Groups with faithful irreducible projective unitary representations

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ABSTRACT. For a countable group Γ and a multiplier $\zeta \in Z^2(\Gamma, \mathbf{T})$, we study the property of Γ having a unitary projective ζ-representation which is both irreducible and projectively faithful. Theorem 1 shows that this property is equivalent to Γ being the quotient of an appropriate group by its centre. Theorem 4 gives a criterion in terms of the minisocle of Γ. Several examples are described to show the existence of various behaviours.

1. Introduction

For a Hilbert space $\mathcal{H}$, we denote by $\mathcal{U}(\mathcal{H})$ the group of its unitary operators. We identify $\mathbf{T} := \{z \in \mathbf{C} \mid |z| = 1\}$ with the centre of $\mathcal{U}(\mathcal{H})$, namely with the scalar multiples of the identity operator $\mathrm{id}_{\mathcal{H}}$, we denote by $\mathcal{PU}(\mathcal{H}) := \mathcal{U}(\mathcal{H})/\mathbf{T}$ the *projective unitary group* of $\mathcal{H}$, and by

$$p_{\mathcal{H}} : \mathcal{U}(\mathcal{H}) \longrightarrow \mathcal{PU}(\mathcal{H})$$

the canonical projection.

Let Γ be a group. A projective unitary representation, or shortly here a *projective representation*, of Γ in $\mathcal{H}$ is a mapping

$$\pi : \Gamma \longrightarrow \mathcal{U}(\mathcal{H})$$

such that $\pi(e) = \mathrm{id}_{\mathcal{H}}$ and such that the composition

$$\underline{\pi} := p_{\mathcal{H}}\pi : \Gamma \longrightarrow \mathcal{PU}(\mathcal{H})$$

is a homomorphism of groups. When we find it useful, we write $\mathcal{H}_\pi$ for the Hilbert space of a projective representation π.

A projective representation π of a group Γ is *projectively faithful*, or shortly *P-faithful*, if the corresponding homomorphism $\underline{\pi}$ is injective.

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The *projective kernel* of π is the normal subgroup

$$\mathrm{Pker}(\pi) = \ker(\underline{\pi}) = \big\{x \in \Gamma \;\big|\; \pi(x) \in \mathbf{T}\big\} \tag{1}$$

of Γ. In case π is a unitary representation, $\ker(\pi)$ is a subgroup of $\mathrm{Pker}(\pi)$, sometimes called the the *quasikernel* of π, which can be a proper subgroup, so that faithfulness of π does not imply P-faithfulness.

A projective representation π is *irreducible* if the only closed $\pi(\Gamma)$-invariant subspaces of $\mathcal{H}_\pi$ are $\{0\}$ and $\mathcal{H}_\pi$.

As a continuation of [BeHa–08], the present paper results from our effort to understand which groups have irreducible P-faithful projective representations. Our first observation is a version in the present context of Satz 4.1 of [Pahl–68]. We denote by $Z(\Gamma)$ the centre of a group Γ.

Theorem 1. *For a group Γ, the following two properties are equivalent:*

(i) *Γ affords an irreducible P-faithful projective representation;*
(ii) *there exists a group Δ which affords an irreducible faithful unitary representation and which is such that $\Delta/Z(\Delta) \approx \Gamma$.*

If, moreover, Γ is countable, these properties are also equivalent to:

(iii) *there exists a* countable *group Δ as in (ii).*

Countable groups which have irreducible faithful unitary representations have been characterised in [BeHa–08], building up on results of [Gasc–54] for finite groups.

A group Γ is *capable* if there exists a group Δ with $\Gamma \approx \Delta/Z(\Delta)$, and *incapable* otherwise. The notion appears in [Baer–38], which contains a criterion of capability for abelian groups which are direct sums of cyclic groups (for this, see also [BeFS–79]), and the terminology "capable" is that of [HaSe-64]. Conditions for capability (several of them being either necessary or sufficient) are given in Chapter IV of [BeTa–82].

The *epicentre* of a group Γ is the largest central subgroup A such that the quotient projection $\Gamma \longrightarrow \Gamma/A$ induces in homology an injective homomorphism $H_2(\Gamma, \mathbf{Z}) \longrightarrow H_2(\Gamma/A, \mathbf{Z})$, where $\mathbf{Z}$ is viewed as a trivial module. This group was introduced in [BeFS–79] and [BeTa–82], with a formally different definition; the terminology is from [Elli–98], and the characterisation given above appears in Theorem 4.2 of [BeFS–79].

Proposition 2 (Beyl-Felgner-Schmid-Ellis)**.** *Let Γ be a group and let $Z^*(\Gamma)$ denote its epicentre.*

(i) Γ is capable if and only if $Z^*(\Gamma) = \{e\}$.
(ii) $\Gamma/Z^*(\Gamma)$ is capable in all cases.
(iii) A perfect group with non trivial centre is incapable.

Corollary 3. *A perfect group with non-trivial centre has no P-faithful projective representation.*

A *multiplier* on Γ is a mapping $\zeta : \Gamma \times \Gamma \longrightarrow \mathbf{T}$ such that

$$\zeta(e,x) \,=\, \zeta(x,e) \,=\, 1 \quad \text{and} \quad \zeta(x,y)\zeta(xy,z) \,=\, \zeta(x,yz)\zeta(y,z) \tag{2}$$

for all $x, y, z \in \Gamma$. We denote by $Z^2(\Gamma, \mathbf{T})$ the set of all these, which is an abelian group for the pointwise product. A projective representation π of Γ in $\mathcal{H}$ determines a unique multiplier $\zeta_\pi = \zeta$ such that

$$\pi(x)\pi(y) \,=\, \zeta(x,y)\pi(xy) \tag{3}$$

for all $x, y \in \Gamma$; we say then that π is a *ζ-representation* of Γ. Conversely, any $\zeta \in Z^2(\Gamma, \mathbf{T})$ occurs in such a way; indeed, ζ is the multiplier determined by the *twisted left regular ζ-representation* λ_ζ of Γ, defined on the Hilbert space $\ell^2(\Gamma)$ by

$$\left(\lambda_\zeta(x)\varphi\right)(y) \,=\, \zeta(x, x^{-1}y)\varphi(x^{-1}y). \tag{4}$$

A good reference for these regular ζ-representations is [Klep–62]. In the special case $\zeta = 1$, a ζ-representation of Γ is just a *unitary representation* of Γ; but we repeat that

First standing assumption. In this paper, *by "representation", we always mean "unitary representation".*

For a projective representation of Γ in $\mathbf{C}$, namely for a mapping $\nu : \Gamma \longrightarrow \mathbf{T}$ with $\nu(e) = 1$, let $\zeta_\nu \in Z^2(\Gamma, \mathbf{T})$ denote the corresponding multiplier, namely the mapping defined by

$$\zeta_\nu(x,y) \,=\, \nu(x)\nu(y)\nu(xy)^{-1}. \tag{5}$$

We denote by $B^2(\Gamma, \mathbf{T})$ the set of all multipliers of the form ζ_ν, which is a subgroup of $Z^2(\Gamma, \mathbf{T})$, and by $H^2(\Gamma, \mathbf{T}) := Z^2(\Gamma, \mathbf{T})/B^2(\Gamma, \mathbf{T})$ the quotient group; as usual, $\zeta, \zeta' \in Z^2(\Gamma, \mathbf{T})$ are *cohomologous* if they have the same image in $H^2(\Gamma, \mathbf{T})$.

Given a ζ-representation π of Γ in $\mathcal{H}$, there is a standard bijection between:

- the set of projective representations $\pi' : \Gamma \longrightarrow \mathcal{U}(\mathcal{H})$ such that $p_\mathcal{H}\pi = p_\mathcal{H}\pi'$, on the one hand,
- and the set of multipliers cohomologous to ζ, on the other hand.

In other terms, a group homomorphism $\underline{\pi}$ of Γ in $\mathcal{PU}(\mathcal{H})$ determines a class[1] in $H^2(\Gamma, \mathbf{T})$, and the set of projective representations covering $\underline{\pi}$ is in bijection with the representatives of this class in $Z^2(\Gamma, \mathbf{T})$. Observe that π and π' above are together irreducible or not, and together P-faithful or not.

[1]This is of course the class associated to the extension of Γ by $\mathbf{T}$ pulled back by $\underline{\pi}$ of the extension $\{1\} \longrightarrow \mathbf{T} \longrightarrow \mathcal{U}(\mathcal{H}) \longrightarrow \mathcal{PU}(\mathcal{H}) \longrightarrow \{1\}$. See for example Section 6.6 in [Weib–94].

For much more on projective representations and multipliers, in the setting of separable locally compact groups, see [Mack–58] and the survey [Pack–08]; for a very short but informative exposition on earlier work, starting with that of Schur, see [Kall–84]. Note that other authors (such as Kleppner) use "multiplier representation" and "projective representation" when Mackey uses "projective representation" and "homomorphism in $\mathcal{PU}(\mathcal{H})$", respectively.

Definition. *Given a group* Γ *and a multiplier* $\zeta \in Z^2(\Gamma, \mathbf{T})$*, the group* Γ *is* irreducibly ζ-represented *if it has an irreducible P-faithful* ζ*-representation.*

This depends only on the class $\underline{\zeta} \in H^2(\Gamma, \mathbf{T})$ of ζ.

For a group Γ, recall that a *foot* of Γ is a minimal normal subgroup, that the *minisocle* is the subgroup $MS(\Gamma)$ of Γ generated by the union of all finite feet of Γ, and that $MA(\Gamma)$ is the subgroup of $MS(\Gamma)$ generated by the union of all finite abelian feet of Γ. It is obvious that $MS(\Gamma)$ and $MA(\Gamma)$ are characteristic subgroups of Γ; it is easy to show that $MA(\Gamma)$ is abelian and is a direct factor of $MS(\Gamma)$. For all this, we refer to Proposition 1 in [BeHa–08].

