On the enumeration of labelled hypertrees and of labelled bipartite trees

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Abstract[1]*: We give a simple formula for the number of hypertrees with k hyperedges of given sizes and $n+1$ labelled vertices with prescribed degrees. A slight generalization of this formula counts labelled bipartite trees with prescribed degrees in each class of vertices.*

1 Main results

1.1 Labelled hypergraphs

A *(finite) hypergraph* is a pair $(\mathcal{V}, \mathcal{E})$ consisting of a finite set $\mathcal{V}$ of *vertices* and of a set $\mathcal{E}$ of *hyperedges* given by subsets of $\mathcal{V}$ containing at least two elements. We define the *size* of a hyperedge E as the number $\mathrm{size}(E)$ of vertices contained in E and the *degree* $\deg v$ of a vertex v as the number of hyperedges containing v. A vertex of degree 1 is also called a *leaf.* The obvious linear relation

$$\sum_{v\in\mathcal{V}} \deg(v) = \sum_{E\in\mathcal{E}} \mathrm{size}(E) \tag{1}$$

links the total sum of vertex-degrees to the total sum of hyperedge-sizes.

Two distinct vertices $v, w \in \mathcal{V}$ are *adjacent* or *neighbours* if they are both contained in some hyperedge E of $\mathcal{E}$. A *path* of *length* k joining two vertices v, w in a hypergraph $(\mathcal{V}, \mathcal{E})$ is a sequence $v_0 = v, v_1, \ldots, v_k = w$ involving only consecutively adjacent vertices. A hypergraph is *connected* if any pair of vertices can be joined by a path. The *(combinatorial) distance* between two vertices v, w of a connected hypergraph is the minimal length of a path in the set of all paths joining v and w. A *cycle* of length k (for $k \geq 3$) is a closed path consisting of k distinct vertices. A *hyperforest* is a hypergraph F such that two distinct hyperedges of F intersect in at most a common vertex and such that every cycle of F is contained in a hyperedge. A *hypertree* is

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a connected hyperforest. Induction on the number k of hyperedges in a hypertree consisting of $n+1$ vertices shows the relation

$$\sum_{E\in\mathcal{E}} \mathrm{size}(E) = n+k \ . \tag{2}$$

Let λ be a partition of $n = \sum_{j=1}^{k} \lambda_j$ having exactly $k \leq n$ nonzero parts $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k \geq 1$. Let $\mu = (\mu_0, \ldots, \mu_n) \in \mathbb{N}^{n+1}$ be a vector with coefficients formed by $n+1$ natural integers $\mu_0, \ldots, \mu_n$ summing up to $k-1 = \sum_{j=0}^{n} \mu_j$.

Theorem 1.1. *The number of hypertrees having $n+1$ vertices $\{0,\ldots,n\}$ of degrees $\deg(i) = 1+\mu_i$ and k hyperedges of sizes $1+\lambda_1,\ldots,1+\lambda_k$ is given by*

$$\binom{n}{\lambda}\frac{1}{\prod_{j=1}^{n}(\nu_j)!}\binom{k-1}{\mu} = \frac{n!}{\left(\prod_{j=1}^{k}\lambda_j!\right)}\frac{1}{\left(\prod_{j=1}^{n}(\nu_j)!\right)}\frac{(k-1)!}{\left(\prod_{j=0}^{n}\mu_j!\right)} \tag{3}$$

with $\nu_j = \sharp\{i \mid \lambda_i = j\}$ counting parts of length j in λ.

Theorem 1.1 has the following equivalent formulation:

Theorem 1.2. *Denoting by $\mathcal{HT}_\lambda(n+1)$ the set of hypertrees with $n+1$ labelled vertices $\{0,\ldots,n\}$ and with k edges of size $1+\lambda_1, 1+\lambda_2,\ldots,1+\lambda_k$, we have*

$$\sum_{T\in\mathcal{HT}_\lambda(n+1)} \prod_{j=0}^{n} x_j^{\deg(j)} = \binom{n}{\lambda}\frac{1}{\prod_{j=1}^{n}(\nu_j)!}\left(x_0+x_1+\cdots+x_n\right)^{k-1} \ . \tag{4}$$

Equivalence between Theorem 1.1 and 1.2 is given by the binomial Theorem.

