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► **To cite this version:**

Bachir Bekka, Jean-Romain Heu. Random products of automorphisms of Heisenberg nilmanifolds and Weil's representation. *Ergodic Theory and Dynamical Systems*, 2011, 31 (5), pp.1277-1286. 10.1017/S014338571000043X . hal-00564615

**HAL Id: hal-00564615**

**<https://hal.science/hal-00564615>**

Submitted on 9 Feb 2011

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# Random products of automorphisms of Heisenberg nilmanifolds and Weil's representation

Bachir Bekka and Jean-Romain Heu

November 1, 2009

## Abstract

For  $n \geq 1$ , let  $H$  be the  $(2n+1)$ -dimensional real Heisenberg group, and let  $\Lambda$  be a lattice in  $H$ . Let  $\Gamma$  be a group of automorphisms of the corresponding nilmanifold  $\Lambda \backslash H$  and  $U$  the associated unitary representation of  $\Gamma$  on  $L^2(\Lambda \backslash H)$ . Denote by  $T$  the maximal torus factor associated to  $\Lambda \backslash H$ . Using Weil's representation (also known as the metaplectic representation), we show that a dense set of matrix coefficients of the restriction of  $U$  to the orthogonal complement of  $L^2(T)$  in  $L^2(\Lambda \backslash H)$  belong to  $\ell^{4n+2+\varepsilon}(\Gamma)$  for every  $\varepsilon > 0$ .

We give the following application to random walks on  $\Lambda \backslash H$  defined by a probability measure  $\mu$  on  $\text{Aut}(\Lambda \backslash H)$ . Denoting by  $\Gamma$  the subgroup of  $\text{Aut}(\Lambda \backslash H)$  generated by the support of  $\mu$  and by  $U^0$  and  $V^0$  the restrictions of  $U$  respectively to the subspaces of  $L^2(\Lambda \backslash H)$  and  $L^2(T)$  with zero mean, we prove the following inequality:

$$\|U^0(\mu)\| \leq \max \left\{ \|V^0(\mu)\|, \|\lambda_\Gamma(\mu)\|^{1/(2n+2)} \right\},$$

where  $\lambda_\Gamma$  is the left regular representation of  $\Gamma$  on  $\ell^2(\Gamma)$ . In particular, the action of  $\Gamma$  on  $\Lambda \backslash H$  has a spectral gap if and only if the corresponding action of  $\Gamma$  on  $T$  has a spectral gap.

## 1 Introduction

Let  $(X, m)$  be a probability space and  $G$  a locally compact group of measure preserving transformations of  $X$ . Given a probability measure  $\mu$  on  $G$ ,

consider a sequence of independent  $\mu$ -distributed random variables  $X_n^\omega$  with values in  $G$  and the corresponding random products  $S_n^\omega = X_n^\omega \dots X_1^\omega$  for  $n \in \mathbf{N}$ . This defines a random walk on  $X$  with initial distribution  $m$  and trajectories  $S_n^\omega(x)$  for  $x \in X$  and  $n \in \mathbf{N}$ . A question of interest is whether this random walk has a spectral gap. To define this notion, let  $U$  be the unitary representation of  $G$  on  $L^2(X, m)$  defined by  $U_g(\xi) = \xi(g^{-1}(x))$  for  $g \in G$ ,  $\xi \in L^2(X, m)$ , and  $x \in X$ . Let  $L_0^2(X, m)$  be the  $U(G)$ -invariant subspace of functions  $\xi$  in  $L^2(X, m)$  with zero mean, that is, with  $\int_X \xi(x) dm(x) = 0$ . Denote by  $U^0$  the restriction of  $U$  to  $L_0^2(X, m)$ . Let  $U^0(\mu)$  be the convolution operator defined on  $L_0^2(X, m)$  by

$$U^0(\mu)\xi = \int_G U_g^0(\xi) d\mu(g) \quad \text{for all } \xi \in L_0^2(X, m).$$

