Exact Simulation of One-dimensional Stochastic Differential Equations involving the local time at zero of the unknown process

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Abstract:

In this article we extend the exact simulation methods of Beskos et al. in [3] to the solutions of one-dimensional stochastic differential equations involving the local time of the unknown process at point zero. In order to perform the method we compute the law of the skew Brownian motion with drift. The method presented in this article covers the case where the solution of the SDE with local time corresponds to a divergence form operator with a discontinuous coefficient at zero. Numerical examples are shown to illustrate the method and the performances are compared with more traditional discretization schemes.

Keywords:

Exact simulation methods ; Skew Brownian motion ; One-dimensional diffusion ; Local Time.

1 Introduction

1.1 Presentation

The implementation of exact simulation methods for one-dimensional SDEs has been a subject of much interest in the last years : see for example [6], [3], [4], [19], [21]. Unlike the classical simulation methods which all involve some kind of discretization error (we mention [2] for the Euler Scheme), the exact simulation methods are constructed in such a way that they do not present any discretization error (when the diffusion coefficient is constant and equal to one). In the last years, the original method presented in the fundamental article [3] has been extended to overcome various limitations of the initial algorithm ; it has been generalized to include the cases of unbounded drifts ([4], [5]), the computation of Greeks in a financial setting [19], and several steps have been made to deal with multidimensional diffusions.

On another hand, the numerical simulation of SDEs corresponding to divergence form operators involving a discontinuous coefficient has been also the subject of various studies in the last years since these SDEs arise in the modelisation of various physical phenomenons. In the one-dimensional context, various Random Walks and an Euler Scheme have been studied for the simulation of the solution of such SDEs : for Random Walks we mention [9], [8], [10], [13] ; for the Euler Scheme see [15], [16] in the case where the discontinuity of the coefficient in the divergence operator appears at

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point 0. Of course, for such SDEs, the order of discretization error of these discretization schemes is usually greater than those obtained in a more classical context.

An important problem comes from the fact that SDEs corresponding to divergence form operators involving a discontinuous coefficient do not enter the classical scope of SDEs covered by the exact simulation methods. The main difficulty is that these SDEs include an additional term, which involves in dimension one the local time of the unknown process (in dimension greater than one, it involves the local time of a one-dimensional auxiliary process; see [7]). In fact, the laws of the solution of such one-dimensional SDEs are no longer absolutely continuous with respect to the Wiener measure.

In this paper we present a first attempt for the adaptation of the exact simulation methods of [3] to one-dimensional SDEs with an additional term that involves the local time of the unknown process at point 0. Namely, our object of study is $(X_t)_{t\geq 0}$ solution of

$$dX_t = \sigma(X_t)dW_t + b(X_t)dt + \beta dL_t^0(X), \tag{1}$$

where $0 \neq |\beta| < 1$ and $L_t^0(X)$ is the symmetric local time of X in zero at time t. Under mild assumptions concerning b and standard ellipticity conditions on σ, it is known that there exists a unique strong solution $(X_t)_{t\geq 0}$ to (1) (see [12] for details).

Note that when σ is identically equal to 1 and b is identically equal to 0, the solution (X_t) of (1) is a standard Skew Brownian Motion (SBM in short).

Let us emphasize that this work includes the situation where b may be discontinuous at 0. So that the results of this paper are also suited for the situation stated in [16], where the solution of (1) corresponds to a divergence form operator whose coefficient is discontinuous at 0 (and is sufficiently smooth elsewhere). We show a numerical example to illustrate this interesting case.

Let us now briefly explain our main idea. When $\sigma \equiv 1$, we show that the law of $(X_t)_{t\geq 0}$ (solution of (1)) is absolutely continuous with respect to the law of some Skew Brownian Motion (SBM) with a drift component. The reason why the SBM with drift appears naturally in our computations is explained in Section 3 (see Remark 3.1).

So, contrary to the already mentioned discretization schemes where the standard SBM is used in force, we do not longer deal with a simple SBM but with a SBM that possesses a drift component. As a consequence, in order to adapt the method of [3] in this setting, we have to be able to draw bridges of the SBM with drift. An important part of this paper is devoted to this matter.

In the last section of the paper, we discuss the limitations of the initial algorithm. The main issue is to relax the boundedness assumptions made on the drift function b, as is done in [4] for "classical" SDEs. In [4], the authors use some kind of factorisations for the sample state space of the standard Brownian Bridge, which are consequences of William's decomposition theorem for Brownian Motion. Proving similar factorisations for the Skew Brownian Bridge with drift seems difficult to us. Nevertheless, we have been able to apply a result stated in Pitman-Yor [18] in the case of the standard Skew Brownian Bridge, which gives a first partial result. Unfortunately, we have not been able to relax the boundedness assumption on the drift function b and we think that much remains to do in this direction.

1.2 Organisation of the paper

The paper is organised as follows :

- Section 2 presents the problem and states our assumptions.
- Section 3 is dedicated to a detailed presentation of the algorithm studied in this paper (under the assumptions of Section 2). We present the method for the simulation of an exact skeleton of $(X_t)_{t\in[0,T]}$ solution of (1) when $\sigma \equiv 1$.

- The title of section 4 is *Simulation of bridges of a Skew Brownian motion with drift*. It is devoted to the computation of the explicit laws needed in order to perform the algorithm presented in section 3. We first compute the transition function of the SBM with drift and then draw the consequences for the computation of the density of the bridges of the SBM with drift. These laws are of critical importance when one wants to implement the methods presented in this article. At the end of the section, we give rejection bounds that are needed for the acceptance/rejection procedure that lies underneath the whole algorithm.

- Section 5 presents numerical results obtained with different examples. The results compare the performance of this method with those coming from other articles. In this section we treat a numerical example involving a diffusion corresponding to a divergence form operator.

- In section 6 we briefly discuss what we shall call "the degenerate case" : the function b is discontinuous at point 0 but we now allow $\beta = 0$. In this case, the solution of the equation is shown to be absolutely continuous w.r.t. a Brownian Motion with Two-Valued Drift. As a consequence, the method extends naturally to this particular case.

- In the first part of Section 7 we discuss the assumptions made in Section 2 regarding the boundedness of function b and explain the difficulties we found in trying to remove them ; in the literature concerning exact simulation algorithms for (classical) diffusions, this problem is strongly related to the decomposition of the trajectories of the Brownian Bridge. Following Pitman-Yor [18], we give insights to such decomposition for the Skew Brownian Bridge. We manage to compute almost explicitly the joint law of the maximum of a Skew Brownian Bridge and the time where this maximum is reached (almost explicitly in this context means up to a Laplace transform inversion). Unfortunately, we have not been able to go further and this problem seems difficult to us. We conclude the paper in the second part of this section and sketch lines for further studies on the subject.

2 Exposition of the problem and statement of our assumptions

2.1 Exposition of the problem

Denote $C = C([0,T],\mathbb{R})$ the set of continuous mappings from $[0,T]$ to $\mathbb{R}$ and $\mathcal{C}$ the Borel σ-field on C induced by the supremum norm.

Let $\mathbb{P}$ be a probability measure on $(C,\mathcal{C})$ and W a Brownian motion under $\mathbb{P}$ together with its completed natural filtration $(\mathcal{F}_t)_{t\geq 0}$.

We seek for an exact simulation algorithm of the paths of the solution of the one-dimensional Stochastic Differential Equation

$$dX_t = dW_t + b(X_t)dt + \beta dL_t^0(X), \tag{2}$$

where $|\beta| < 1$, $L_t^0(X)$ is the symmetric local time of X in zero at time t.

2.2 Notations and assumptions

Throughout the whole paper, we make the following assumptions

- The function $b : \mathbb{R} \to \mathbb{R}$ is bounded and differentiable on $\mathbb{R}^{*,+}$ and $\mathbb{R}^{*,-}$ with a possible discontinuity at point $\{0\}$. We suppose that both limits $\lim_{z\to 0+} b(z) := b(0+)$ and $\lim_{z\to 0-} b(z) := b(0-)$ exist and are finite. The value $b(0)$ of the function b at 0 is of no importance and can be fixed arbitrarily to some constant (possibly different from either $b(0+)$ or $b(0-)$).

- If $\beta \neq 0$, we set $\mu := \frac{1+\beta}{2\beta} b(0+) - \frac{1-\beta}{2\beta} b(0-)$ and define $\bar{b}(z) := b(z) - \mu$. We suppose that the function
$$z \mapsto \bar{\phi}(z) := \frac{\bar{b}^2(z) + \bar{b}'(z) + 2\mu\bar{b}(z)}{2} \mathbb{1}_{\mathbb{R}^{*,+} \cup \mathbb{R}^{*,-}}(z)$$
is bounded.

  We set $\tilde{\phi}(z) := \bar{\phi}(z) - m$ with $m = \inf_{z \in \mathbb{R}} \bar{\phi}(z)$; the constant K denotes an upper bound of the function $\tilde{\phi}$.

- The function $u \mapsto \exp[\bar{B}(u) - (u-x)^2/2T]$, where $\bar{B}(u) := \int_0^u \bar{b}(y)dy$ is integrable.

Except in Section 6 (corresponding to what we decide to call "the degenerate case" when $\beta = 0$ and b is still discontinuous at 0), throughout the paper we make the strong additional assumption

- $\beta \neq 0$.

From [12], we know that strong existence and uniqueness holds for the solution of equation (2) under the above assumptions.

Our goal is to sample exactly with respect to the law of X under $\mathbb{P}$ following the ideas of [3].

3 Presentation of the algorithm

3.1 Application of Girsanov's theorem

Recall that $\bar{b}(z) := b(z) - \mu$ where μ is the constant defined by

$$\mu := \frac{1+\beta}{2\beta} b(0+) - \frac{1-\beta}{2\beta} b(0-). \tag{3}$$

Note that since $\beta \neq 0$ by assumption, this constant is well-defined. In the case where b is continuous at point $\{0\}$, note that μ reduces to $b(0)$.

