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# Lattices with and lattices without spectral gap 

Bachir Bekka and Alexander Lubotzky *

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For Fritz Grunewald on his 60th birthday


#### Abstract

Let $G=\mathbf{G}(\mathbf{k})$ be the $\mathbf{k}$-rational points of a simple algebraic group $\mathbf{G}$ over a local field $\mathbf{k}$ and let $\Gamma$ be a lattice in $G$. We show that the regular representation $\rho_{\Gamma \backslash G}$ of $G$ on $L^{2}(\Gamma \backslash G)$ has a spectral gap, that is, the restriction of $\rho_{\Gamma \backslash G}$ to the orthogonal of the constants in $L^{2}(\Gamma \backslash G)$ has no almost invariant vectors. On the other hand, we give examples of locally compact simple groups $G$ and lattices $\Gamma$ for which $L^{2}(\Gamma \backslash G)$ has no spectral gap. This answers in the negative a question asked by Margulis [Marg91, Chapter III, 1.12]. In fact, $G$ can be taken to be the group of orientation preserving automorphisms of a $k$-regular tree for $k>2$.


## 1 Introduction

Let $G$ be a locally compact group. Recall that a unitary representation $\pi$ of $G$ on a Hilbert space $\mathcal{H}$ has almost invariant vectors if, for every compact subset $Q$ of $G$ and every $\varepsilon>0$, there exists a unit vector $\xi \in \mathcal{H}$ such that $\sup _{x \in Q}\|\pi(x) \xi-\xi\|<\varepsilon$. If this holds, we also say that the trivial representation $1_{G}$ is weakly contained in $\pi$.

Recall that a lattice $\Gamma$ in $G$ is a discrete subgroup such that there exists a finite $G$-invariant regular Borel measure $\mu$ on $\Gamma \backslash G$. Denote by $\rho_{\Gamma \backslash G}$ the unitary representation of $G$ given by right translation on the Hilbert space

[^0]$L^{2}(\Gamma \backslash G, \mu)$ of the square integrable measurable functions on $\Gamma \backslash G$. The subspace $\mathbb{C}_{\Gamma \backslash G}$ of the constant functions on $\Gamma \backslash G$ is $G$-invariant as well as its orthogonal complement
$$
L_{0}^{2}(\Gamma \backslash G)=\left\{\xi \in L^{2}(\Gamma \backslash G): \int_{\Gamma \backslash G} \xi(x) d \mu(x)=0\right\} .
$$

Denote by $\rho_{\Gamma \backslash G}^{0}$ the restriction of $\rho_{\Gamma \backslash G}$ to $L_{0}^{2}(\Gamma \backslash G, \mu)$. We say that $\rho_{\Gamma \backslash G}$ (or $L^{2}(\Gamma \backslash G, \mu)$ ) has a spectral gap if $\rho_{\Gamma \backslash G}^{0}$ has no almost invariant vectors. (In [Marg91, Chapter III., 1.8], $\Gamma$ is then called weakly cocompact.) It is well-known that $L^{2}(\Gamma \backslash G)$ has a spectral gap when $\Gamma$ is cocompact in $G$ (see [Marg91, Chapter III, 1.10]). Margulis (op.cit, 1.12) asks whether this result holds more generally when $\Gamma$ is a subgroup of finite covolume.

The goal of this note is to prove the following results:
Theorem 1 Let $\mathbf{G}$ be a simple algebraic group over a local field $\mathbf{k}$ and $G=$ $\mathbf{G}(\mathbf{k})$, the group of $\mathbf{k}$-rational points in $\mathbf{G}$. Let $\Gamma$ be a lattice in $G$. Then the unitary representation $\rho_{\Gamma \backslash G}$ on $L^{2}(\Gamma \backslash G)$ has a spectral gap.

Theorem 2 For an integer $k>2$, let $X$ be the $k$-regular tree and $G=$ $\operatorname{Aut}(X)$. Then $G$ contains a lattice $\Gamma$ for which the unitary representation $\rho_{\Gamma \backslash G}$ on $L^{2}(\Gamma \backslash G)$ has no spectral gap.

So, Theorem 2 answers in the negative Margulis' question mentioned above.

Theorem 1 is known in case $\mathbf{k}=\mathbf{R}$ ([Bekk98]). It holds, more generally, when $G$ is a real Lie group ([BeCo08]). Observe also that when $\mathbf{k}-\operatorname{rank}(\mathbf{G}) \geq$ 2, the group $G$ has Kazhdan's Property (T) (see [BHV]) and Theorem 1 is clear in this case. When $\mathbf{k}$ is non-archimedean with characteristic 0 , every lattice $\Gamma$ in $\mathbf{G}(\mathbf{k})$ is uniform (see [Serr, p.84]) and hence the result holds as mentioned above. By way of contrast, $G$ has many non uniform lattices when the characteristic of $\mathbf{k}$ is non zero (see [Serr] and [Lubo91]). So, in order to prove Theorem 1, it suffices to consider the case where the characteristic of $\mathbf{k}$ is non-zero and where $\mathbf{k}-\operatorname{rank}(\mathbf{G})=1$.

Recall that when $\mathbf{k}$ is non-archimedean and $\mathbf{k}-\operatorname{rank}(\mathbf{G})=1$, the group $\mathbf{G}(\mathbf{k})$ acts by automorphisms on the associated Bruhat-Tits tree $X$ (see [Serr]). This tree is either the $k$-regular tree $X_{k}$ (in which every vertex has constant degree $k$ ) or is the bi-partite bi-regular tree $X_{k_{0}, k_{1}}$ (where every vertex has either degree $k_{0}$ or degree $k_{1}$ and where all neighbours of a vertex
of degree $k_{i}$ have degree $\left.k_{1-i}\right)$. The proof of Theorem 1 will use the special structure of a fundamental domain for the action of $\Gamma$ on $X$ as described in [Lubo91] (see also [Ragh89] and [Baum03]).