Observe that  $\|U^0(\mu)\| \leq 1$ . We say that  $\mu$  has a spectral gap in  $(X, m)$  if  $\|U^0(\mu)\| < 1$ . This spectral gap property has several interesting applications; the most immediate one is the exponentially fast convergence of the sequence of functions  $x \mapsto \mathbf{E}(\xi(S_n^\omega(x)))$  to  $\int_X \xi dm$  in the  $L^2$ -norm for every  $\xi \in L^2(X, m)$ . Other applications include the existence of a rate of convergence for random ergodic theorems, a central limit theorem, and the uniqueness of invariant means on  $L^\infty(X, m)$ ; see [FuSh99], [Guiv05], [Lubo94], [Sarn90].

The spectral gap property can be formulated in terms of weak containment of group representations (see [BeHV08, G.4.2]). Assume that the subgroup generated by the support of  $\mu$  is dense in  $G$ . Assume moreover that  $\mu$  is aperiodic (that is, the support of  $\mu$  is not contained in the coset of a proper closed subgroup of  $G$ ). Then  $\mu$  has a spectral gap in  $(X, m)$  if and only if there is no  $G$ -almost invariant vectors in  $L_0^2(X, m)$ . If this is the case, we say for short that the  $G$ -action on  $X$  has a spectral gap.

We emphasize that the existence of a spectral gap property is a phenomenon which can occur only in the context of non-amenable groups: when  $G$  is a discrete amenable group and  $m$  is non-atomic, then  $G$  has *never* a spectral gap on  $X$  (see [JuRo79] or [Schmi80]).

When  $X$  is the  $n$ -dimensional torus  $\mathbf{R}^n/\mathbf{Z}^n$ , equipped with the normalized Lebesgue measure  $m$ , sufficient conditions were given in [FuSh99] for the existence of a spectral gap for the action of a subgroup of  $GL_n(\mathbf{Z})$  by automorphisms on  $\mathbf{R}^n/\mathbf{Z}^n$  (see also Example 4 below). In this paper, we will consider the case where  $X$  is a Heisenberg nilmanifold and  $G$  a group of automorphisms of  $X$ .

For  $n \geq 1$ , let  $H = H_{2n+1}(\mathbf{R})$  be the  $(2n+1)$ -dimensional real Heisenberg group. This is a two step nilpotent Lie group with one-dimensional centre  $Z$  (see Section 2 below). Let  $\Lambda$  be a lattice in  $H$ :  $\Lambda$  is a discrete subgroup of  $H$  such that there exists a (unique) probability measure  $m$  on the Borel sets of the corresponding nilmanifold  $\Lambda \backslash H$  which is invariant under right translation by elements from  $H$ . (Observe that  $\Lambda$  is cocompact in  $H$ .) Denote by  $\text{Aut}(H)$  the group of continuous automorphisms of  $H$  and by  $\text{Aut}(\Lambda \backslash H)$  the subgroup of all  $g \in \text{Aut}(H)$  such that  $g(\Lambda) = \Lambda$ ; every automorphism  $g \in \text{Aut}(\Lambda \backslash H)$  induces a homeomorphism of  $\Lambda \backslash H$ .

Let  $\Gamma$  be a subgroup of  $\text{Aut}(\Lambda \backslash H)$ . The action of  $\Gamma$  on  $\Lambda \backslash H$  preserves the  $H$ -invariant probability measure  $m$  on  $\Lambda \backslash H$ . Let  $U$  be the associated unitary representation of  $\Gamma$  on  $L^2(\Lambda \backslash H, m)$ . Let  $T = \Lambda Z \backslash H$  be the maximal torus factor of  $\Lambda \backslash H$ . Observe that  $T \cong \mathbf{R}^{2n} / \mathbf{Z}^{2n}$  and  $\text{Aut}(T) \cong GL_{2n}(\mathbf{Z})$ . Since  $\text{Aut}(\Lambda \backslash H)$  preserves  $Z\Lambda$ , we have a homomorphism  $p : \text{Aut}(\Lambda \backslash H) \rightarrow \text{Aut}(T)$  and an induced action of  $\Gamma$  on  $T$ . This defines a unitary representation of  $\Gamma$  on  $L^2(T)$ , where  $T$  is equipped with normalized Lebesgue measure. We can (and will) identify  $L^2(T)$ , as  $\Gamma$ -space, with a closed  $U(\Gamma)$ -invariant subspace of  $L^2(\Lambda \backslash H)$ . Denote by  $\mathcal{H}$  the orthogonal complement of  $L^2(T)$  in  $L^2(\Lambda \backslash H)$ , so that we have an orthogonal decomposition