Let N be a normal subgroup of Γ and σ a ζ-representation of N, for some $\zeta \in Z^2(N, \mathbf{T})$. If $\zeta = 1$ (the case of ordinary representations), define the Γ-*kernel of* σ by

$$\ker_\Gamma(\sigma) \,=\, \ker\Big(\bigoplus_{\gamma\in\Gamma} \sigma^\gamma\Big)$$

where $\sigma^\gamma(x) := \sigma(\gamma x \gamma^{-1})$; say, as in [BeHa–08], that σ is Γ-*faithful* if this Γ-kernel is reduced to $\{e\}$; when ζ is the restriction to N of a multiplier (usually denoted by ζ again) in $Z^2(\Gamma, \mathbf{T})$, there is an analogous notion for the general case ($\zeta \neq 1$), called Γ-*P-faithfulness*, used in Theorem 4, but defined only in Section 3 below. Before the next result, we find it convenient to define one more property.

Definition. *A group* Γ *has* Property (Fab) *if any normal subgroup of* Γ *generated by one conjugacy class has a finite abelianisation.*

Examples of groups which enjoy Property (Fab) include finite groups, $\mathrm{SL}_n(\mathbf{Z})$ for $n \geq 3$, and more generally lattices in a finite product $\prod_{\alpha\in A} G_\alpha$ of simple groups G_α over (possibly different) local fields k_α when $\sum_{\alpha\in A} k_\alpha - \mathrm{rank}(G_\alpha) \geq 2$ (see [Marg–91], IV.4.10, and Example VI below). They also include abelian locally finite groups, and more generally torsion groups which are FC, namely which are such that all their conjugacy classes are finite; in particular, they include groups of the form $MS(\Gamma)$ and $MA(\Gamma)$.

Theorem 4. *Let Γ be a countable group and let $\zeta \in Z^2(\Gamma, \mathbf{T})$. Consider the following conditions:*

(i) *Γ is irreducibly ζ-represented;*
(ii) *$MS(\Gamma)$ has a Γ-P-faithful irreducible ζ-representation;*
(iii) *$MA(\Gamma)$ has a Γ-P-faithful irreducible ζ-representation.*

Then $(i) \Longrightarrow (ii) \Longleftrightarrow (iii)$.

If, moreover, Γ has Property (Fab), then $(ii) \Longrightarrow (i)$, so that (i), (ii), and (iii) are equivalent.

The hypothesis "Γ countable" is essential because our arguments use measure theory and direct integrals; in fact, Theorem 4 fails in general for uncountable groups (see Example (VII) in [BeHa–08], Page 863). About the converse of $(ii) \Longrightarrow (i)$, see Example I below.

Recall that a group Γ has *infinite conjugacy classes*, or is *icc*, if $\Gamma \neq \{e\}$ and if any conjugacy class in $\Gamma \smallsetminus \{e\}$ is infinite. For example, a lattice in a centreless connected semisimple Lie group without compact factors is icc, as a consequence of Borel Density Theorem (see Example VI).

Corollary 5. *Let Γ be a countable group which has Property (Fab) and which fulfills at least one of the three following conditions:*

(i) *Γ is torsion free;*
(ii) *Γ is icc;*
(iii) *Γ has a faithful primitive action on an infinite set.*

Then, for any $\zeta \in Z^2(\Gamma, \mathbf{T})$, the group Γ is irreducibly ζ-represented.

Indeed, any of Conditions (i) to (iii) implies that $MS(\Gamma) = \{e\}$. Recall that, if Γ fulfills (iii) on an infinite set X, any normal subgroup $N \neq \{e\}$ acts transitively on X, and therefore is infinite (see [GeGl–08]).

A group can be either irreducibly represented or not, and also either irreducibly ζ-represented or not (for some ζ). These dichotomies separate groups in four classes, each one illustrated in Section 2 by one of Examples I to IV below. Examples V and VI illustrate the same class as Example I.

Section 3 contains standard material on multipliers, and the definition of Γ-P-faithfulness; mind the "second standing assumption" on the normalisation of multipliers, which applies to all other sections. In Section 4, we review central extensions and prove Theorem 1. Sections 5 and 6 contain the proof of Theorem 4, respectively the part which does not involve our "Property (Fab)" and the part where it appears.

Section 7 contains material on (in)capability, and the proof of Proposition 2. Section 8 describes a construction of irreducible P-faithful projective representations of a class of abelian groups, and expands on Example II. The last section is a digression to point out a fact from homological algebra which in our opinion is not quoted often enough in the literature on projective representations.

2. **Examples**

Example I. The implication $(ii) \Longrightarrow (i)$ of Theorem 4 does not hold for a free abelian group $\mathbf{Z}^n$ $(n \geq 1)$ and the unit multiplier[2] $\zeta = 1 \in Z^2(\mathbf{Z}^n, \mathbf{T})$. Indeed, on the one hand, Condition (ii) of Theorem 4 is satisfied since $MS(\mathbf{Z}^n) = \{0\}$. On the other hand, since $\mathbf{Z}^n$ is abelian, any irreducible ζ-representation (that is any ordinary irreducible representation) is one-dimensional, so that its projective kernel is the whole of $\mathbf{Z}^n$, and therefore $\mathbf{Z}^n$ is *not* irreducibly ζ-represented. Moreover, $\mathbf{Z}^n$ being for any $n \geq 1$ a (dense) subgroup of $\mathbf{T}$, it has an irreducible faithful representation of dimension one.

Example II. There are groups which do not afford any irreducible faithful representation but which do have projective representations which are irreducible and P-faithful.

The Vierergruppe $\mathbf{V} = \mathbf{Z}/2\mathbf{Z} \times \mathbf{Z}/2\mathbf{Z}$, being finite abelian non-cyclic, does not have any irreducible faithful representation. If $\zeta \in Z^2(\mathbf{V}, \mathbf{T})$ is a cocycle representing the non-trivial cohomology class in $H^2(\mathbf{V}, \mathbf{T}) \approx \mathbf{Z}/2\mathbf{Z}$, then $\mathbf{V}$ has a ζ-representation of degree 2 which is both irreducible and P-faithful, essentially given by the Pauli matrices (see Section IV.3 in [Simo–96]).

Part of this carries over to any non-trivial finite abelian group of the form $L \times L$. More on this in Section 8.

Example III. Let us first recall a few basic general facts about irreducible projective representations of a finite group Γ. The cohomology group $H^2(\Gamma, \mathbf{T})$ is isomorphic to the homology group $H_2(\Gamma, \mathbf{Z})$, and is finite. Choose a multiplier $\zeta \in Z^2(\Gamma, \mathbf{T})$, say normalised (see Section 3 below). An element $x \in \Gamma$ is *ζ-regular* if $\zeta(x, y) = \zeta(y, x)$ whenever $y \in \Gamma$ commutes with x; it can be checked that a conjugate of a regular element is again regular. Let $h(\zeta)$ denote the number of conjugacy classes of ζ-regular elements in Γ. Then it is known that Γ has exactly $h(\zeta)$ irreducible ζ-representations, up to unitary equivalence, say

[2]Recall that $H^2(\mathbf{Z}, \mathbf{T}) = \{0\}$, because $\mathbf{Z}$ is free, so that

$$Z^2(\mathbf{Z}, \mathbf{T}) = B^2(\mathbf{Z}, \mathbf{T}) = \mathrm{Mapp}(\mathbf{Z}, \mathbf{T}) / \mathrm{Hom}(\mathbf{Z}, \mathbf{T}).$$

Also $H^2(\mathbf{Z}^n, \mathbf{T}) = \mathbf{Z}^{n(n-1)/2}$ for all $n \geq 1$.

of degrees $d_1, \ldots, d_{h(\zeta)}$; moreover each d_j divides the order of Γ, and $\sum_{j=1}^{h(\zeta)} d_j^2 = |\Gamma|$. See Chapter 6 in [BeZh–98], in particular Corollary 10 and Theorem 13 Page 149.

Clearly $h(\zeta) \leq h(1)$ for all $\zeta \in Z^2(\Gamma, \mathbf{T})$. It follows from Lemma 11 below that, if $\zeta \neq 1$, then $d_j \geq 2$ for all $j \in \{1, \ldots, h(\zeta)\}$.

Now, for the gist of this Example III, assume that Γ is a nonabelian finite simple group. Then, except for the unit character, any representation of Γ is faithful and any projective representation of Γ is P-faithful.

Example IV. Let Γ be a perfect group. Its *universal central extension* $\widetilde{\Gamma}$ is a perfect group with centre the[3] *Schur multiplier* $H_2(\Gamma, \mathbf{Z})$ and central quotient Γ [Kerv–70]. If this Schur multiplier is not $\{0\}$, $\widetilde{\Gamma}$ is incapable, and therefore does not have any irreducible P-faithful projective representation (Corollary 3). If Γ is as in (i) or (ii) below, $\widetilde{\Gamma}$ is moreover not irreducibly represented (by [Gasc–54] and [BeHa–08]):

(i) Γ is a finite simple group with $H_2(\Gamma, \mathbf{Z})$ not cyclic[4]. The complete list of such groups is given in Theorem 4.236, Page 301 of [Gore–82], and includes the finite simple group $\mathrm{PSL}_3(\mathbf{F}_4)$, also denoted by $A_2(4)$, one of the two finite simple groups of order 20160.

(ii) Γ is one of the Steinberg groups $\mathrm{St}_3(\mathbf{Z})$ and $\mathrm{St}_4(\mathbf{Z})$, which are the universal central extensions of $\mathrm{SL}_3(\mathbf{Z})$ and $\mathrm{SL}_4(\mathbf{Z})$, respectively. Indeed, van der Kallen[5] [Kall–74] has shown that

$$H_2(\mathrm{SL}_3(\mathbf{Z}), \mathbf{Z}) \approx H_2(\mathrm{SL}_4(\mathbf{Z}), \mathbf{Z}) \approx \mathbf{Z}/2\mathbf{Z} \times \mathbf{Z}/2\mathbf{Z}.$$

Thus, these groups are not irreducibly represented by [BeHa–08]. (For $n \geq 5$, it is known that $H_2(\mathrm{SL}_n(\mathbf{Z}), \mathbf{Z}) \approx \mathbf{Z}/2\mathbf{Z}$; see [Miln–71], Page 48. And $H_2(\mathrm{SL}_2(\mathbf{Z}), \mathbf{Z}) = \{0\}$, see the comments after Proposition 18.)