We have

$$dX_t = dW_t + \bar{b}(X_t)dt + \mu dt + \beta dL_t^0(X).$$

In particular, we perform Girsanov's theorem and we write

$$dX_t = dW_t^{SD} + \mu dt + \beta dL_t^0(X), \tag{4}$$

where $W_t^{SD} := W_t + \int_0^t \bar{b}(X_s)ds$ is a Brownian motion under the new probability $\mathbb{W}^{SD}$ defined by

$$\frac{d\mathbb{P}}{d\mathbb{W}^{SD}} = \exp\Big\{ \int_0^T \bar{b}(X_t)dW_t^{SD} - \frac{1}{2}\int_0^T \bar{b}^2(X_t)dt \Big\}. \tag{5}$$

From our assumptions on b, we are in position to apply the symmetric Itô-Tanaka formula to the function $\bar{B}(u) := \int_0^u \bar{b}(y)dy$ and $(X_t)_{t\geq 0}$.

Applying the occupation's time formula, we obtain

$$\begin{aligned}
\bar{B}(X_T) - \bar{B}(x) &= \int_0^T \frac{\bar{b}(X_t+) + \bar{b}(X_t-)}{2} dX_t + \frac{1}{2}\int_0^T \bar{b}'(X_t)\mathbb{1}_{X_t \neq 0}dt + \frac{\bar{b}(0+) - \bar{b}(0-)}{2} L_t^0(X) \\
&= \int_0^T \bar{b}(X_t)\mathbb{1}_{X_t\neq 0} dW_t^{SD} + \mu \int_0^T \bar{b}(X_t)\mathbb{1}_{X_t \neq 0}dt \\
&\quad + \frac{1}{2}\int_0^T \bar{b}'(X_t)\mathbb{1}_{X_t\neq 0}dt + \underbrace{\left(\frac{\bar{b}(0+) + \bar{b}(0-)}{2}\beta + \frac{b(0+) - b(0-)}{2} \right)}_{=0} L_t^0(X)
\end{aligned} \tag{6}$$

where the last line comes from the definition of $\bar{b}$ and μ. From the fact that $\ell\{t \in [0,T] : X_t = 0\} = 0$ (where ℓ stands for the Lebesgue measure), we see that

$$\bar{B}(X_T) - \bar{B}(x) = \int_0^T \bar{b}(X_t) dW_t^{SD} + \mu \int_0^T \bar{b}(X_t) dt + \frac{1}{2} \int_0^T \bar{b}'(X_t) dt. \tag{7}$$

Thus, (5) implies that

$$\mathbb{E}_{\mathbb{P}}[F(X)] = \mathbb{E}_{\mathbb{W}^{SD}}\left[F(X) \exp\left\{\bar{B}(X_T) - \bar{B}(x) - \int_0^T \bar{\phi}(X_t) dt\right\}\right],$$

where $\bar{\phi}(z) = \dfrac{\bar{b}^2(z) + \bar{b}'(z) + 2\mu\bar{b}(z)}{2}$.

Remark 3.1 Note that, because of the definition of $\bar{b}$, there is no local time appearing in equality (7) after the application of the Itô-Tanaka formula. This ensures that there is no local time involved in the exponential martingale of Girsanov's theorem, which makes it tractable from a numerical perspective.

Restrospectively, this explains why in the sequel we have to deal with a Skew Brownian Motion with drift instead of a simple standard SBM as it is usually done for the discretization schemes in this context.

3.2 Exact simulation algorithm for skew diffusions

Let $B^{\beta,\mu}$ be the SBM of parameter β and drift μ. That is to say $B^{\beta,\mu}$ is the strong solution of

$$dB_t^{\beta,\mu} = dW_t + \mu dt + \beta dL_t^0(B^{\beta,\mu}). \tag{8}$$

Notation : in the sequel, we denote $p^{\beta,\mu}(t,x,y)$ the transition probability density of $B^{\beta,\mu}$. Considering (4) and (5) it is clear that the law of X under $\mathbb{W}^{SD}$ is given by $p^{\beta,\mu}(t,x,y)dy$.

Following the lines of Beskos et al. in [3], and considering the computations performed in the above section, we give an algorithm that returns an exact drawing of a skeleton of $(X_t)_{t\in[0,T]}$ solution of (2) starting from x_0 :

Step 1 - Simulate a random variable Z according to the density

$$h(y) = C \exp\left(\bar{B}(y) - \bar{B}(x_0)\right) p^{\beta,\mu}(T, x_0, y).$$

Keep in memory the value z of Z.

Step 2 - Simulate a Poisson Point Process with unit density on $[0,T] \times [0,K]$. The result is a random number n of points of coordinates $(t_1, z_1), \ldots, (t_n, z_n)$.

Step 3 - Simulate $(B_{t_1}^{\beta,\mu}, \ldots, B_{t_n}^{\beta,\mu})$ conditioned on $B_0^{\beta,\mu} = x_0$ and $B_T^{\beta,\mu} = z$.

Step 4 - If $\forall i \in \{1, \ldots, n\}$ $\tilde{\phi}(B_{t_i}^{\beta,\mu}) \leq z_i$ accept the trajectory. Else return to step 1.

This algorithm returns a drawing of $(X_{t_1}, \ldots, X_{t_n}, X_T)$ (in particular we get an exact drawing of X_T, it is the value z of Z used for an accepted trajectory).

Note that in order to apply the methodology of [3] we have to be able to draw bridges of a drifted Skew Brownian Motion $B^{\beta,\mu}$. Indeed, this is crucial in order to perform the Step 3. The next section is entirely devoted to this matter.

4 Simulation of bridges of a Skew Brownian motion with drift

4.1 Notations

Throughout this section we will use the following notations :

- recall that $p^{\beta,\mu}(t,x,y)dy$ is the transition probability density of $B^{\beta,\mu}$
- $\left(B_t^\beta\right)_{t\geq 0}$ is the solution of $B_t^\beta = x + W_t + \beta L_t^0(B^\beta)$, that is to say B^β is a (standard) Skew Brownian motion with parameter β starting from x.
- $\alpha := (\beta+1)/2$ is the probability of partial reflection above 0 (for B^β).
- $\tau_0 := \inf\{t \geq 0 : B_t^\beta = 0\}$ and $T_0 := \inf\{t \geq 0 : W_t = 0\}$.
- $N^c(z) := \dfrac{1}{\sqrt{2\pi}} \displaystyle\int_z^\infty e^{-\frac{z^2}{2}} dz$ is the complementary normal cumulative function.

Let us notice that τ_0 and T_0 have common distribution and let us denote $h(x,.)$ their probability density knowing $B_0^\beta = x$ (or $W_0 = x$). That is to say, if W is a Brownian motion under $\mathbb{P}$ then $\mathbb{P}^x(T_0 \in ds) = h(x,s)ds$.

4.2 Computation of the density $p^{\beta,\mu}(t,x,y)$

Remark 4.1 The results of Proposition 4.1 and Proposition 4.2 below differ from the ones given by T. Appuhamillage et al. in the recent article [1]. In Theorem 1.4 of [1] the authors state a slightly different formula for $p^{\beta,\mu}(t,0,y)$ from ours : the product $\beta\mu$ appears with an absolute value which is not the case for the formula we give in Proposition 4.1.

The result of our Proposition 4.2 has to be compared with Corollary 3.3 in [1]. Since we think there is a computational error in [1], we explain in this section how we derive the explicit formula for the joint density of $(B_t^\beta, L_t^0(B^\beta))$. Note that we use a different method of proof than in [1].

We have the following proposition.

Proposition 4.1 *We have that,*

$$p^{\beta,\mu}(t,x,y) = \begin{cases} \frac{1}{\sqrt{2\pi t}} \exp\{\mu(y-x) - \frac{1}{2}\mu^2 t\}\big(\exp\{-\frac{(y-x)^2}{2t}\} - \exp\{-\frac{(y+x)^2}{2t}\}\big) \\ + \frac{2\alpha}{\sqrt{2\pi t}} \exp\Big\{ -\frac{(x+y)^2}{2t} + \mu(y-x) - \frac{1}{2}\mu^2 t\Big\} \\ \qquad\qquad \times\big[1 - \beta\mu\sqrt{2\pi t}\exp\Big\{\frac{(x+y+t\beta\mu)^2}{2t}\Big\} N^c(\frac{\beta\mu t + x + y}{\sqrt{t}})\big], \\ \text{if } x \geq 0, y \geq 0, \\ \\ \frac{2(1-\alpha)}{\sqrt{2\pi t}} \exp\Big\{ -\frac{(x-y)^2}{2t} + \mu(y-x) - \frac{1}{2}\mu^2 t\Big\} \\ \qquad\qquad \times\big[1 - \beta\mu\sqrt{2\pi t}\exp\Big\{\frac{(x-y+t\beta\mu)^2}{2t}\Big\} N^c(\frac{\beta\mu t + x - y}{\sqrt{t}})\big], \\ \text{if } x \geq 0, y < 0. \\ \\ \frac{1}{\sqrt{2\pi t}} \exp\{\mu(y-x) - \frac{1}{2}\mu^2 t\}\big(\exp\{-\frac{(y-x)^2}{2t}\} - \exp\{-\frac{(y+x)^2}{2t}\}\big) \\ + \frac{2(1-\alpha)}{\sqrt{2\pi t}} \exp\Big\{ -\frac{(x+y)^2}{2t} + \mu(y-x) - \frac{1}{2}\mu^2 t\Big\} \\ \qquad\qquad \times\big[1 - \beta\mu\sqrt{2\pi t}\exp\Big\{\frac{(-x-y+t\beta\mu)^2}{2t}\Big\} N^c(\frac{\beta\mu t - y - x}{\sqrt{t}})\big], \\ \text{if } x < 0, y < 0, \\ \\ \frac{2\alpha}{\sqrt{2\pi t}} \exp\Big\{ -\frac{(x-y)^2}{2t} + \mu(y-x) - \frac{1}{2}\mu^2 t\Big\} \\ \qquad\qquad \times\big[1 - \beta\mu\sqrt{2\pi t}\exp\Big\{\frac{(y-x+t\beta\mu)^2}{2t}\Big\} N^c(\frac{\beta\mu t + y - x}{\sqrt{t}})\big], \\ \text{if } x < 0, y \geq 0. \end{cases}$$

Remark 4.2 It can be shown that the quantity $1-\beta\mu\sqrt{2\pi t}\exp\big\{\frac{(|x|+|y|+t\beta\mu)^2}{2t}\big\}N^c(\frac{\beta\mu t+|x|+|y|}{\sqrt{t}})$ involved in $p^{\beta,\mu}(t,x,y)$ remains strictly positive, whatever the sign of $\beta\mu$.