Theorems 1 and 2 provide a further illustration of the different behaviour of general tree lattices as compared to lattices in rank one simple Lie groups over local fields; for more on this topic, see [Lubo95].

The proofs of Theorems 1 and 2 will be given in Sections 3 and 4; they rely in a crucial way on Proposition 6 from Section 2, which relates the existence of a spectral gap with expander diagrams. In turn, Proposition 6 is based, much in the spirit of [Broo81], on analogues for diagrams proved in [Mokh03] and [Morg94] of the inequalities of Cheeger and Buser between the isoperimeric constant and the bottom of the spectrum of the Laplace operator on a Riemannian manifold (see Proposition 5). This connection between the combinatorial expanding property and representation theory is by now a very popular theme; see [Lubo94] and the references therein. While most applications in this monograph are from representation theory to combinatorics, we use in the current paper this connection in the opposite direction: the existence or absence of a spectral gap is deduced from the existence of an expanding diagram or of a non-expanding diagram, respectively.

## 2 Spectral gap and expander diagrams

We first show how the existence of a spectral gap for groups acting on trees is related with the bottom of the spectrum of the Laplacian for an associated diagram.

A graph $X$ consists of a set of vertices $V X$, a set of oriented edges $E X$, a fix-point free involution ${ }^{-}: E X \rightarrow E X$, and end point mappings $\partial_{i}: E X \rightarrow$ $V X$ for $i=0,1$ such that $\partial_{i}(\bar{e})=\partial_{1-i}(e)$ for all $e \in E X$. Assume that $X$ is locally finite, that is, for every $x \in V X$, the degree $\operatorname{deg}(x)$ of $x$ is finite, where $\operatorname{deg}(x)$ is the cardinality of the set

$$
\partial_{0}^{-1}(x)=\left\{e \in E X: \quad \partial_{0}(e)=x\right\} .
$$

The group $\operatorname{Aut}(X)$ of automorphisms of the graph $X$ is a locally compact group in the topology of pointwise convergence on $X$, for which the stabilizers of vertices are compact open subgroups.

We will consider infinite graphs called diagrams of finite volume. An edgeindexed graph $(D, i)$ is a graph $D$ equipped with a function $i: E D \rightarrow \mathbf{R}^{+}$
(see [BaLu01, Chapter 2]). A measure $\mu$ for an edge-indexed graph $(D, i)$ is a function $\mu: V D \cup E D \rightarrow \mathbf{R}^{+}$with the following properties (see [Mokh03] and [BaLu01, 2.6]):

- $i(e) \mu\left(\partial_{0} e\right)=\mu(e)$
- $\mu(e)=\mu(\bar{e})$ for all $e \in V D$, and
- $\sum_{x \in V D} \mu(x)<\infty$.

Following [Morg94], we will say that $D=(D, i, \mu)$ is a diagram of finite volume. The in-degree $\operatorname{indeg}(x)$ of a vertex $x \in V D$ is defined by

$$
\operatorname{indeg}(x)=\sum_{e \in \partial_{0}^{-1}(x)} i(e)=\sum_{e \in \partial_{0}^{-1}(x)} \frac{\mu(e)}{\mu(x)}
$$

The diagram $D$ is $k$-regular if indeg $(x)=k$ for all $x \in V D$.
Let $D=(D, i, \mu)$ be a connected diagram of finite volume. Observe that $\mu$ is determined, up to a multiplicative constant, by the weight function $i$. Indeed, fix $x_{0} \in V D$ and set $\Delta(e)=i(e) / i(\bar{e})$ for $e \in E D$. Then

$$
\mu\left(\partial_{1} e\right)=\frac{\mu(\bar{e})}{i(\bar{e})}=\frac{\mu(e)}{i(\bar{e})}=\mu\left(\partial_{0} e\right) \Delta(e)
$$

for every $e \in E D$. Hence $\mu(x)=\Delta\left(e_{1}\right) \Delta\left(e_{2}\right) \ldots \Delta\left(e_{n}\right) \mu\left(x_{0}\right)$ for every path $\left(e_{1}, e_{2}, \ldots, e_{n}\right)$ from $x_{0}$ to $x \in V D$.

Let $D=(D, i, \mu)$ be a diagram of finite volume. An inner product is defined for functions on $V D$ by

$$
\langle f, g\rangle=\sum_{x \in V D} f(x) \overline{g(x)} \mu(x)
$$

The Laplace operator $\Delta$ on functions $f$ on $V D$ is defined by

$$
\Delta f(x)=f(x)-\frac{1}{\operatorname{indeg}(x)} \sum_{e \in \partial_{0}^{-1}(x)} \frac{\mu(e)}{\mu(x)} f\left(\partial_{1}(e)\right)
$$