The identities (1) and (2) show that the conditions $\sum_{i=1}^{k}\lambda_k = n$ and $\sum_{i=0}^{n}\mu_i = k-1$ are necessary for the existence of hypertrees with edge-sizes $1+\lambda_1,\ldots,1+\lambda_k$ and vertex-degrees $1+\mu_0,\ldots,1+\mu_n$. Theorem 1.1 or 1.2 shows that they are also sufficient.

Multiplication by $(n+1)$ of the results given by Theorem 1.1 and 1.2 gives enumerative results for rooted labelled hypertrees.

Removal of the vertex 0 in trees enumerated by Theorem 1.1 gives enumerative results for planted forests with μ_0+1 connected components (inducing an error of 1 in the degree of root vertices and in the size of hyperedges containing a root).

Counting trees or hypertrees with labelled vertices is a fairly old sport and started with Sylvester [11] and Cayley [3] (according to the notes of Chapter 5 in [10]) mentionning the total number

$$(n+1)^{n-1}$$

of labelled trees on $n+1$ vertices. This corresponds of course to the specialization $x_0 = \cdots = x_n = 1$ in Theorem 1.2 for $\lambda = (1, 1, \ldots, 1)$ the trivial partition of n. In [4], Erdély and Etherington refined Cayley's theorem by enumerating labelled (ordinary) trees with given vertex-degrees (corresponding to the case of the trivial partition $\lambda = (1, 1, \ldots, 1)$ of n in Theorem 1.1 or 1.2), see also Theorem 5.3.10 in [10]. For trees, one can also consult the monograph [8] or the numerous more recent literature.

Denoting by $S_2(n, k)$ the Stirling number of the second kind enumerating partitions of $\{1, \ldots, n\}$ into k non-empty subsets, summing identity (4) over all partitions of $\{1, \ldots, n\}$ into exactly k parts and setting $x_0 = x_1 = \cdots = x_n = 1$ yields the number

$$(n+1)^{k-1} S_2(n, k) \tag{5}$$

of hypertrees with $n+1$ labelled vertices and k hyperedges given by Husimi in [6], see also [7], [12], [5] and [1] for other treatments and related results.

It is perhaps worthwile to mention the following two corollaries of Theorem 1.1:

The first result counts weighted hypertrees and is a kind of counterpart of Husimi's result (5) in the sense that it involves Stirling numbers of the first kind counting the number $(-1)^{n+k} S_1(n, k)$ of permutations of $\{1, \ldots, n\}$ involving k disjoint cycles:

Corollary 1.3. *We have*

$$\sum_{T \in \mathcal{HT}_k(n+1)} w(T) = (-1)^{n+k} (n+1)^{k-1} S_1(n, k)$$

or more precisely

$$\sum_{T \in \mathcal{HT}_k(n+1)} w(T) \prod_{j=0}^{n} x_j^{\deg(j)} = (-1)^{n+k} (x_0 + x_1 + \cdots + x_n)^{k-1} S_1(n, k)$$

where $\mathcal{HT}_k(n+1)$ denotes the set of all labelled hypertrees with k hyperedges and vertices $\{0, \ldots, n\}$, where $w(T) = \prod_{j=1}^{k} (\lambda_j - 1)!$ for a labelled hypertree T with k hyperedges of size $1+\lambda_1, \ldots, 1+\lambda_k$ and where $S_1(n, k)$ is the Stirling number of the first kind defined by $\sum_{k=0}^{n} S_1(n, k) x^k = \prod_{j=0}^{n-1} (x - j)$.