To prove Proposition 4.1 we will use a Cameron-Martin formula and the probability density of $(B_t^\beta, L_t^0(B^\beta))$, given in the following proposition.

Proposition 4.2 *Let W be a Brownian motion defined on $(C,\mathcal{C},\mathbb{P})$ and B^β the strong solution of (8) with $\mu=0$.*

We have for all $t>0$,

$$\mathbb{P}^x\big[\,B_t^\beta\in dy; L_t^0(B^\beta)\in dl\,\big]=\begin{cases}\frac{2\alpha(l+y+x)}{\sqrt{2\pi t^3}}\exp\big\{-\frac{(l+y+x)^2}{2t}\big\}dydl\\ +\frac{1}{\sqrt{2\pi t}}\big(\exp\{-\frac{(y-x)^2}{2t}\}-\exp\{-\frac{(y+x)^2}{2t}\}\big)dy\delta_0(dl)\\ \text{if } x\geq 0, y\geq 0, l\geq 0,\\ \\ \frac{2(1-\alpha)(l-y+x)}{\sqrt{2\pi t^3}}\exp\big\{-\frac{(l-y+x)^2}{2t}\big\}dydl\\ x\geq 0, y<0, l\geq 0.\end{cases}$$

To prove Proposition 4.2 we will use the following lemmas.

Lemma 4.1 *Let W be a Brownian motion defined on $(C,\mathcal{C},\mathbb{P})$ and B^β the strong solution of (8) with $\mu=0$.*

We have for all $t,l>0$, and $y\geq 0$,

$$\mathbb{P}^0\big[\,|B_t^\beta|\in dy; L_t^0(B^\beta)\in dl\,\big]=\mathbb{P}^0\big[\,|W_t|\in dy; L_t^0(W)\in dl\,\big]. \tag{9}$$

For $x\neq 0$ we have for all $t,l>0$ and $y\geq 0$,

$$\mathbb{P}^x\big[\,|B_t^\beta|\in dy; L_t^0(B^\beta)\in dl; t\geq\tau_0\,\big]=\mathbb{P}^x\big[\,|W_t|\in dy; L_t^0(W)\in dl; t\geq T_0\,\big]. \tag{10}$$

Remark 4.3 We even have that the process $(|B_t^\beta|, L_t^0(B^\beta))_{t\geq 0}$ is distributed as $(|W_t|, L_t^0(W))_{t\geq 0}$ under $\mathbb{P}^0$. Indeed their common distribution is the one of $(M_t^W-W_t, M_t^W)_{t\geq 0}$ under $\mathbb{P}^0$, where $M_t^W=\max_{0\leq s\leq t}W_s$.

This is related to the Lévy theorem, as stated for instance in Theorem 3.6.17 in [11], where it is proved by using the Skorokhod method. We use the same technique to prove Lemma 4.1.

Proof of Lemma 4.1. Let be $x\neq 0$, and assume $W_0=x$ a.s so that $B_0^\beta=x$ a.s. Using the symmetric Tanaka formula we have

$$\begin{aligned}|B_t^\beta| &= |x|+\textstyle\int_0^t \mathrm{sgn}(B_s^\beta)dB_s^\beta + L_t^0(B^\beta)\\ &= |x|+\textstyle\int_0^t \mathrm{sgn}(B_s^\beta)dW_s + \beta\int_0^t \mathrm{sgn}(B_s^\beta)dL_s^0(B^\beta) + L_t^0(B^\beta)\\ &= |x|+\textstyle\int_0^t \mathrm{sgn}(B_s^\beta)dW_s + L_t^0(B^\beta),\end{aligned}$$

where $\mathrm{sgn}(x)=\mathbf{1}_{x>0}-\mathbf{1}_{x<0}$ is the symmetric sign function (note that $\mathrm{sgn}(0)=0$).

As we have $|B_t^\beta|\geq 0$ and $dL_t^0(B^\beta)=\mathbf{1}_{B_t^\beta=0}dL_t^0(B^\beta)$, it follows from Lemma 3.6.14 in [11], that,

$$L_t^0(B^\beta)=\max\big[0,\max_{0\leq s\leq t}\{-(|x|+\int_0^s \mathrm{sgn}(B_r^\beta)dW_r)\}\big].$$

We note $\mathcal{W}_t^{\beta,(-|x|)}=-|x|-\int_0^t \mathrm{sgn}(B_s^\beta)dW_s$ and notice that this is a Brownian motion starting from $-|x|$. Along the event $\{t\geq\tau_0\}$ we have $\max_{0\leq s\leq t}\mathcal{W}_s^{\beta,(-|x|)}\geq 0$, and thus

$$\begin{pmatrix}|B_t^\beta|\\ L_t^0(B^\beta)\end{pmatrix}=\begin{pmatrix}\max_{0\leq s\leq t}\mathcal{W}_s^{\beta,(-|x|)}-\mathcal{W}_t^{\beta,(-|x|)}\\ \max_{0\leq s\leq t}\mathcal{W}_s^{\beta,(-|x|)}\end{pmatrix}. \tag{11}$$

But the Tanaka formula applied to W gives

$$|W_t| = |x| + \int_0^t \mathrm{sgn}(W_s) dW_s + L_t^0(W).$$

Thus setting $\mathcal{W}_t^{(-|x|)} = -|x| - \int_0^t \mathrm{sgn}(W_s) dW_s$, and applying the same method as above, we get that on the event $\{t \geq T_0\}$,

$$\begin{pmatrix} |W_t| \\ L_t^0(W) \end{pmatrix} = \begin{pmatrix} \max_{0\leq s\leq t} \mathcal{W}_s^{(-|x|)} - \mathcal{W}_t^{(-|x|)} \\ \max_{0\leq s\leq t} \mathcal{W}_s^{(-|x|)} \end{pmatrix}. \tag{12}$$

As $\mathcal{W}^{\beta,(-|x|)}$ and $\mathcal{W}^{(-|x|)}$ are both Brownian motion starting from $-|x|$ and τ_0 and T_0 have the same distribution, we can infer, comparing (11) and (12), that (10) holds. We proceed in a similar manner for $x = 0$, and noticing that $\mathbb{P}^0[t \geq \tau_0] = \mathbb{P}^0[t \geq T_0] = 1$ we get (9).

□

Lemma 4.2 *Let W be a Brownian motion on $(C, \mathcal{C}, \mathbb{P})$. We have for all $x \in \mathbb{R}$, all $t, l > 0$ and $y \geq 0$,*

$$\mathbb{P}^x[\, |W_t| \in dy; L_t^0(W) \in dl\,] = \frac{2(l + y + |x|)}{\sqrt{2\pi t^3}} \exp\big\{ - \frac{(l + y + |x|)^2}{2t} \big\} dy dl.$$

Proof. First, by adapting the proof of Proposition 2.8.1 in [11], we get, for all $x_0 \in \mathbb{R}$, $a \leq b$ and $b \geq x_0$,

$$\mathbb{P}^{x_0}[\, W_t \in da; \max_{0\leq s\leq t} W_s \in db\,] = \frac{2(2b - a - x_0)}{\sqrt{2\pi t^3}} \exp\big\{ - \frac{(2b - a - x_0)^2}{2t} \big\} da db.$$

Let $\Phi(a, b) := (b - a, b)$. As we have seen in the proof of Lemma 4.1 we have $(|W_t|, L_t^0(W)) = \Phi(\mathcal{W}_t^{(-|x|)}, \max_{0\leq s\leq t} \mathcal{W}_s^{(-|x|)})$ on the event $\{t \geq T_0\}$, where $\mathcal{W}^{(-|x|)}$ is a Brownian motion starting from $-|x|$ (under $\mathbb{P}^x$). But for $l > 0$ we have $\{L_t^0(W) \in dl\} \subset \{t \geq T_0\}$. A change of variable formula gives the desired result.

□

Lemma 4.3 *We have for all $t > 0$ and $y, l \geq 0$,*

$$\mathbb{P}^0\big[\, B_t^\beta \in dy; L_t^0(B^\beta) \in dl\,\big] = \alpha \mathbb{P}^0\big[\, |B_t^\beta| \in dy; L_t^0(B^\beta) \in dl\big].$$

For all $t > 0$ and $y < 0$,

$$\mathbb{P}^0\big[\, B_t^\beta \in dy; L_t^0(B^\beta) \in dl\,\big] = (1 - \alpha) \mathbb{P}^0\big[\, |B_t^\beta| \in (-dy); L_t^0(B^\beta) \in dl\big].$$

Proof. Our starting point is the construction of the Skew Brownian motion from a reflecting Brownian Motion with a change of sign of each excursion with probability $1 - \alpha$ as explained in [20] page 487 exercise 2.16 (we use the same notations as [20] in the explanations below). More precisely, let $(Y_n)_{n\geq 0}$ be a sequence of independent r.v.'s taking the values 1 and -1 with probabilities α and $1 - \alpha$ and independent of some Brownian Motion B. We note $\mathcal{H} := \sigma(Y_n : n \geq 0)$ the corresponding σ-algebra generated by the whole sequence $(Y_n)_{n\geq 0}$. We denote $\mathcal{E} := \sigma(\mathrm{e}_s : s \geq 0)$ the σ-algebra generated by all the excursions of B. For each ω in the set on which B is defined, the set of excursions $\mathrm{e}_s(\omega)$ is countable and may be ordered. Define a process B_t^β by putting $B_t^\beta(\omega) = Y_{n_s(\mathbf{e})}(\omega) |\mathrm{e}_s(t - \tau_{s-}(\omega), \omega)|$ if $\tau_{s-} \leq t \leq \tau_s$ and where e_s is the $n_s(\mathbf{e})$-th excursion in the above ordering ($n_s(\mathbf{e})$ is a random variable measurable w.r.t. $\mathcal{E}$, which depends on the whole excursion process $\mathbf{e} = (\mathrm{e}_u)_{u>0}$ and the time variable s in the local time scale). It may be proved that the process thus obtained is a Markov process and that it is a Skew Brownian motion of parameter α.

