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Pascal Maillard

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A characterisation of superposable random measures

PASCAL MAILLARD*

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Summary. Let Z be a point process on \mathbb{R} and $T_\alpha Z$ its translation by $\alpha \in \mathbb{R}$. Let Z' be an independent copy of Z . We say that Z is *superposable*, if $T_\alpha Z + T_\beta Z'$ and Z are equal in law for every $\alpha, \beta \in \mathbb{R}$, such that $e^\alpha + e^\beta = 1$. We prove a characterisation of superposable point processes in terms of decorated Poisson processes, which was conjectured by Brunet and Derrida [A branching random walk seen from the tip, 2010, <http://arxiv.org/abs/1011.4864v1>]. We further prove a generalisation to random measures.

Keywords. Superposable random measures, superposable point processes, infinitely divisible random measures, branching Brownian motion.

MSC2010. 60G55, 60G57

1 Introduction

Let Z be a *random measure* on \mathbb{R} , i.e. a random variable taking values in the space \mathcal{M} of boundedly finite measures on \mathbb{R} , i.e. measures, which assign finite mass to every bounded Borel set in \mathbb{R} . In addition, let $\mathcal{N} \subset \mathcal{M}$ be the space of boundedly finite counting measures on \mathbb{R} . If Z takes values in \mathcal{N} , we also call Z a *point process*.

For every $x \in \mathbb{R}$ define the translation operator $T_x : \mathcal{M} \rightarrow \mathcal{M}$, by $(T_x \mu)(A) = \mu(A - x)$ for every Borel set $A \subset \mathbb{R}$. Note that $T_x(\mathcal{N}) = \mathcal{N}$. Denote equality in law by $\stackrel{(d)}{=}$. Let Z' be an independent copy of Z . We say that Z is *superposable*, if

$$T_\alpha Z + T_\beta Z' \stackrel{(d)}{=} Z, \quad \text{for every } \alpha, \beta \in \mathbb{R} \text{ such that } e^\alpha + e^\beta = 1.$$

The main result of this article states informally: *Every superposable point process can be represented as a Poisson process of intensity measure $e^{-x} dx$ on \mathbb{R} , decorated by an auxiliary point process.* We further prove a generalisation of this result to superposable random measures.

The content of the article is organised as follows: In Section 2 we recall a connection with branching Brownian motion that motivated our study. In Section 3, results about infinitely divisible random measures are recalled. Decorated Poisson point processes are defined and studied in Section 4. Section 5 contains the main results. In Section 6, some results about spaces of measures are recalled. The main theorems are proven in Section 7. Section 8 contains a discussion of the results.

*Laboratoire de Probabilités et Modèles Aléatoires, UMR 7599, Université Paris VI, Case courrier 188, 4 Place Jussieu, 75252 PARIS Cedex 05, France, e-mail: pascal.maillard@upmc.fr

2 A connection with branching Brownian motion

Consider a particle system on the real line where, starting with one particle at the origin, particles move according to Brownian motion with variance 2 and branch at rate 1. This system is called the *branching Brownian motion*. Let $X(t) = \sum_i \delta_{X_i(t)}$ be the point process formed by the particles alive at time t . Lalley and Sellke [7] showed that the *derivative martingale* $W_t = \sum_i (2t - X_i(t)) \exp(X_i(t) - 2t)$ converges almost surely to a limit $W > 0$ as $t \rightarrow \infty$ and that for every $x \in \mathbb{R}$,

$$\lim_{s \rightarrow \infty} \lim_{t \rightarrow \infty} P(\max_i X_i(t) < m_t + x | \mathcal{F}_s) = \exp(-CW e^{-x}) = \exp(-C e^{-x + \log W}),$$

where $m_t = 2t - \frac{3}{2} \log t$ and $C > 0$. Brunet and Derrida [2] refined this result and provided convincing arguments that for any finite number of initial particles at arbitrary positions, conditioning on W , $T_{-m_t} X(t)$ converges in law as $t \rightarrow \infty$ to $T_{\log W} Z$, where Z is a point process independent from W whose law does not depend on the initial configuration. In other words, the point process formed by the right-most particles of branching Brownian motion at a large time t is a translation by $m_t + \log W$ of an independent point process Z .

Brunet and Derrida deduced from this a property they called *superposability*: Let X' be another branching Brownian motion independent from X , let $X'' = X + X'$ be the superposition of X and X' and let W' and W'' be the limits of their respective derivative martingales. Then $W'' = W + W'$ by definition and conditionally on W and W' ,

$$T_{\log(W+W')} Z'' \stackrel{(d)}{=} \text{d-lim}_{t \rightarrow \infty} T_{-m_t} X''(t) \stackrel{(d)}{=} \text{d-lim}_{t \rightarrow \infty} T_{-m_t} X(t) + T_{-m_t} X''(t) \stackrel{(d)}{=} T_{\log W} Z + T_{\log W'} Z',$$

where Z' and Z'' are defined analogously to Z and d-lim means limit in law. Since W and W' can take any positive value (for example by choosing the initial configurations accordingly), this gives $T_\gamma Z \stackrel{(d)}{=} T_\alpha Z + T_\beta Z'$ for every $\alpha, \beta, \gamma \in \mathbb{R}$, such that $e^\alpha + e^\beta = e^\gamma$. Thus, Z is superposable, in the sense defined in the introduction.

Brunet and Derrida now conjectured (p. 18 of the same paper) that Z could be represented as a Poisson process of intensity measure $e^{-x} dx$ on \mathbb{R} , *decorated* by an auxiliary point process. Furthermore, they asked the question whether this is the case for every superposable point process. As mentioned above, we show in this article that this conjecture is indeed true for every superposable point process.