Example V. This example and the next one will show, besides the $\mathbf{Z}^n$'s of Example I, groups which are irreducibly represented, but which do not have any irreducible P-faithful representation.

Any finite perfect group Γ with centre $Z(\Gamma)$ cyclic and not $\{0\}$ has these properties, by Gaschütz theorem and by Corollary 3. This is for example the case of the quasi simple group $\mathrm{SL}_n(\mathbf{F}_q)$ whenever the finite

[3]Recall that, for a perfect group Γ and a trivial Γ-module A, we have $H^2(\Gamma, A) \approx \mathrm{Hom}(H_2(\Gamma, \mathbf{Z}), A)$ as a consequence of the universal coefficient theorem for cohomology; in particular, $H^2(\Gamma, \mathbf{T})$ is the Pontryagin dual of the Schur multiplier. In case $H_2(\Gamma, \mathbf{Z})$ is moreover finite, e.g. if Γ is perfect and finite, $H^2(\Gamma, \mathbf{T})$ is isomorphic to the Schur multiplier (non-canonically).

[4]If $H_2(\Gamma, \mathbf{Z})$ is cyclic not $\{0\}$, see Example V.

[5]Thanks to Andrei Rapinchuk for this reference.

field $\mathbf{F}_q$ has non-trivial nth roots of unity, so that $Z(\mathrm{SL}_n(\mathbf{F}_q)) = \{\lambda \in \mathbf{F}_q \mid \lambda^n = 1\}$ is cyclic and not $\{e\}$. (As usual, $\mathrm{SL}_2(\mathbf{F}_2)$ and $\mathrm{SL}_2(\mathbf{F}_3)$ are ruled out.)

The groups $\mathrm{SL}_{2n}(\mathbf{Z})$, for $2n \geq 4$, are perfect with centre cyclic of order 2, and therefore incapable, so that Corollary 3 applies; the group $\mathrm{SL}_2(\mathbf{Z})$, which is not perfect, is also incapable (see Section 7). On the other hand, since for all $n \geq 1$ the minisocle of $\mathrm{SL}_{2n}(\mathbf{Z})$ coincides with its centre, of order 2, these groups do have representations which are irreducible and faithful. These considerations hold also for the symplectic groups $\mathrm{Sp}_{2n}(\mathbf{Z})$, $2n \geq 6$, which are perfect [Rein–95].

Example VI.a. Let B be a finite set. For $\beta \in B$, let $\mathbf{k}_\beta$ be a local field and $\mathbf{G}_\beta$ be a nontrivial connected semi-simple group defined over $\mathbf{k}_\beta$, without $\mathbf{k}_\beta$-anisotropic factor. Set $G = \prod_{\beta \in B} \mathbf{G}_\beta(\mathbf{k}_\beta)$, with its locally compact topology which makes it a σ-compact, metrisable, compactly generated group. Let Γ be an irreducible lattice in G.

If N is a finite normal subgroup of Γ, we claim that N is central in Γ. If there are several factors ($|B| \geq 2$), the claim is a consequence of the fact that the projection of the lattice in each factor is dense, by irreducibility. If $|B| = 1$, consider $x \in N$. The centraliser $Z_\Gamma(x)$ of x in Γ is also a lattice in G because it is of finite index in Γ. By the Borel-Wang density theorem (Corollary 4.4 of Chapter II in [Marg–91]), $Z_\Gamma(x)$ is Zariski-dense in G, so that x commutes with every element of G, and this proves the claim.

If follows that, if moreover the centre of G is finite cyclic, then $MS(\Gamma) = MA(\Gamma)$ is also a finite cyclic group, so that Γ is irreducibly represented by [BeHa–08].

Example VI.b. To continue this same example, let us particularise the situation to the case of a non-compact semi-simple real Lie group G, which is connected, not simply connected, and with a non-trivial centre. Let Γ be a lattice in G with a non-trivial centre $Z(\Gamma)$. Denote by $\widetilde{\Gamma}$ the inverse image of Γ in the universal cover of G and by $p : \widetilde{\Gamma} \to \Gamma$ the canonical projection. Observe that $Z(\widetilde{\Gamma}) = p^{-1}(Z(\Gamma))$. Choose a set-theoretical section $s : \Gamma \to \widetilde{\Gamma}$ for p with $s(e) = e$ and a character $\chi \in \mathrm{Hom}(Z(\widetilde{\Gamma}), \mathbf{T})$. Define a mapping

$$\zeta : \Gamma \times \Gamma \longrightarrow \mathbf{T}, \qquad \zeta(x, y) = \chi\left(s(x)s(y)s(xy)^{-1}\right).$$

It is straightforward to check that ζ is a multiplier, namely that $\zeta \in Z^2(\Gamma, \mathbf{T})$.

[Classes of multipliers of this kind are not arbitrary. They correspond precisely to those classes in $H^2(\Gamma, \mathbf{T})$ which are restrictions of classes in the appropriately defined group $H^2(G, \mathbf{T})$. The latter group is known

to be isomorphic to $\mathrm{Hom}(\pi_1(G), \mathbf{T})$; see Proposition 3.4 in [Moor–64] and [BaMi–00].]

It is obvious that, if χ extends to a unitary character $\widetilde{\chi}$ of $\widetilde{\Gamma}$, then ζ belongs to $B^2(\Gamma, \mathbf{T})$. Indeed, in this case $\zeta = \zeta_\nu$ for $\nu : \Gamma \longrightarrow \mathbf{T}$ defined by $\nu(x) = \widetilde{\chi}(s(x))$. Conversely, assume that $\zeta = \zeta_\nu$ for some $\nu : \Gamma \longrightarrow \mathbf{T}$ with $\nu(e) = 1$. As in the proof of Theorem 1.1 in [BaMi–00], define $v : \widetilde{\Gamma} \longrightarrow \mathbf{T}$ by $v(x) = \chi(x^{-1}s(p(x)))^{-1}$ for $x \in \widetilde{\Gamma}$ (observe that $x^{-1}s(p(x)) \in Z(\widetilde{\Gamma})$, so that $v(x)$ is well-defined). One checks that

$$\zeta(p(x), p(y)) \;=\; v(x)v(y)v(xy))^{-1} \qquad \forall x, y \in \widetilde{\Gamma}.$$

It follows that the function $\widetilde{\chi} : \widetilde{\Gamma} \longrightarrow \mathbf{T}$, defined by $\widetilde{\chi}(x) = \nu(p(x))v(x)^{-1}$, is a character of $\widetilde{\Gamma}$ which extends χ.

As a consequence, if the intersection of $Z(\widetilde{\Gamma})$ with the commutator subgroup $[\widetilde{\Gamma}, \widetilde{\Gamma}]$ is not reduced to $\{e\}$, we can find χ such that the corresponding multiplier ζ does not belong to $B^2(\Gamma, \mathbf{T})$. We provide in VI.c below an example for which this does occur.

We claim that Γ has no P-faithful irreducible ζ-representation. By Theorem 4, it suffices to show that $MS(\Gamma)$ has no Γ-P-faithful irreducible ζ-representation.

The group $MS(\Gamma)$ coincides with $Z(\Gamma)$ as we have shown in Part VI.a of the present example. Observe that $s(x) \in Z(\widetilde{\Gamma})$ for every $x \in Z(\Gamma)$. We have therefore $\zeta(x, y) = \zeta(y, x)$ for all $x \in Z(\Gamma)$ and $y \in \Gamma$. In particular, it follows that the restriction of ζ to $Z(\Gamma)$ is trivial (see Lemma 7.2 in [Klep–65]). Upon changing ζ inside its cohomology class, we can assume that $\zeta(x, y) = 1$ for all $x, y \in Z(\Gamma)$.

Let σ be an irreducible ζ-representation of $Z(\Gamma)$. Since the restriction of ζ to $Z(\Gamma)$ is trivial, we have $\mathrm{Pker}\,\sigma = Z(\Gamma)$. From the fact that $\zeta(x, y) = \zeta(y, x)$ for all $x \in Z(\Gamma)$ and $y \in \Gamma$, it follows that $\mathrm{Pker}_\Gamma\,\sigma = Z(\Gamma)$; see Remark (B) after Proposition 8. Hence, σ is not Γ-P-faithful since $Z(\Gamma)$ is non-trivial by assumption.

Example VI.c. Let Δ be the fundamental group of a closed surface of genus 2, viewed as a subgroup of $PSL_2(\mathbf{R})$. Let Γ be the inverse image of Δ in $SL_2(\mathbf{R})$; observe that $Z(\Gamma)$ is the two-element group. The group $\widetilde{\Gamma}$, the discrete subgroup of the universal cover of $SL_2(\mathbf{R})$ defined in VI.b, has a presentation with (see IV.48 in [Harp–00])

generators: a_1, a_2, b_1, b_2, c
and relations: c is central, and $[a_1, b_1][a_2, b_2] = c^2$.

In particular, the intersection of $Z(\widetilde{\Gamma})$ with $[\widetilde{\Gamma}, \widetilde{\Gamma}]$ is non trivial .

3. Γ-P-faithfulness for projective representations of normal subgroups of Γ

Let N be a normal subgroup of a group Γ and let $\zeta \in Z^2(N, \mathbf{T})$. Let σ be a ζ-representation of N.

For $\gamma \in \Gamma$, the mapping

$$N \ni x \longmapsto \sigma(\gamma x \gamma^{-1}) \in \mathcal{U}(\mathcal{H}_\sigma)$$

is in general not a ζ-representation of N, but is a ζ^γ-representation of N, where $\zeta^\gamma \in Z^2(N, \mathbf{T})$ is defined by

$$\zeta^\gamma(x, y) = \zeta(\gamma x \gamma^{-1}, \gamma y \gamma^{-1}) \tag{6}$$

for all $x, y \in N$.