By construction $\mathcal{H} := \sigma(Y_n \ : \ n \geq 0)$ and $\mathcal{E} := \sigma(\mathrm{e}_s \ : \ s \geq 0)$ are independent ; in particular, if we denote by $R(\mathrm{e})$ the end point of excursion e, then we have $\tau_t(\omega) = \sum_{s \leq t} R(\mathrm{e}_s(\omega))$ and thus τ_t is measurable w.r.t. $\mathcal{E}$. Because $L_t(B^\beta)$ is recovered as the r.c.l inverse of τ_t, it is measurable w.r.t. $\mathcal{E}$ and independent of $\mathcal{H}$.

Let $\mathcal{S}$ be the space of real sequences $(a_k)_{k\in\mathbb{N}}$ and denote $\Phi : \mathcal{S} \times \mathbb{N} \to \mathbb{R}$ the coordinate function defined by $\Phi((a_k)_{k\in\mathbb{N}}, n) = a_n$. From the independence of $\mathcal{H}$ and $\mathcal{E}$, the properties of the conditional expectation, and since $y \geq 0$,

$$
\begin{aligned}
&\mathbb{P}^0\big[\, B^\beta_t \in dy; L^0_t(B^\beta) \in dl \,\big] = \mathbb{E}^0\Big[\mathbb{P}^0\big[\, B^\beta_t \in dy; L^0_t(B^\beta) \in dl \,|\, \mathcal{E} \,\big]\Big] \\
&= \mathbb{E}^0\Big[\mathbb{P}^0\big[\, Y_{n_s(\mathbf{e})} > 0; |\mathrm{e}_s(t - \tau_{s-}(\omega), \omega)| \in dy; L^0_t(B^\beta) \in dl \,|\, \mathcal{E} \,\big]\Big] \\
&= \mathbb{E}^0\Big[\mathbb{P}^0\big[\, \Phi((Y_k)_{k\in\mathbb{N}}(\omega), n_s(\mathrm{e}(\omega))) > 0 \,|\, \mathcal{E} \,\big]; |\mathrm{e}_s(t - \tau_{s-}(\omega), \omega)| \in dy;\ L^0_t(B^\beta) \in dl\Big] \\
&= \mathbb{E}^0\Big[\mathbb{P}^0\big[Y_n > 0\big] \ |_{n=n_s(\mathrm{e}(\omega))}; |\mathrm{e}_s(t - \tau_{s-}(\omega), \omega)| \in dy; L^0_t(B^\beta) \in dl\Big] \\
&= \alpha\mathbb{P}^0\big[\, |B^\beta_t| \in dy; L^0_t(B^\beta) \in dl\big].
\end{aligned}
$$

□

Proof of Proposition 4.2. *Step 1.* Using Lemmas 4.1 and 4.3 we have for $y \geq 0$

$$
\begin{aligned}
\mathbb{P}^0\big[\, B^\beta_t \in dy; L^0_t(B^\beta) \in dl \,\big] &= \alpha\mathbb{P}^0\big[\, |B^\beta_t| \in dy; L^0_t(B^\beta) \in dl\big] \\
&= \alpha\mathbb{P}^0\big[\, |W_t| \in dy; L^0_t(W) \in dl \,\big].
\end{aligned}
$$

Note that, therefore, using Lemma 4.2 we have the result for $x = 0$ (we use similar arguments for $y < 0$).

Step 2. Let $x, l > 0$, $y \geq 0$. As $\mathbb{P}[B^\beta_t \in dy; L^0_t(B^\beta) \in dl; t < \tau_0] = 0$, we have, using the strong Markov property,

$$
\begin{aligned}
\mathbb{P}^x[\, B^\beta_t \in dy; L^0_t(B^\beta) \in dl \,] &= \mathbb{P}^x[\, |B^\beta_t| \in dy; L^0_t(B^\beta) \in dl; B^\beta_t > 0; t \geq \tau_0 \,] \\
&= \mathbb{E}^x[\mathbf{1}_{\{t\geq\tau_0\}}\mathbb{P}^x[\, |B^\beta_t| \in dy; L^0_t(B^\beta) \in dl; B^\beta_t > 0 \,|\mathcal{F}_{\tau_0}]] \qquad (13) \\
&= \int_0^t \mathbb{P}^0[|B^\beta_{t-s}| \in dy; L^0_{t-s}(B^\beta) \in dl; B^\beta_{t-s} > 0]h(x, s)ds,
\end{aligned}
$$

as $h(x, .)$ is the density of τ_0 under $\mathbb{P}^x$. But $h(x, .)$ is also the density of T_0. And using the first step of the proof we have

$$
\mathbb{P}^0[|B^\beta_{t-s}| \in dy; L^0_{t-s}(B^\beta) \in dl; B^\beta_{t-s} > 0] = \alpha\mathbb{P}^0[\, |W_{t-s}| \in dy; L^0_{t-s}(W) \in dl \,]. \qquad (14)
$$

Using again the strong Markov property we get

$$
\begin{aligned}
\mathbb{P}^x[\, B^\beta_t \in dy; L^0_t(B^\beta) \in dl \,] &= \mathbb{E}^x[\mathbf{1}_{\{t\geq T_0\}}\alpha\mathbb{P}^x[\, |W_t| \in dy; L^0_t(W) \in dl \,|\mathcal{F}_{T_0}]] \\
&= \alpha\mathbb{P}^x[\, |W_t| \in dy; L^0_t(W) \in dl; t \geq T_0 \,] \qquad (15) \\
&= \alpha\mathbb{P}^x[\, |W_t| \in dy; L^0_t(W) \in dl \,]
\end{aligned}
$$

Step 3. It is a consequence of the reflexion principle that

$$
\begin{aligned}
\mathbb{P}^x[\, B^\beta_t \in dy; L^0_t(B^\beta) = 0 \,] &= \mathbb{P}^x[\, B^\beta_t \in dy; t < \tau_0 \,] \\
&= \tfrac{1}{\sqrt{2\pi t}}\big(\exp\{-\tfrac{(y-x)^2}{2t}\} - \exp\{-\tfrac{(y+x)^2}{2t}\}\big).
\end{aligned}
$$

Using Step 1 to 3 we have the result for $x \geq 0$ and $y \geq 0$. For $y < 0$ we use Step 1 and 2 with α replaced by $1-\alpha$ (for $l > 0$), and the fact that $\mathbb{P}^x[\, B_t^\beta \in dy; L_t^0(B^\beta) = 0\,] = \mathbb{P}^x[\, B_t^\beta \in dy; t < \tau_0\,] = 0$. □

Proof of Proposition 4.1. We have

$$dB_t^{\beta,\mu} = dW_t^\mu + \beta dL_t^0(B^{\beta,\mu}),$$

with $W_t^\mu = W_t + \mu t$ a Brownian motion under $\mathbb{Q}^\mu$ defined by $\dfrac{d\mathbb{Q}^\mu}{d\mathbb{P}} = \exp\{-\mu W_t - \dfrac{1}{2}\mu^2 t\}$. Note that under $\mathbb{Q}^\mu$ the process $B^{\beta,\mu}$ starting from 0 is distributed as B^β starting from 0 under $\mathbb{P}$.

For any bounded continuous function f and any $t \geq 0$, we have

$$\begin{aligned}
\mathbb{E}^x_{\mathbb{P}}[f(B_t^{\beta,\mu})] &= \mathbb{E}^0_{\mathbb{P}}[f(B_t^{\beta,\mu} + x)] \\
&= \mathbb{E}^0_{\mathbb{Q}^\mu}[f(B_t^{\beta,\mu} + x)\exp\{\mu W_t^\mu - \tfrac{1}{2}\mu^2 t\}] \\
&= \textstyle\int\int_{\mathbb{R}^2} f(y+x)\exp\{\mu w - \frac{1}{2}\mu^2 t\}\mathbb{P}^0[B_t^\beta \in dy;\, W_t \in dw] \\
&= \textstyle\int\int_{\mathbb{R}^2} f(y)\exp\{\mu w - \frac{1}{2}\mu^2 t\}\mathbb{P}^{x,0}[B_t^\beta \in dy;\, W_t \in dw]
\end{aligned} \tag{16}$$

Suppose $\beta > 0$.

We set $\Phi_x(z,l) = (z, z-x-\beta l)$ which defines a bijection $\Phi_x : \mathbb{R}\times\mathbb{R}_+ \to D_x$ where $D_x = \{(y,w) \in \mathbb{R}^2 : y - x \geq w\}$. Note that $(B_t^\beta, W_t) = \Phi_x(B_t^\beta, L_t^0(B^\beta))$. Besides, almost surely, $(B^\beta, L^0(B^\beta)) \in \mathbb{R}\times\mathbb{R}_+$ and $(B^\beta, W) \in D_x$.

For $x > 0$, Proposition 4.2 ensures that the measure $\mathbb{P}^x[\, B_t^\beta \in dy; L_t^0(B^\beta) \in dl\,]$ has a density with respect to $dy\,dl$ on $\mathbb{R}\times\mathbb{R}_+^*$, and gives mass to the segments of $\mathbb{R}_+ \times \{0\}$ with the density

$$\mathbb{P}^x[\, B_t^\beta \in dy; L_t^0(B^\beta) = 0\,] = \frac{1}{\sqrt{2\pi t}}\big(\exp\{-\frac{(y-x)^2}{2t}\} - \exp\{-\frac{(y+x)^2}{2t}\}\big)dy. \tag{17}$$

Let us denote $\Delta_x = \{(y,w) \in \mathbb{R}_+\times\mathbb{R} : y = w + x\}$. The measure $\mathbb{P}^{x,0}[B_t^\beta \in dy;\, W_t \in dw]$ has a density $g^{x,0}_{B^\beta,W}(y,w)$ with respect to $dy\,dw$ on $D_x \setminus \Delta_x = \Phi_x(\mathbb{R}\times\mathbb{R}_+^*)$. But it gives mass to the segments of the line Δ_x. Let us denote $\Phi_x^{-1}(y,w) = (\Phi_1^{-1}(y,w), \Phi_2^{-1}(y,w))$ and notice that $\Phi_1^{-1}(y,w) = y$. Let $A_1 \subset \mathbb{R}_+$ and $A = \{(y,w) \in \mathbb{R}^2 : y \in A_1, y = w + x\} \subset \Delta_x$. As $\Phi_x^{-1}(A) \subset \mathbb{R}\times\{0\}$ we have

$$\begin{aligned}
\mathbb{P}^{x,0}[(B_t^\beta, W_t) \in A] &= \mathbb{P}^x[(B_t^\beta, L_t^0(B^\beta)) \in \Phi_x^{-1}(A)] \\
&= \mathbb{P}^x[B_t^\beta \in \Phi_1^{-1}(A); L_t^0(B^\beta) = 0\,] \\
&= \mathbb{P}^x[B_t^\beta \in A_1; L_t^0(B^\beta) = 0\,].
\end{aligned}$$