3 Infinitely divisible random measures

A random measure Z is said to be *infinitely divisible* if for every $n \in \mathbb{N}$ there exist iid random measures $Z^{(1)}, \dots, Z^{(n)}$ such that

$$Z \stackrel{(d)}{=} Z^{(1)} + \dots + Z^{(n)}.$$

The random measure Z is said to be infinitely divisible as a point process, if for every $n \in \mathbb{N}$ there exist iid point processes $Z^{(1)}, \dots, Z^{(n)}$ such that the above equality holds. In particular, Z is a point process. Note that a point process can be infinitely divisible as a random measure, but not as a point process (example: if $\mu \in \mathcal{N}$ is non-random and non-zero, then $\mu = \frac{1}{n}\mu + \dots + \frac{1}{n}\mu$, but μ is not infinitely divisible as a point process).

For every non-negative measurable function $f : \mathbb{R} \rightarrow \mathbb{R}_+$ define the *Laplace functional*

$$L(f) = L_Z(f) = \mathbb{E}[\exp(-\langle Z, f \rangle)] \in [0, 1],$$

where $\langle \mu, f \rangle = \int_{\mathbb{R}} f(x)\mu(dx)$ for any $\mu \in \mathcal{M}$ and any μ -integrable function f . The Laplace functional uniquely characterises Z ([3], p. 161). We further define the *cumulant*

$$K(f) = -\log L(f) \in [0, \infty].$$

The main result about infinitely divisible random measures is the following (see [6], Theorem 6.1 or [4], Proposition 10.2.IX, however, note the error in the theorem statement of the latter reference: F_1 may be infinite as it is defined). Here, we define $\mathcal{M}^* = \mathcal{M} \setminus \{0\}$ and $\mathcal{N}^* = \mathcal{N} \setminus \{0\}$.

Fact 3.1. *The random measure Z is infinitely divisible if and only if*

$$K(f) = \langle \lambda, f \rangle + \int_{\mathcal{M}^*} [1 - \exp(-\langle \mu, f \rangle)] \Lambda(d\mu),$$

where

- $\lambda \in \mathcal{M}$ is a (non-random) measure and
- Λ is a measure on \mathcal{M}^* satisfying

$$\int_0^\infty (1 - e^{-x}) \Lambda(\mu(A) \in dx) < \infty, \quad (3.1)$$

for every bounded Borel set $A \subset \mathbb{R}$.

The probabilistic interpretation ([6], Lemma 6.5) of this fact is that Z is the superposition of the non-random measure λ and of the atoms of a Poisson process on \mathcal{M}^* with intensity Λ . It has the following analogous result in the case of point processes ([4], Proposition 10.2.V):

Fact 3.2. *A point process Z is infinitely divisible if and only if its cumulant K satisfies*

$$K(f) = \int_{\mathcal{N}^*} [1 - \exp(-\langle \mu, f \rangle)] \tilde{Q}(d\mu),$$

for some measure \tilde{Q} on \mathcal{N}^* which satisfies $\tilde{Q}(\mu(A) > 0) < \infty$ for every bounded Borel set $A \subset \mathbb{R}$. In other words, an infinitely divisible random measure is infinitely divisible as a point process if and only if $\lambda = 0$ and Λ is concentrated on \mathcal{N}^* , where λ and Λ are the measures from Fact 3.1.

Lemma 3.3. *Let Z be a superposable random measure. Then Z is infinitely divisible. If Z is a point process, then it is infinitely divisible as a point process.*

Proof. Let $\alpha, \beta, \gamma \in \mathbb{R}$ such that $e^\alpha + e^\beta = e^\gamma$. Let Z' be an independent copy of Z . The superposability of Z implies that

$$T_\gamma Z \stackrel{(d)}{=} T_\gamma (T_{\alpha-\gamma} Z + T_{\beta-\gamma} Z') = T_\alpha Z + T_\beta Z'.$$

Now one easily shows by induction that if $n \in \mathbb{N}$ and $\alpha_1, \dots, \alpha_n \in \mathbb{R}$ such that $e^{\alpha_1} + \dots + e^{\alpha_n} = 1$ and if $Z^{(1)}, \dots, Z^{(n)}$ are independent copies of Z , then

$$Z \stackrel{(d)}{=} T_{\alpha_1} Z^{(1)} + \dots + T_{\alpha_n} Z^{(n)}.$$

In particular, choosing $\alpha_i = -\log n$, $i = 1, \dots, n$ proves both statements. \square

4 Decorated Poisson processes

Let D be a random measure on \mathbb{R} (the *decoration*), and ξ a Poisson point process on \mathbb{R} with intensity measure $m(dx)$ (also denoted by $\text{PPP}(m)$). Denote its atoms by ξ_1, ξ_2, \dots . Let $D^{(1)}, D^{(2)}, \dots$ be independent copies of D and independent of ξ . We set

$$Z = \sum_{i=1}^{\infty} T_{\xi_i} D^{(i)}. \quad (4.1)$$

We say that Z *exists*, if it is almost surely boundedly finite. In this case, Z is called a *decorated Poisson point process with Poisson intensity m and decoration D* (denoted by $\text{DPPP}(m, D)$ for short). It is a particular version of the independent Poisson cluster process ([3], Section 6.3).

Proposition 4.1. *The $\text{DPPP}(m, D)$ exists if and only if*

$$\int^{\infty-} \left[\int_0^{\infty} (1 - e^{-y}) \mathbb{P}(D(A - x) \in dy) \right] m(dx) < \infty, \quad (4.2)$$

for every bounded Borel set $A \subset \mathbb{R}$. In this case, its cumulant satisfies

$$K(f) = \int_{\mathbb{R}} (1 - \mathbb{E}[\exp(-\langle T_x D, f \rangle)]) m(dx). \quad (4.3)$$

Remark 4.2. If D is a point process, then (4.2) is equivalent to

$$\int^{\infty-} \mathbb{P}(D(A - x) > 0) m(dx) < \infty. \quad (4.4)$$

If $m(dx) = \varphi(x) dx$, we also write $\text{DPPP}(\varphi(x), D)$ for short. In this paper, we will mostly consider the case where $\varphi(x) = e^{-x}$.