Suppose moreover that ζ is the restriction to N of some multiplier on Γ. Then the multiplier ζ^γ is cohomologous to ζ; more precisely:

Lemma 6 (Mackey)**.** *Let $\zeta \in Z^2(\Gamma, \mathbf{T})$, let σ be a ζ-representation of N, and let $\gamma \in \Gamma$.*

(i) Define a mapping $\nu_\gamma : N \longrightarrow \mathbf{T}$ by $\nu_\gamma(x) = \frac{\zeta(\gamma,x)\zeta(\gamma x,\gamma^{-1})}{\zeta(\gamma^{-1},\gamma)}$. Then

$$\frac{\zeta^\gamma(x, y)}{\zeta(x, y)} = \frac{\nu_\gamma(xy)}{\nu_\gamma(x)\nu_\gamma(y)} \tag{7}$$

for all $x, y \in N$. In particular, $\underline{\zeta^\gamma} = \underline{\zeta} \in H^2(\Gamma, \mathbf{T})$.

(ii) Define a mapping $\sigma^\gamma : N \longrightarrow \mathcal{U}(\mathcal{H}_\sigma)$ by

$$\sigma^\gamma(x) = \zeta(\gamma, x)\zeta(\gamma x, \gamma^{-1})\, \sigma(\gamma x \gamma^{-1}). \tag{8}$$

Then

$$\sigma^\gamma(x)\sigma^\gamma(y) = \zeta(\gamma^{-1}, \gamma)\zeta(x, y)\, \sigma^\gamma(xy) \tag{9}$$

for all $x, y \in N$.

Proof. For (i), we refer to Lemma 4.2 in [Mack–58], of which the proof uses (2) from Section 1. For (ii), we have

$$\begin{aligned}
&\sigma^\gamma(x)\sigma^\gamma(y) = \\
&\quad \zeta(\gamma, x)\zeta(\gamma x, \gamma^{-1})\zeta(\gamma, y)\zeta(\gamma y, \gamma^{-1})\zeta(\gamma x \gamma^{-1}, \gamma y \gamma^{-1})\, \sigma(\gamma x y \gamma^{-1}) = \\
&\quad \frac{\zeta(\gamma, x)\zeta(\gamma x, \gamma^{-1})\zeta(\gamma, y)\zeta(\gamma y, \gamma^{-1})}{\zeta(\gamma, xy)\zeta(\gamma xy, \gamma^{-1})}\, \frac{\zeta^\gamma(x, y)}{\zeta(x, y)}\, \zeta(x, y)\, \sigma^\gamma(xy) = \\
&\quad \zeta(\gamma^{-1}, \gamma) \frac{\nu_\gamma(x)\nu_\gamma(y)}{\nu_\gamma(xy)}\, \frac{\zeta^\gamma(x, y)}{\zeta(x, y)}\, \zeta(x, y)\, \sigma^\gamma(xy) = \\
&\quad \zeta(\gamma^{-1}, \gamma)\zeta(x, y)\, \sigma^\gamma(xy),
\end{aligned}$$

where we have used (i) in the last equality. □

Equation (9) makes it convenient to restrict the discussion to normalised multipliers.

Definition. *A multiplier ζ on a group Γ is* normalised *if $\zeta(x, x^{-1}) = 1$ for all $x \in \Gamma$. A projective representation π of a group Γ is* normalised *if $\pi(x^{-1}) = \pi(x)^{-1}$ for all $x \in \Gamma$.*

(Some authors, see e.g. Page 142 of [BeZh–98], use "normalised" for multipliers in a different meaning.)

Lemma 7. *(i) Any multiplier ζ' on a group is cohomologous to a normalised mutliplier ζ.*

(ii) If ζ is a normalised multiplier on a group Γ, then

$$\zeta(y^{-1}, x^{-1}) \,=\, \zeta(x, y) \tag{10}$$

for all $x, y \in \Gamma$.

Proof. (i) Let π' be an arbitrary ζ'-representation of Γ on a Hilbert space $\mathcal{H}$. Define $J = \{\gamma \in \Gamma \mid \gamma^2 = e\}$ and choose a partition $\Gamma = J \sqcup K \sqcup L$ such that $\ell \in L$ if and only if $\ell^{-1} \in K$. For each $\gamma \in J$, choose $z_\gamma \in \mathbf{T}$ such that $(z_\gamma \pi'(\gamma))^2 = \mathrm{id}_{\mathcal{H}}$. Define a projective representation π of Γ on $\mathcal{H}$ by $\pi(\gamma) = z_\gamma \pi'(\gamma)$ if $\gamma \in J$, by $\pi(\gamma) = \pi'(\gamma)$ if $\gamma \in K$, and by $\pi(\gamma) = \pi'(\gamma^{-1})^{-1}$ if $\gamma \in L$. Then π is a normalised projective representation of which the multiplier ζ is normalised and cohomologous to ζ'.

(ii) This is a consequence of the identity

$$\pi(x^{-1})\pi(y^{-1}) \,=\, \frac{1}{\zeta(x, y)} \pi(y^{-1}x^{-1}),$$

which is a way of writing Equation (3) when π is normalised. □

Proposition 8. *Let Γ be a group, $\zeta \in Z^2(\Gamma, \mathbf{T})$ a normalised multiplier, and N a normal subgroup of Γ. Let σ be a ζ-representation of N; for $\gamma \in \Gamma$, define σ^γ as in Lemma 6.*

(i) The mapping

$$\sigma^\gamma : N \longrightarrow \mathcal{U}(\mathcal{H}_\sigma)$$

is a ζ-representation of N.

(ii) We have

$$\sigma^{\gamma_1 \gamma_2} \,=\, (\sigma^{\gamma_1})^{\gamma_2} \tag{11}$$

for all $\gamma_1, \gamma_2 \in \Gamma$.

Proof. Claim (i) follows from Lemma 6, because ζ is normalised, and checking Claim (ii) is straightforward. □

Second standing assumption. *All multipliers appearing from now on in this paper are assumed to be normalised.*

It is convenient to define now projective analogues of Γ-kernels, and Γ-faithfulness, a notion already used in the formulation of Theorem 4.

Definitions. *Let Γ be a group, N a normal subgroup, $\zeta \in Z^2(\Gamma, \mathbf{T})$ a multiplier, and $\sigma : N \longrightarrow \mathcal{U}(\mathcal{H})$ a ζ-representation of N.*

(i) *The* projective Γ-kernel *of σ is the normal subgroup*

$$\begin{aligned} \mathrm{P}\ker_\Gamma(\sigma) &= \big\{x \in \mathrm{Pker}(\sigma) \,\big|\, \sigma^\gamma(x) = \sigma(x) \text{ for all } \gamma \in \Gamma\big\} \\ &= \mathrm{Pker}\Big(\bigoplus_{\gamma\in\Gamma} \sigma^\gamma\Big). \end{aligned} \tag{12}$$

of Γ.

(ii) *The projective representation σ is* Γ-P-faithful *if* $\mathrm{P}\ker_\Gamma(\sigma) = \{e\}$.

Remarks. (A) In the particular case $\zeta = 1$, observe that $\mathrm{P}\ker_\Gamma(\sigma)$ is a subgroup of $\ker_\Gamma(\sigma)$ which can be a proper subgroup.

(B) Suppose that N is a central subgroup in Γ. For a ζ-representation σ of N, we have

$$\mathrm{P}\ker_\Gamma(\sigma) = \big\{x \in \mathrm{Pker}(\sigma) \,\big|\, \zeta(\gamma x, \gamma^{-1})\zeta(\gamma, x) = 1 \text{ for all } \gamma \in \Gamma\big\}.$$

Since

$$\zeta(\gamma x, \gamma^{-1})\zeta(x, \gamma) = \zeta(x, \gamma)\zeta(x\gamma, \gamma^{-1}) = \zeta(x, 1)\zeta(\gamma, \gamma^{-1}) = 1$$

for every $x \in Z(\Gamma)$ and $\gamma \in \Gamma$ (recall that ζ is normalised), we have $\zeta(\gamma x, \gamma^{-1}) = \zeta(x, \gamma)^{-1}$ and therefore also

$$\mathrm{P}\ker_\Gamma(\sigma) = \big\{x \in \mathrm{Pker}(\sigma) \,\big|\, \zeta(x, \gamma) = \zeta(\gamma, x) \text{ for all } \gamma \in \Gamma\big\}. \tag{13}$$

4. Extensions of groups associated to multipliers Proof of Theorem 1

Consider a group Γ, a multiplier $\zeta \in Z^2(\Gamma, \mathbf{T})$, and a subgroup A of $\mathbf{T}$ containing $\zeta(\Gamma \times \Gamma)$.

We define a group $\Gamma(\zeta)$ with underlying set $A \times \Gamma$ and multiplication

$$(s, x)(t, y) = (st\zeta(x, y), xy) \tag{14}$$

for all $s, t \in A$ and $x, y \in \Gamma$; observe that $(s, x)^{-1} = (s^{-1}, x^{-1})$, because ζ is normalised. This fits naturally in a central extension

$$\{e\} \longrightarrow A \overset{s\mapsto(s,e)}{\longrightarrow} \Gamma(\zeta) \overset{(s,x)\mapsto x}{\longrightarrow} \Gamma \longrightarrow \{e\}. \tag{15}$$

We insist on the fact that $\Gamma(\zeta)$ depends on the choice of A, even if the notation does not show it. Whenever H is a subgroup of Γ, we identify $H(\zeta)$ with the appropriate subgroup of $\Gamma(\zeta)$.

To any ζ-representation π of Γ on some Hilbert space $\mathcal{H}$ corresponds a representation π^0 of $\Gamma(\zeta)$ on the same space defined by

$$\pi^0(s,x) \;=\; s\pi(x) \tag{16}$$

for all $(s,x) \in \Gamma(\zeta)$. Conversely, to any representation π^0 of $\Gamma(\zeta)$ on $\mathcal{H}$ which is the identity on A (namely which is such that $\pi^0(a) = a\,\mathrm{id}_{\mathcal{H}}$ for all $a \in A$) corresponds a ζ-representation π of Γ defined by

$$\pi(x) \;=\; \pi^0(1,x). \tag{17}$$

Lemma 9. *(i) The correspondance $\pi \leftrightsquigarrow \pi^0$ given by Equations (16) and (17) is a bijection between ζ-representations of Γ on $\mathcal{H}$ and representations of $\Gamma(\zeta)$ on $\mathcal{H}$ which are the identity on A.*

(ii) If π and π^0 correspond to each other in this way, π is irreducible if and only if π^0 is so.