The operator $\Delta$ is a self-adjoint positive operator on $L^{2}(V D)$. Let

$$
L_{0}^{2}(V D)=\left\{f \in L^{2}(V D):\left\langle f, 1_{V D}\right\rangle=0\right\}
$$

and set

$$
\lambda(D)=\inf _{f}\langle\Delta f, f\rangle,
$$

where $f$ runs over the unit sphere in $L_{0}^{2}(V D)$. Observe that

$$
\lambda(D)=\inf \{\lambda: \lambda \in \sigma(\Delta) \backslash\{0\}\}
$$

where $\sigma(\Delta)$ is the spectrum of $\Delta$.
Let now $X$ be a locally finite tree, and let $G$ be a closed subgroup of $\operatorname{Aut}(X)$. Assume that $G$ acts with finitely many orbits on $X$. Let $\Gamma$ be a discrete subgroup of $G$ acting without inversion on $X$. Then the quotient graph $\Gamma \backslash X$ is well-defined. Since $\Gamma$ is discrete, for every vertex $x$ and every edge $e$, the stabilizers $\Gamma_{x}$ and $\Gamma_{e}$ are finite. Moreover, $\Gamma$ is a lattice in $G$ if and only if $\Gamma$ is a lattice in $\operatorname{Aut}(X)$ and this happens if and only if

$$
\sum_{x \in D} \frac{1}{\left|\Gamma_{x}\right|}<\infty
$$

where $D$ is a fundamental domain of $\Gamma$ in $X$ (see [Serr]). The quotient graph $\Gamma \backslash X \cong D$ is endowed with the structure of an edge-indexed graph given by the weight function $i: E D \rightarrow \mathbf{R}^{+}$where $i(e)$ is the index of $\Gamma_{e}$ in $\Gamma_{x}$ for $x=\partial_{0}(e)$. A measure $\mu: V D \cup E D \rightarrow \mathbf{R}^{+}$is defined by

$$
\mu(x)=\frac{1}{\left|\Gamma_{x}\right|} \quad \text { and } \quad \mu(e)=\frac{1}{\left|\Gamma_{e}\right|}
$$

for $x \in V D$ and $e \in E D$. Observe that $\mu(V D)=\sum_{x \in D} 1 /\left|\Gamma_{x}\right|<\infty$. So, $D=(D, i, \mu)$ is a diagram of finite volume.

Let $G$ be a group acting on a tree $X$. As in [BuMo00, 0.2], we say that the action of $G$ on $X$ is locally $\infty$-transitive if, for every $x \in V X$ and every $n \geq 1$, the stabilizer $G_{x}$ of $x$ acts transitively on the sphere $\{y \in X: d(x, y)=n\}$.

Proposition 3 Let $X$ be either the $k$-regular tree $X_{k}$ or the bi-partite biregular tree $X_{k_{0}, k_{1}}$ for $k \geq 3$ or $k_{0} \geq 3$ and $k_{1} \geq 3$. Let $G$ be a closed subgroup of $\operatorname{Aut}(X)$. Assume that the following conditions are both satisfied:

- $G$ acts transitively on $V X$ in the case $X=X_{k}$ and $G$ acts transitively on the set of vertices of degree $k_{0}$ as well as on the set of vertices of degree $k_{1}$ in the case $X=X_{k_{0}, k_{1}}$;
- the action of $G$ on $X$ is locally $\infty$-transitive.

Let $\Gamma$ be a lattice in $G$ and let $D=\Gamma \backslash X$ be the corresponding diagram of finite volume. The following properties are equivalent:
(i) the unitary representation $\rho_{\Gamma \backslash G}$ on $L^{2}(\Gamma \backslash G)$ has a spectral gap;
(ii) $\lambda(D)>0$.

For the proof of this proposition, we will need a few general facts. Let $G$ be a second countable locally compact group and $U$ a compact subgroup of $G$. Let $C_{c}(U \backslash G / U)$ be the space of continuous functions $f: G \rightarrow \mathbf{C}$ which have compact support and which are constant on the double cosets $U g U$ for $g \in G$.

Fix a left Haar measure $\mu$ on $G$. Recall that $L^{1}(G, \mu)$ is a Banach algebra under the convolution product, the $L^{1}$-norm and the involution $f^{*}(g)=$ $\overline{f\left(g^{-1}\right)}$; observe that $C_{c}(U \backslash G / U)$ is a $*$-subalgebra of $L^{1}(G, \mu)$. Let $\pi$ be a (strongly continuous) unitary representation of $G$ on a Hilbert space $\mathcal{H}$. A continuous *-representation of $L^{1}(G)$, still denoted by $\pi$, is defined on $\mathcal{H}$ by

$$
\pi(f) \xi=\int_{G} f(x) \pi(x) \xi d \mu(x), \quad f \in L^{1}(G), \quad \xi \in \mathcal{H}
$$

Assume that the closed subspace $\mathcal{H}^{U}$ of $U$-invariant vectors in $\mathcal{H}$ is non-zero. Then $\pi(f) \mathcal{H}^{U} \subset \mathcal{H}^{U}$ for all $f \in C_{c}(U \backslash G / U)$. In this way, a continuous *-representation $\pi_{U}$ of $C_{c}(U \backslash G / U)$ is defined on $\mathcal{H}^{U}$.

Proposition 4 With the previous notation, let $f \in C_{c}(U \backslash G / U)$ be a function with the following properties: $f(x) \geq 0$ for all $x \in G, \int_{G} f d \mu=1$, and the subgroup generated by the support of $f$ is dense in $G$. The following conditions are equivalent:
(i) the trivial representation $1_{G}$ is weakly contained in $\pi$;
(ii) 1 belongs to the spectrum of the operator $\pi_{U}(f)$.