Proposition 4.3. *Let Z be a $\text{DPPP}(e^{-x}, D)$. Then Z is superposable. Moreover, Z has finite intensity (i.e. $E[Z(A)] < \infty$ for every bounded Borel set $A \subset \mathbb{R}$) if and only if $\mathbb{E}[\langle D, e^x \rangle] < \infty$.*

Proof of Proposition 4.1. Let Z be defined by (4.1). Conditioning on ξ we get for any non-negative measurable function f ,

$$\begin{aligned} \mathbb{E}[\exp(-\langle Z, f \rangle)] &= \mathbb{E} \left[\prod_{i \in \mathbb{N}} \mathbb{E}[\exp(-\langle T_{\xi_i} D^{(i)}, f \rangle) \mid \xi] \right] \\ &= \exp \left(\int_{\mathbb{R}} (\mathbb{E}[\exp(-\langle T_x D, f \rangle)] - 1) m(dx) \right), \end{aligned}$$

since the probability generating functional of a $\text{PPP}(m)$ is given by ([4], Example 9.4(c), p. 60)

$$\mathbb{E} \left[\prod_{i \in \mathbb{N}} g(\xi_i) \right] = \exp \left(\int_{\mathbb{R}} (g(x) - 1) m(dx) \right),$$

for any measurable function g taking values in $[0, 1]$. It follows that $K(f)$ is of the form (4.3) if the process exists.

Let Q denote the law of the decoration D restricted to \mathcal{M}^* . If Z exists, it is infinitely divisible by Fact 3.1 with $\lambda = 0$ and $\Lambda = \int_{\mathbb{R}} T_x Q m(dx)$. Equation (4.2) then follows from (3.1). On the other hand, if (4.2) is fulfilled, then (3.1) is verified as well and Z exists by Fact 3.1. \square

Proof of Proposition 4.3. Let $f : \mathbb{R} \rightarrow \mathbb{R}^+$ be measurable and set $g(x) = \mathbb{E}[\exp(-\langle T_x D, f \rangle)]$. For every $\alpha \in \mathbb{R}$, we have

$$\begin{aligned} \mathbb{E}[\exp(-\langle T_\alpha Z, f \rangle)] &= \exp\left(\int_{\mathbb{R}} e^{-x}(g(x+\alpha) - 1) dx\right) = \exp\left(e^\alpha \int_{\mathbb{R}} e^{-x}(g(x) - 1) dx\right) \\ &= \mathbb{E}[\exp(-\langle Z, f \rangle)]^{e^\alpha}. \end{aligned}$$

Now let $\alpha, \beta \in \mathbb{R}$ such that $e^\alpha + e^\beta = 1$ and let Z' be an independent copy of Z . Then

$$\mathbb{E}[\exp(-\langle T_\alpha Z + T_\beta Z', f \rangle)] = \mathbb{E}[\exp(-\langle Z, f \rangle)]^{e^\alpha + e^\beta} = \mathbb{E}[\exp(-\langle Z, f \rangle)],$$

hence Z is superposable. For the second statement, note that we have by Tonelli's theorem,

$$E[Z(A)] = \mathbb{E}\left[\sum_{i \in \mathbb{N}} \mathbb{E}[T_{\xi_i} D(A) | \xi]\right] = \int_{\mathbb{R}} \mathbb{E}[D(A-y)e^{-y}] dy = \mathbb{E}\left[\int_{\mathbb{R}} D(A-y)e^{-y} dy\right],$$

for every bounded Borel set $A \subset \mathbb{R}$. Again by Tonelli's theorem we have

$$\int_{\mathbb{R}} D(A-y)e^{-y} dy = \int_{\mathbb{R}} \int_{\mathbb{R}} \mathbb{1}_{A-y}(x)e^{-y} dy D(dx) = \langle D, \int_{\mathbb{R}} \mathbb{1}_{A-y}(\cdot)e^{-y} dy \rangle.$$

For $x \in \mathbb{R}$, $x \in A-y$ implies $y \in [\min A - x, \max A - x]$. Since e^{-y} is decreasing, we therefore have

$$|A|e^{-\max A}e^x \leq \int_{\mathbb{R}} \mathbb{1}_{A-y}(x)e^{-y} dy \leq |A|e^{-\min A}e^x,$$

where $|A|$ denotes the Lebesgue measure of A . We conclude that $E[Z(A)] < \infty$ if and only if $\mathbb{E}[\langle D, e^x \rangle] < \infty$. \square

5 Main results

For $\mu \in \mathcal{M}^*$, define the measurable function

$$M(\mu) = \inf\{x \in \mathbb{R} : \mu((x, \infty)) < \min(1, \mu(\mathbb{R})/2)\}.$$

By definition, $M(\mu) \in \mathbb{R} \cup \{+\infty\}$ for all $\mu \in \mathcal{M}^*$. Note that if μ is a counting measure, then $M(\mu)$ is the position of the right-most atom of μ , i.e. $M(\mu) = \sup \text{supp } \mu$.

Theorem 5.1. *Let Z be a random measure and let $K(f)$ be its cumulant. Then Z is superposable if and only if for every measurable non-negative function $f : \mathbb{R} \rightarrow \mathbb{R}_+$,*

$$K(f) = c \int_{\mathbb{R}} e^{-x} f(x) dx + \int_{\mathbb{R}} e^{-x} \int_{\mathcal{M}^*} [1 - \exp(-\langle \mu, f \rangle)] T_x \Delta(d\mu) dx, \quad (5.1)$$

for some constant $c \geq 0$ and some measure Δ on \mathcal{M}^* , such that

$$\int_{\mathbb{R}} e^x \int_0^\infty (1 - e^{-y}) \Delta(\mu(A+x) \in dy) dx < \infty, \quad (5.2)$$

for every bounded Borel set $A \subset \mathbb{R}$. Furthermore, for any $m \in \mathbb{R}$, Δ can be chosen such that $\Delta(M(\mu) \neq m) = 0$, and as such, it is unique.