Assume moreover that the subgroup A of $\mathbf{T}$ contains both the image $\zeta(\Gamma\times\Gamma)$ of the multiplier and the subset

$$T_\pi \;:=\; \{z \in \mathbf{T} \;\mid\; z\,\mathrm{id}_{\mathcal{H}} = \pi(x) \text{ for some } x \in \mathrm{Pker}(\pi)\} \tag{18}$$

of $\mathbf{T}$.

(iii) If π and π^0 are as above, π is P-faithful if and only if π^0 is faithful.

Observation. If Γ is countable, $\zeta(\Gamma\times\Gamma)$ and T_π are countable subsets of $\mathbf{T}$, so that there exists a countable group A as in (15) which contains both T_π and the image of ζ.

Proof. Claims (i) and (ii) are obvious. The generalisation of Claim (i) for continuous representations of locally compact groups appears as a corollary to Theorem 1 in [Klep–74]; see also Theorem 2.1 in [Mack–58].

For Claim (iii), suppose first that π is P-faithful. If $(s,x) \in \ker(\pi^0)$, namely if $s\pi(x) = 1$, then $x \in \mathrm{Pker}(\pi)$, so that $x = e$; it follows that $s = 1$, so that $(s,x) = (1,e)$. Thus π^0 is faithful.

Suppose now that π^0 is faithful. If $x \in \mathrm{Pker}(\pi)$, namely if there exists $s \in \mathbf{T}$ such that $s\pi(x) = 1$, then $(s,x) \in \ker(\pi^0)$, so that $s = 1$ and $x = e$. Thus π is P-faithful. □

Proof of Theorem 1. Let π be an irreducible P-faithful projective representation of Γ, of multiplier ζ. Choose a subgroup A of $\mathbf{T}$ containing $\zeta(\Gamma\times\Gamma)$ and T_π (as defined in Lemma 9). Let $\Gamma(\zeta)$ be as in (14) and π^0 be as in (16). Since π^0 is irreducible and faithful (Lemma 9), Schur's Lemma implies that A is the centre of $\Gamma(\zeta)$, so that $\Gamma \approx \Gamma(\zeta)/Z(\Gamma(\zeta))$.

If Γ is countable, $\Gamma(\zeta)$ can be chosen countable, by the observation just after Lemma 9.

Conversely, let Δ be a group such that $\Gamma \approx \Delta/Z(\Delta)$ and let π^0 be a representation of Δ which is irreducible and faithful. Again by Schur's Lemma, the subgroup $(\pi^0)^{-1}(\mathbf{T})$ coincides with the centre of Δ. Let $\mu : \Gamma \longrightarrow \Delta$ be any set-theoretical section of the projection $\Delta \longrightarrow \Delta/Z(\Delta) \approx \Gamma$, with $\mu(e_\Gamma) = e_\Delta$. The assignment $\pi : \gamma \longmapsto \pi^0(\mu(\gamma))$ defines a projective representation of Γ which is irreducible and P-faithful, by Lemma 9. □

Our next lemma reduces essentially to Lemma 9.iii if $N = \Gamma$. It will be used in the proof of Lemma 13.

Lemma 10. *Consider a normal subgroup N of Γ and a ζ-representation σ of N in some Hilbert space $\mathcal{H}$. Let A be a subgroup of $\mathbf{T}$ containing both $\zeta(N \times N)$ and the subset*

$$T_{\sigma,\Gamma} := \{z \in \mathbf{T} \mid z\,\mathrm{id}_{\mathcal{H}} = \sigma(x) \text{ for some } x \in \mathrm{Pker}_\Gamma(\sigma)\} \tag{19}$$

of $\mathbf{T}$ (compare with Equation (18)). Define $N(\zeta)$ and $\Gamma(\zeta)$ as in the beginning of the present section. Then

(i) $(\sigma^\gamma)^0 = (\sigma^0)^\gamma$ *for all* $\gamma \in \Gamma$;

(ii) $\mathrm{P}\ker_\Gamma(\sigma) = \left\{ x \in N \;\middle|\; \begin{array}{l} \text{there exists } s \in A \\ \text{with } (s,x) \in \ker_{\Gamma(\zeta)}(\sigma^0) \end{array} \right\}$, *so that, in particular, σ is Γ-P-faithful if and only if σ^0 is $\Gamma(\zeta)$-faithful.*

Proof. Checking (i) is straightforward.

To show (ii), let $x \in \mathrm{P}\ker_\Gamma(\sigma)$. Thus there exists $s \in A$ such that $\sigma^\gamma(x) = s^{-1}\,\mathrm{id}_{\mathcal{H}}$ for all $\gamma \in \Gamma$. Then, for all $\gamma \in \Gamma$, we have

$$(\sigma^0)^\gamma(s,x) = (\sigma^\gamma)^0(s,x) = s\sigma^\gamma(x) = ss^{-1}\,\mathrm{id}_{\mathcal{H}} = \mathrm{id}_{\mathcal{H}},$$

that is, $(s,x) \in \ker_{\Gamma(\zeta)}(\sigma^0)$.

Conversely, let $x \in N$ be such that there exists $s \in A$ with $(s,x) \in \ker_{\Gamma(\zeta)} \sigma^0$. Then, for all $\gamma \in \Gamma$, we have

$$\sigma^\gamma(x) = (\sigma^\gamma)^0(1,x) = s^{-1}(\sigma^\gamma)^0(s,x) = s^{-1}(\sigma^0)^\gamma(s,x) = s^{-1}\,\mathrm{id}_{\mathcal{H}},$$

that is, $x \in \mathrm{P}\ker_\Gamma(\sigma)$. □

Given a group Γ and a multiplier $\zeta \in Z^2(\Gamma, \mathbf{T})$, a ζ-*character* of Γ is a ζ-representation $\chi : \Gamma \longrightarrow \mathbf{T} = \mathcal{U}(\mathbf{C})$; we denote by $X^\zeta(\Gamma)$ the set of all these. Observe that, for $\chi_1, \chi_2 \in X^\zeta(\Gamma)$, the product $\chi_1\overline{\chi_2}$ is a character of Γ in the usual sense, namely a homomorphism from Γ to $\mathbf{T}$. Such a homomorphism factors via the *abelianisation* $\Gamma/[\Gamma,\Gamma]$, that we denote by Γ^{ab}. We denote by $\widehat{\Gamma^{\mathrm{ab}}}$ the *character group* $\mathrm{Hom}(\Gamma, \mathbf{T})$. For further reference, we state here the following straightforward observations.

Lemma 11. *Let Γ be a group and let $\zeta \in Z^2(\Gamma, \mathbf{T})$.*
(i) If $\zeta \notin B^2(\Gamma, \mathbf{T})$, then $X^\zeta(\Gamma) = \emptyset$.
(ii) If $\zeta \in B^2(\Gamma, \mathbf{T})$, there exists a bijection between $X^\zeta(\Gamma)$ and $\widehat{\Gamma^{\mathrm{ab}}}$.

Proof. (i) Suppose that there exists $\chi \in X^\zeta(\Gamma)$; then $\zeta(x,y) = \frac{\chi(x)\chi(y)}{\chi(xy)}$ for all $x, y \in \Gamma$, so that ζ is a coboundary.
(ii) If $\zeta \in B^2(\Gamma, \mathbf{T})$, there exists a mapping $\nu : \Gamma \longrightarrow \mathbf{T}$ such that ζ is related to ν as in Fromula (5), so that $\nu \in X^\zeta(\Gamma)$. For any $\chi \in X^\zeta(\Gamma)$, observe that $\chi\overline{\nu}$ is an ordinary character of Γ, so that $\chi \longmapsto \chi\overline{\nu}$ is a bijection $X^\zeta(\Gamma) \longrightarrow X^1(\Gamma) = \widehat{\Gamma^{\mathrm{ab}}}$. □

5. Proof of $(i) \Longrightarrow (ii) \Longleftrightarrow (iii)$ in Theorem 4

The first proposition of this section is a reminder of Section 3 of [Mack–58].

Proposition 12 (Mackey)**.** *A ζ-representation π of a countable group Γ has a direct integral decomposition in irreducible ζ-representations, of the form*

$$\pi = \int_\Omega^\oplus \pi_\omega \, d\mu(\omega). \tag{20}$$

Proof. Consider a subgroup A of $\mathbf{T}$ containing $\zeta(\Gamma\times\Gamma)$ and the subset T_π defined in (18), the resulting extension $\Gamma(\zeta)$, and the representation π^0 of $\Gamma(\zeta)$ defined in (16). There exists a direct integral decomposition in irreducible representations

$$\pi^0 = \int_\Omega^\oplus (\pi^0)_\omega \, d\mu(\omega)$$

with respect to a measurable field $\omega \longmapsto (\pi^0)_\omega$ of irreducible representations of $\Gamma(\zeta)$ on a measure space (Ω, μ); see [Di–69C*], Sections 8.5 and 18.7.

Since $\pi^0(s,x) = s\pi(x)$ for all $(s,x) \in \Gamma(\zeta)$, we have $(\pi^0)_\omega(s,x) = s\,(\pi^0)_\omega(1,x)$ for all $(s,x) \in \Gamma(\zeta)$ and for almost all $\omega \in \Omega$. It follows that, for almost all $\omega \in \Omega$, the mapping $\pi_\omega : \Gamma \longrightarrow \mathcal{U}(\mathcal{H}_\omega)$ defined by $\pi_\omega(x) = (\pi^0)_\omega(1,x)$ is a ζ-representation of Γ which is irreducible, and $(\pi_\omega)^0 = (\pi^0)_\omega$. Hence we have a decomposition as in (20). □

We isolate in the next lemma an argument that we will use in the proofs of Propositions 14, 15, and 16.

Notation. Let Γ be a group and N a normal subgroup. We denote by $(C_j)_{j\in J}$ the Γ-conjugacy classes contained in N and, for each $j \in J$, by N_j the normal subgroup of Γ generated by C_j.