Proof Assume that $1_{G}$ is weakly contained in $\pi$. There exists a sequence of unit vectors $\xi_{n} \in \mathcal{H}$ such that

$$
\lim _{n}\left\|\pi(x) \xi_{n}-\xi_{n}\right\|=0
$$

uniformly over compact subsets of $G$. Let

$$
\eta_{n}=\int_{U} \pi(u) \xi_{n} d u
$$

where $d u$ denotes the normalized Haar measure on $U$. It is easily checked that $\eta_{n} \in \mathcal{H}^{U}$ and that

$$
\lim _{n}\left\|\pi(f) \eta_{n}-\eta_{n}\right\|=0
$$

Since

$$
\left\|\eta_{n}-\xi_{n}\right\| \leq \int_{U}\left\|\pi(u) \xi_{n}-\xi_{n}\right\| d u
$$

we have $\left\|\eta_{n}\right\| \geq 1 / 2$ for sufficiently large $n$. This shows that 1 belongs to the spectrum of the operator $\pi_{U}(f)$.

For the converse, assume that 1 belongs to the spectrum of $\pi_{U}(f)$. Hence, 1 belongs to the spectrum of $\pi(f)$, since $\pi_{U}(f)$ is the restriction of $\pi(f)$ to the invariant subspace $\mathcal{H}^{U}$. As the subgroup generated by the support of $f$ is dense in $G$, this implies that $1_{G}$ is weakly contained in $\pi$ (see [BHV, Proposition G.4.2]).

Proof of Proposition 3 We give the proof only in the case where $X$ is the bi-regular tree $X_{k_{0}, k_{1}}$. The case where $X$ is the regular tree $X_{k}$ is similar and even simpler.

Let $X_{0}$ and $X_{1}$ be the subsets of $X$ consisting of the vertices of degree $k_{0}$ and $k_{1}$, respectively. Fix two points $x_{0} \in X_{0}$ and $x_{1} \in X_{1}$ with $d\left(x_{0}, x_{1}\right)=1$. So, $X_{0}$ is the set of vertices $x$ for which $d\left(x_{0}, x\right)$ is even and $X_{1}$ is the set of vertices $x$ for which $d\left(x_{0}, x\right)$ is odd. Let $U_{0}$ and $U_{1}$ be the stabilizers of $x_{0}$ and $x_{1}$ in $G$. Since $G$ acts transitively on $X_{0}$ and on $X_{1}$, we have $G / U_{0} \cong X_{0}$. and $G / U_{1} \cong X_{1}$.

We can view the normed $*$-algebra $C_{c}\left(U_{0} \backslash G / U_{0}\right)$ as a space of finitely supported functions on $X_{0}$. Since $U_{0}$ acts transitively on every sphere around $x_{0}$, it is well-known that the pair $\left(G, U_{0}\right)$ is a Gelfand pair, that is, the algebra $C_{c}\left(U_{0} \backslash G / U_{0}\right)$ is commutative (see for instance [BLRW09, Lemma 2.1]). Observe that $C_{c}\left(U_{0} \backslash G / U_{0}\right)$ is the linear span of the characteristic functions $\delta_{n}^{(0)}$ (lifted to $G$ ) of spheres of even radius $n$ around $x_{0}$. Moreover, $C_{c}\left(U_{0} \backslash G / U_{0}\right)$ is generated by $\delta_{2}^{(0)}$; indeed, this follows from the formulas (see
[BLRW09, Theorem 3.3])

$$
\begin{aligned}
\delta_{4}^{(0)} & =\delta_{2}^{(0)} * \delta_{2}^{(0)}-k_{0}\left(k_{1}-1\right) \delta_{0}^{(0)}-\left(k_{1}-2\right) \delta_{2}^{(0)} \\
\delta_{2 n+2}^{(0)} & =\delta_{2}^{(0)} * \delta_{2 n}^{(0)}-\left(k_{0}-1\right)\left(k_{1}-1\right) \delta_{2 n-2}^{(0)}-\left(k_{1}-2\right) \delta_{2 n}^{(0)} \quad \text { for } \quad n \geq 2
\end{aligned}
$$

Let $f_{0}=\frac{1}{\left\|\delta_{2}^{(0)}\right\|_{1}} \delta_{2}^{(0)}$. We claim that $f_{0}$ has all the properties listed in Proposition 4.

Indeed, $f_{0}$ is a non-negative and $U_{0}$-bi-invariant function on $G$ with $\int_{G} f_{0}(x) d x=1$. Moreover, let $H$ be the closure of the subgroup generated by the support of $f_{0}$. Assume, by contradiction, that $H \neq G$. Then there exists a function in $C_{c}\left(U_{0} \backslash G / U_{0}\right)$ whose support is disjoint from $H$. This is a contradiction, as the algebra $C_{c}\left(U_{0} \backslash G / U_{0}\right)$ is generated by $f_{0}$. This shows that $H=G$.

Let $\pi$ be the unitary representation of $G$ on $L_{0}^{2}(\Gamma \backslash G)$ defined by right translations. Observe that the space of $\pi\left(U_{0}\right)$-invariant vectors is $L_{0}^{2}\left(\Gamma \backslash X_{0}\right)$. So, we have a $*$-representation $\pi_{U_{0}}$ of $C_{c}\left(U_{0} \backslash G / U_{0}\right)$ on $L^{2}\left(\Gamma \backslash X_{0}, \mu\right)$, where $\mu$ is the measure on the diagram $D=\Gamma \backslash X$, as defined above.

