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# On the enumeration of labelled hypertrees and of labelled bipartite trees

Roland Bacher

February 13, 2011

*Abstract*<sup>1</sup>: We give a simple formula for the number of hypertrees with  $k$  hyperedges of given sizes and  $n + 1$  labelled vertices with prescribed degrees. A slight generalization of this formula counts labelled bipartite trees with prescribed degrees in each class of vertices.

## 1 Main results

### 1.1 Labelled hypergraphs

A (*finite*) *hypergraph* is a pair  $(\mathcal{V}, \mathcal{E})$  consisting of a finite set  $\mathcal{V}$  of *vertices* and of a set  $\mathcal{E}$  of *hyperedges* given by subsets of  $\mathcal{V}$  containing at least two elements. We define the *size* of a hyperedge  $E$  as the number  $\text{size}(E)$  of vertices contained in  $E$  and the *degree*  $\deg v$  of a vertex  $v$  as the number of hyperedges containing  $v$ . A vertex of degree 1 is also called a *leaf*. The obvious linear relation

$$\sum_{v \in \mathcal{V}} \deg(v) = \sum_{E \in \mathcal{E}} \text{size}(E) \quad (1)$$

links the total sum of vertex-degrees to the total sum of hyperedge-sizes.

Two distinct vertices  $v, w \in \mathcal{V}$  are *adjacent* or *neighbours* if they are both contained in some hyperedge  $E$  of  $\mathcal{E}$ . A *path* of length  $k$  joining two vertices  $v, w$  in a hypergraph  $(\mathcal{V}, \mathcal{E})$  is a sequence  $v_0 = v, v_1, \dots, v_k = w$  involving only consecutively adjacent vertices. A hypergraph is *connected* if any pair of vertices can be joined by a path. The (*combinatorial*) *distance* between two vertices  $v, w$  of a connected hypergraph is the minimal length of a path in the set of all paths joining  $v$  and  $w$ . A *cycle* of length  $k$  (for  $k \geq 3$ ) is a closed path consisting of  $k$  distinct vertices. A *hyperforest* is a hypergraph  $F$  such that two distinct hyperedges of  $F$  intersect in at most a common vertex and such that every cycle of  $F$  is contained in a hyperedge. A *hypertree* is

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a connected hyperforest. Induction on the number  $k$  of hyperedges in a hypertree consisting of  $n + 1$  vertices shows the relation

$$\sum_{E \in \mathcal{E}} \text{size}(E) = n + k . \quad (2)$$

Let  $\lambda$  be a partition of  $n = \sum_{j=1}^k \lambda_j$  having exactly  $k \leq n$  nonzero parts  $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_k \geq 1$ . Let  $\mu = (\mu_0, \dots, \mu_n) \in \mathbb{N}^{n+1}$  be a vector with coefficients formed by  $n + 1$  natural integers  $\mu_0, \dots, \mu_n$  summing up to  $k - 1 = \sum_{j=0}^n \mu_j$ .

**Theorem 1.1.** *The number of hypertrees having  $n + 1$  vertices  $\{0, \dots, n\}$  of degrees  $\deg(i) = 1 + \mu_i$  and  $k$  hyperedges of sizes  $1 + \lambda_1, \dots, 1 + \lambda_k$  is given by*

$$\binom{n}{\lambda} \frac{1}{\prod_{j=1}^n (\nu_j)!} \binom{k-1}{\mu} = \frac{n!}{\left(\prod_{j=1}^k \lambda_j!\right)} \frac{1}{\left(\prod_{j=1}^n (\nu_j)!\right)} \frac{(k-1)!}{\left(\prod_{j=0}^n \mu_j!\right)} \quad (3)$$

with  $\nu_j = \#\{i \mid \lambda_i = j\}$  counting parts of length  $j$  in  $\lambda$ .

Theorem 1.1 has the following equivalent formulation:

**Theorem 1.2.** *Denoting by  $\mathcal{HT}_\lambda(n + 1)$  the set of hypertrees with  $n + 1$  labelled vertices  $\{0, \dots, n\}$  and with  $k$  edges of size  $1 + \lambda_1, 1 + \lambda_2, \dots, 1 + \lambda_k$ , we have*

$$\sum_{T \in \mathcal{HT}_\lambda(n+1)} \prod_{j=0}^n x_j^{\deg(j)} = \binom{n}{\lambda} \frac{1}{\prod_{j=1}^n (\nu_j)!} (x_0 + x_1 + \dots + x_n)^{k-1} . \quad (4)$$

Equivalence between Theorem 1.1 and 1.2 is given by the binomial Theorem.