Using this and (17) in (16) we get

$$\begin{aligned}
\mathbb{E}^x_{\mathbb{P}}[f(B_t^{\beta,\mu})] &= \textstyle\int\int_{D_x\setminus\Delta_x} f(y)\exp\{\mu w - \frac{1}{2}\mu^2 t\}\mathbb{P}^{x,0}[B_t^\beta \in dy;\, W_t \in dw] \\
&\quad + \textstyle\int\int_{\Delta_x} f(y)\exp\{\mu w - \frac{1}{2}\mu^2 t\}\mathbb{P}^{x,0}[B_t^\beta \in dy;\, W_t \in dw] \\
&= \textstyle\int_{\mathbb{R}} f(y)\int_{-\infty}^{y-x}\exp\{\mu w - \frac{1}{2}\mu^2 t\}g^{x,0}_{B^\beta,W}(y,w)dw\,dy \\
&\quad + \textstyle\int_{\mathbb{R}_+} f(y)\frac{1}{\sqrt{2\pi t}}\exp\{\mu(y-x) - \frac{1}{2}\mu^2 t\}\big(\exp\{-\frac{(y-x)^2}{2t}\} - \exp\{-\frac{(y+x)^2}{2t}\}\big)dy.
\end{aligned}$$

We now compute $\int_{-\infty}^{y-x}\exp\{\mu w - \frac{1}{2}\mu^2 t\}g^{x,0}_{B^\beta,W}(y,w)dw$ with a change of variable and an integration by parts. We have for $y \geq 0$,

$$\int_{-\infty}^{y-x}\exp\{\mu w - \frac{1}{2}\mu^2 t\}g^{x,0}_{B^\beta,W}(y,w)dw = \frac{e^{-\frac{1}{2}\mu^2 t}}{\beta}\int_{-\infty}^{y-x} e^{\mu w}\frac{2\alpha(\frac{y-w-x}{\beta} + x + y)}{\sqrt{2\pi t^3}}e^{-\frac{(\frac{y-w-x}{\beta}+x+y)^2}{2t}}dw.$$

And,

$$\int_{-\infty}^{y-x} e^{\mu w}(\tfrac{y-w-x}{\beta}+x+y)e^{-\frac{(\frac{y-w-x}{\beta}+x+y)^2}{2t}}dw = \beta e^{\mu(y-x)}\int_0^{\infty} e^{-\beta\mu w'}(w'+x+y)e^{-\frac{(w'+x+y)^2}{2t}}dw'$$

$$= \beta e^{\mu(y-x)}\left(te^{-\frac{(x+y)^2}{2t}} - \beta\mu t\int_0^{\infty} e^{-\beta\mu w'-\frac{(w'+x+y)^2}{2t}}dw'\right)$$
$$= \beta e^{\mu(y-x)}\left(te^{-\frac{(x+y)^2}{2t}} - \sqrt{2\pi}\beta\mu t^{3/2}e^{\frac{\beta^2}{2}\mu^2 t}e^{\beta\mu(x+y)}N^c(\tfrac{x+y+t\beta\mu}{\sqrt{t}})\right)$$
$$= \beta\, t\, e^{\mu(y-x)}e^{-\frac{(x+y)^2}{2t}}\left(1-\sqrt{2\pi t}\beta\mu e^{\frac{(x+y+\beta\mu t)^2}{2t}}N^c(\tfrac{x+y+t\beta\mu}{\sqrt{t}})\right)$$

which yields the desired result. The cases $y<0$ and $\beta<0$ are treated in a similar way.

For the case $x<0$, we perform the change of variable $x\to -x$, $y\to -y$, $\alpha\to 1-\alpha$ (which implies the change of variable $\beta\to-\beta$) and $\mu\to-\mu$.

□

4.3 Simulation of bridges of $B^{\beta,\mu}$

For $0<t<T$ let us denote $q^{\beta,\mu}(t,T,a,b,y)$ the density of $B_t^{\beta,\mu}$ knowing that $B_0^{\beta,\mu}=a$ and $B_T^{\beta,\mu}=b$. That is to say

$$\mathbb{P}[B_t^{\beta,\mu}\in dy \,|\, B_0^{\beta,\mu}=a\,,\, B_T^{\beta,\mu}=b] = q^{\beta,\mu}(t,T,a,b,y)dy.$$

Remark 4.4 Remember that for all $\mu\in\mathbb{R}$,

$$\forall 0<t<T,\ \forall a,b,y\in\mathbb{R},\quad q^{0,\mu}(t,T,a,b,y)=q^{0,0}(t,T,a,b,y).$$

One may use the two following lemmas in order to sample along the law given by $q^{\beta,\mu}(t,T,a,b,y)$ using a standard acceptance/rejection algorithm of Brownian bridges random values.

Lemma 4.4 *Let $a,b\in\mathbb{R}$, $0<t<T$.*
For $(\beta,\mu)\in(-1,1)\times\mathbb{R}$, we have

$$\forall y\in\mathbb{R},\quad q^{\beta,\mu}(t,T,a,b,y)=\frac{p^{\beta,\mu}(t,a,y)p^{\beta,\mu}(T-t,y,b)}{p^{\beta,\mu}(T,a,b)}. \tag{18}$$

Lemma 4.5 *Let $a,b\in\mathbb{R}$, $0<t<T$.*

- *For $(\beta,\mu)\in(-1,1)\times\mathbb{R}$ with $\beta\mu\geq 0$, we have*

$$\forall y\in\mathbb{R},\quad q^{\beta,\mu}(t,T,a,b,y)\leq K_{T,a,b}\,q^{0,0}(t,T,a,b,y), \tag{19}$$

where

$$K_{T,a,b}:=4\overline{\alpha}^2\frac{p^{0,\mu}(T,a,b)}{p^{\beta,\mu}(T,a,b)},$$

and $\overline{\alpha}:=\max(\alpha,1-\alpha)$.

- *For $(\beta,\mu)\in(-1,1)\times\mathbb{R}$ with $\beta\mu<0$, set*

$$\gamma(t,z):=1-\beta\mu\sqrt{2\pi t}\exp(\frac{(z+t\beta\mu)^2}{2t})N^c(\frac{\beta\mu t+z}{\sqrt{t}}).$$

Then,

$$p^{\beta,\mu}(t,x,y)\leq 2\overline{\alpha}\gamma(t,|x|)p^{0,\mu}(t,x,y) \tag{20}$$

and

$$p^{\beta,\mu}(t,x,y) \le 2\overline{\alpha}\gamma(t,|y|)p^{0,\mu}(t,x,y). \tag{21}$$

In particular,

$$\forall y \in \mathbb{R}, \quad q^{\beta,\mu}(t,T,a,b,y) \le K_{T,a,b}\, q^{0,0}(t,T,a,b,y), \tag{22}$$

where

$$K_{T,a,b} := 4\overline{\alpha}^2\gamma(t,|a|)\gamma(T-t,|b|)\frac{p^{0,\mu}(T,a,b)}{p^{\beta,\mu}(T,a,b)}.$$

Proof. • Case $\beta\mu \ge 0$. Let $t > 0$ and $x \ge 0$. Looking at Proposition 4.1 it is clear that for $y < 0$,

$$p^{\beta,\mu}(t,x,y) \le 2(1-\alpha)p^{0,\mu}(t,x,y).$$

For $y \ge 0$ we have

$$\begin{aligned} p^{\beta,\mu}(t,x,y) &\le p^{0,\mu}(t,x,y) + (2\alpha-1)\exp\big\{-\tfrac{(x+y)^2}{2t} + \mu(y-x) - \tfrac{1}{2}\mu^2 t\big\} \\ &\le 2\alpha\, p^{0,\mu}(t,x,y), \end{aligned}$$

where we have used $(y-x)^2 \le (y+x)^2$ (because $x, y > 0$). We can proceed in a similar way for $x < 0$ and finally, we get that

$$\forall t > 0,\ \forall x, y \in \mathbb{R}, \quad p^{\beta,\mu}(t,x,y) \le 2\overline{\alpha}\, p^{0,\mu}(t,x,y). \tag{23}$$

Thus, using the previous inequality gives

$$\begin{aligned} q^{\beta,\mu}(t,T,a,b,y) &= \frac{p^{\beta,\mu}(t,a,y)p^{\beta,\mu}(T-t,y,b)}{p^{\beta,\mu}(T,a,b)} \\ &\le 4\overline{\alpha}^2\frac{p^{0,\mu}(T,a,b)}{p^{\beta,\mu}(T,a,b)}\frac{p^{0,\mu}(t,a,y)p^{0,\mu}(T-t,y,b)}{p^{0,\mu}(T,a,b)} \\ &\le 4\overline{\alpha}^2\frac{p^{0,\mu}(T,a,b)}{p^{\beta,\mu}(T,a,b)}q^{0,0}(t,T,a,b,y). \end{aligned} \tag{24}$$

• Case $\beta\mu < 0$. Let us note $\Gamma(t,x,y) := 1 - \beta\mu\sqrt{2\pi t}\exp\big\{\frac{(|x|+|y|+t\beta\mu)^2}{2t}\big\}N^c(\frac{\beta\mu t+|x|+|y|}{\sqrt{t}})$. For fixed $x \in \mathbb{R}$, $y \mapsto \Gamma(t,x,y)$ is an even function. As we have

$$\forall z > 0, \quad z\, e^{\frac{z^2}{2}}\int_z^\infty e^{-\frac{u^2}{2}}du < 1, \tag{25}$$

the function $z \mapsto \sqrt{2\pi}\exp(\frac{z^2}{2})N^c(z)$ has negative first derivative on $\mathbb{R}^+$. Therefore $y \mapsto \Gamma(t,x,y)$ is decreasing on $\mathbb{R}^+$ and we have $\max_{y\in\mathbb{R}}\Gamma(t,x,y) = \gamma(t,|x|)$. Using this and the same kind of computations than in the previous case we get (20). As the roles of x and y are symmetric in $\Gamma(t,x,y)$, we get (21). We then obtain (22), using the same computations than for (24). □

Remark 4.5 Note that (25) also allows to prove that $p^{\beta,\mu}(t,x,y)$ remains strictly positive (see Remark 4.2).