Similar facts are also true for the algebra $C_{c}\left(U_{1} \backslash G / U_{1}\right)$ : this is a commutative normed $*$-algebra, it is generated by the characteristic function $\delta_{2}^{(1)}$ of the sphere of radius 2 around $x_{1}$, and the representation $\pi$ of $G$ on $L_{0}^{2}(\Gamma \backslash G)$ induces a $*$-representation $\pi_{U_{1}}$ of $C_{c}\left(U_{1} \backslash G / U_{1}\right)$ on $L_{0}^{2}\left(\Gamma \backslash X_{1}, \mu\right)$. Likewise, the function $f_{1}=\frac{1}{\left\|\delta_{2}^{(1)}\right\|_{1}} \delta_{2}^{(1)}$ has all the properties listed in Proposition 4.

Let $A_{X}$ be the adjacency operator defined on $\ell^{2}(X)$ by

$$
A_{X} f(x)=\frac{1}{\operatorname{deg}(x)} \sum_{e \in \partial_{0}^{-1}(x)} f\left(\partial_{1}(e)\right), \quad f \in \ell^{2}(X)
$$

Since $A_{X}$ commutes with automorphisms of $X$, it induces an operator $A_{D}$ on $L^{2}(V D, \mu)$ given by

$$
A_{D} f(x)=\frac{1}{\operatorname{indeg}(x)} \sum_{e \in \partial_{0}^{-1}(x)} \frac{\mu(e)}{\mu(x)} f\left(\partial_{1}(e)\right), \quad f \in L^{2}(V D, \mu),
$$

where $D$ is the diagram obtained from the quotient graph $\Gamma \backslash X$. So, $\Delta=$ $I-A_{D}$, where $\Delta$ is the Laplace operator on $D$.

Let $B_{D}$ denote the restriction of $A_{D}$ to the space $L_{0}^{2}(V D, \mu)$. It follows that $\lambda(\Delta)>0$ if and only if 1 does not belong to the spectrum of $B_{D}$.

Proposition 3 will be proved, once we have shown the following
Claim: 1 belongs to the spectrum of $B_{D}$ if and only if $1_{G}$ is weakly contained in $\pi$.

For this, we consider the squares of the operators $A_{X}$ and $A_{D}$ and compute

$$
A_{X}^{2} f(x)=\frac{1}{k_{0} k_{1}} \operatorname{deg}(x) f(x)+\frac{1}{k_{0} k_{1}} \sum_{d(x, y)=2} f(y), \quad f \in \ell^{2}(X)
$$

The subspaces $\ell^{2}\left(X_{0}\right)$ and $\ell^{2}\left(X_{1}\right)$ of $\ell^{2}(X)$ are invariant under $A_{X}^{2}$ and the restrictions of $A_{X}^{2}$ to $\ell^{2}\left(X_{0}\right)$ and $\ell^{2}\left(X_{1}\right)$ are given by right convolution with the functions

$$
\begin{aligned}
& g_{0}=\frac{1}{k_{0} k_{1}} \delta_{e}+\left(1-\frac{1}{k_{0} k_{1}}\right) f_{0} \\
& g_{1}=\frac{1}{k_{0} k_{1}} \delta_{e}+\left(1-\frac{1}{k_{0} k_{1}}\right) f_{1},
\end{aligned}
$$

where $\delta_{e}$ is the Dirac function at the group unit $e$ of $G$.
It follows that the restrictions of $B_{D}^{2}$ to the subspaces $L_{0}^{2}\left(\Gamma \backslash X_{0}, \mu\right)$ and $L_{0}^{2}\left(\Gamma \backslash X_{1}, \mu\right)$ coincide with the operators $\pi_{U_{0}}\left(g_{0}\right)$ and $\pi_{U_{1}}\left(g_{1}\right)$, respectively.

For $i=0,1$, the spectrum $\sigma\left(\pi_{U_{i}}\left(g_{i}\right)\right)$ of $\pi_{U_{i}}\left(g_{i}\right)$ is the set

$$
\sigma\left(\pi_{U_{i}}\left(g_{i}\right)\right)=\left\{\frac{1}{k_{0} k_{1}}+\left(1-\frac{1}{k_{0} k_{1}}\right) \lambda: \lambda \in \sigma\left(\pi_{U_{i}}\left(f_{i}\right)\right)\right\} .
$$

Thus, 1 belongs to the spectrum of $\pi_{U_{0}}\left(f_{i}\right)$ if and only if 1 belongs to the spectrum of $\pi_{U_{0}}\left(g_{i}\right)$.

To prove the claim above, assume that 1 belongs to the spectrum of $B_{D}$. Then 1 belongs to the spectrum of $B_{D}^{2}$. Hence 1 belongs to the spectrum of either $\pi_{U_{0}}\left(g_{0}\right)$ or $\pi_{U_{1}}\left(g_{1}\right)$ and therefore 1 belongs to the spectrum of either $\pi_{U_{0}}\left(f_{0}\right)$ or $\pi_{U_{1}}\left(f_{1}\right)$. It follows from Proposition 4 that $1_{G}$ is weakly contained in $\pi$.

Conversely, suppose that $1_{G}$ is weakly contained in $\pi$. Then, again by Proposition 4, 1 belongs to the spectra of $\pi_{U_{0}}\left(f_{0}\right)$ and $\pi_{U_{1}}\left(f_{1}\right)$. Hence, 1 belongs to the spectra of $\pi_{U_{0}}\left(g_{0}\right)$ and $\pi_{U_{1}}\left(g_{1}\right)$. We claim that 1 belongs to the spectrum of $B_{D}$.