$$L^2(\Lambda \backslash H) = L^2(T) \oplus \mathcal{H}$$

into  $U(\Gamma)$ -invariant subspaces. Here is our main result.

**Theorem 1** *The matrix coefficients of the restriction of  $U$  to  $\mathcal{H}$  are strongly  $L^{4n+2+\varepsilon}$ : there are dense subspaces  $D_1$  and  $D_2$  of  $\mathcal{H}$  such that, for any  $v \in D_1$  and  $w \in D_2$ , the matrix coefficient  $\gamma \mapsto \langle U_\gamma v, w \rangle$  belongs to  $\ell^{4n+2+\varepsilon}(\Gamma)$ , for every  $\varepsilon > 0$ .*

Concerning the proof of the previous theorem, we first show that the representation  $U$  is linked with Weil's representation, which is also known as Segal-Shale-Weil, metaplectic, or oscillator representation (see [Shal62], [Weil64]). The crucial tool is then a result from [HoMo79] about the decay of the matrix coefficients of Weil's representation.

Here is an immediate consequence of Theorem 1. Recall that, if  $X$  is a locally compact space,  $C_0(X)$  denotes the space of complex-valued continuous functions on  $X$  which tend to zero at infinity.

**Corollary 2** *The restriction of the unitary representation  $U$  to  $\mathcal{H}$  is mixing: the matrix coefficients  $\gamma \mapsto \langle U_\gamma v, w \rangle$  belong to  $c_0(\Gamma)$  for all  $v, w \in \mathcal{H}$ .*

The previous corollary immediately implies that the ergodicity or mixing of the  $\Gamma$ -action on  $\Lambda \backslash H$  is equivalent to the ergodicity or mixing of the  $\Gamma$ -action on  $T$  (see Corollary 6 below).

We apply Theorem 1 to the existence of a spectral gap for the random walk on  $\Lambda \backslash H$  associated to a probability measure  $\mu$  on  $\text{Aut}(\Lambda \backslash H)$ .

**Theorem 3** *Let  $\mu$  be a probability measure on  $\text{Aut}(\Lambda \backslash H)$ . Denote by  $\Gamma$  be the subgroup of  $\text{Aut}(\Lambda \backslash H)$  generated by the support of  $\mu$ . Let  $U^0$  and  $V^0$  be the associated unitary representations of  $\Gamma$  on  $L_0^2(\Lambda \backslash H)$  and  $L_0^2(T)$  respectively. Then*

$$\|U^0(\mu)\| \leq \max\{\|V^0(\mu)\|, \|\lambda_\Gamma(\mu)\|^{1/(2n+2)}\},$$

where  $\lambda_\Gamma$  is the left regular representation of  $\Gamma$  on  $\ell^2(\Gamma)$ . In particular, the action of  $\Gamma$  on  $\Lambda \backslash H$  has a spectral gap if and only if the corresponding action of  $\Gamma$  on  $T$  has a spectral gap.