The decomposition of the cumulant in Theorem 5.1 is reminiscent of (4.3), and indeed, if Δ has unit mass, then the theorem says that Z is a superposition of a measure $\lambda(dx) = ce^{-x}dx$ and a DPPP(e^{-x}, D), where D follows the law Δ . In general, however, Δ need not have finite mass, in which case Z cannot be represented as a DPPP(e^{-x}, D). This corresponds to an accumulation of an infinite number of “small” measures in a bounded set and is therefore not possible if we restrict ourselves to point processes. The following theorem is the consequence:

Corollary 5.2. *A point process Z is superposable if and only if it is equal in law to a DPPP(e^{-x}, D) for some point process D satisfying (4.4). In this case, there exists a unique pair (m, D) , where $m \in \mathbb{R}$ is a constant and D a point process satisfying (4.4) and $\mathbb{P}(M(D) = m) = 1$, such that Z is equal in law to DPPP(e^{-x}, D).*

6 Spaces of measures

Let $(\mathcal{X}, d_{\mathcal{X}})$ be a complete separable metric space with a fixed origin $x_0 \in \mathcal{X}$ and let $\mathcal{M}_{\mathcal{X}}$ be the space of all boundedly finite positive measures on \mathcal{X} . We equip $\mathcal{M}_{\mathcal{X}}$ with the metric

$$d(\mu, \nu) = \int_0^\infty e^{-r} \frac{d_r(\mu, \nu)}{1 + d_r(\mu, \nu)} dr, \quad (6.1)$$

where

$$d_r(\mu, \nu) = \inf\{\varepsilon \geq 0 : \mu(F) \leq \nu(F^\varepsilon) + \varepsilon \text{ and } \nu(F) \leq \mu(F^\varepsilon) + \varepsilon \text{ for all closed } F \subset B_r(x_0)\}. \quad (6.2)$$

Here, $B_r(x_0)$ the open ball of radius r around x_0 and $F^\varepsilon = \{x \in \mathcal{X} : d_{\mathcal{X}}(x, F) < \varepsilon\}$. It is known ([3], p. 403ff¹) that the metric space $(\mathcal{M}_{\mathcal{X}}, d)$ is complete and separable and that $d(\mu_n, \mu) \rightarrow 0$ as $n \rightarrow \infty$ if and only if $\langle \mu_n, f \rangle \rightarrow \langle \mu, f \rangle$ for every bounded continuous function f vanishing outside a bounded set. If \mathcal{X} is locally compact and the closed bounded sets are compact, d induces the vague topology on $\mathcal{M}_{\mathcal{X}}$. Note however that $\mathcal{M}_{\mathcal{X}}$ is in general *not* locally compact, even if \mathcal{X} is.

The subset $\mathcal{M}_{\mathcal{X}}^* = \mathcal{M}_{\mathcal{X}} \setminus \{0\}$ being open it is a complete separable metric space as well ([1], IX.6.1, Proposition 2) when endowed with the metric

$$d^*(\mu, \nu) = d(\mu, \nu) + \left| \frac{1}{d(\mu, 0)} - \frac{1}{d(\nu, 0)} \right|, \quad (6.3)$$

which is equivalent to d on $\mathcal{M}_{\mathcal{X}}^*$. Note that since $d(\mu, \nu) \leq 1$ for all $\mu, \nu \in \mathcal{M}_{\mathcal{X}}$, a set $\mathcal{A} \subset \mathcal{M}_{\mathcal{X}}^*$ is bounded in $(\mathcal{M}_{\mathcal{X}}^*, d^*)$ if and only if $\inf\{d(\mu, 0) : \mu \in \mathcal{A}\} > 0$.

During the rest of the paper, \mathcal{X} will either be the real line \mathbb{R} , equipped with the usual distance $d_{\mathbb{R}}(x, y) = |x - y|$ and the origin $x_0 = 0$, or the space $(\mathcal{M}_{\mathbb{R}}^*, d^*)$ with an arbitrary fixed origin $\mu_0 \in \mathcal{M}_{\mathbb{R}}^*$. Note that with our previous notation, we have $\mathcal{M} = \mathcal{M}_{\mathbb{R}}$ and $\mathcal{M}^* = \mathcal{M}_{\mathbb{R}}^*$. The symbols d and d^* will always refer to the metrics of \mathcal{M} and \mathcal{M}^* . We further define $\mathfrak{M} = \mathcal{M}_{\mathcal{M}^*}$.

We now prove the Lipschitz and local Lipschitz properties of the translation operator T_x in \mathcal{M} and \mathcal{M}^* .

¹Their definition is slightly different but the results hold for our definition as well. Indeed, their definition seems to be erroneous, since their $d_r(\mu^{(r)}, \nu^{(r)})$ is not necessarily increasing in r , as stated on page 403. Our $d_r(\mu, \nu)$, however, is increasing in r .

Proposition 6.1. 1. We have $d(T_x\mu, T_y\mu) \leq |x - y|$ for every $x, y \in \mathbb{R}$ and $\mu \in \mathcal{M}$.

2. For every bounded $\mathcal{A} \subset \mathcal{M}^*$ there exists $L_{\mathcal{A}} \in (0, \infty)$, such that

$$d^*(T_x\mu, T_y\mu) \leq L_{\mathcal{A}}|x - y| \quad \text{for every } x, y \in [0, 1] \text{ and } \mu \in \mathcal{A}.$$

In proving the second part of Proposition 6.1, we need the following lemma, which is useful by itself, too.

Lemma 6.2. Let $\mathcal{A} \subset \mathcal{M}^*$ be bounded and $a \geq 0$. Then the set

$$\mathcal{B} = \bigcup_{x \in [-a, a]} T_x\mathcal{A}$$

is bounded as well.