Indeed, assume by contradiction that 1 does not belong to the spectrum of $B_{D}$, that is, $B_{D}-I$ has a bounded inverse on $L_{0}^{2}(V D, \mu)$. Since 1 belongs to the spectrum of the self-adjoint operator $\pi_{U_{0}}\left(g_{0}\right)$, there exists a sequence of unit vectors $\xi_{n}^{(0)}$ in $L_{0}^{2}\left(\Gamma \backslash X_{0}, \mu\right)$ with

$$
\lim _{n}\left\|\pi_{U_{0}}\left(g_{0}\right) \xi_{n}^{(0)}-\xi_{n}^{(0)}\right\|=0
$$

As the restriction of $B_{D}^{2}$ to $L_{0}^{2}\left(\Gamma \backslash X_{0}, \mu\right)$ coincides with $\pi_{U_{0}}\left(g_{0}\right)$, we have

$$
\begin{aligned}
\left\|\pi_{U_{0}}\left(g_{0}\right) \xi_{n}^{(0)}-\xi_{n}^{(0)}\right\| & =\left\|\left(B_{D}^{2}-I\right) \xi_{n}^{(0)}\right\| \\
& =\left\|\left(B_{D}-I\right)\left(B_{D}+I\right) \xi_{n}^{(0)}\right\| \\
& \geq \frac{1}{\left\|\left(B_{D}-I\right)^{-1}\right\|}\left\|\left(B_{D}+I\right) \xi_{n}^{(0)}\right\|
\end{aligned}
$$

So, $\lim _{n}\left\|B_{D} \xi_{n}^{(0)}+\xi_{n}^{(0)}\right\|=0$. On the other hand, observe that $B_{D}$ maps $L_{0}^{2}\left(\Gamma \backslash X_{0}, \mu\right)$ to the subspace $L^{2}\left(\Gamma \backslash X_{1}, \mu\right)$ and that these subspaces are orthogonal to each other. Hence,

$$
\left\|B_{D} \xi_{n}^{(0)}+\xi_{n}^{(0)}\right\|^{2}=\left\|B_{D} \xi_{n}^{(0)}\right\|^{2}+\left\|\xi_{n}^{(0)}\right\|^{2}
$$

This is a contradiction since $\left\|\xi_{n}^{(0)}\right\|=1$ for all $n$. The proof of Proposition 3 is now complete.

Next, we rephrase Proposition 3 in terms of expander diagrams. Let $(D, i, w)$ be a diagram with finite volume. For a subset $S$ of $V D$, set

$$
E\left(S, S^{c}\right)=\left\{e \in E D: \partial_{0}(e) \in S, \partial_{1}(e) \notin S\right\}
$$

We say that $D$ is an expander diagram if there exists $\varepsilon>0$ such that

$$
\frac{\mu\left(E\left(S, S^{c}\right)\right)}{\mu(S)} \geq \varepsilon
$$

for all $S \subset V D$ with $\mu(S) \leq \mu(D) / 2$. The motivation for this definition comes from expander graphs (see [Lubo94]).

We quote from [Mokh03] and [Morg94] the following result which is standard in the case of finite graphs.

Proposition 5 ([Mokh03], [Morg94]) Let $(D, i, w)$ be a diagram with finite volume. Assume that $\sup _{e \in E D} i(\bar{e}) / i(e)<\infty$ and that $\sup _{x \in V D} \operatorname{indeg}(x)<$ $\infty$ The following conditions are equivalent:
(i) $D$ is an expander diagram;
(ii) $\lambda(D)>0$.

As an immediate consequence of Propositions 3 and 5 , we obtain the following result which relates the existence of a spectral gap to an expanding property of the corresponding diagram.

Proposition 6 Let $X$ be either the $k$-regular tree $X_{k}$ or the bi-partite biregular tree $X_{k_{0}, k_{1}}$ for $k \geq 3$ or $k_{0} \geq 3$ and $k_{1} \geq 3$. Let $G$ be a closed subgroup of $\operatorname{Aut}(X)$ satisfiying both conditions from Proposition 3. Let $\Gamma$ be a lattice in $G$ and let $D=\Gamma \backslash X$ be the corresponding diagram of finite volume. The following properties are equivalent.
(i) The unitary representation $\rho_{\Gamma \backslash G}$ on $L^{2}(\Gamma \backslash G)$ has a spectral gap;
(ii) $D$ is an expander diagram.

## 3 Proof of Theorem 1

Let $G=\mathbf{G}(\mathbf{k})$ be the $\mathbf{k}$-rational points of a simple algebraic group $\mathbf{G}$ over a local field $\mathbf{k}$ and let $\Gamma$ be a lattice in $G$. As explained in the Introduction, we may assume that $\mathbf{k}$ is non-archimedean and that $\mathbf{k}-\operatorname{rank}(\mathbf{G})=1$. By the Bruhat-Tits theory, $G$ acts on a regular or bi-partite bi-regular tree $X$ with one or two orbits. Moreover, the action of $G$ on $X$ is locally $\infty$-transitive (see [Chou94, p.33]).

Passing to the subgroup $G^{+}$of index at most two consisting of orientation preserving automorphisms, we can assume that $G$ acts without inversion. Indeed, assume that $L^{2}\left(\Gamma \cap G^{+} \backslash G^{+}\right)$has a spectral gap. If $\Gamma$ is contained in $G^{+}$, then $L^{2}(\Gamma \backslash G)$ has a spectral gap since $G^{+}$has finite index (see [BeCo08, Proposition 6]). If $\Gamma$ is not contained in $G^{+}$, then $\Gamma \cap G^{+} \backslash G^{+}$may be identified as a $G^{+}$-space with $\Gamma \backslash \Gamma G^{+}=\Gamma \backslash G$. Hence, $1_{G^{+}}$is not weakly contained in the $G^{+}$-representation defined on $L_{0}^{2}(\Gamma \backslash G)$.