Proof Observe that

$$\binom{n}{\lambda} \frac{1}{\prod_{j=1}^{n} (\nu_j)!} \prod_{j=1}^{k} (\lambda_j - 1)!$$

counts the number of permutations of $\{1, \ldots, n\}$ in the conjugacy class consisting of products of k disjoint cycles with lengths $\lambda_1, \ldots, \lambda_k$. Summing over all partitions of $\{1,\ldots,n\}$ into k non-empty subsets and applying Theorem 1.2 (with $x_0 = \cdots = x_n = 1$ for the first formula) yields the result since $(-1)^{n+k}S_1(n,k)$ counts the total number of permutations of $\{1,\ldots,n\}$ consisting of k disjoint cycles. $\Box$

Corollary 1.4. *Let $\mathcal{HT}_k(n+1)$ be the set of all labelled hypertrees with k hyperedges and $n+1$ vertices. The two random variables given by the sizes of hyperedges and by the degrees of vertices are independent for a random hypertree T choosen with uniform probability in $\mathcal{HT}_k(n+1)$.*

More precisely, a random hypertree, choosen with uniform probability among all labelled hypertrees with k hyperedges and vertices $\{0,\ldots,n\}$, has edge-sizes $1+\lambda_1,\ldots,1+\lambda_k$ associated to a partition λ of n with probability

$$\frac{1}{S_2(n,k)}\binom{n}{\lambda}\frac{1}{\prod_{j=1}^k(\nu_j)!}$$

with $S_2(n,k)$ denoting the Stirling number of the second kind counting partitions of $\{1,\ldots,n\}$ into k non-empty subsets and with $\nu_j = \sharp\{i \mid \lambda_i = j\}$ counting the number of parts equal to j in λ.

Such a random tree has vertices $0,\ldots,n$ of degrees $1+\mu_0,\ldots,1+\mu_n$ with probability $\frac{1}{(n+1)^{k-1}}\binom{k-1}{\mu}$.

Proof This is an immediate consequence of the fact that the right side of formula (3) factors into a product of two terms depending only on hyperedge-sizes, respectively vertex-degrees. $\Box$

1.2 Labelled bipartite trees

A bipartite graph is an ordinary graph with vertices $\mathcal{V} = \mathcal{V}_1 \cup \mathcal{V}_2$ partitioned into two subsets such that no pair of adjacent vertices is in the same class.

Hypergraphs are in one-to-one correspondence with certain bipartite graphs as follows (see for example page 5 of [2]): To a hypergraph $(\mathcal{V},\mathcal{E})$ we associate the bipartite graph with vertices in the first class representing elements of $\mathcal{V}$, vertices of the second class representing elements of $\mathcal{E}$ and with edges encoding incidence (ie. there is an egde relating a vertex v to a hyperedge E if v belongs to E). Vertices of edge-type (representing elements of $\mathcal{E}$) have to be of degree at least 2 and every bipartite graph having only vertices of degree at least 2 in its second class of vertices corresponds to a hypergraph. Hypertrees are encoded by (ordinary) trees with no leaves in their second bipartite class of vertices.

Theorem 1.5. *Given two natural integers a,b and two integral vectors $\alpha = (\alpha_0,\ldots,\alpha_a) \in \mathbb{N}^{a+1}$ and $\beta = (\beta_0,\ldots,\beta_b)$ such that $b = \sum_{i=0}^a \alpha_i$ and $a = \sum_{i=0}^b \beta_i$, the number of labelled bipartite trees with vertex bipartition $\mathcal{U} \cup \mathcal{V}$*

with vertices $\mathcal{U} = \{u_0, \ldots, u_a\}$ *of degree* $\deg u_i = 1 + \alpha_i$ *and vertices* $\mathcal{V} = (v_0, \ldots, v_b)$ *of degree* $\deg v_i = 1 + \beta_i$ *is given by*

$$\binom{a}{\beta}\binom{b}{\alpha} = \frac{a!\ b!}{\left(\prod_{i=0}^{b} \beta_i!\right)\left(\prod_{i=0}^{a} \alpha_i!\right)} \ . \tag{6}$$

Equivalently, we have

$$\sum_{T \in \mathcal{T}(a+1,b+1)} \left(\prod_{i=0}^{a} x_i^{\deg v_i}\right)\left(\prod_{i=0}^{b} y_i^{\deg u_i}\right) = (x_0 + \cdots + x_a)^b (y_0 + \cdots + y_b)^a \tag{7}$$

where $\mathcal{T}(a+1, b+1)$ *denotes the set of labelled trees with* $a+1$ *vertices in the first class and* $b+1$ *vertices in the second class of its vertex-partition.*

Setting $x_0 = \cdots = x_a = y_0 = \cdots = y_b = 1$ in Formula (7) yields the well-known number $(a+1)^b(b+1)^a$ of spanning trees in the complete bipartite graph $K_{a+1,b+1}$.