**Example 4** Let  $\Gamma$  be a subgroup of  $\text{Aut}(\Lambda \backslash H)$  such that its image  $p(\Gamma) \subset GL_{2n}(\mathbf{Z})$  under the homomorphism  $p : \text{Aut}(\Lambda \backslash H) \rightarrow \text{Aut}(T) \cong GL_{2n}(\mathbf{Z})$  acts irreducibly on  $\mathbf{R}^{2n}$  and does not have an abelian subgroup of finite index (this is for instance the case if  $p(\Gamma)$  is Zariski dense in  $GL_{2n}(\mathbf{R})$ ). Then, as shown in [FuSh99, Theorem 6.5], the action of  $\Gamma$  on  $T$  has a spectral gap.

In the case  $n = 1$ , we have the following more precise result.

**Corollary 5** *Let  $H = H_3(\mathbf{R})$  be the 3-dimensional Heisenberg group and  $\Lambda$  a lattice in  $H$ . Let  $\mu$  be a probability measure on  $\text{Aut}(\Lambda \backslash H)$ . Then*

$$\|U^0(\mu)\| \leq \|\lambda_\Gamma(\mu)\|^{1/4},$$

where  $\Gamma$  is the subgroup generated by the support of  $\mu$ . In particular, if  $\mu$  is aperiodic,  $\|U^0(\mu)\| < 1$  if and only if  $\Gamma$  is non-amenable.

**Acknowledgments** We are grateful to J.-P. Conze, A. Gamburd and Y. Guivarc'h for interesting discussions.

## 2 Proofs

We first recall the definition of the Heisenberg group  $H = H_{2n+1}(\mathbf{R})$ ; we then describe the automorphism group of  $H$  as well as its irreducible unitary representations.

Let  $n \geq 1$  be an integer. Consider the symplectic form  $\beta$  on  $\mathbf{R}^{2n}$  given by

$$\beta((x, y), (x', y')) = (x, y)^t J (x', y') \quad \text{for all } (x, y), (x', y') \in \mathbf{R}^{2n},$$

where  $J$  is the  $(2n \times 2n)$ -matrix

$$J = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$$

and  $I_n$  is the  $n \times n$ -identity matrix. The  $(2n + 1)$ -dimensional Heisenberg group (over  $\mathbf{R}$ ) is the group  $H = H_{2n+1}(\mathbf{R})$  with underlying set  $\mathbf{R}^{2n} \times \mathbf{R}$  and product

$$((x, y), s)((x', y'), t) = \left( (x + x', y + y'), s + t + \frac{1}{2}\beta((x, y), (x', y')) \right),$$

for  $(x, y), (x', y') \in \mathbf{R}^{2n}$ ,  $s, t \in \mathbf{R}$ . This is a two-step nilpotent Lie group. Its centre  $Z$  coincides with its commutator subgroup and is given by

$$Z = \{((0, 0), s) : s \in \mathbf{R}\}.$$

The symplectic group  $Sp_{2n}(\mathbf{R})$ , which is the subgroup of  $GL_{2n}(\mathbf{R})$  of all matrices  $g$  with  ${}^t g J g = J$ , acts by automorphisms on  $H = H_{2n+1}(\mathbf{R})$ :

$$g((x, y), t) = (g(x, y), t) \quad \text{for all } g \in Sp_{2n}(\mathbf{R}), (x, y) \in \mathbf{R}^{2n}, t \in \mathbf{R}.$$

As is well-known (see [Foll89, 1.22]), the automorphism group  $\text{Aut}(H)$  of  $H$  is generated by:

- the inner automorphisms,
- the automorphisms defined by matrices from  $Sp_{2n}(\mathbf{R})$  as above,
- the dilations  $((x, y), t) \mapsto ((rx, ry), r^2t)$  for  $r > 0$ , and
- the inversion  $i : ((x, y), t) \mapsto ((y, x), -t)$ .