Exact	Euler ($\Delta t = 10^{-n}$, $n = 2, 4$)	Random Walk ($h = \frac{1}{10}, \frac{1}{200}$)
239s	17s	3.52s
	1680s	1411s

Table 1: CPU times for 10^6 drawings of X_T.

5 Numerical results and comparison with other methods

Example 1. We first deal with a toy example. We consider the following SDE

$$dX_t = dW_t - \frac{\pi}{2}\cos(\frac{\pi}{5}X_t)dt + \beta dL_t^0(X), \quad X_0 = x_0, \tag{26}$$

with $\beta = 0.6$, and $x_0 = 0.2$. Note that, here, the drift $b(x) = -\frac{\pi}{2}\cos(\frac{\pi}{5}x)$ is bounded and of class $\mathcal{C}^\infty$ on the whole real line. The constant drift involved in Section 3, equals $\mu = b(0) = -\frac{\pi}{2}$. So we will have to sample bridges of SBM with non zero drift μ, using the results of Section 4.

Our goal is to sample values of X_T with $T = 1$ and $(X_t)_{0\le t\le T}$ following (26), and to draw a histogram of them. To this end we have first to sample X_T from

$$\begin{aligned} h(y) &= C\exp\big(\bar{B}(y) - \bar{B}(x_0)\big)\, p^{\beta,\mu}(T, x_0, y) \\ &= C\exp\big(\tfrac{5}{2}\big(\sin(\tfrac{\pi}{5}x_0) - \sin(\tfrac{\pi}{5}y)\big) - \mu(y - x_0)\big)\, p^{\beta,\mu}(T, x_0, y) \end{aligned}$$

(Step 1 of the Algorithm). This can be done by rejecting standard normal random variables with mean x_0. Indeed, using (20), we have here

$$\frac{h(y)}{C} \le 2\overline{\alpha}\gamma(T, |x_0|)\exp(5 - \frac{\mu^2 T}{2})\, p^{0,0}(T, x_0, y).$$

Then we accept or reject the proposed value X_T, using Steps 2 to 4 of the algorithm, with bridges of $B^{\beta,\mu}$,

$$\tilde{\phi}(x) = \frac{\pi^2}{8}\cos^2(\frac{\pi}{5}x) + \frac{\pi^2}{20}\sin(\frac{\pi}{5}x) + \frac{\pi^2}{20},$$

and $K = \frac{9\pi^2}{20}$.

We plot on Figure 1 the histogram obtained with 10^6 drawings of X_T, sampled with our exact procedure. On the same figure we plot the histogram obtained with 10^6 drawings of the Euler Scheme used in [15] and [16], for decreasing time steps. We can observe the convergence of Euler type simulations to exact ones. Note that to have the Euler scheme fitting the exact procedure we have to take a fine time step (namely $\Delta t = 10^{-4}$). This is because, as shown in [15], the rate of weak convergence of the Euler scheme in this situation is of order $(\Delta t)^{1/2-\epsilon}$, for smooth initial condition.

On Figure 2 we plot again the histogram of our 10^6 exact drawings of X_T, but this time together with 10^6 drawings of the random walk based approximation of X_T studied in [10], for decreasing space steps. Again we can observe the convergence of the process with discretization error.

In Table 1 we report the CPU times needed to get the 10^6 drawings, with the three different methods (and with the different discretization steps we have used). Programs were written in C-language and executed on a personal computer equipped with an Intel Core 2 duo processor, running at 2.23 Ghz. On this example exact simulations is competitive, compared to schemes with very fine grids.

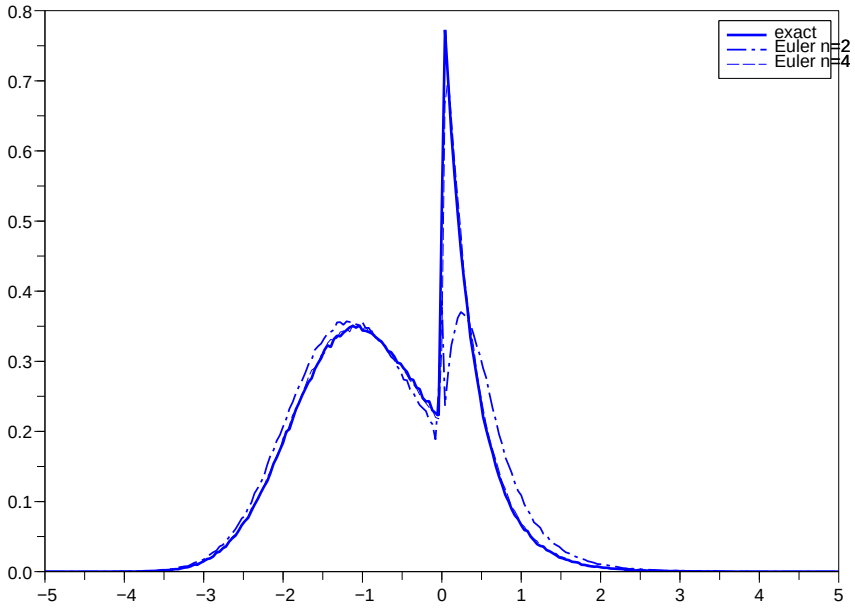

Figure 1: Histogram of the positions at time $T = 1.0$ of 10^6 paths of the solution of (26) starting from $x_0 = 0.2$: exact versus Euler with time step $\Delta t = 10^{-n}$, for $n = 2, 4$.

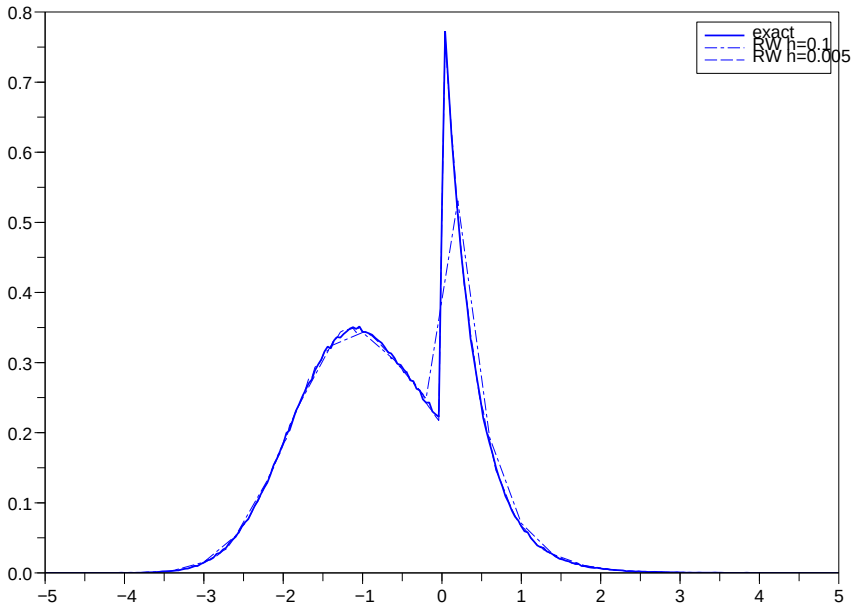

Figure 2: Histogram of the positions at time $T = 1.0$ of 10^6 paths of the solution of (26) starting from $x_0 = 0.2$: exact versus random walk with space steps $h = \frac{1}{10}, \frac{1}{200}$.

Example 2. We want now to sample along the law of the continuous Markov process X generated by

$$L = \frac{1}{2}\frac{\mathrm{d}}{dx}\big(a\frac{\mathrm{d}}{dx}\cdot\big) \tag{27}$$

with

$$a(x) = \begin{cases} \frac{x^2+x+1}{(2x+1)^2} & \text{if } x \geq 0 \\ \\ \frac{3x^2-x+2}{(6x-1)^2} & \text{if } x < 0. \end{cases}$$

Note that $a(0+) = 1 \neq 2 = a(0-)$. The coefficient $a(x)$ is of class $\mathcal{C}^1$ on $\mathbb{R}^{*,-}$ and $\mathbb{R}^{*,+}$, and uniformly strictly positive and bounded, which ensures the existence of X; in addition X solves

$$dX_t = \sqrt{a(X_t)}dW_t + \frac{a'(X_t)}{2}dt + \frac{a(0+)-a(0-)}{a(0+)+a(0-)}dL_t^0(X), \tag{28}$$

(see [13], [9]). We define the Lamperti transformation $\Phi(x) = \int_0^x dz/\sqrt{a(z)}$ and set $Y_t := \Phi(X_t)$. Then

$$dY_t = dW_t + \frac{1}{2}(\sqrt{a})' \circ \Phi^{-1}(Y_t)dt + \frac{\sqrt{a(0+)}-\sqrt{a(0-)}}{\sqrt{a(0+)}+\sqrt{a(0-)}}dL_t^0(Y), \tag{29}$$

(this follows from Proposition 3.1 in [9]; see also [13] and [17]). Firstly, note that $\left|\frac{\sqrt{a(0+)}-\sqrt{a(0-)}}{\sqrt{a(0+)}+\sqrt{a(0-)}}\right| < 1$. Secondly, we have

$$(\sqrt{a})'(x) = \begin{cases} \frac{1}{2\sqrt{x^2+x+1}} - 2\frac{\sqrt{x^2+x+1}}{(2x+1)^2} & \text{if } x \geq 0 \\ \\ -\frac{1}{2\sqrt{3x^2-x+1}} + 6\frac{\sqrt{3x^2-x+2}}{(6x-1)^2} & \text{if } x < 0, \end{cases}$$

$$\Phi(x) = \begin{cases} 2\sqrt{x^2+x+1} - 2 & \text{if } x \geq 0 \\ \\ -2\sqrt{3x^2-x+1} + 2\sqrt{2} & \text{if } x < 0, \end{cases} \quad \text{and} \quad \Phi^{-1}(y) = \begin{cases} \frac{-1+\sqrt{(y+2)^2-3}}{2} & \text{if } y \geq 0 \\ \\ \frac{1-\sqrt{1-12[2-(\sqrt{2}-y/2)^2]}}{6} & \text{if } y < 0. \end{cases}$$

As $(\sqrt{a})'(x)$ is bounded with bounded first derivative on $\mathbb{R}^{*,-}$ and $\mathbb{R}^{*,+}$, the explicitly known coefficients $\beta = \frac{\sqrt{a(0+)}-\sqrt{a(0-)}}{\sqrt{a(0+)}+\sqrt{a(0-)}}$ and $b(y) = \frac{1}{2}(\sqrt{a})' \circ \Phi^{-1}(y)$ satisfy the assumptions of section 2. Thus we can perform exact sampling from (29), and, applying the exact inverse transformation Φ^{-1}, get samples from (28) with absolutely no discretization error.