Remark 1.6. *Theorem 1.5 implies that a bipartite random tree (choosen with uniform probability) with vertex-bipartition* $\{u_0, \ldots, u_a\} \cup \{v_0, \ldots, v_b\}$ *has vertices of degree* $\deg(u_i) = 1 + \alpha_i$ *for* $i = 0, \ldots, a$ *with probability* $\frac{1}{(1+a)^b}\binom{b}{\alpha}$ *independently of the degrees of the vertices* $v_0, \ldots, v_b$.

2 Proof of Theorem 1.1

We give a bijective proof of Theorem 1.1. More precisely, we construct a map $T \longmapsto (\mathcal{P}(T), W(T))$ which associates to a hypertree with k hyperedges of size $1 + \lambda_1, \ldots, 1 + \lambda_k$ and vertices $\{0, \ldots, n\}$ of degrees $\mu_0, \ldots, \mu_n$ a pair $(\mathcal{P}(T), W(T))$ formed by a partition $\mathcal{P}(T)$ of $\{1, \ldots, n\}$ into k subsets of cardinalities $\lambda_1, \ldots, \lambda_k$ and by a word $W(T) \in \{0, \ldots, n\}^{k-1}$ of length $k-1$ involving a letter i of the alphabet $\{0, \ldots, n\}$ with multiplicity μ_i. Theorem 1.1 follows from the fact that the map $T \longmapsto (\mathcal{P}(T), W(T)$ is one-to-one, since there are $\binom{n}{\lambda}\frac{1}{\prod_{j=1}^{n}(\nu_j)!}$ possibilities for $\mathcal{P}(T)$ and $\binom{k-1}{\mu}$ possibilities for $W(T)$.

2.1 The map $T \longmapsto \mathcal{P}(T)$

A hyperedge E of a hypertree with vertices $\{0, \ldots, n\}$ contains a unique vertex $m(E)$ at minimal distance to the vertex 0. In particular, we have $m(E) = 0$ if E contains 0. We call $m(E)$ the *marked vertex of* E.

Removing the marked vertex $m(E)$ from every hyperedge $E \in \mathcal{E}(T)$ of a hypertree T with vertices $\{0, \ldots, n\}$ yields a partition $\mathcal{P}(T)$ of $\{1, \ldots, n\}$ with parts $E \setminus \{m(E)\}$ of cardinalities $(\text{size}(E) - 1)$ indexed by the set $\mathcal{E}(T)$ of all hyperedges in T.

2.2 Construction of $T \longmapsto W(T)$

Given a hypertree T with vertices $\{0,\dots,n\}$ and k hyperedges, we construct recursively a word $W(T) \in \{0,\dots,n\}^{k-1}$ which encodes exactly the loss of information induced by the map $T \longmapsto \mathcal{P}(T)$.

The construction of the word $W(T)$ is somehow dual to the Prüfer code encoding labelled planted forests (see for example the first proof of Theorem 5.3.2 in [10]). Prüfer codes are defined by keeping track of neighbours of successively removed largest leaves in ordinary trees, the construction of the word $W(T)$ is based on local simplifications around largest non-leaves in hypertrees.

We start with a few useful definitions and notations:

A *hyperstar* is a hypertree containing a vertex v, called a *center* of the hyperstar, at distance at most 1 from all other vertices. A center of a hyperstar is unique (and given by the intersection of two arbitrary hyperedges) except in the degenerate case where the hyperstar consists of a unique hyperedge.

As above, we denote by $m(E)$ the marked vertex realizing the distance to 0 of a hyperedge E. The remaining vertices of E are *unmarked* vertices.

Given i in $\{1,\dots,n\}$, we denote by $U(i) \in \mathcal{E}(T)$ the unique hyperedge of T containing i as an unmarked vertex. Similarly, we denote by $P(i)$ the unique set $U(i) \setminus m(U(i))$ containing i of the partition $\mathcal{P}(T)$. The set $P(i)$ can be constructed by removing the marked vertex from the unique hyperedge $U(i)$ containing i as an unmarked vertex.