The identities (1) and (2) show that the conditions  $\sum_{i=1}^k \lambda_k = n$  and  $\sum_{i=0}^n \mu_i = k - 1$  are necessary for the existence of hypertrees with edge-sizes  $1 + \lambda_1, \dots, 1 + \lambda_k$  and vertex-degrees  $1 + \mu_0, \dots, 1 + \mu_n$ . Theorem 1.1 or 1.2 shows that they are also sufficient.

Multiplication by  $(n + 1)$  of the results given by Theorem 1.1 and 1.2 gives enumerative results for rooted labelled hypertrees.

Removal of the vertex 0 in trees enumerated by Theorem 1.1 gives enumerative results for planted forests with  $\mu_0 + 1$  connected components (inducing an error of 1 in the degree of root vertices and in the size of hyperedges containing a root).

Counting trees or hypertrees with labelled vertices is a fairly old sport and started with Sylvester [11] and Cayley [3] (according to the notes of Chapter 5 in [10]) mentioning the total number

$$(n + 1)^{n-1}$$

of labelled trees on  $n + 1$  vertices. This corresponds of course to the specialization  $x_0 = \dots = x_n = 1$  in Theorem 1.2 for  $\lambda = (1, 1, \dots, 1)$  the trivial partition of  $n$ . In [4], Erdélyi and Etherington refined Cayley's theorem by enumerating labelled (ordinary) trees with given vertex-degrees (corresponding to the case of the trivial partition  $\lambda = (1, 1, \dots, 1)$  of  $n$  in Theorem 1.1 or 1.2), see also Theorem 5.3.10 in [10]. For trees, one can also consult the monograph [8] or the numerous more recent literature.

Denoting by  $S_2(n, k)$  the Stirling number of the second kind enumerating partitions of  $\{1, \dots, n\}$  into  $k$  non-empty subsets, summing identity (4) over all partitions of  $\{1, \dots, n\}$  into exactly  $k$  parts and setting  $x_0 = x_1 = \dots = x_n = 1$  yields the number

$$(n + 1)^{k-1} S_2(n, k) \tag{5}$$

of hypertrees with  $n + 1$  labelled vertices and  $k$  hyperedges given by Husimi in [6], see also [7], [12], [5] and [1] for other treatments and related results.

It is perhaps worthwhile to mention the following two corollaries of Theorem 1.1:

The first result counts weighted hypertrees and is a kind of counterpart of Husimi's result (5) in the sense that it involves Stirling numbers of the first kind counting the number  $(-1)^{n+k} S_1(n, k)$  of permutations of  $\{1, \dots, n\}$  involving  $k$  disjoint cycles:

**Corollary 1.3.** *We have*

$$\sum_{T \in \mathcal{HT}_k(n+1)} w(T) = (-1)^{n+k} (n + 1)^{k-1} S_1(n, k)$$

or more precisely

$$\sum_{T \in \mathcal{HT}_k(n+1)} w(T) \prod_{j=0}^n x_j^{\deg(j)} = (-1)^{n+k} (x_0 + x_1 + \dots + x_n)^{k-1} S_1(n, k)$$

where  $\mathcal{HT}_k(n+1)$  denotes the set of all labelled hypertrees with  $k$  hyperedges and vertices  $\{0, \dots, n\}$ , where  $w(T) = \prod_{j=1}^k (\lambda_j - 1)!$  for a labelled hypertree  $T$  with  $k$  hyperedges of size  $1 + \lambda_1, \dots, 1 + \lambda_k$  and where  $S_1(n, k)$  is the Stirling number of the first kind defined by  $\sum_{k=0}^n S_1(n, k) x^k = \prod_{j=0}^{n-1} (x - j)$ .