Here we have,

$$\mu = \frac{1}{4}\frac{a'(0+)-a'(0-)}{\sqrt{a(0+)}-\sqrt{a(0-)}} = -\frac{26}{4(1-\sqrt{2})}.$$

As we have

$$\bar{B}(y) = \begin{cases} -\mu y + \frac{1}{2}\log(\sqrt{a} \circ \Phi^{-1}(y)) & \text{if } y \geq 0 \\ \\ -\mu y + \frac{1}{2}[\log(\sqrt{a} \circ \Phi^{-1}(y)) - \log(\sqrt{2})] & \text{if } y < 0, \end{cases}$$

we can show (using again (20)), that for all $y_0, y \in \mathbb{R}$ and $T > 0$,

$$\exp\big(\bar{B}(y) - \bar{B}(y_0)\big)\, p^{\beta,\mu}(T, y_0, y) \leq \sqrt{\sqrt{2}}\sqrt{\sqrt{24}}\, 2\bar{\alpha}\gamma(T, |y_0|)\, e^{-\frac{1}{2}\mu^2 T} p^{0,0}(T, y_0, y).$$

This allows to sample Y_T from $h(y) = C\exp\big(\bar{B}(y) - \bar{B}(y_0)\big)\, p^{\beta,\mu}(T, y_0, y)$, by rejecting normal variables with mean y_0 and variance T.

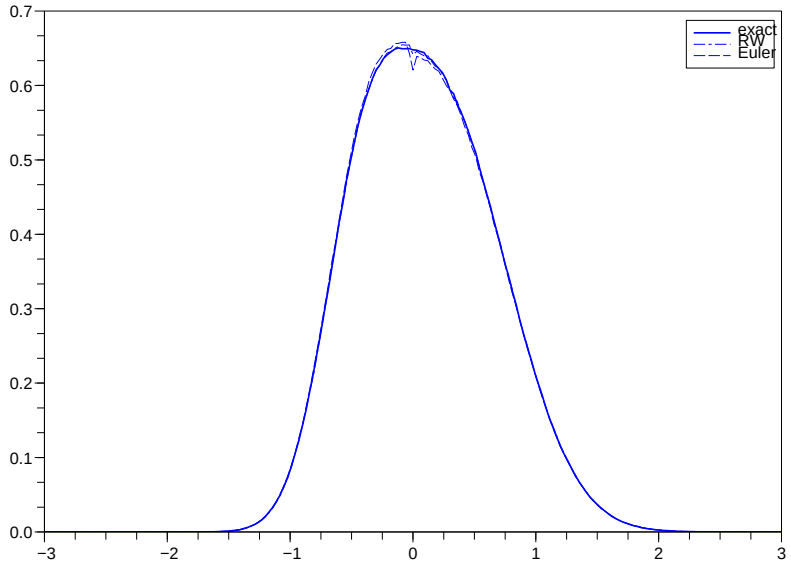

Figure 3: Histogram of the positions at time $T = 1$ of 10^7 paths of the solution of (28) starting from $x_0 = 0.0$: exact versus random walk with space step $h = 3.10^{-3}$ and Euler scheme with $\Delta t = 10^{-4}$.

We then accept or reject the proposed value Y_T by using bridges of $B^{\beta,\mu}$ and

$$\tilde{\phi}(y) = \frac{((1/2)(\sqrt{a})' \circ \Phi^{-1}(y))^2 + (1/2)((\sqrt{a})''\sqrt{a}) \circ \Phi^{-1}(y)}{2},$$

with

$$(\sqrt{a})''(x) \begin{cases} -\frac{2x+1}{4(x^2+x+1)^{3/2}} - \frac{1}{(2x+1)\sqrt{x^2+x+1}} + 8\frac{\sqrt{x^2+x+1}}{(2x+1)^3} & \text{if } x \geq 0 \\ \frac{6x-1}{4(3x^2-x+2)^{3/2}} - \frac{3}{(2x+1)\sqrt{3x^2-x+2}} + 72\frac{\sqrt{3x^2-x+2}}{(6x-1)^3} & \text{if } x < 0. \end{cases}$$

We take $K = \frac{(6\sqrt{2}-1/2)^2/4+(141-1/8)/2}{2}$ as an upper bound for $\tilde{\phi}$. We plot on Figure 3 the histogram of 10^7 drawings of X_T for $x_0 = 0.0$ and $T = 1$, obtained from the exact procedure. We plot on the same figure the histograms obtained with the Euler scheme and the random walk approximation mentioned in Example 1.

Remark 5.1 Note that, at least graphically and contrary to what we can see on Figures 1 and 2, the transition density plotted on Figure 3 seems to be continuous at 0 : this matches the well-known theoretical result, which asserts that the transition density of diffusion semigroups corresponding to elliptic divergence form operator of the form (27) is always continuous. We refer to Stroock [22] for a proof based on the self-adjoint properties of these semi-groups and Nash's inequality.

6 The degenerate case $\beta = 0$

6.1 Application of Girsanov's theorem

In this section, we briefly treat the case where $\beta = 0$. More precisely, in this section we study the solution (X_t) of the SDE

$$dX_t = dW_t + b(X_t)dt \tag{30}$$

where the function $b : \mathbb{R} \to \mathbb{R}$ is bounded and differentiable on $\mathbb{R}^{*,+}$ and $\mathbb{R}^{*,-}$ with a possible discontinuity at point $\{0\}$. In this particular case, the method of the previous section cannot be applied directly because the constant μ ceases to be well-defined. Nevertheless, this case remains tractable with the help of the results of [11] pp.440-441 concerning the transition probabilities of Brownian Motion with Two-Valued Drift.

It is not difficult to prove that $\ell\{t \in [0,T] \ : \ X_t = 0\} = 0$, so that, we may transform equation (30) as follows :

$$\begin{aligned} dX_t &= dW_t + (b(X_t)\mathbb{1}_{X_t \geq 0} + b(X_t)\mathbb{1}_{X_t<0})\, dt \\ &= dW_t + (b(X_t) - b(0+))\, \mathbb{1}_{X_t>0} dt + (b(X_t) - b(0-))\, \mathbb{1}_{X_t<0} dt + (b(0+)\mathbb{1}_{X_t \geq 0} + b(0-)\mathbb{1}_{X_t<0})\, dt \\ &= dW_t + \bar{b}(X_t)dt + (b(0+)\mathbb{1}_{X_t>0} + b(0-)\mathbb{1}_{X_t<0})\, dt \end{aligned}$$

where we have set

$$\bar{b}(z) := (b(z) - b(0+))\, \mathbb{1}_{z>0} + (b(z) - b(0-))\, \mathbb{1}_{z<0}.$$

Note that with this definition $\bar{b}$ is continuous at 0 and $\bar{b}(0) = 0$.

Following the lines of the previous sections, we suppose

- the function
$$z \mapsto \bar{\phi}(z) := \frac{\bar{b}^2(z) + \bar{b}'(z)}{2} \mathbb{1}_{\mathbb{R}^{*,+} \cup \mathbb{R}^{*,-}}(z)$$
is bounded.

  We set $\tilde{\phi}(z) := \bar{\phi}(z) - m$ with $m = \inf_{z \in \mathbb{R}} \bar{\phi}(z)$; the constant K denotes an upper bound of function $\tilde{\phi}$.

- the function $u \mapsto \exp[\bar{B}(u) - (u-x)^2/2T]$, where $\bar{B}(u) := \displaystyle\int_0^u \bar{b}(y)dy$ is integrable.

Since $\bar{b}$ is continuous at 0 and $\bar{b}(0) = 0$ applying the Itô-Tanaka formula, we have

$$\int_0^T \bar{b}(X_s)dW_s - \frac{1}{2}\int_0^t \bar{b}^2(X_s)ds = \bar{B}(X_t) - \bar{B}(x) - \int_0^t \bar{\phi}(X_s)ds,$$

and no local time of X is involved in this equality.

Performing Girsanov's theorem, we see that

$$dX_t = dW_t^{TVD} + (b(0+)\mathbb{1}_{X_t>0} + b(0-)\mathbb{1}_{X_t<0})\, dt, \tag{31}$$

where $W_t^{TVD} := W_t + \displaystyle\int_0^t \bar{b}(X_s)ds$ is a Brownian motion under the new probability $\mathbb{W}^{TVD}$ defined by

$$\frac{d\mathbb{P}}{d\mathbb{W}^{TVD}} = \exp\Big\{ \int_0^T \bar{b}(X_t)dW_t^{TVD} - \frac{1}{2}\int_0^T \bar{b}^2(X_t)dt \Big\}. \tag{32}$$

Thus,

$$\mathbb{E}_{\mathbb{P}}[F(X)] = \mathbb{E}_{\mathbb{W}^{TVD}}\big[F(X)\exp\big\{\bar{B}(X_T) - \bar{B}(x) - \int_0^T \bar{\phi}(X_t)dt\big\}\big],$$

where (X_t) under the measure $\mathbb{W}^{TVD}$ is solution of equation (31) : under $\mathbb{W}^{TVD}$, (X_t) a Brownian Motion with Two-Valued Drift (See [11] pp.440-441 for details concerning these particular Brownian Motions).

Let us now briefly sketch the algorithm that may be derived from these facts for the simulation of an exact skeleton of (X_t) (under the initial probability $\mathbb{P}$).

6.2 Algorithm

Notation : Set $\theta = (\theta_0, \theta_1) := (b(0+), b(0-))$; in the sequel and following [11] we drop any reference to θ and we denote $\tilde{p}(t,x,y)$ the transition probability density of (X_t) under $\mathbb{W}^{TVD}$. Considering (31) and (32), this density can be explicitly computed : the formulas are given for example in [11] pp.440-441.