Proof. Let $\mu \in \mathcal{M}^*$. Inspecting the definition of d_r in (6.2), we see that

$$d_r(T_x\mu, 0) \geq d_{r-|x|}(\mu, 0),$$

for every $x \in \mathbb{R}$ and $r \geq |x|$. The function $g(x) = x/(1+x)$ being monotone on \mathbb{R}_+ , we therefore get

$$d(T_x\mu, 0) = \int_0^\infty e^{-r} g(d_r(T_x\mu, 0)) dr \geq \int_{|x|}^\infty e^{-r} g(d_{r-|x|}(\mu, 0)) dr = e^{-|x|} d(\mu, 0).$$

In particular, $d(T_x\mu, 0) \geq e^{-a} d(\mu, 0)$ for $x \in [-a, a]$. The statement now follows from the fact that a set $\mathcal{A} \subset \mathcal{M}^*$ is bounded if and only if $\inf\{d(\mu, 0) : \mu \in \mathcal{A}\} > 0$. \square

Proof of Proposition 6.1. Inspecting (6.2) gives $d_r(T_x\mu, T_y\mu) \leq |x - y|$ for all $\mu \in \mathcal{M}$ and $r \geq 0$, whence

$$d(T_x\mu, T_y\mu) \leq \int_0^\infty e^{-r} d_r(T_x\mu, T_y\mu) \leq |x - y|.$$

This proves the first part. For the second part, let $\mathcal{B} = \{T_x\mu : \mu \in \mathcal{A}, x \in [0, 1]\}$, which is bounded by Lemma 6.2. Hence, $\alpha = \inf\{d(\mu, 0) : \mu \in \mathcal{B}\} > 0$. For $\mu, \nu \in \mathcal{B}$ we therefore have

$$\left| \frac{1}{d(\mu, 0)} - \frac{1}{d(\nu, 0)} \right| \leq \frac{1}{\alpha^2} |d(\nu, 0) - d(\mu, 0)| \leq \frac{1}{\alpha^2} d(\mu, \nu),$$

by the triangle inequality, whence $d^*(\mu, \nu) \leq (1 + \alpha^{-2})d(\mu, \nu)$. The statement now follows from the first part (with $L_{\mathcal{A}} = 1 + \alpha^{-2}$). \square

7 Proofs

The “if” part of Corollary 5.2 is the first statement of Proposition 4.3. The “if” part of Theorem 5.1 is proven in the same way. It remains to prove the “only if” parts. Let Z be a superposable random measure. Then, for $\alpha, \beta \in \mathbb{R}$, such that $e^\alpha + e^\beta = 1$, we have

$$\begin{aligned} K(f) &= -\log \mathbb{E}[\exp(-\langle Z, f \rangle)] = -\log \mathbb{E}[\exp(-\langle T_\alpha Z, f \rangle)] - \log \mathbb{E}[\exp(-\langle T_\beta Z, f \rangle)] \\ &= K(f(\cdot + \alpha)) + K(f(\cdot + \beta)), \end{aligned}$$

since Z is superposable. Setting $\varphi(x) = K(f(\cdot + x))$ for $x \in \mathbb{R}$ and replacing f by $f(\cdot + x)$ in the above equation, we get $\varphi(x) = \varphi(x + \alpha) + \varphi(x + \beta)$ for all $x \in \mathbb{R}$. The following lemma now shows that $\varphi(x) = e^x \varphi(0)$ for every $x \in \mathbb{R}$:

Lemma 7.1. *Let $g : \mathbb{R} \rightarrow \mathbb{R}_+$ be a non-negative function with the property*

$$g(x) = g(x + \alpha) + g(x + \beta), \quad \text{for all } \alpha, \beta \in \mathbb{R} \text{ such that } e^\alpha + e^\beta = 1.$$

Then $g(x) = e^x g(0)$ for every $x \in \mathbb{R}$.

Proof. As in the proof of Lemma 3.3, one easily shows by induction that for every real numbers $\alpha_1, \dots, \alpha_n$, such that $\sum_{i=1}^n e^{\alpha_i} = 1$, we have $g(x) = \sum_{i=1}^n g(x + \alpha_i)$. In particular, taking $\alpha_i = -\log n$, $i = 1, \dots, n$, we get

$$g(x) = ng(x - \log n). \tag{7.1}$$

In particular, $g(-\log n) = g(0)/n$.

Now, g is an increasing function, since for every $\alpha < 0$ and $e^\beta = 1 - e^\alpha$, we have $g(x) - g(x + \alpha) = g(x + \beta) \geq 0$. Since $g(-\log n) = g(0)/n \rightarrow 0$ as $n \rightarrow \infty$, $g(x)$ goes to 0 as $x \rightarrow -\infty$. It follows that $g(x) - g(x + \alpha) = g(x + \beta) \rightarrow 0$ as $\alpha \uparrow 0$, since $\beta \rightarrow -\infty$ as $\alpha \uparrow 0$, hence g is left-continuous.

Now fix $x \in \mathbb{R}$ and let $q_n = \frac{p_n}{r_n}$ be an increasing sequence of rational numbers converging to e^x . Then $g(x) = \lim_{n \rightarrow \infty} g(\log q_n)$. By (7.1),

$$g(\log q_n) = g(\log p_n - \log r_n) = p_n g(-\log r_n) = \frac{p_n}{r_n} g(0),$$

which goes to $e^x g(0)$ as $n \rightarrow \infty$. This proves the statement. \square

Corollary 7.2.

$$K(f(\cdot + x)) = e^x K(f), \quad \text{for all } x \in \mathbb{R}.$$

By Lemma 3.3, Z is infinitely divisible. Let λ, Λ be the corresponding measures from Fact 3.1.