We associate to every hypertree T with k hyperedges and vertices $\{0,\dots,n\}$ of degrees $1+\mu_0,\dots,1+\mu_n$ a word $W(T)$ of $\{0,\dots,n\}^{k-1}$ involving μ_i copies of a letter i in $\{0,\dots,n\}$. The word $W(T) = w_1 \dots w_{k-1}$ is defined recursively as follows:

We set $W(T) = 0^{k-1}$ if T is a hyperstar centered at 0.

Otherwise, there exists a largest integer a in $\{1,\dots,n\}$ such that $\mu_a > 0$ and $\mu_{a+1} = \mu_{a+2} = \dots = \mu_n = 0$. We denote by $A_1,\dots,A_{k-1} \subset \{1,\dots,n\} \setminus \{a\}$ all $k-1$ elements of $\mathcal{P}(T) \setminus \{P(a)\}$ not containing a with indices determined by requiring $\min A_i < \min A_j$ if $i < j$. Exactly μ_a elements among $A_1,\dots,A_{k-1}$ correspond to the set $m^{-1}(a)$ of hyperedges with marked vertex a. Let $\alpha_1,\dots,\alpha_{\mu_a}$ be the associated indices. Otherwise stated, $A_{\alpha_j} \cup \{a\}$ is a hyperedge with marked vertex a of T for $j = 1,\dots,\mu_a$. The indices $\alpha_1,\dots,\alpha_{\mu_a} \in \{1,\dots,k-1\}$ define the μ_a letters $w_{\alpha_1} = \dots = w_{\alpha_{\mu_a}} = a$ of the word $W(T) = w_1 w_2 \dots w_{k-1}$. Removing these μ_a letters from the word $W(T)$ leaves a word W' which we define recursively by the identity $W' = W(T_a)$ where T_a is obtained from T by merging all hyperedges of T containing a into a unique hyperedge. More precisely, T_a is constructed by removing first all hyperedges containing a from of T, followed by the adjunction of one new hyperedge consisting of a and of all its neighbours in T. The hypertree T_a has strictly fewer hyperedges than T. All vertices except

a have the same degree in T and in T_a and a is a leaf in T_a.

Since all vertices $a, a+1, \ldots, n$ are leaves of T_a, we have $W' \in \{0, 1, \ldots, a-1\}^{k-1-\mu_a}$ and this processus stops eventually.

Remark that the μ_a positions of all letters equal to a in $W(T)$ encode exactly the elements of $\mathcal{P}(T)$ associated to hyperedges with marked vertex a in a hypertree T. This implies that the map $T \longmapsto (\mathcal{P}(T), W(T))$ is into.

2.3 Construction of the reciprocal map $(\mathcal{P}, W) \longmapsto T$

We claim that the map $T \longmapsto (\mathcal{P}(T), W(T))$ is one-to-one: Indeed, let $\mathcal{P}$ be a partition of $\{1, \ldots, n\}$ into k non-empty subsets and let $W \in \{0, \ldots, n\}^{k-1}$ be a word of length $k-1$ with letters in the alphabet $\{0, \ldots, n\}$. If $W = 0^{k-1}$, the pair $(\mathcal{P}, W)$ corresponds to the hyperstar centered at 0 with hyperedges obtained by adding the central vertex 0 to every subset $A \subset \{1, \ldots, n\}$ involved in the partition $\mathcal{P}$.