Following the lines of Beskos and al [3], the computation performed in the above section leads naturally to the following algorithm :

Step 1 - Simulate a random variable Z according to the density

$$h(y) = C \exp\left(\bar{B}(y) - \bar{B}(x_0)\right) \tilde{p}(T, x_0, y).$$

Keep in memory the value z of Z.

Step 2 - Simulate a Poisson Point Process with unit density on $[0,T] \times [0,K]$. The result is a random number n of points $(t_1, z_1), \dots (t_n, z_n)$

Step 3 - Simulate the values of a Brownian Motion with Two-Valued Drift $B^{\text{tvd}}_{t_1}, \dots B^{\text{tvd}}_{t_n}$ conditionally on $B^{\text{tvd}}_0 = x_0$ and $B^{\text{tvd}}_T = z$.

Step 4 - If $\forall i \in \{1, \dots, n\}$ $\tilde{\phi}(B^{\text{tvd}}_{t_i}) \leq z_i$ accept the trajectory. Else return to step 1.

This algorithm returns a drawing of $(X_{t_1}, \dots, X_{t_n}, X_T)$.

7 Discussion and concluding remarks

7.1 An open problem : the path decomposition of a skew Brownian bridge

An important issue for the extension of the initial exact simulation method is to overcome the restraining assumptions made on the drift function b (see section 2.2) : namely, the assumption of boundedness for b.

For example, it is frustrating that these assumptions do not allow us to simulate exactly what one may call the "Skewed Ornstein-Uhlenbeck" diffusion process. This difficulty appears even in the classical case (solutions of non skewed SDEs) and the fundamental reason is that we do not know how to simulate exactly a Poisson Process with σ-finite intensity on the whole space $\mathbb{R}$.

In the classical case, where b is everywhere differentiable and no local time is involved (non skewed SDEs), this problem is solved by decomposing the trajectory of the standard Brownian bridge on $[0,T]$ w.r.t. the space-time point where it attains its maximum or both its maximum and is minimum : we refer to [4] for a detailed presentation of this problem in the classical setting.

Consequently, if one wants to overcome the restraining assumptions of section 2.2 concerning the drift function b, one has to search for such kind of decompositions for (at least) the Skew Brownian Motion (not to mention the drifted Skew Brownian Motion). Unfortunately, we did not manage to obtain this decomposition. Up to our knowledge, no results can be found in the literature concerning this decomposition and this open problem seems difficult to us. However, we give below some insight concerning this problem thanks to an application of a theoretical result stated in [18].

Set $\alpha := \frac{\beta+1}{2}$. Let $\tau_z^{\beta,\mu} := \inf(s \geq 0 \ : \ B_s^{\beta,\mu} = z)$. Set $u_\lambda(x;z) := \mathbb{E}_x\left(\mathrm{e}^{-\lambda \tau_z^{\beta,\mu}}\right)$ which gives the Laplace transform of $\tau_z^{\beta,\mu}$ at $\lambda > 0$ (with $B_0^{\beta,\mu} = x$).

Proposition 7.1 *(case $\mu = 0$)*

In the simple case where $\mu = 0$, the function u_λ is given by

$$u_\lambda(x;z) = \begin{cases} \dfrac{\sinh\left(\sqrt{2\lambda}(z-x)\right)}{\sinh(\sqrt{2\lambda}z)} \dfrac{\alpha}{\cosh(\sqrt{2\lambda}z) - (1-\alpha)\mathrm{e}^{-\sqrt{2\lambda}z}} & \\ \quad + \dfrac{\sinh(\sqrt{2\lambda}x)}{\sinh(\sqrt{2\lambda}z)} & \text{if } z \geq x > 0, \\ \mathrm{e}^{-\sqrt{2\lambda}(x-z)} & \text{if } x \geq z \geq 0, \\ \mathrm{e}^{\sqrt{2\lambda}x} \dfrac{\alpha}{\cosh(\sqrt{2\lambda}z) - (1-\alpha)\mathrm{e}^{-\sqrt{2\lambda}z}} & \text{if } x < 0 < z, \\ \mathrm{e}^{\sqrt{2\lambda}(x-z)} & \text{if } 0 \geq z \geq x, \\ \mathrm{e}^{-\sqrt{2\lambda}x} \dfrac{1-\alpha}{\cosh(\sqrt{2\lambda}z) - \alpha\mathrm{e}^{\sqrt{2\lambda}z}} & \text{if } z < 0 < x, \\ \dfrac{\sinh\left(\sqrt{2\lambda}(z-x)\right)}{\sinh(\sqrt{2\lambda}z)} \dfrac{1-\alpha}{\cosh(\sqrt{2\lambda}z) - \alpha\mathrm{e}^{\sqrt{2\lambda}z}} & \\ \quad + \dfrac{\sinh(\sqrt{2\lambda}x)}{\sinh(\sqrt{2\lambda}z)} & \text{if } 0 > x \geq z. \end{cases} \tag{33}$$

Remark 7.1 Note that if $\alpha = 1/2$, we retrieve after easy computations the well known result that gives the Laplace Transform of the law of the hitting time of z by a standard Brownian Motion starting from x.

Proof. We only sketch the proof. The different cases may be easily conjectured from the description of the excursion measure for the SBM $\left(B_s^{\beta,0}\right)_{s\geq 0}$ and the known facts concerning the standard Brownian Motion (decomposition of the different cases when a skew Brownian Motion reaches z starting from x). In order to check rigorously the validity of the result, one may verify that the formulas (33) yield a solution of Dynkin's problem associated to the generator of $\left(B_s^{\beta,0}\right)_{s\geq 0}$ namely :

$$\begin{cases} \dfrac{1}{2}\dfrac{d^2}{dx^2}u_\lambda(.;z) = \lambda u_\lambda(.;z) \\ u_\lambda(z;z) = 1, \end{cases} \tag{34}$$

with

$$u_\lambda(.;z) \in \{g \in C^0(\mathbb{R}) \cap C^2\left((-\infty,0)\cup(0,\infty)\right) \ : \ \alpha g'(0+) = (1-\alpha)g'(0-)\}.$$

□

A scale function s and the corresponding integrated speed measure m of a Skew Brownian Motion are given by

$$s(x) = \begin{cases} \frac{2}{\beta+1}x & \text{if } x \geq 0 \\ \frac{2}{1-\beta}x & \text{if } x < 0 \end{cases} \quad ; \quad m(x) = \begin{cases} (\beta+1)x & \text{if } x \geq 0 \\ (1-\beta)x & \text{if } x < 0. \end{cases}$$

(see [14]). In particular, the density $\ell^{\beta,0}(t,x,y)dy$ of the SBM w.r.t. the speed measure $m(dy)$ is

given by

$$\ell^{\beta,0}(t,x,y) = \begin{cases} \dfrac{1}{2\alpha\sqrt{2\pi t}}\big(\exp\{-\dfrac{(y-x)^2}{2t}\} - \exp\{-\dfrac{(y+x)^2}{2t}\}\big) \\ +\dfrac{1}{\sqrt{2\pi t}}\exp\big\{-\dfrac{(x+y)^2}{2t}\big\}, \text{ if } x>0, y>0; \\ \dfrac{1}{\sqrt{2\pi t}}\exp\big\{-\dfrac{(x-y)^2}{2t}\big\}, \text{ if } x>0, y<0 \text{ or if } x<0, y>0; \\ \dfrac{1}{2(1-\alpha)\sqrt{2\pi t}}\big(\exp\{-\dfrac{(y-x)^2}{2t}\} - \exp\{-\dfrac{(y+x)^2}{2t}\}\big) \\ +\dfrac{1}{\sqrt{2\pi t}}\exp\big\{-\dfrac{(x+y)^2}{2t}\big\}, \text{ if } x<0, y<0. \end{cases}$$

(of course $\ell^{\beta,0}(t,x,y) = \ell^{\beta,0}(t,y,x)$).

Let

$$M_T^{\beta,0} := \sup_{0\le s\le t} B_s^{\beta,0} \; ; \quad \rho_T^{\beta,0} := \inf\{s \ge 0 \; : \; B_s^{\beta,0} = M_s^{\beta,0}\}.$$

Then, applying the results of Theorem 2 in Pitman-Yor [18], we have the following proposition :

Proposition 7.2

1. *For any $a, b \le z < \infty$, $\lambda > 0$, we have that*

$$\mathbb{E}\left(\mathrm{e}^{-\lambda\rho_T^{\beta,0}}\mathbb{1}_{M_T^{\beta,0}\in dz} \;\middle|\; B_0^{\beta,0} = a, B_T^{\beta,0} = b\right) = \frac{u_\lambda(a;z)u_\lambda(z;b)}{\ell^{\beta,0}(T,a,b)}s(dz). \tag{35}$$

2. *Moreover, under $\mathbb{P}\left(. \;\middle|\; B_0^{\beta,0} = a, B_T^{\beta,0} = b, M_T^{\beta,0} = z, \rho_T^{\beta,0} = u\right)$, the path fragments*

$$\left(B_s^{\beta,0} \; : \; 0 \le s \le u\right) \quad \left(B_{T-s}^{\beta,0} \; : \; 0 \le s \le T-u\right)$$

are independent, distributed respectively like

$$\left(B_s^{\beta,0} \; : \; 0 \le s \le \tau_z^{\beta,0}\right) \text{ under } \mathbb{P}\left(. \;\middle|\; B_0^{\beta,0} = a\right) \text{ given } \tau_z^{\beta,0} = u$$

and

$$\left(B_s^{\beta,0} \; : \; 0 \le s \le \tau_z^{\beta,0}\right) \text{ under } \mathbb{P}\left(. \;\middle|\; B_0^{\beta,0} = b\right) \text{ given } \tau_z^{\beta,0} = T-u.$$

An open problem is to find a description of these laws and to give a procedure in order to simulate these laws exactly.

7.2 Conclusion

In this paper we presented an extension of the exact simulation method of [3] that permits to produce an exact sample skeleton of a one dimensional diffusion process skewed at 0. This method may be applied to diffusions related to strongly elliptic divergence form operators that possess a discontinuous coefficient at 0.

In our opinion, this first work should be extended in several directions :

- How can we overcome the restraining boundedness assumption on the drift function b ?
- What about one dimensional diffusion process skewed at a finite number of points ?
- How do we extend the method to the multidimensional case ?

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