Lemma 13. *Let Γ be a group, $\zeta \in Z^2(\Gamma, \mathbf{T})$, and N a normal subgroup of Γ. Let $(C_j)_{j\in J}$ and $(N_j)_{j\in J}$ be as above. Let A be a subgroup of $\mathbf{T}$ containing $\zeta(\Gamma \times \Gamma)$, as well as $\chi(x)$ for every $\chi \in X^\zeta(N_j)$, $j \in J$, and $x \in N_j$. Let $N(\zeta)$ be the central extension of N corresponding to ζ and A as in (15).*

Then, for every ζ-representation σ of N, we have: σ is Γ-P-faithful if and only if the corresponding representation σ^0 of $N(\zeta)$ is $\Gamma(\zeta)$-faithful.

Remark. This lemma will be applied in situations where Γ is a countable group. Observe that, if Γ is countable, there exists a countable group A as in the previous lemma as soon as Γ has Property (Fab), or more generally as soon as N_j^{ab} is finite for all $j \in J$ such that the restriction of ζ to N_j is in $B^2(N_j, \mathbf{T})$.

Proof. In view of Lemma 10, it suffices to prove that A contains $T_{\sigma,\Gamma}$ for every ζ-representation σ of N.

Let $z \in T_{\sigma,\Gamma}$; choose $x \in \mathrm{Pker}_\Gamma(\sigma)$ such that $\sigma(x) = \mathrm{id}_{\mathcal{H}_\sigma}$. Let $j \in J$ be such that $x \in C_j$; we have $N_j \subset \mathrm{Pker}_\Gamma(\sigma)$, because the latter group is normal in Γ. The restriction of σ to N_j defines a ζ-character $\chi \in X^\zeta(N_j)$ such that $z = \chi(x)$. This shows that $z \in A$, by the choice of A. □

Implications $(i) \Longrightarrow (ii)$ and $(i) \Longrightarrow (iii)$ of Theorem 4 are particular cases of the following proposition, because the minisocle $MS(\Gamma)$ and the subgroup $MA(\Gamma)$ of a countable group Γ have the properties assumed for the group N below (Proposition 1 in [BeHa–08]).

Proposition 14. *Let Γ be a countable group, N a normal subgroup, and $\zeta \in Z^2(\Gamma, \mathbf{T})$. Let $(C_j)_{j\in J}$ and $(N_j)_{j\in J}$ be as just before Lemma 13. Assume that the abelianised group N_j^{ab} is finite for all $j \in J$ such that the restriction to N_j of ζ is in $B^2(N_j, \mathbf{T})$.*

Let π be a ζ-representation of Γ and let

$$\sigma := \pi|_N = \int_\Omega^\oplus \sigma_\omega \, d\mu(\omega) \tag{21}$$

be a direct integral decomposition of the restriction of π to N in irreducible ζ-representations σ_ω of N.

If π is irreducible and P-faithful, then σ_ω is Γ-P-faithful for almost all $\omega \in \Omega$.

Proof. The strategy is to reduce the proof to the case of ordinary representations and to use Lemma 9 of [BeHa–08].

By hypothesis and by Lemma 11, $X^\zeta(N_j)$ is finite (possibly empty) for all $j \in J$. Since J is countable, we can choose a *countable* subgroup

A of $\mathbf{T}$ containing $\zeta(\Gamma \times \Gamma)$, as well as $\chi(x)$ for every $\chi \in X^{\zeta}(N_j)$, $j \in J$, and $x \in N_j$.

Let $\Gamma(\zeta)$ and $N(\zeta)$ be as in (14); let π^0 and σ^0_ω be the representations of $\Gamma(\zeta)$ and $N(\zeta)$ corresponding to the ζ-representations π and σ_ω, respectively. Because π is P-faithful, the subset T_π defined in (18) is reduced to $\{e\}$ and therefore π^0 is faithful (Lemma 9).

Since (see the proof of Proposition 12)

$$\sigma^0 \,=\, \pi^0|_N \,=\, \int_\Omega^{\oplus} \sigma^0_\omega d\mu(\omega),$$

the representation σ^0_ω of $N(\zeta)$ is $\Gamma(\zeta)$-faithful for almost all ω (Lemma 9 of [BeHa–08]). Therefore, by Lemma 13, σ_ω is Γ-P-faithful for almost all ω. □

The equivalence $(ii) \Longleftrightarrow (iii)$ of Theorem 4 is a particular case of the following Proposition.

Proposition 15. *Assume that the normal subgroup N of Γ is a direct product $B \times S$ of normal subgroups of Γ, and that $S = \prod_{i\in I} S_i$ is a restricted direct product of finite simple nonabelian subgroups S_i. Assume moreover that any Γ-invariant subgroup of B generated by one Γ-conjugacy class has finite abelianisation.*

The following conditions are equivalent:

(α) *N has a Γ-P-faithful irreducible ζ-representation;*
(β) *B has a Γ-P-faithful irreducible ζ-representation.*

Proof The proof of the implication $(\alpha) \Rightarrow (\beta)$ follows closely the proof of Proposition 14, with one difference: one has to use the more general version of Lemma 9 in [BeHa–08] which is mentioned at the bottom of page 866 of this article.

For the converse implication, we assume now that B has a Γ-P-faithful irreducible ζ-representation σ. The group S has a faithful irreducible (unitary) representation, say ρ, such that $\rho(x) \notin \mathbf{T}$ for all $x \in S, x \neq e$, namely ρ is P-faithful; see the proof of Lemma 13 in [BeHa–08] (this Lemma 13 contains a hypothesis "A abelian", but it is redundant for the part of the proof we need here). The tensor product $\sigma \otimes \rho$ is an irreducible ζ-representation of N. Since σ is Γ-P-faithful, it follows from Lemma 12 of [BeHa–08] that $\sigma \otimes \rho$ is Γ-P-faithful. □

6. **End of proof of Theorem 4**

Let us first recall the definition of induction for projective representations, from Section 4 in [Mack–58].

Let Γ be a group, $\zeta \in Z^2(\Gamma, \mathbf{T})$ a multiplier, H a subgroup of Γ, and $\sigma : H \longrightarrow \mathcal{U}(\mathcal{K})$ a ζ-representation. Let $\mathcal{H}$ be the Hilbert space of mappings $f : \Gamma \longrightarrow \mathcal{K}$ with[6] the two following properties:

- $f(hx) = \zeta(h, x)\, \sigma(h)(f(x))$ for all $x \in \Gamma$ and $h \in H$,
- $\sum_{x \in \Gamma \backslash H} \|f(x)\|^2 < \infty$.

The ζ-representation $\mathrm{Ind}_H^\Gamma(\sigma)$ of Γ is the multiplier representation of Γ in $\mathcal{H}$ defined by

$$\big(\mathrm{Ind}_H^\Gamma(\sigma)\,(x)\,f\big))(y) \;=\; f(yx) \tag{22}$$

for all $x, y \in \Gamma$.

It can be checked (see [Mack–58], Pages 273-4) that the representation $\big(\mathrm{Ind}_H^\Gamma(\sigma)\big)^0$ of $\Gamma(\zeta)$ associated to $\mathrm{Ind}_H^\Gamma(\sigma)$ is the representation $\mathrm{Ind}_{H(\zeta)}^{\Gamma(\zeta)}(\sigma^0)$ induced by the representation σ^0 from $H(\zeta)$ to $\Gamma(\zeta)$.

The last claim of Theorem 4 follows from the next proposition.

Proposition 16. *Let Γ be a countable group and let $\zeta \in Z^2(\Gamma, \mathbf{T})$. Let $(C_j)_{j \in J}$ and $(N_j)_{j \in J}$ be as just before Lemma 13, with $N = \Gamma$. Assume that the abelianised group N_j^{ab} is finite for all $j \in J$ such that the restriction to N_j of ζ is in $B^2(N_j, \mathbf{T})$.*[7]

Let σ be a ζ-representation of the minisocle $MS(\Gamma)$. Set $\pi := \mathrm{ind}_{MS(\Gamma)}^\Gamma(\sigma)$ and let

$$\pi \;=\; \int_\Omega^\oplus \pi_\omega d\mu(\omega) \tag{23}$$

be a direct integral decomposition of π in irreducible ζ-representations of Γ.

If σ is irreducible and Γ-P-faithful, then π_ω is P-faithful for almost all $\omega \in \Omega$.

Proof. As for Proposition 14, the strategy is to reduce the proof to the case of ordinary representations, and to use this time Lemma 10 of [BeHa–08]. We write M for $MS(\Gamma)$.

By hypothesis and by Lemma 11, we can choose a countable subgroup A be a $\mathbf{T}$ containing the sets $\zeta(\Gamma \times \Gamma)$ and $X^\zeta(N_j)(x)$ for every $j \in J$ and every $x \in N_j$. We consider the corresponding extension $\Gamma(\zeta)$ of Γ. Denote by π^0 and π_ω^0 the representations of $\Gamma(\zeta)$ corresponding to the ζ-representations π and π_ω of Γ, and similarly σ^0 for the representation of $M(\zeta)$ corresponding to the ζ-representation σ of M.

[6] We use $H\backslash\Gamma$, rather than Γ/H as in [BeHa–08], which provides easier formulas.

[7] This assumption holds whenever Γ has Property (Fab).

We have

$$\pi^0 \,=\, \mathrm{ind}_{M(\zeta)}^{\Gamma(\zeta)}\sigma^0 \,=\, \int_\Omega^\oplus \pi^0_\omega d\mu(\omega).$$

In view of Lemma 13 applied to $N = \Gamma$, it suffices to show that π^0_ω is P-faithful for almost all ω.

Since σ is Γ-P-faithful, we have that σ^0 is Γ-faithful (again by Lemma 13). It will follow from Lemma 10 in [BeHa–08] that π^0_ω is faithful for almost all ω provided we show that $M(\zeta) \cap L \neq \{e\}$, for every finite foot L in $\Gamma(\zeta)$.

In order to check this condition, let L be finite foot in $\Gamma(\zeta)$. We claim that $L \subset M(\zeta)$. Indeed, recall that, set-theoretically, we have $\Gamma(\zeta) = A \times \Gamma$ and $M(\zeta) = A \times M$; thus, for any $(t, y) \in L$ with $(t, y) \neq e$, we have $y \in M(= MS(\Gamma))$, and therefore $(t, y) \in M(\zeta)$. □

7. Capable and incapable groups Proof of Proposition 2

In Proposition 2, Claims (i) and (ii) are respectively Corollary 2.3 and part of Corollary 2.2 of [BeFS–79]. Claim (iii) is a consequence of Claim (i), in a formulation and with a proof shown to us by Graham Ellis [Ellis], see below. Corollary 3 is an immediate consequence of Theorem 1 and Proposition 2.