**Proof** Observe that

$$\binom{n}{\lambda} \frac{1}{\prod_{j=1}^n (\nu_j)!} \prod_{j=1}^k (\lambda_j - 1)!$$

counts the number of permutations of  $\{1, \dots, n\}$  in the conjugacy class consisting of products of  $k$  disjoint cycles with lengths  $\lambda_1, \dots, \lambda_k$ . Summing

over all partitions of  $\{1, \dots, n\}$  into  $k$  non-empty subsets and applying Theorem 1.2 (with  $x_0 = \dots = x_n = 1$  for the first formula) yields the result since  $(-1)^{n+k} S_1(n, k)$  counts the total number of permutations of  $\{1, \dots, n\}$  consisting of  $k$  disjoint cycles.  $\square$

**Corollary 1.4.** *Let  $\mathcal{HT}_k(n+1)$  be the set of all labelled hypertrees with  $k$  hyperedges and  $n+1$  vertices. The two random variables given by the sizes of hyperedges and by the degrees of vertices are independent for a random hypertree  $T$  chosen with uniform probability in  $\mathcal{HT}_k(n+1)$ .*

*More precisely, a random hypertree, chosen with uniform probability among all labelled hypertrees with  $k$  hyperedges and vertices  $\{0, \dots, n\}$ , has edge-sizes  $1 + \lambda_1, \dots, 1 + \lambda_k$  associated to a partition  $\lambda$  of  $n$  with probability*

$$\frac{1}{S_2(n, k)} \binom{n}{\lambda} \frac{1}{\prod_{j=1}^k (\nu_j)!}$$

*with  $S_2(n, k)$  denoting the Stirling number of the second kind counting partitions of  $\{1, \dots, n\}$  into  $k$  non-empty subsets and with  $\nu_j = \#\{i \mid \lambda_i = j\}$  counting the number of parts equal to  $j$  in  $\lambda$ .*

*Such a random tree has vertices  $0, \dots, n$  of degrees  $1 + \mu_0, \dots, 1 + \mu_n$  with probability  $\frac{1}{(n+1)^{k-1}} \binom{k-1}{\mu}$ .*

**Proof** This is an immediate consequence of the fact that the right side of formula (3) factors into a product of two terms depending only on hyperedge-sizes, respectively vertex-degrees.  $\square$

## 1.2 Labelled bipartite trees

A bipartite graph is an ordinary graph with vertices  $\mathcal{V} = \mathcal{V}_1 \cup \mathcal{V}_2$  partitioned into two subsets such that no pair of adjacent vertices is in the same class.

Hypergraphs are in one-to-one correspondence with certain bipartite graphs as follows (see for example page 5 of [2]): To a hypergraph  $(\mathcal{V}, \mathcal{E})$  we associate the bipartite graph with vertices in the first class representing elements of  $\mathcal{V}$ , vertices of the second class representing elements of  $\mathcal{E}$  and with edges encoding incidence (ie. there is an edge relating a vertex  $v$  to a hyperedge  $E$  if  $v$  belongs to  $E$ ). Vertices of edge-type (representing elements of  $\mathcal{E}$ ) have to be of degree at least 2 and every bipartite graph having only vertices of degree at least 2 in its second class of vertices corresponds to a hypergraph. Hypertrees are encoded by (ordinary) trees with no leaves in their second bipartite class of vertices.

**Theorem 1.5.** *Given two natural integers  $a, b$  and two integral vectors  $\alpha = (\alpha_0, \dots, \alpha_a) \in \mathbb{N}^{a+1}$  and  $\beta = (\beta_0, \dots, \beta_b)$  such that  $b = \sum_{i=0}^a \alpha_i$  and  $a = \sum_{i=0}^b \beta_i$ , the number of labelled bipartite trees having vertex bipartition  $\mathcal{U} \cup \mathcal{V}$*

with vertices  $\mathcal{U} = \{u_0, \dots, u_a\}$  of degree  $\deg u_i = 1 + \alpha_i$  and vertices  $\mathcal{V} = \{v_0, \dots, v_b\}$  of degree  $\deg v_i = 1 + \beta_i$  is given by

$$\binom{a}{\beta} \binom{b}{\alpha} = \frac{a! b!}{\left(\prod_{i=0}^b \beta_i!\right) \left(\prod_{i=0}^a \alpha_i!\right)}. \quad (6)$$

Equivalently, we have

$$\sum_{T \in \mathcal{T}(a+1, b+1)} \left( \prod_{i=0}^a x_i^{\deg v_i} \right) \left( \prod_{i=0}^b y_i^{\deg u_i} \right) = (x_0 + \dots + x_a)^b (y_0 + \dots + y_b)^a \quad (7)$$

where  $\mathcal{T}(a+1, b+1)$  denotes the set of labelled trees with  $a+1$  vertices in the first class and  $b+1$  vertices in the second class of its vertex-partition.

Setting  $x_0 = \dots = x_a = y_0 = \dots = y_b = 1$  in Formula (7) yields the well-known number  $(a+1)^b (b+1)^a$  of spanning trees in the complete bipartite graph  $K_{a+1, b+1}$ .