Lemma 7.3. *There exists a constant $c \geq 0$, such that $\lambda = ce^{-x} dx$. Furthermore, for every $x \in \mathbb{R}$, the Radon-Nikodym derivative of $T_x \Lambda$ with respect to Λ is*

$$\frac{dT_x \Lambda}{d\Lambda} = e^x.$$

Proof. The measures $T_x \lambda, T_x \Lambda$ are the measures corresponding to the infinitely divisible random measure $T_x Z$ by Fact 3.1. But by Corollary 7.2, the measures $e^x \lambda$ and $e^x \Lambda$ correspond to $T_x Z$, as well. Since these measures are unique, we have $T_x \lambda = e^x \lambda$ and $T_x \Lambda = e^x \Lambda$. The second statement follows immediately. For the first statement, note that $c_1 = \lambda([0, 1]) < \infty$, since $[0, 1]$ is a bounded set. It follows that

$$\lambda([0, \infty)) = \sum_{n \geq 0} \lambda([n, n+1)) = \sum_{n \geq 0} c_1 e^{-n} = \frac{c_1 e}{e-1} =: c,$$

hence $\lambda([x, \infty)) = ce^{-x}$ for every $x \in \mathbb{R}$. The first statement of the lemma follows. \square

In order to obtain the decomposition (5.1) of the measure Λ , we are going to disintegrate it with respect to the measurable map M . The next lemma ensures that M is Λ -almost surely finite.

Lemma 7.4. *There exists a strictly positive increasing function $h : \mathbb{R} \rightarrow \mathbb{R}_+$, such that $h(x) = 1$ for $x \geq 0$ and*

$$\Lambda(\langle \mu, h \rangle = \infty) = 0.$$

In particular, $\Lambda(M(\mu) = +\infty) = 0$.

Proof. We first show that $\Lambda(\mu((0, \infty)) = \infty) = 0$. For $n \in \mathbb{Z}$, let $I_n = [n, n+1)$ and $I = [0, 1)$. By (3.1), we have

$$\int_0^1 \Lambda(\mu(I) > x) dx = \int_0^1 x \Lambda(\mu(I) \in dx) < \infty.$$

By monotonicity, the first integral is greater than or equal to $x \Lambda(\mu(I) > x)$ for every $x \in [0, 1]$, hence $\Lambda(\mu(I) > x) \leq C/x$ for some constant $0 \leq C < \infty$. By Lemma 7.3, it follows that

$$\Lambda(\mu(I_n) > e^{-n/2}) = e^{-n} \Lambda(\mu(I) > e^{-n/2}) \leq C e^{-n/2},$$

for every $n \in \mathbb{N}$. Hence, $\sum_{n \in \mathbb{N}} \Lambda(\mu(I_n) > e^{-n/2}) < \infty$. By the Borel-Cantelli lemma,

$$\Lambda\left(\limsup_{n \rightarrow \infty} \{\mu(I_n) > e^{-n/2}\}\right) = 0,$$

which implies $\Lambda(\mu((0, \infty)) = \infty) = 0$.

Now let (h_n) be a decreasing sequence of strictly positive numbers, such that

$$\Lambda(h_n \mu(I_{-n}) > e^{-n}) < e^{-n}.$$

Set $h(x) = h_n$ for $x \in I_{-n}$ and $h(x) = 1$ for $x \geq 0$. The statement now follows from the Borel-Cantelli lemma and the first part. \square

For a Borel set $A \subset \mathbb{R}$ define $\mathcal{M}_A^* = \{\mu \in \mathcal{M}^* : M(\mu) \in A\}$. Then $\mathcal{M}_A^* \cap \mathcal{M}_{A'}^* = \emptyset$, if $A \cap A' = \emptyset$. Furthermore, by Lemma 7.4, we have $\Lambda(M(\mu) = \infty) = 0$, hence Λ is concentrated on $\mathcal{M}_{\mathbb{R}}^*$.

For $n \in \mathbb{N}_0$ define

$$\Delta_n = 2^n \Lambda|_{\mathcal{M}_{[0, 2^{-n})}^*}.$$

We are going to show that the sequence $(\Delta_n)_{n \in \mathbb{N}}$ converges in \mathfrak{M} to a limit Δ and that

$$\Lambda = \int_{\mathbb{R}} e^{-x} T_x \Delta dx. \quad (7.2)$$

We first need to show that the sequence (Δ_n) is relatively compact in \mathfrak{M} . By [3], Proposition A2.6.IV, p. 405, this is true if and only if the restrictions $(\Delta_n^{\mathcal{F}})$ to all bounded closed $\mathcal{F} \subset \mathcal{M}^*$ are relatively compact in the weak topology. By Prokhorov's theorem, this is the case for a fixed \mathcal{F} if and only if

$$\sup_{n \in \mathbb{N}} \Delta_n(\mathcal{F}) < \infty, \quad (7.3)$$

and for all $\delta > 0$ there exists a compact $\mathcal{K} \subset \mathcal{F}$, such that

$$\forall n \in \mathbb{N} : \Delta_n(\mathcal{F} \setminus \mathcal{K}) < \delta. \quad (7.4)$$

First of all, we note that by Lemma 7.3 we have for every $n \geq 0$:

$$\Lambda|_{\mathcal{M}_{[0,1]}^*} = \sum_{k=0}^{2^n-1} e^{-k2^{-n}} T_{k2^{-n}} \Lambda|_{\mathcal{M}_{[0,2^{-n}]}^*},$$

hence

$$\Delta_0 = 2^{-n} \sum_{k=0}^{2^n-1} e^{-k2^{-n}} T_{k2^{-n}} \Delta_n. \quad (7.5)$$

Now let $\mathcal{A} \subset \mathcal{M}^*$ be a bounded Borel set and set $\mathcal{B} = \bigcup_{x \in [0,1]} T_x \mathcal{A}$, which is bounded by Lemma 6.2. By (7.5), we have for every $n \in \mathbb{N}$:

$$\Delta_0(\mathcal{B}) = 2^{-n} \sum_{k=0}^{2^n-1} e^{-k2^{-n}} T_{k2^{-n}} \Delta_n(\mathcal{B}).$$

But since $T_{k2^{-n}} \Delta_n(\mathcal{B}) = \Delta_n(T_{-k2^{-n}} \mathcal{B}) \geq \Delta_n(\mathcal{A})$, it follows that

$$S_{\mathcal{A}} = \sup_{n \in \mathbb{N}} \Delta_n(\mathcal{A}) < \infty \quad \text{for every bounded } \mathcal{A} \subset \mathcal{M}^*, \quad (7.6)$$

whence (7.3) follows.