Otherwise, let $a \in \{1, \ldots, n\}$ be the largest strictly positive integer involved with strictly positive multiplicity $\mu_a > 0$ in $W = w_1 \ldots w_{k-1}$ and let $1 \leq \alpha_1 < \cdots < \alpha_{\mu_a} \leq k-1$ denote the μ_a indices defined by $w_{\alpha_1} = \cdots = w_{\alpha_{\mu_a}} = a$. We denote by $P(a) \in \mathcal{P}$ the unique subset of $\mathcal{P}$ containing the element a. Let $A_1, \ldots, A_{k-1}$ be the $k-1$ remaining elements of $\mathcal{P}$ corresponding to subsets of $\{1, \ldots, \ldots, n\} \setminus \{a\}$. Indices of $A_1, \ldots, A_{k-1}$ are defined by the requirement $\min A_i < \min A_j$ if $i < j$. The sets $A_{\alpha_j} \cup \{a\}$, $j = 1, \ldots, \mu_a$, are then by construction the μ_a hyperedges with marked vertex a of a tree T such that $\mathcal{P} = \mathcal{P}(T)$ and $W = W(T)$. The remaining hyperedges of such a tree T are defined as follows: merge the $\mu_a + 1$ elements $P(a), A_{\alpha_1}, \ldots, A_{\alpha_{\mu_a}}$ of $\mathcal{P}$ into a unique subset $A = P(a) \cup \bigcup_{j=1}^{\mu_a} A_{\alpha_j}$ of $\{1, \ldots, n\}$ and complete this subset to a partition $\mathcal{P}'$ of $\{1, \ldots, n\}$ by adjoining all elements of $\mathcal{P}$ not contained in A. Similarly, define a word W' obtained from W by removing all μ_a occurences of the letter a. The pair $(\mathcal{P}', W')$ defines then recursively a hypertree T'. Hyperedges of the tree T not containing a are then given by hyperedges of T' not containing a. The unique hyperedge $U(a)$ of T containing the vertex a as an unmarked vertex is obtained by adjoining to the set $P(a)$ the marked vertex of the unique hyperedge in T' with unmarked vertices given by $A = P(a) \cup \bigcup_{j=1}^{\mu_a} A_{\alpha_j} \in \mathcal{P}'$. Remark that the tree T' is simpler than the final tree T in the sense that a is a non-leaf of T (the vertices $a+1, a+2, \ldots, n$ are however leaves of T), but is a leaf (together with $a+1, \ldots, n$) of the tree T'. Thus the construction stops eventually.

A tree T constructed in this way is the unique tree satisfying $\mathcal{P} = \mathcal{P}(T)$ and $W = W(T)$. The map $T \longmapsto (\mathcal{P}(T), W(T))$ is thus also onto. This ends the proof of Theorem 1.1. $\square$

Remark 2.1. *The bijection $T \longmapsto (\mathcal{P}(T), W(T))$ is not completely natural in the sense that it depends on the choice of a particular vertex v (given by* *the vertex* 0 *in our case), on the choice of a linear order of the remaining vertices and on the choice of a suitable order relation for subsets of partitions of* $\mathcal{V} \setminus \{v\}$.

Remark 2.2. *The action of the symmetric group* $\mathcal{S}_n$ *on partitions of* $\{1, \ldots, n\}$ *and the action of* $\mathcal{S}_{k-1}$ *permuting letters in a word of length* $k-1$ *induce a transitive action of* $\mathcal{S}_n \times \mathcal{S}_{k-1}$ *on labelled hypertrees with* k *hyperedges of given sizes and vertices* $0, \ldots, n$ *of given degrees.*

3 Proofs of Theorem 1.5

We give two proofs of Theorem 1.5. The first proof consists in showing that it is essentially equivalent to Theorem 1.1. The second proof is obtained by a minor modification of the bijective proof given above for Theorem 1.1.

First proof Suppose first that $\beta_0, \ldots, \beta_b$ are all strictly positive. We consider thus bipartite graphs having a vertex-bipartition $\mathcal{U} \cup \mathcal{V}$ with no leaves in $\mathcal{V}$. Interpreting vertices of $\mathcal{V}$ as hyperedges, ordered by size, the number of such graphs is obtained by multiplying the corresponding number of labelled hypertree (given by formula (3) with $n = a, \lambda = \{\beta_0, \ldots, \beta_b\}$ in decreasing order, $k = b+1$ and $\mu = \{\alpha_0, \ldots, \alpha_a\}$ in decreasing order) by $\prod_{j=1}^{b} \nu_j!$ where $\nu_j = \sharp\{i \mid \beta_i = j\}$ counts the number of vertices of degree $j+1$ in $\mathcal{V}$. We get thus in this case the equivalence between Theorem 1.5 and Theorem 1.1. The general case is by induction on the number of leaves in $\mathcal{V}$. Indeed, the last such leave can be adjacent to any non-leaf in $\mathcal{U}$ and we get thus the recursion