Proof of Claim (iii) in Proposition 2. For a central extension

$$\{e\} \,\to\, A \,\to\, \Gamma \,\to\, \Gamma/A \,\to\, \{e\},$$

the Ganea extension of the Hochschild-Serre exact sequence in homology with trivial coefficients $\mathbf{Z}$ is

$$\begin{array}{ccccccc} A \otimes_{\mathbf{Z}} \Gamma^{\mathrm{ab}} & \longrightarrow & H_2(\Gamma, \mathbf{Z}) & \longrightarrow & H_2(\Gamma/A, \mathbf{Z}) & \longrightarrow & \\ A & \longrightarrow & H_1(\Gamma, \mathbf{Z}) & \longrightarrow & H_1(\Gamma/A, \mathbf{Z}) & \longrightarrow & \{0\} \end{array}$$

(see for example [EcHS–72]). If Γ is perfect (so that $\Gamma^{\mathrm{ab}} = \{0\}$), this reduces to

$$\{0\} \;\longrightarrow\; H_2(\Gamma, \mathbf{Z}) \;\longrightarrow\; H_2(\Gamma/A, \mathbf{Z}) \;\longrightarrow\; A \;\longrightarrow\; \{0\}$$

and it follows from the definition of the epicentre of Γ that $Z^*(\Gamma) = Z(\Gamma)$. Thus Claim (iii) is a straightforward consequence of Claim (i) of Proposition 2. □

It is well-known that any cyclic group $C \neq \{e\}$ is incapable. Indeed, suppose *ab absurdo* that $C = \Delta/Z(\Delta)$. Choose a generator s of C and a preimage t of s in Δ; any $\delta \in \Delta$ can be written as $\delta = zt^j$ for some $z \in Z(\Delta)$ and $j \in \mathbf{Z}$, and two elements of this kind commute with each other, so that Δ is abelian, hence $Z(\Delta) = \Delta$, incompatible with

$C \neq \{e\}$. The next lemma, which appears on Page 137 of [Hall–40], rests on an elaboration of the same argument.

Lemma 17. *Let Γ be a group containing an element $s_0 \neq e$ such that the set*

$$\{s \in \Gamma \mid \text{there exists } n \in \mathbf{Z} \text{ with } s^n = s_0\}$$

(where n can depend on s) generates Γ. Then Γ is incapable.

Proof. It suffices to show that, given any central extension

$$\{e\} \longrightarrow A \longrightarrow \Delta \xrightarrow{\pi} \Gamma \longrightarrow \{e\},$$

s_0 has a preimage t_0 in Δ which is central.

Let $\delta \in \Delta$. There exists

$$s_1, \dots, s_k \in \Gamma \quad \text{and} \quad j_1, \dots, j_k, n_1, \dots, n_k \in \mathbf{Z}$$

such that

$$\pi(\delta) = s_1^{j_1} \cdots s_k^{j_k} \quad \text{and} \quad s_1^{n_1} = \cdots = s_k^{n_k} = s_0.$$

For $i = 0, \dots, k$, choose a preimage t_i of s_i in Δ. There exist $a, a_1, \dots, a_k \in A$ with

$$\delta = a t_1^{j_1} \cdots t_k^{j_k} \quad \text{and} \quad a_i t_i^{n_i} = t_0 \text{ for } i = 1, \dots, k.$$

It follows that t_0 commutes with t_i for $i = 1, \dots, k$, and thus that t_0 commutes with δ, as was to be shown. □

Claims (i) to (v) of the following proposition are straightforward consequences of Lemma 17, and the last claim follows from Theorem 1.

Proposition 18. *The following groups are incapable:*

(i) *cyclic groups, quasicyclic groups $\mathbf{Z}(p^\infty)$, and the groups $\mathbf{Z}[1/m]$ for all integers $m \geq 2$;*
(ii) *finite abelian groups $\mathbf{Z}/d_1\mathbf{Z} \times \cdots \times \mathbf{Z}/d_m\mathbf{Z}$, (where $n \geq 2$, $d_1, \dots, d_m \geq 2$, $d_1 | d_2 | \cdots | d_m$) with $d_{m-1} < d_m$;*
(iii) *subgroups of $\mathbf{Q}$;*
(iv) $\mathrm{SL}_2(\mathbf{Z}) = \langle s, t \mid s^2 = t^3 \text{ is central of order } 2 \rangle$;
(v) $\langle s, t \mid s^m = t^n \text{ and } (s^m)^k = 1 \rangle$ *for $m, n \geq 1$, $k \geq 2$, as well as $\langle s, t \mid s^m = t^n \rangle$.*

In particular, these groups do not afford any irreducible P-faithful projective representations.

Remarks. About (i): for any prime p, the quasicyclic group $\mathbf{Z}(p^\infty)$ is the subgroup of $\mathbf{T}$ of roots of 1 of order some power of p; equivalently, $\mathbf{Z}(p^\infty)$ is the quotient $\mathbf{Q}_p/\mathbf{Z}_p$ of the p-adic numbers by the p-adic integers.

About (iii): there is a classification of the subgroups of $\mathbf{Q}$, which is is standard; see for example Chapter 10 of [Rotm–95].

About (iv), let us recall that $\mathrm{SL}_2(\mathbf{Z})$ is generated by a square root $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and a cubic root $\begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}$ of the central matrix $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$. Next, since $\mathrm{SL}_2(\mathbf{Z}) = \langle s, t \mid s^2 = t^3,\ s^4 = 1 \rangle$ has deficiency ≥ 0 and finite abelianisation (indeed $\mathrm{SL}_2(\mathbf{Z})^{\mathrm{ab}} \approx \mathbf{Z}/12\mathbf{Z}$), it follows from Philip Hall's Inequality[8] that $H_2(\mathrm{SL}_2(\mathbf{Z}), \mathbf{Z}) = \{0\}$.

Similarly, $H_2(\mathrm{PSL}_2(\mathbf{Z}), \mathbf{Z}) = \{0\}$. This follows alternatively from the formula $H_n(\Gamma_1 * \Gamma_2, \mathbf{Z}) \approx H_n(\Gamma_1, \mathbf{Z}) \oplus H_n(\Gamma_2, \mathbf{Z})$ for $n \geq 1$, see Corollary 6.2.10 in [Weib–94]. But $\mathrm{PSL}_2(\mathbf{Z})$ is capable, since its centre is trivial.

About (v): the group $\mathrm{SL}_2(\mathbf{Z})$ is of course a particular case of groups in (v); if m and n are coprime and at least 2, the group $\langle s, t \mid s^m = t^n \rangle$ is a *torus knot group*.

8. **On abelian groups**

The next proposition rests on a construction which appears in many places, including [Mack–49] and [Weil–64]. It is part of the *Stone–von Neumann–Mackey Theorem*, see the beginning of [MuNN–91].

Let L be an abelian group, written multiplicatively. Consider the group $X(L) = \mathrm{Hom}(L, \mathbf{T})$ of characters of L, with the topology of the simple convergence, which makes it a locally compact abelian group. By Pontryagin duality, we can (and do) identify L to the group of continuous characters on $X(L)$. Consider also a dense subgroup M of $X(L)$ and the direct product group $L \times M$. The mapping

$$\zeta : (L \times M) \times (L \times M) \longrightarrow \mathbf{T},\ \ ((\ell, m), (\ell', m')) \longmapsto m'(\ell)$$

is a multiplier on $L \times M$. Let A be a subgroup of $\mathbf{T}$ containing the image of ζ. By definition, the corresponding *generalised Heisenberg group* is

$$H^A_{L,M} \ = \ A \times L \times M$$

[8]Namely: for a finitely presented group Γ, the deficiency of Γ is bounded by the difference $\dim_{\mathbf{Q}}((\Gamma)^{\mathrm{ab}} \otimes_{\mathbf{Z}} \mathbf{Q}) - s(H_2(\Gamma, \mathbf{Z}))$, where $s(H)$ stands for the minimum number of generators of the group H; see e.g. Lemma 1.2 in [Epst–61]. Recall also that the *deficiency* of a finite presentation of a group is the number of its generators minus the number of its relations, and the deficiency of a finitely presented group the maximum of the deficiencies of its finite presentations.

with product defined by

$$(z, \ell, m)\,(z', \ell', m') \;=\; (zz'\ell(m')\,,\; \ell\ell'\,,\; mm').$$

It is routine to check that the centre of $H^A_{L,M}$ is A.

Proposition 19. *Any abelian group of the form $L \times M$, with M dense in $X(L)$ as above, affords a projective representation which is irreducible and P-faithful.*

Proof. Let us sketch the definition and some properties of the "Stone–von Neumann–Mackey representation" of $H^A_{L,M}$ on $\ell^2(L)$; the latter is a Hilbert space, with scalar product defined by $\langle \xi \mid \eta \rangle = \sum_{\ell \in L} \overline{\xi(\ell)}\eta(\ell)$.

For $(z, \ell, m) \in H^A_{L,M}$ and $\xi \in \ell^2(L)$, set

$$(R(z, \ell, m)\xi)(x) \;=\; zm(x)\xi(x\ell) \quad \text{for all } x \in L.$$

It can be checked that $R(z, \ell, m)$ is a unitary operator on $\mathcal{H} = \ell^2(L)$ and that

$$R \;:\; H^A_{L,M} \longrightarrow \mathcal{U}(\mathcal{H})$$

is a representation of $H^A_{L,M}$ on $\mathcal{H}$.