Now let $\mathcal{F} \subset \mathcal{M}^*$ be bounded and closed. Boundedness implies that \mathcal{F} is a subset of $\{\mu \in \mathcal{M} : d(\mu, 0) \geq \varepsilon\}$ for some $\varepsilon > 0$, which is closed in \mathcal{M} , hence \mathcal{F} is closed in \mathcal{M} as well. Let h be the function from Lemma 7.4, such that Λ and hence all Δ_n are concentrated on

$$\mathcal{M}^h = \{\mu \in \mathcal{M} : \langle \mu, h \rangle < \infty\}.$$

For $c \geq 0$, define

$$\mathcal{M}_c^h = \{\mu \in \mathcal{M} : \langle \mu, h \rangle \leq c\},$$

which is compact ([6], 15.7.5, p. 170). \mathcal{F} being closed in \mathcal{M} , the set $\mathcal{F}_c = \mathcal{F} \cap \mathcal{M}_c^h$ is then compact as well. Set $\mathcal{G}_c = \mathcal{F} \setminus \mathcal{F}_c$. Since h is increasing, we have $T_x \mathcal{G}_c \subset \mathcal{G}_c$ for every $x \geq 0$. It follows that

$$\Delta_n(\mathcal{G}_c) \leq \Delta_n(T_{-x} \mathcal{G}_c) = T_x \Delta_n(\mathcal{G}_c).$$

Equation (7.5) now implies that $\Delta_n(\mathcal{G}_c) \leq e \Delta_0(\mathcal{G}_c)$ for every $n \in \mathbb{N}$. Since Δ_0 is concentrated on \mathcal{M}^h , $\Delta_0(\mathcal{G}_c)$ converges to 0 as $c \rightarrow \infty$, whence (7.4) follows. As mentioned above, it follows that (Δ_n) is relatively compact in \mathfrak{M} .

Let $\Delta \in \mathfrak{M}$ be any accumulation point of the sequence (Δ_n) . We show that $\Delta_0 = \int_{[0,1]} e^{-x} T_x \Delta \, dx$. Note that this proves (7.2), since by Lemma 7.3 we have

$$\Lambda = \sum_{k \in \mathbb{Z}} e^{-k} T_k \Delta_0.$$

Let $f : \mathcal{M}^* \rightarrow \mathbb{R}_+$ be bounded, Lipschitz-continuous with Lipschitz constant L_f and supported on a bounded set. Set $\mathcal{A} = \bigcup_{x \in [-1,0]} T_x \text{supp } f$ and $\mathcal{B} = \bigcup_{x \in [-1,1]} T_x \text{supp } f$, which are bounded sets by Lemma 6.2. Note that for $\mu \in \mathcal{A}$ and $x, y \in [0, 1]$, we have $T_x \mu, T_y \mu \in \mathcal{B}$ and therefore

$$|f(T_x \mu) - f(T_y \mu)| \leq L_f d^*(T_x \mu, T_y \mu) \leq L_f L_{\mathcal{B}} |x - y|, \quad (7.7)$$

by Proposition 6.1. Writing $\Gamma(f) = \int_{\mathcal{M}^*} f(\mu) \Gamma(d\mu)$ for any measure $\Gamma \in \mathfrak{M}$, we get by (7.5),

$$\begin{aligned} \left| \Delta_0(f) - \int_0^1 e^{-x} T_x \Delta(f) dx \right| &\leq \left| \sum_{k=0}^{2^n-1} 2^{-n} e^{-k2^{-n}} T_{k2^{-n}} (\Delta_n - \Delta)(f) \right| \\ &+ \left| \sum_{k=0}^{2^n-1} 2^{-n} e^{-k2^{-n}} T_{k2^{-n}} \Delta(f) - \int_0^1 e^{-x} T_x \Delta(f) dx \right|. \end{aligned} \quad (7.8)$$

Define $g_n(x) = T_x(\Delta_n - \Delta)(f)$, $x \in [0, 1]$. Since $\Delta_n \rightarrow \Delta$ in \mathfrak{M} , $g_n(x)$ converges pointwise to 0 as $n \rightarrow \infty$. Moreover, by (7.7)

$$\begin{aligned} |g_n(x) - g_n(y)| &\leq \left| \int_{\mathcal{A}} f(T_x \mu) - f(T_y \mu) \Delta_n(d\mu) \right| + \left| \int_{\mathcal{A}} f(T_x \mu) - f(T_y \mu) \Delta(d\mu) \right| \\ &\leq (S_{\mathcal{A}} + \Delta(\mathcal{A})) L_f L_{\mathcal{B}} |x - y|, \end{aligned}$$

hence the convergence is uniform. This implies that the first term of the right hand side of (7.8) goes to 0 as $n \rightarrow \infty$.

For the second term, let $x, y \in [0, 1]$ and $\mu \in \mathcal{A}$. By (7.7),

$$\begin{aligned} |e^{-x} f(T_x \mu) - e^{-y} f(T_y \mu)| &\leq |f(T_x \mu) - f(T_y \mu)| + |(e^{-|x-y|} - 1) f(T_x \mu)| \\ &\leq (L_f L_{\mathcal{B}} + \|f\|_{\infty}) |x - y| =: K_f |x - y|, \end{aligned}$$

hence

$$|e^{-x} T_x \Delta(f) - e^{-y} T_y \Delta(f)| \leq \Delta(\mathcal{A}) K_f |x - y| \quad (7.9)$$

The second term of the right hand side of (7.8) is then smaller than $\Delta(\mathcal{A}) K_f 2^{-(n+1)}$, which converges to 0 as $n \rightarrow \infty$. It follows that

$$\Delta_0(f) = \int_0^1 e^{-x} T_x \Delta(f) dx.$$

Since every characteristic function of a bounded open set can be approximated by an increasing sequence of bounded Lipschitz-continuous functions with bounded support, this proves that the two measures are equal, from which (7.2) and therefore (5.1) follow. Equation (5.2) follows from (7.2) and (3.1).