Let $X$ be the Bruhat-Tits tree associated to $G$. It is shown in [Lubo91, Theorem 6.1] (see also [Baum03]) that $\Gamma$ has fundamental domain $D$ in $X$ of the following form: there exists a finite set $F \subset D$ such that $D \backslash F$ is a union of finitely many disjoint rays $r_{1}, \ldots, r_{s}$. (Recall that a ray in $X$ is an infinite
path beginning at some vertex and without backtracking.) Moreover, for every ray $r_{j}=\left\{x_{0}^{j}, x_{1}^{j}, x_{2}^{j}, \ldots\right\}$ in $D \backslash F$, the stabilizer $\Gamma_{x_{i}^{j}}$ of $x_{i}^{j}$ is contained in the stabilizer $\Gamma_{x_{i+1}^{j}}$ of $x_{i+1}^{j}$ for all $i$.

To prove Theorem 1, we apply Proposition 6. So, we have to prove that $D$ is an expander diagramm.

Choose $i \in\{0,1, \ldots\}$ such that, with

$$
D_{1}=F \cup \bigcup_{j=1}^{s}\left\{x_{0}^{j}, \ldots, x_{i}^{j}\right\}
$$

we have $\mu\left(D_{1}\right)>1 / 2$.
Let $S$ be a subset of $D$ with $\mu(S) \leq \mu(D) / 2$. Then $D_{1} \nsubseteq S$. Two cases can occur.

- First case: $S \cap D_{1}=\emptyset$. Thus, $S$ is contained in

$$
\bigcup_{j=1}^{s}\left\{x_{i+1}^{j}, x_{i+2}^{j}, \ldots\right\} .
$$

Fix $j \in\{1, \ldots, s\}$. Let $i(j) \in\{0,1, \ldots\}$ be minimal with the property that $x_{i(j)+1}^{j} \in S$. Then $e_{j}:=\left(x_{i(j)+1}^{j}, x_{i(j)}^{j}\right) \in E\left(S, S^{c}\right)$. Observe that $\left|\Gamma_{x_{l+1}^{j}}\right|=$ $\operatorname{deg}\left(x_{l}^{j}\right)\left|\Gamma_{x_{l}^{j}}\right|$ for all $l \geq 0$. Let $k$ be the minimal degree for vertices in $X$ (so, $k=\min \left\{k_{0}, k_{1}\right\}$ if $\left.X=X_{k_{0}, k_{1}}\right)$.Then $\mu\left(x_{l+1}^{j}\right) \leq \mu\left(x_{l}^{j}\right) / k$ for all $l$ and

$$
\mu\left(e_{j}\right)=\frac{1}{\left|\Gamma_{e_{j}}\right|} \geq \frac{k}{\left|\Gamma_{x_{i(j)}^{j}}\right|}=k \mu\left(x_{i(j)}^{j}\right) .
$$

Therefore, we have

$$
\begin{aligned}
\frac{\mu\left(E\left(S, S^{c}\right)\right)}{\mu(S)} & \geq \frac{\sum_{j=1}^{s} \mu\left(e_{j}\right)}{\sum_{j=1}^{s} \mu\left(\left\{x_{i(j)+1}^{j}, x_{i(j)+1}^{j}, \ldots,\right\}\right)} \\
& \geq k \frac{\sum_{j=1}^{s} \mu\left(x_{i(j)}^{j}\right)}{\sum_{j=1}^{s} \sum_{l=0}^{\infty} \mu\left(x_{i(j)+l}^{j}\right)} \\
& \geq k \frac{\sum_{j=1}^{s} \mu\left(x_{i(j)}^{j}\right)}{\sum_{j=1}^{s} \mu\left(x_{i(j)}^{j}\right) \sum_{l=0}^{\infty} k^{-l}} \\
& =k \frac{\sum_{j=1}^{s} \mu\left(x_{i(j)}^{j}\right)}{\frac{1}{1-k^{-1}} \sum_{j=1}^{s} \mu\left(x_{i(j)}^{j}\right)} \\
& =k \frac{1}{\frac{1}{1-k^{-1}}}=k-1 .
\end{aligned}
$$

- Second case: $S \cap D_{1} \neq \emptyset$. Then there exist $x \in S \cap D_{1}$ and $y \in D_{1} \backslash S$. Since $D_{1}$ is a connected subgraph, there exists a path $\left(e_{1}, e_{2}, \ldots, e_{n}\right)$ in $E D_{1}$ from $x$ to $y$. Let $l \in\{1, \ldots, n\}$ be minimal with the property $\partial_{0}\left(e_{l}\right) \in S$ and $\partial_{1}\left(e_{l}\right) \notin S$. Then $e_{l} \in E\left(S, S^{c}\right)$. Hence, with $\left.C=\min \left\{\mu(e): e \in E D_{1}\right)\right\}>0$, we have

$$
\frac{\mu\left(E\left(S, S^{c}\right)\right)}{\mu(S)} \geq \frac{C}{\mu(D)} .
$$

This completes the proof of Theorem 1.

## 4 Proof of Theorem 2

Let $(D, i, \mu)$ be a $k$-regular diagram. By the "inverse Bass-Serre theory" of groups acting on trees, there exists a lattice $\Gamma$ in $G=\operatorname{Aut}\left(X_{k}\right)$ for which $D=\Gamma \backslash X_{k}$. Indeed, we can find a finite grouping of ( $D, i$ ), that is, a graph of finite groups $\mathbf{D}=(D, \mathcal{D})$ such that $i(e)$ is the index of $\mathcal{D}_{e}$ in $\mathcal{D}_{\partial_{0} e}$ for all $e \in E D$. Fix an origin $x_{0}$. Let $\Gamma=\pi_{1}\left(\mathbf{D}, x_{0}\right)$ be the fundamental group of $\left(\mathbf{D}, x_{0}\right)$. The universal covering of $\left(\mathbf{D}, x_{0}\right)$ is the $k$-regular tree $X_{k}$ and the diagram $D$ can identified with the diagram associated to $\Gamma \backslash X_{k}$. For all this, see (2.5), (2.6) and (4.13) in [BaLu01].