**Remark 1.6.** *Theorem 1.5 implies that a bipartite random tree (chosen with uniform probability) with vertex-bipartition  $\{u_0, \dots, u_a\} \cup \{v_0, \dots, v_b\}$  has vertices of degree  $\deg(u_i) = 1 + \alpha_i$  for  $i = 0, \dots, a$  with probability  $\frac{1}{(1+a)^b} \binom{b}{\alpha}$  independently of the degrees of the vertices  $v_0, \dots, v_b$ .*

We give two proofs of Theorem 1.5. The first proof shows that it is essentially equivalent to Theorem 1.1. The second proof is bijective: We construct a map  $T \mapsto (W(T), W'(T))$  from trees with vertex-bipartition  $\mathcal{U} \times \mathcal{V}$  into  $\mathcal{V}^{\#\mathcal{U}-1} \times \mathcal{U}^{\#\mathcal{V}-1}$  which is one-to-one and respects degrees: A vertex  $u \in \mathcal{U}$ , respectively  $v \in \mathcal{V}$ , of  $T$  is involved with multiplicity  $\deg(u) - 1$  in  $W'(T)$ , respectively with multiplicity  $\deg(v) - 1$  in  $W(T)$ .

## 2 Proof of Theorem 1.1

We give a bijective proof of Theorem 1.1. More precisely, we construct a map  $T \mapsto (\mathcal{P}(T), W(T))$  which associates to a hypertree with  $k$  hyperedges of size  $1 + \lambda_1, \dots, 1 + \lambda_k$  and vertices  $\{0, \dots, n\}$  of degrees  $\mu_0, \dots, \mu_n$  a pair  $(\mathcal{P}(T), W(T))$  formed by a partition  $\mathcal{P}(T)$  of  $\{1, \dots, n\}$  into  $k$  subsets of cardinalities  $\lambda_1, \dots, \lambda_k$  and by a word  $W(T) \in \{0, \dots, n\}^{k-1}$  of length  $k-1$  involving a letter  $i$  of the alphabet  $\{0, \dots, n\}$  with multiplicity  $\mu_i$ . Theorem 1.1 follows from the fact that the map  $T \mapsto (\mathcal{P}(T), W(T))$  is one-to-one, since there are  $\binom{n}{\lambda} \frac{1}{\prod_{j=1}^k (\nu_j)!}$  possibilities for  $\mathcal{P}(T)$  and  $\binom{k-1}{\mu}$  possibilities for  $W(T)$ .

## 2.1 The map $T \mapsto \mathcal{P}(T)$

A hyperedge  $E$  of a hypertree with vertices  $\{0, \dots, n\}$  contains a unique vertex  $m(E)$  at minimal distance to the vertex 0. In particular, we have  $m(E) = 0$  if  $E$  contains 0. We call  $m(E)$  the *marked vertex of  $E$* .

Removing the marked vertex  $m(E)$  from every hyperedge  $E \in \mathcal{E}(T)$  of a hypertree  $T$  with vertices  $\{0, \dots, n\}$  yields a partition  $\mathcal{P}(T)$  of  $\{1, \dots, n\}$  with parts  $E \setminus \{m(E)\}$  of cardinalities  $(\text{size}(E) - 1)$  indexed by the set  $\mathcal{E}(T)$  of all hyperedges in  $T$ .

## 2.2 Construction of $T \mapsto W(T)$

Given a hypertree  $T$  with vertices  $\{0, \dots, n\}$  and  $k$  hyperedges, we construct recursively a word  $W(T) \in \{0, \dots, n\}^{k-1}$  which encodes exactly the loss of information induced by the map  $T \mapsto \mathcal{P}(T)$ .

The construction of the word  $W(T)$  is somehow dual to the Prüfer code encoding labelled planted forests (see for example the first proof of Theorem 5.3.2 in [10]). Prüfer codes are defined by keeping track of neighbours of successively removed largest leaves in ordinary trees, the construction of the word  $W(T)$  is based on local simplifications around largest non-leaves in hypertrees.

We start with a few useful definitions and notations:

A *hyperstar* is a hypertree containing a vertex  $v$ , called a *center* of the hyperstar, at distance at most 1 from all other vertices. A center of a hyperstar is unique (and given by the intersection of two arbitrary hyperedges) except in the degenerate case where the hyperstar consists of a unique hyperedge.

As above, we denote by  $m(E)$  the marked vertex realizing the distance to 0 of a hyperedge  $E$ . The remaining vertices of  $E$  are *unmarked* vertices.