$$\sum_{k, \alpha_k \geq 1} \binom{a}{\beta} \binom{b-1}{\alpha_0, \ldots, \alpha_{k-1}, \alpha_k - 1, \alpha_{k+1}, \ldots, \alpha_a} = \binom{a}{\beta} \binom{b}{\alpha}$$

for the total number of possible bipartite trees. $\Box$

Second proof We establish a bijection between trees described by Theorem 1.5 and pairs of words (W, W') where $W \in \{0, \ldots, b)^a$ and $W' \in \{0, \ldots, a\}^b$ with i involved in W, respectively W' with multiplicity β_i, respectively α_i. Given a suitable tree T, the word $W(T) = w_1 w_2 \ldots w_a$ is constructed as follows. Given a vertex $v_i \in \mathcal{V}$, the set $\mathcal{N}(v_i) \subset \mathcal{U}$ of its neighbours contains a unique vertex $n(i) \in \mathcal{U}$ separating u_0 from v_i (where u_0 separates u_0 from from v_i by convention) and a subset $\tilde{\mathcal{N}}(v_i) = \mathcal{N}(v_i) \setminus \{n(i)\}$ of β_i vertices in $\mathcal{U}$ which are separated by v_i from u_0. The sets $\tilde{\mathcal{N}}(v_0), \ldots, \tilde{\mathcal{N}}(v_b)$ form a partition of $\{u_1, \ldots, u_a\}$. We set $w_i = j$ if $u_i \in \tilde{\mathcal{N}}(v_j)$. Otherwise stated, for $i = 1, \ldots, a$, the $i-$th letter w_i encodes the unique neighbour of u_i in $\mathcal{V}$ which separates u_i from u_0.

The recursive construction of the word $W' = W'(T) = w'_1 \ldots w'_b$ is more involved: If $\alpha_1 = \alpha_2 = \cdots = \alpha_a = 0$, set $W' = 0^{\alpha_0} = 0^b$. Otherwise, let $c \geq 1$ be the largest integer such that $\alpha_c > 0$. Remove from $\mathcal{V}$ the unique neighbour

$m(c)$ of u_c which separates u_c from u_0. The remaining α_c neighbours $\tilde{\mathcal{N}}(u_c)$ of u_c form a subset of α_c elements in $\mathcal{V}\setminus\{m(c)\}$, totally ordered by $v_i < v_j$ if $i < j$. Using the unique order-preserving bijection between $\mathcal{V}\setminus\{m(c)\}$ and $\{1,\dots,b\}$, the set $\tilde{\mathcal{N}}(u_c)$ defines a set of α_c indices corresponding to the letter c in the word $W' = w'_1 \dots w'_b$. The recursive construction of W' proceeds now as follows: Suppose that the position of all letters strictly larger than c in W' is already determined for some integer c. It is thus enough to describe the subword W'_c obtained by removing these $\alpha_{c+1}+\dots+\alpha_a$ letters from W'. If $\alpha_1 = \alpha_2 = \dots = \alpha_c = 0$ we set $W'_c = 0^{\alpha_0}$ and we are finished. Otherwise, we can suppose $\alpha_c \geq 1$ and we consider the forest F_c obtained by considering the subgraph of T encoded by the word $W(T)$ and by all letters $> c$ of the (only partially known) word $W'(T)$. The forest F_c has u_0 as an isolated vertex and contains a set $\tilde{F}_c$ of $b+1-(\alpha_{c+1}+\dots+\alpha_a)$ other connected components all containing a unique vertex in $\mathcal{V}$ which is closest to u_0. We root these other connected components at this unique vertex. The vertex u_c belongs to a unique connected component T'_c of $\tilde{F}_c$ and is adjacent to the roots of exactly α_c other connected components in $\tilde{F}_c$. The roots of these α_c connected components define a subset of α_c positions among all roots of trees in $\tilde{F}_c \setminus T'_c$, totally ordered in the obvious way by their indices. We get in this way the α_c positions of the letter c in W'_c.