By construction, $\Delta(M(\mu) \neq 0) = 0$. Given $m \in \mathbb{R}$, one easily checks that (7.2) holds as well with Δ replaced by $e^{-m} T_m \Delta$, which is concentrated on $\{\mu \in \mathcal{M}^* : M(\mu) = m\}$. Finally, the uniqueness statement of Theorem 5.1 follows from the next lemma:

Lemma 7.5. *Let $\Lambda_1, \Lambda_2 \in \mathfrak{M}$, such that*

$$\int_{\mathbb{R}} e^{-x} T_x \Lambda_1 dx = \Lambda = \int_{\mathbb{R}} e^{-x} T_x \Lambda_2 dx. \quad (7.10)$$

Suppose further that there exist $y_1, y_2 \in \mathbb{R}$, such that $\Lambda_1(M(\mu) \neq y_1) = \Lambda_2(M(\mu) \neq y_2) = 0$. Then $\Lambda_2 = e^{y_1 - y_2} T_{y_2 - y_1} \Lambda_1$.

Proof. Set $\Lambda'_1 = e^{y_1 - y_2} T_{y_2 - y_1} \Lambda_1$. Then $\Lambda'_1(M(\mu) \neq y_2) = 0$ and changing variables in (7.10) gives $\int_{\mathbb{R}} e^{-x} \Lambda'_1 dx = \Lambda$. We can therefore assume w.l.o.g. that $y_1 = y_2 = y$. For $\varepsilon \in (0, 1)$, we then have

$$\Lambda|_{\mathcal{M}_{[y, y+\varepsilon]}} = \int_{[0, \varepsilon]} e^{-x} T_x \Lambda_i dx, \quad i = 1, 2. \quad (7.11)$$

Now let $f, L_f, \mathcal{A}, \mathcal{B}$ and $L_{\mathcal{B}}$ be as above. By (7.9), with Δ replaced by Λ_i , we have

$$\left| \int_{[0, \varepsilon]} [e^{-x} T_x \Lambda_i(f) - \Lambda_i(f)] dx \right| \leq \varepsilon^2 K_f \Lambda_i(\mathcal{A})/2, \quad (7.12)$$

for $i = 1, 2$. Equations (7.11) and (7.12) now imply

$$\Lambda_1(f) = \lim_{\varepsilon \rightarrow 0} \varepsilon^{-1} \Lambda_{|\mathcal{M}_{[y, y+\varepsilon]}}(f) = \Lambda_2(f),$$

which gives $\Lambda_1 = \Lambda_2$. \square

If Z is a point process, we know from Lemma 3.3 that it is infinitely divisible as a point process. Fact 3.2 then implies that $\lambda = 0$ and that Λ is concentrated on \mathcal{M}^* , hence Δ as well. Equation (5.2) then implies that $\Delta(\mu(A) > 0) < \infty$ for any bounded Borel set $A \subset \mathbb{R}$. In particular, this holds for $A = [-1, 1]$. But since $M(\mu) = 0$ implies $\mu([-1, 1]) > 0$ and since Δ is concentrated on $\mathcal{M}_{\{0\}}^*$, it has finite mass. If Δ has mass zero, then $Z = 0$ almost surely and Z is a DPPP($e^{-x}, 0$). If Δ has positive mass, set $m = \log \Delta(\mathcal{M}^*)$. The measure $\Delta' = e^{-m} T_m \Delta$ is then a probability measure and (7.2) holds with Δ replaced by Δ' . By Proposition 4.1, Z is a DPPP(e^{-x}, D), where D follows the law Δ' . Uniqueness of the pair (m, D) follows from Lemma 7.5.

8 Discussion

- The decomposition (7.2) is really a disintegration of Λ with respect to the measurable map $M : \mathcal{M}^* \rightarrow \mathbb{R}$. One could ask whether the above proof could be simplified by using abstract measure disintegration theorems (e.g. [5]). If these could be applied, they would give a family of measures $(\Gamma_x)_{x \in \mathbb{R}}$, such that $\Lambda = \int_{\mathbb{R}} \Gamma_x dx$, and applying Lemma 7.3, one could hope to obtain $\Gamma_x = e^{-x} T_x \Delta$ for some measure Δ . However, *a priori* this would only be true for *almost every* $x \in \mathbb{R}$ and not for *all* x . Thus, we cannot simply set $\Delta = \Gamma_0$. It is this problem that had us choose the much more elementary approach used in the preceding proof.
- If one does not require Δ to be concentrated on $\{M(\mu) = m\}$ for some $m \in \mathbb{R}$, then the uniqueness property of Theorem 5.1 does not hold. For example, any linear combination $\int T_x \Delta \nu(dx)$ of the $T_x \Delta$ (with some measure ν on \mathbb{R}) satisfies (7.2) and (5.1) (modulo some multiplicative constant).
- Call a random measure Z ρ -superposable ($\rho \neq 0$), if

$$T_\alpha Z + T_\beta Z' \stackrel{(d)}{=} Z, \quad \text{for every } \alpha, \beta \in \mathbb{R} \text{ such that } e^{\rho\alpha} + e^{\rho\beta} = 1,$$

where Z' is an independent copy of Z . Define the scaling operator $S_r : \mathcal{M} \rightarrow \mathcal{M}$ for $r \neq 0$ by $S_r \mu(A) = \mu(A/r)$ for any Borel set $A \subset \mathbb{R}$. One easily checks that $T_\alpha S_r = S_r T_{\alpha/r}$, and as a consequence, the random measure $S_\rho Z$ is superposable. Thus, the study of ρ -superposable measures reduces easily to superposable measures.

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