Given  $i$  in  $\{1, \dots, n\}$ , we denote by  $U(i) \in \mathcal{E}(T)$  the unique hyperedge of  $T$  containing  $i$  as an unmarked vertex. Similarly, we denote by  $P(i)$  the unique set  $U(i) \setminus m(U(i))$  containing  $i$  of the partition  $\mathcal{P}(T)$ . The set  $P(i)$  can be constructed by removing the marked vertex from the unique hyperedge  $U(i)$  containing  $i$  as an unmarked vertex.

We associate to every hypertree  $T$  with  $k$  hyperedges and vertices  $\{0, \dots, n\}$  of degrees  $1 + \mu_0, \dots, 1 + \mu_n$  a word  $W(T)$  of  $\{0, \dots, n\}^{k-1}$  involving  $\mu_i$  copies of a letter  $i$  in  $\{0, \dots, n\}$ . The word  $W(T) = w_1 \dots w_{k-1}$  is defined recursively as follows:

We set  $W(T) = 0^{k-1}$  if  $T$  is a hyperstar centered at 0.

Otherwise, there exists a largest integer  $a$  in  $\{1, \dots, n\}$  such that  $\mu_a > 0$  and  $\mu_{a+1} = \mu_{a+2} = \dots = \mu_n = 0$ . We denote by  $A_1, \dots, A_{k-1} \subset \{1, \dots, n\} \setminus \{a\}$  all  $k-1$  elements of  $\mathcal{P}(T) \setminus \{P(a)\}$  not containing  $a$  with indices determined by requiring  $\min A_i < \min A_j$  if  $i < j$ . Exactly  $\mu_a$  elements among  $A_1, \dots, A_{k-1}$  correspond to the set  $m^{-1}(a)$  of hyperedges with marked vertex

$a$ . Let  $\alpha_1, \dots, \alpha_{\mu_a}$  be the associated indices. Otherwise stated,  $A_{\alpha_j} \cup \{a\}$  is a hyperedge with marked vertex  $a$  of  $T$  for  $j = 1, \dots, \mu_a$ . The indices  $\alpha_1, \dots, \alpha_{\mu_a} \in \{1, \dots, k-1\}$  define the  $\mu_a$  letters  $w_{\alpha_1} = \dots = w_{\alpha_{\mu_a}} = a$  of the word  $W(T) = w_1 w_2 \dots w_{k-1}$ . Removing these  $\mu_a$  letters from the word  $W(T)$  leaves a word  $W'$  which we define recursively by the identity  $W' = W(T_a)$  where  $T_a$  is obtained from  $T$  by merging all hyperedges of  $T$  containing  $a$  into a unique hyperedge. More precisely,  $T_a$  is constructed by removing first all hyperedges containing  $a$  from  $T$ , followed by the adjunction of one new hyperedge consisting of  $a$  and of all its neighbours in  $T$ . The hypertree  $T_a$  has strictly fewer hyperedges than  $T$ . All vertices except  $a$  have the same degree in  $T$  and in  $T_a$  and  $a$  is a leaf in  $T_a$ .

Since all vertices  $a, a+1, \dots, n$  are leaves of  $T_a$ , we have  $W' \in \{0, 1, \dots, a-1\}^{k-1-\mu_a}$  and this processus stops eventually.

Remark that the  $\mu_a$  positions of all letters equal to  $a$  in  $W(T)$  encode exactly the elements of  $\mathcal{P}(T)$  associated to hyperedges with marked vertex  $a$  in a hypertree  $T$ . This implies that the map  $T \mapsto (\mathcal{P}(T), W(T))$  is into.

### 2.3 Construction of the reciprocal map $(\mathcal{P}, W) \mapsto T$

We claim that the map  $T \mapsto (\mathcal{P}(T), W(T))$  is one-to-one: Indeed, let  $\mathcal{P}$  be a partition of  $\{1, \dots, n\}$  into  $k$  non-empty subsets and let  $W \in \{0, \dots, n\}^{k-1}$  be a word of length  $k-1$  with letters in the alphabet  $\{0, \dots, n\}$ . If  $W = 0^{k-1}$ , the pair  $(\mathcal{P}, W)$  corresponds to the hyperstar centered at 0 with hyperedges obtained by adding the central vertex 0 to every subset  $A \subset \{1, \dots, n\}$  involved in the partition  $\mathcal{P}$ .