The space $\mathcal{H}$ has a natural orthonormal basis $(\delta_u)_{u \in L}$. It is easy to check that

$$R(z, \ell, m)\delta_u \;=\; zm(u\ell^{-1})\delta_{u\ell^{-1}},$$

so that the representation R is faithful. Observe that, for all $u \in L$ and $m \in M$, the vector δ_u is an eigenvector of $R(1, 1, m)$ with eigenvalue $m(u)$. If

$$V_u \;=\; \{\xi \in \mathcal{H} \mid R(1, 1, m)\xi = m(u)\xi \text{ for all } m \in M\},$$

then $V_u = \mathbf{C}\delta_u$ is an eigenspace of dimension 1 and $\mathcal{H} = \bigoplus_{u \in L} V_u$ (Hilbert sum).

Let now $S \in \mathcal{L}(\mathcal{H})$ be an operator commuting with $R(1, 1, m)$ for all $m \in M$. Since M is dense in $X(L)$, for every $u, v \in L$ with $u \neq v$, there exists $m \in M$ such that $m(u) \neq m(v)$. As is easily checked, this implies that S is diagonal with respect to the basis $(\delta_u)_{u \in L}$, namely that there exist complex numbers s_u such that $S(\delta_u) = s_u\delta_u$ for all $u \in M$. Suppose moreover that S commutes with $R(1, \ell, 1)$ for all $\ell \in L$; since $R(1, \ell, 1)\delta_u = \delta_{u\ell^{-1}}$, we have $s_u = s_{u\ell^{-1}}$ for all $u, \ell \in L$. Thus S is a scalar multiple of the identity operator. It follows from Schur's lemma that the representation R is irreducible.

The representation R of $H^A_{L,M}$ provides a projective representation of $L \times M$ which is irreducible and P-faithful. □

In particular, the following groups afford projective representations which are irreducible and P-faithful:

- $\mathbf{Z}^n$ for any $n \geq 2$, as $\mathbf{Z}^{n-1}$ is a dense subgroup of $X(\mathbf{Z}) \approx \mathbf{T}$.
- $\mathbf{Z}(p^\infty) \times \mathbf{Z}$, as $\mathbf{Z}$ is a dense subgroup of $X(\mathbf{Z}(p^\infty)) \approx \mathbf{Z}_p$. For the latter isomorphism, see e.g. [Bour–67], chap. 2, § 1, no. 9, cor. 4 of prop. 12; $\mathbf{Z}(p^\infty)$ is as just after Proposition 18.
- $\mathbf{Q}^n$ for any $n \geq 2$. Indeed, let us check this for $n = 2$, the general case being entirely similar. The group $X(\mathbf{Q})$ can be identified with $\mathbf{A}/\varphi(\mathbf{Q})$; here $\mathbf{A}$ is the group of adeles of $\mathbf{Q}$ and $\varphi : \mathbf{Q} \longrightarrow \mathbf{A}$ is the diagonal embedding of $\mathbf{Q}$ in $\mathbf{A}$ (recall that $\varphi(\mathbf{Q})$ is discrete and cocompact in $\mathbf{A}$). More precisely, let χ_0 be a non-trivial character of $\mathbf{A}$ with $\chi_0|_{\varphi(\mathbf{Q})} = 1$. Then the mapping

$$\Phi : \mathbf{A} \longrightarrow X(\mathbf{Q}), \qquad a \longmapsto (q \mapsto \chi_0(a\varphi(q)))$$

factorizes to an isomorphism $\mathbf{A}/\varphi(\mathbf{Q}) \to X(\mathbf{Q})$ (see Chapter 3 in [GGPS–90]). Fix $a_0 \in \mathbf{A}$ with $a_0 \notin \varphi(\mathbf{Q})$ and define a group homomorphism

$$f : \mathbf{Q} \longrightarrow X(\mathbf{Q}), \qquad f(q) = \Phi(a_0\varphi(q)).$$

Then f is injective since $a_0\varphi(q) \notin \varphi(\mathbf{Q})$ for all $q \in \mathbf{Q}^*$. We claim that the range of f is dense. Indeed, assume that this is not the case. By Pontryagin duality, there exists $q_0 \in \mathbf{Q}^*$ such that $f(q)(\varphi(q_0)) = 1$ for all $q \in \mathbf{Q}$. This means that $\chi_0(a_0\varphi(q_0q)) = 1$ for all $q \in \mathbf{Q}$, that is, $\Phi(a_0\varphi(q_0))$ is the trivial character of $\mathbf{Q}$. This is a contradiction, since $a_0\varphi(q_0) \notin \varphi(\mathbf{Q})$.

Note that Proposition 19 carries over to dense subgroups of groups of the form $B \times X(B)$, with B a locally compact abelian group.

The case of finite groups is covered by a result of Frucht [Fruc–31]. For a modern exposition (and improvements[9]), see Page 166 of [BeZh–98].

Proposition 20 (Frucht)**.** *For a finite abelian group Γ, the two following properties are equivalent:*

(i) *Γ affords a projective representation which is irreducible and P-faithful;*
(ii) *there exists a (finite abelian) group L such that Γ is isomorphic to the direct sum $L \times L$.*

[9]Any finite abelian group affords two irreducible projective representations of which the direct sum is P-faithful. For a characterisation of those finite groups which have a faithful linear representation which is a direct sum of k irreducible representations, see Page 245, and indeed all of Chapter 9, in [BeZh–98].

Observation, from [Sury–08]. Consider a prime p, the "Heisenberg group" H below, and its noncyclic centre $Z(H)$:

$$H = \begin{pmatrix} 1 & \mathbf{F}_p & \mathbf{F}_{p^2} \\ 0 & 1 & \mathbf{F}_{p^2} \\ 0 & 0 & 1 \end{pmatrix} \supset Z(H) = \begin{pmatrix} 1 & 0 & \mathbf{F}_{p^2} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \approx \mathbf{F}_{p^2} \approx \mathbf{F}_p \oplus \mathbf{F}_p$$

(where the last $\approx$ indicates of course an isomorphism of additive groups, not of rings!). The quotient $H/Z(H) \approx \mathbf{F}_p \oplus \mathbf{F}_{p^2}$ is an abelian group of which the order p is not a square, and therefore which does not have the properties of Proposition 20, but which is however a capable group.

Recall (from just after Proposition 19) that $\mathbf{Z}^3$ affords a projective representation which is irreducible and P-faithful, and compare with Claim (ii) of Proposition 20.

9. **Final remarks**

In some sense, what follows goes back for finite groups to papers by Schur, from 1904 and 1907. For the general case, see Sections V.5 and V.6 in [Stam–73], [EcHS–72], and [Kerv–70].

A *stem cover* of a group Γ is a group $\widetilde{\Gamma}$ given with a surjection p onto Γ such that $\ker(p)$ is central in $\widetilde{\Gamma}$, contained in $[\widetilde{\Gamma}, \widetilde{\Gamma}]$, and isomorphic to $H_2(\Gamma, \mathbf{Z})$. Any group has a stem cover. The isomorphism type of $\widetilde{\Gamma}$ is uniquely determined in case Γ is perfect, but not in general. For example, the dihedral group of order 8 and the quaternion group both qualify for $\widetilde{\Gamma}$ if Γ is the Vierergruppe.

To check the existence of stem covers, consider $H_2(\Gamma) := H_2(\Gamma, \mathbf{Z})$ as a trivial Γ-module, the short exact sequence

$$\{0\} \to \mathrm{Ext}(\Gamma^{\mathrm{ab}}, H_2(\Gamma)) \to H^2(\Gamma, H_2(\Gamma)) \to \mathrm{Hom}(H_2(\Gamma), H_2(\Gamma)) \to \{0\}$$

of the universal coefficient theorem in cohomology, and a multiplier ζ in $Z^2(\Gamma, H_2(\Gamma))$ of which the cohomology class $\underline{\zeta}$ is mapped onto the identity homomorphism of $H_2(\Gamma)$ to itself. Then the corresponding central extension

$$\{0\} \longrightarrow H_2(\Gamma) \longrightarrow \widetilde{\Gamma} \xrightarrow{p} \Gamma \longrightarrow \{1\},$$

in other words the central extension of characteristic class $\underline{\zeta}$, is a stem cover of Γ. Stem covers of Γ are classified (as central extensions of Γ by $H_2(\Gamma)$) by the group $\mathrm{Ext}(\Gamma^{\mathrm{ab}}, H_2(\Gamma, \mathbf{Z}))$; see Proposition V.5.3 of [Stam–73], and Theorem 2.2 of [EcHS–72]. In particular, if Γ is perfect, it has a unique stem cover, also called its *universal central extension*. If Γ is finite, its stem covers are also called its *Schur representation groups*.

Let $p : \widetilde{\Gamma} \longrightarrow \Gamma$ be a stem cover. For any central extension

$$\{0\} \longrightarrow A \longrightarrow \widetilde{\Delta} \xrightarrow{q} \Delta \longrightarrow \{1\}$$

with divisible kernel A (more generally with A such that $\mathrm{Ext}(\Gamma^{\mathrm{ab}}, A) = \{0\}$) and for any homomorphism $\underline{\rho} : \Gamma \longrightarrow \Delta$, there exists a homomorphism $\rho^0 : \widetilde{\Gamma} \longrightarrow \widetilde{\Delta}$ such that $\underline{\rho}(p(\tilde{\gamma})) = q(\rho^0(\tilde{\gamma}))$ for all $\tilde{\gamma} \in \widetilde{\Gamma}$; see Proposition V.5.5 of [Stam–73]. In particular, for a Hilbert space $\mathcal{H}$ and a homomorphism $\underline{\pi} : \Gamma \longrightarrow \mathcal{PU}(\mathcal{H})$, there exists a unitary representation $\pi^0 : \widetilde{\Gamma} \longrightarrow \mathcal{U}(\mathcal{H})$ such that $\underline{\pi}(p(\tilde{\gamma})) = p_{\mathcal{H}}(\pi^0(\tilde{\gamma}))$ for all $\tilde{\gamma} \in \widetilde{\Gamma}$.

Observe that, if Γ is countable, $H_2(\Gamma)$ is countable (this follows for example from the Schur-Hopf formula $H_2(\Gamma) = R \cap [F, F]/[F, R]$ where $\Gamma = F/R$ with F free), so that $\widetilde{\Gamma}$ is also countable.

It would be interesting to understand, say for the proof of Theorem 4, if and how one could use the stem cover(s) of Γ instead of the groups $\Gamma(\zeta)$ which appear in Section 4.

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