It is easy to check that the map $T \longmapsto (W(T), W'(T))$ is into. We leave it to the reader to check that it is onto by working out the reciprocal map $(W, W') \longmapsto T$. It defines thus a one-to-one map finishing the proof of Theorem 1.5. $\Box$

Remark 3.1. *(i) The map $T \longmapsto (W(T), W'(T))$ depends on total orders of the vertices. It depends also an order on the sets $\mathcal{U}, \mathcal{V}$ involved in the vertex-bipartition $\mathcal{U}\cup\mathcal{V}$. This map is of course very similar to the map considered in the proof of Theorem 1.1. There is however a subtle difference: The forests F_c used in the recursive construction of $W'(T)$ are naturally rooted (by the choice of the root-vertex u_0 in $\mathcal{U}$). This is not the case by its counterpart given by the partition $\mathcal{P}'$ in the proof for hypertrees.*

(ii) The obvious action (by permuting the positions of the letters in W and W') of $\mathcal{S}_a \times \mathcal{S}_b$ induces a transitive action on the set of labelled bipartite trees enumerated by formula (6).

(iii) Combining the Prüfer code with the map $T \longmapsto (W(T), W'(T))$ yields a one-to-one map between $\{0,\dots,n\}^{n-1}$ and

$$\bigcup_{A\subsetneq\{1,\dots,n\}} \{\{0\}\cup A\}^{n-1-\sharp(A)} \times \{\{1,\dots,n\}\setminus A\}^{\sharp(A)}$$

(where we agree that the first vertex of a tree belongs always to the first

subset of the vertex-bipartition) illustrating the identity

$$(x_0 + \cdots + x_n)^{n-1} = \sum_{A \subsetneq \{1,\dots,n\}} \left(x_0 + \sum_{j \in A} x_j \right)^{n-1-\sharp(A)} \left(\sum_{j \in \{1,\dots,n\} \setminus A} x_j \right)^{\sharp(A)}$$

(for $n \geq 1$) having the specialization

$$\sum_{k=0}^{n-1} \binom{n}{k} (k+1)^{n-1-k} (n-k)^k = (n+1)^{n-1} \ ,$$

see [9] for similar identities. Is there an easier way to describe such a bijection?

References

[1] A. Bedini, S. Caracciolo, A. Sportiello, "Hyperforests on the Complete Hypergraph by Grassmann Integral Representation", *J. Phys. A* **41** (2008), 205003, 28 pp.

[2] B. Bollobás, *Graph theory. An introductory course*, Graduate Texts in Mathematics, New York-Berlin, Springer, 1979.

[3] A. Cayley, "A theorem on trees", *Quart. J. Math.* **23** (1889), 376–378; *Collected Papers*, vol. 13, Cambridge University Press, 1897, pp. 26–28.

[4] A. Erdély, I.M.H. Etherington, "Some problems of non-associative combinations (2)", *Edinburgh Math. Notes* **32** (1940), 7–12.

[5] I.M. Gessel, L.H. Kalikow, "Hypergraphs and a functional equation of Bouwkamp and de Bruijn", *J. Combin. Theory Ser. A 110* (2005), no. 2, 275–289.

[6] K. Husimi, "Note on Mayer's theory of cluster integrals", *Journal of Chemical Physics* **18** (1950), 682–684.

[7] Kreweras, "Counting problems in dendroids" in *Combinatorial Structures and their Applications*, R.K. Guy et al. editors, 223-226, Gordon and Breach, New York (1970).

[8] J.W. Moon, *Counting labelled trees*, Canadian Mathematical Monographs, No. 1.

[9] T.D. Porter, "Binomial Identities Generated by Counting Spanning Trees", *Ars Combinatoria* **82** (2007), 159–163.

[10] R. P. Stanley, *Enumerative Combinatorics, Volume 2*, Cambridge University Press, Cambridge, 1999.

[11] J.J. Sylvester, “On the change of systems of independent variables”, *Quart. J. Math.* **1** (1857), 42–56; *Collected Mathematical Papers*, vol. 2, Cambridge, 1908, pp. 65–85.

[12] L. Takàcs, “On the number of distinct forests”, *SIAM J.Disc. Math.* **3** (1990), 574–581.

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