Otherwise, let  $a \in \{1, \dots, n\}$  be the largest strictly positive integer involved with strictly positive multiplicity  $\mu_a > 0$  in  $W = w_1 \dots w_{k-1}$  and let  $1 \leq \alpha_1 < \dots < \alpha_{\mu_a} \leq k-1$  denote the  $\mu_a$  indices defined by  $w_{\alpha_1} = \dots = w_{\alpha_{\mu_a}} = a$ . We denote by  $P(a) \in \mathcal{P}$  the unique subset of  $\mathcal{P}$  containing the element  $a$ . Let  $A_1, \dots, A_{k-1}$  be the  $k-1$  remaining elements of  $\mathcal{P}$  corresponding to subsets of  $\{1, \dots, n\} \setminus \{a\}$ . Indices of  $A_1, \dots, A_{k-1}$  are defined by the requirement  $\min A_i < \min A_j$  if  $i < j$ . The sets  $A_{\alpha_j} \cup \{a\}$ ,  $j = 1, \dots, \mu_a$ , are then by construction the  $\mu_a$  hyperedges with marked vertex  $a$  of a tree  $T$  such that  $\mathcal{P} = \mathcal{P}(T)$  and  $W = W(T)$ . The remaining hyperedges of such a tree  $T$  are defined as follows: merge the  $\mu_a + 1$  elements  $P(a), A_{\alpha_1}, \dots, A_{\alpha_{\mu_a}}$  of  $\mathcal{P}$  into a unique subset  $A = P(a) \cup \bigcup_{j=1}^{\mu_a} A_{\alpha_j}$  of  $\{1, \dots, n\}$  and complete this subset to a partition  $\mathcal{P}'$  of  $\{1, \dots, n\}$  by adjoining all elements of  $\mathcal{P}$  not contained in  $A$ . Similarly, define a word  $W'$  obtained from  $W$  by removing all  $\mu_a$  occurrences of the letter  $a$ . The pair  $(\mathcal{P}', W')$  defines then recursively a hypertree  $T'$ . Hyperedges of the tree  $T$  not containing  $a$  are then given by hyperedges of  $T'$  not containing  $a$ . The unique hyperedge  $U(a)$  of  $T$  containing the vertex  $a$  as an unmarked vertex is obtained by adjoining to the set  $P(a)$  the marked vertex of the unique hyperedge in  $T'$  with unmarked vertices given by  $A = P(a) \cup \bigcup_{j=1}^{\mu_a} A_{\alpha_j} \in \mathcal{P}'$ .



Remark that the tree  $T'$  is simpler than the final tree  $T$  in the sense that  $a$  is a non-leaf of  $T$  (the vertices  $a + 1, a + 2, \dots, n$  are however leaves of  $T$ ), but is a leaf (together with  $a + 1, \dots, n$ ) of the tree  $T'$ . Thus the construction stops eventually.

A tree  $T$  constructed in this way is the unique tree satisfying  $\mathcal{P} = \mathcal{P}(T)$  and  $W = W(T)$ . The map  $T \mapsto (\mathcal{P}(T), W(T))$  is thus also onto. This ends the proof of Theorem 1.1.  $\square$

**Remark 2.1.** *The bijection  $T \mapsto (\mathcal{P}(T), W(T))$  is not completely natural in the sense that it depends on the choice of a particular vertex  $v$  (given by the vertex 0 in our case), on the choice of a linear order of the remaining vertices and on the choice of a suitable order relation for subsets of partitions of  $\mathcal{V} \setminus \{v\}$ .*

**Remark 2.2.** *The action of the symmetric group  $\mathcal{S}_n$  on partitions of  $\{1, \dots, n\}$  and the action of  $\mathcal{S}_{k-1}$  permuting letters in a word of length  $k - 1$  induce a transitive action of  $\mathcal{S}_n \times \mathcal{S}_{k-1}$  on labelled hypertrees with  $k$  hyperedges of given sizes and vertices  $0, \dots, n$  of given degrees.*

### 3 Proofs of Theorem 1.5

We give two proofs of Theorem 1.5. The first proof consists in showing that it is essentially equivalent to Theorem 1.1. The second proof is obtained by a minor modification of the bijective proof given above for Theorem 1.1.

