11):2890–2908, 2008.

[8] Juan Casado-Díaz, Enrique Fernández-Cara, and Jacques Simon. Why viscous fluids adhere to rugose walls: a mathematical explanation. *J. Differential Equations*, 189(2):526–537, 2003.

[9] Vivette Girault and Pierre-Arnaud Raviart. *Finite element methods for Navier-Stokes equations*, volume 5 of *Springer Series in Computational Mathematics*. Springer-Verlag, Berlin, 1986. Theory and algorithms.

[10] Willi Jäger and Andro Mikelić. On the roughness-induced effective boundary conditions for an incompressible viscous flow. *J. Differential Equations*, 170(1):96–122, 2001.

[11] Bijan Mohammadi, Olivier Pironneau, and Frederic Valentin. Rough boundaries and wall laws. *Internat. J. Numer. Methods Fluids*, 27(1-4, Special Issue):169–177, 1998. Finite elements in fluids.