In view of Proposition 6, Theorem 2 will be proved once we present examples of $k$-regular diagrams with finite volume which are not expanders.

An example of such a diagram appears in [Mokh03, Example 3.4]. For the convenience of the reader, we review the construction.

Fix $k \geq 3$ and let $q=k-1$. For every integer $n \geq 1$, let $D_{n}$ be the finite graph with $2 n+1$ vertices:

$$
\stackrel{\circ}{x_{1}^{(n)}}-\underset{x_{2}^{(n)}}{\circ}-\circ-\cdots \circ-\underset{x_{2 n}^{(n)}}{\circ}-\underset{\substack{(n) \\ x_{2 n+1}}}{\circ}
$$

Let $D$ be the following infinite ray:

$$
\stackrel{\circ}{x_{0}}-\stackrel{\circ}{x_{1}}-D_{1}-\underset{x_{2}}{\circ}-\underset{x_{3}}{\circ}-D_{2}-\circ-\circ-\cdots--\underset{x_{2 n-2}}{\circ}-\underset{x_{2 n-1}}{\circ}-D_{n}-\circ-\circ \cdots
$$

We first define a weight function $i_{n}$ on $E D_{n}$ as follows:

- $i_{n}(e)=1$ if $e=\left(x_{1}^{(n)}, x_{2}^{(n)}\right)$ or $e=\left(x_{2}^{(n)}, x_{1}^{(n)}\right)$
- $i_{n}(e)=q$ if $e=\left(x_{m}^{(n)}, x_{m+1}^{(n)}\right)$ for $m$ even
- $i_{n}(e)=1$ if $e=\left(x_{m}^{(n)}, x_{m+1}^{(n)}\right)$ for $m$ odd
- $i_{n}(e)=q$ if $e=\left(x_{m+1}^{(n)}, x_{m}^{(n)}\right)$ for $m$ even
- $i_{n}(e)=1$ if $e=\left(x_{m+1}^{(n)}, x_{m}^{(n)}\right)$ for $m$ odd.

Observe that $i_{n}(e) / i_{n}(\bar{e})=1$ for all $e \in E D_{n}$. Define now a weight function $i$ on $E D$ as follows:

- $i(e)=q+1$ if $e=\left(x_{0}, x_{1}\right)$
- $i(e)=q$ if $e=\left(x_{1}, x_{0}\right)$
- $i(e)=1$ if $e=\left(x_{m}, x_{m+1}\right)$ for $m \geq 1$
- $i(e)=q$ if $e=\left(x_{m+1}, x_{m}\right)$ for $m \geq 1$
- $i(e)=i_{n}(e)$ if $e \in E D_{n}$.

One readily checks that, for every vertex $x \in D$,

$$
\sum_{e \in \partial_{0}^{-1}(x)} i(e)=q+1=k
$$

that is, $(D, i)$ is $k$-regular. The measure $\mu: V D \rightarrow \mathbf{R}^{+}$corresponding to $i$ (see the remark at the beginning of Section 2) is given by

- $\mu\left(x_{0}\right)=1 /(q+1)$
- $\mu\left(x_{2 m-2}\right)=1 / q^{m-1}$ for $m \geq 2$
- $\mu\left(x_{2 m-1}\right)=1 / q^{m}$ for $m \geq 1$
- $\mu(x)=1 / q^{n}$ if $x \in D_{n}$.

One checks that, if we define $\mu(e)=i(e) \mu\left(\partial_{0} e\right)$ for all $e \in E D$, we have $\mu(\bar{e})=\mu(e)$. Moreover,

$$
\mu\left(D_{n}\right)=(2 n+1) \frac{1}{q^{n}}
$$

and hence

$$
\mu(D) \leq \frac{1}{q+1}+2 \sum_{n \geq 0} \frac{1}{q^{n}}+\sum_{n \geq 1} \mu\left(D_{n}\right)<\infty .
$$

We have also

$$
E\left(D_{n}, D_{n}^{c}\right)=\left\{\left(x_{2 n-1}, x_{2 n-2}\right),\left(x_{2 n}, x_{2 n+1}\right)\right\}
$$

so that

$$
\mu\left(E\left(D_{n}, D_{n}^{c}\right)\right)=q \frac{1}{q^{n}}+\frac{1}{q^{n}}=\frac{q+1}{q^{n}} .
$$

Hence

$$
\frac{\mu\left(E\left(D_{n}, D_{n}^{c}\right)\right)}{\mu\left(D_{n}\right)}=\frac{\frac{q+1}{q^{n}}}{(2 n+1) \frac{1}{q^{n}}}=\frac{q+1}{2 n+1}
$$

and

$$
\lim _{n} \frac{\mu\left(E\left(D_{n}, D_{n}^{c}\right)\right)}{\mu\left(D_{n}\right)}=0 .
$$

Observe that, since $\lim _{n} \mu\left(D_{n}\right)=0$, we have $\mu\left(D_{n}\right) \leq \mu(D) / 2$ for sufficiently large $n$. This completes the proof of Theorem 2 .

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