**First proof** Suppose first that  $\beta_0, \dots, \beta_b$  are all strictly positive. We consider thus bipartite graphs having a vertex-bipartition  $\mathcal{U} \cup \mathcal{V}$  with no leaves in  $\mathcal{V}$ . Interpreting vertices of  $\mathcal{V}$  as hyperedges, ordered by size, the number of such graphs is obtained by multiplying the corresponding number of labelled hypertree (given by formula (3) with  $n = a, \lambda = \{\beta_0, \dots, \beta_b\}$  in decreasing order,  $k = b + 1$  and  $\mu = \{\alpha_0, \dots, \alpha_a\}$  in decreasing order) by  $\prod_{j=1}^b \nu_j!$  where  $\nu_j = \#\{i \mid \beta_i = j\}$  counts the number of vertices of degree  $j + 1$  in  $\mathcal{V}$ . We get thus in this case the equivalence between Theorem 1.5 and Theorem 1.1. The general case is by induction on the number of leaves in  $\mathcal{V}$ . Indeed, the last such leaf can be adjacent to any non-leaf in  $\mathcal{U}$  and we get thus the recursion

$$\sum_{k, \alpha_k \geq 1} \binom{a}{\beta} \binom{b-1}{\alpha_0, \dots, \alpha_{k-1}, \alpha_k - 1, \alpha_{k+1}, \dots, \alpha_a} = \binom{a}{\beta} \binom{b}{\alpha}$$

for the total number of possible bipartite trees.  $\square$

**Second proof** We construct a map which associates to a tree  $T$  having bipartite vertices  $\mathcal{U} \cup \mathcal{V}$ , with vertices  $\mathcal{U} = \{u_0, \dots, u_a\}$  of degrees  $\alpha_0, \dots, \alpha_a$  in the first class and vertices  $\mathcal{V} = \{v_0, \dots, v_b\}$  of degrees  $\beta_0, \dots, \beta_b$  in the second class, a word  $(W, W') = (W(T), W'(T)) \in \mathcal{V}^a \times \mathcal{U}^b$  such that a vertex  $v_i$  is involved  $\beta_i$  times in  $W$  and a vertex  $u_i$  is involved  $\alpha_i$  times in  $W'$ . This

implies the result since there are  $\binom{a}{\beta}$  possibilities for  $W$  and  $\binom{b}{\alpha}$  possibilities for  $W'$ .

We root a tree  $T$  with vertex-bipartition  $\mathcal{U} \times \mathcal{V}$  as above at the vertex  $u_0$  of  $\mathcal{U}$  and we orient all edges of  $T$  away from the root vertex  $u_0$ . An edge joining two neighbouring vertices  $s, t$  with  $s$  closer to  $u_0$  than  $t$  is thus oriented from  $s$  to  $t$ . We call the vertex  $s$  of such an edge the *parent* of  $t$  and we write  $s = p(t)$ . Similarly, we call  $t$  a *child* of  $s$ . Every vertex other than  $u_0$  has a unique parent and edges of  $T$  are in bijection with  $\{u_1, \dots, u_a\} \cup \mathcal{V}$  by considering the unique edge joining a vertex  $s \neq u_0$  to its parent  $p(s)$ .

We consider the total order induced by indices on both sets  $\mathcal{U}$  and  $\mathcal{V}$ .

The word  $W$  encodes all edges of  $T$  starting at a vertex of  $\mathcal{V}$  and ending at a vertex in  $\{u_1, \dots, u_a\}$ . More precisely,  $W$  is given by the word

$$p(u_1)p(u_2)\dots p(u_a)$$

encoding the parents of  $u_1, u_2, \dots, u_a$ . Since every element  $v_i$  in  $\mathcal{V}$  is the parent of exactly  $\beta_i$  vertices in  $\mathcal{U}$ , the word  $W$  involves a vertex  $v_i \in \mathcal{V}$  with multiplicity  $\beta_i$ .

The recursive definition of the word  $W'$  encoding all edges starting at an element of  $\mathcal{U}$  and ending at an element of  $\mathcal{V}$  is more involved. More precisely, for  $i \geq 1$ , the positions of a letter  $u_i$  in  $W'$  encode the  $\alpha_i$  edges of the form  $\{v, u_i\}$  with  $p(v) = u_i$  in the following way:

Suppose that for some integer  $c \in \{1, \dots, a\}$  all positions of the letters  $u_{c+1}, u_{c+2}, \dots, u_a$  in  $W'$  are known. The  $\alpha_{c+1} + \alpha_{c+2} + \dots + \alpha_a$  positions of the letters  $u_{c+1}, u_{c+2}, \dots, u_a$  in  $W'$  and the word  $W$  define a subforest  $F_c$  of  $T$  consisting of all edges involving a vertex in  $\{u_{c+1}, u_{c+2}, \dots, u_a\}$  together with the  $c$  edges of the form  $\{u_i, p(u_i)\}$  joining the vertices  $u_1, \dots, u_c$  to their predecessors  $p(u_1), \dots, p(u_c)$  in  $\mathcal{V}$ . The forest  $F_c$  has  $u_0$  as an isolated vertex and contains

$$b + 1 - (\alpha_{c+1} + \alpha_{c+2} + \dots + \alpha_a) = 1 + \alpha_0 + \alpha_1 + \dots + \alpha_c$$

other connected components which are all rooted at a an element of  $\mathcal{V}$  having no parent in its connected component of  $F_c$ . We denote by  $\tilde{F}_c$  the subforest of  $F_c$  obtained by removing the isolated vertex  $u_0$  and the connected component containing  $u_c$  from  $F_c$ . The forest  $\tilde{F}_c$  has exactly  $\alpha_0 + \dots + \alpha_c$  connected components which we order totally accordingly to the total order of the corresponding root-vertices. The  $\alpha_c$  children of  $u_c$  are roots of  $\alpha_c$  connected components of  $\tilde{F}_c$ . The relative positions of these  $\alpha_c$  connected components among the totally ordered set of all connected components of  $\tilde{F}_c$ , determine the  $\alpha_c$  positions of the letter  $u_c$  among the  $\alpha_0 + \dots + \alpha_c$  letters of  $W'$  which are different from  $u_{c+1}, u_{c+2}, \dots, u_a$ .

If all letters except  $u_0$  of  $W'$  are known, we complete  $W'$  with  $\alpha_0$  copies of the letter  $u_0$  in the unique possible way.

It is easy to see that the map  $T \mapsto (W(T), W'(T))$  is into. We leave it to the reader to check that it is onto by working out the obvious reciprocal map  $(W, W') \mapsto T$  (only the last step involving the tree  $F_0$  is not contained in the above description: one gets  $T$  from the forest  $F_0$  by joining  $u_0$  to the root-vertices of the remaining  $\alpha_0 + 1$  connected components of  $F_0$ ). It defines thus a one-to-one map finishing the proof of Theorem 1.5.  $\square$

**Remark 3.1.** (i) The map  $T \mapsto (W(T), W'(T))$  depends on total orders of the vertices. It depends also an order on the sets  $\mathcal{U}, \mathcal{V}$  involved in the vertex-bipartition  $\mathcal{U} \cup \mathcal{V}$ . This map is of course very similar to the map considered in the proof of Theorem 1.1. There is however a subtle difference: The forests  $F_c$  used in the recursive construction of  $W'(T)$  are naturally rooted (by the choice of the root-vertex  $u_0$  in  $\mathcal{U}$ ). This is not the case by its counterpart given by the partition  $\mathcal{P}'$  in the proof for hypertrees.

(ii) The obvious action (by permuting the positions of the letters in  $W$  and  $W'$ ) of  $\mathcal{S}_a \times \mathcal{S}_b$  induces a transitive action on the set of labelled bipartite trees enumerated by formula (6).

(iii) Combining the Prüfer code with the map  $T \mapsto (W(T), W'(T))$  yields a one-to-one map between  $\{0, \dots, n\}^{n-1}$  and

$$\bigcup_{A \subsetneq \{1, \dots, n\}} \{\{0\} \cup A\}^{n-1-\#(A)} \times \{\{1, \dots, n\} \setminus A\}^{\#(A)}$$

(where we agree that the first vertex of a tree belongs always to the first subset of the vertex-bipartition) illustrating the identity

$$(x_0 + \dots + x_n)^{n-1} = \sum_{A \subsetneq \{1, \dots, n\}} \left( x_0 + \sum_{j \in A} x_j \right)^{n-1-\#(A)} \left( \sum_{j \in \{1, \dots, n\} \setminus A} x_j \right)^{\#(A)}$$

(for  $n \geq 1$ ) having the specialization

$$(n+1)^{n-1} = \sum_{k=0}^{n-1} \binom{n}{k} (k+1)^{n-1-k} (n-k)^k,$$

see [9] for similar identities. Is there an easier way to describe such a bijection?

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Roland BACHER, Université Grenoble I, CNRS UMR 5582, Institut Fourier,  
 100 rue des maths, BP 74, F-38402 St. Martin d’Hères, France.  
 e-mail: Roland.Bacher@ujf-grenoble.fr