The connected component  $\text{Aut}(H)_0$  of the identity in  $\text{Aut}(H)$  is a subgroup of index two and can be viewed as the group of  $(2n+1) \times (2n+1)$ -matrices of the form

$$\begin{pmatrix} rA & 0 \\ a^t & r^{2n} \end{pmatrix}$$

with  $A \in Sp_{2n}(\mathbf{R})$ ,  $r > 0$ , and  $a$  a column vector in  $\mathbf{R}^{2n}$  (the action on  $H$  corresponding to the usual action on  $\mathbf{R}^{2n+1}$ ). The subgroup of automorphisms of  $H$  fixing pointwise the centre can be identified with the group of matrices of the form

$$\begin{pmatrix} A & 0 \\ a^t & 1 \end{pmatrix}$$

and is hence isomorphic to the semi-direct product  $Sp_{2n}(\mathbf{R}) \ltimes \mathbf{R}^{2n}$ , for the standard action of  $Sp_{2n}(\mathbf{R})$  on  $\mathbf{R}^{2n}$ .

The unitary dual  $\widehat{H}$  of  $H$  (that is, the set of classes of irreducible unitary representations of  $H$  under unitary equivalence) consists of the equivalence classes of the following representations (see [Foll89, 1.50]):

- the unitary characters of the abelianized group  $H/Z$ ;
- for every  $t \in \mathbf{R} \setminus \{0\}$ , the infinite dimensional representation  $\pi_t$  defined on  $L^2(\mathbf{R}^n)$  by the formula

$$\pi_t((a, b), s)\xi(x) = \exp(2\pi its) \exp\left(2\pi\left\langle a, x - \frac{b}{2} \right\rangle\right) \xi(x - b)$$

for  $((a, b), s) \in H$ ,  $\xi \in L^2(\mathbf{R}^n)$ , and  $x \in \mathbf{R}^n$ .

For  $t \neq 0$ , the representation  $\pi_t$  is, up to unitary equivalence, the unique irreducible unitary representation of  $H$  whose restriction to the centre  $Z$  is a multiple of the unitary character  $s \mapsto \exp(2\pi its)$ .

The group  $\text{Aut}(H)$  acts on  $\widehat{H}$  by

$$\pi^g(h) = \pi(g^{-1}(h)) \quad \text{for all } \pi \in \widehat{H}, g \in \text{Aut}(H), h \in H.$$

Let  $g \in Sp_{2n}(\mathbf{R})$ . For  $t \in \mathbf{R} \setminus \{0\}$ , the representation  $\pi_t^g$  is unitary equivalent to  $\pi_t$ , since both representations have the same restriction to  $Z$ . Therefore, there exists a unitary operator  $\sigma(g)$  on  $L^2(\mathbf{R}^n)$  such that

$$\sigma(g)\pi_t(g^{-1}(h))\sigma(g)^{-1} = \pi_t(h) \quad \text{for all } h \in H.$$

By Schur's lemma,  $\sigma(g)$  is unique up to a scalar multiple of the identity operator. Hence, for  $g_1, g_2 \in Sp_{2n}(\mathbf{R})$ , there exists a complex number  $c(g_1, g_2)$  of modulus one such that  $\sigma(g_1)\sigma(g_2) = c(g_1, g_2)\sigma(g_1g_2)$ . This means that  $g \mapsto \sigma(g)$  is a projective unitary representation of  $Sp_{2n}(\mathbf{R})$ . We extend  $\sigma$  to a projective unitary representation  $\omega_t$ , called Weil's representation, of  $Sp_{2n}(\mathbf{R}) \ltimes \mathbf{R}^{2n}$  by setting

$$\omega_t(g, a) = \sigma(g)\pi_t(a) \quad \text{for all } (g, a) \in Sp_{2n}(\mathbf{R}) \ltimes \mathbf{R}^{2n}.$$

Although we will not need this fact, it is worth mentioning that  $\omega_t$  lifts to an ordinary representation of a two-fold cover of  $Sp_{2n}(\mathbf{R}) \ltimes \mathbf{R}^{2n}$  (see [Foll89, Chapter 4]).

How, let  $\Lambda$  be a lattice in  $H$ . (As an example,  $\Lambda$  can be the standard lattice  $\{((x, y), s/2) : x, y \in \mathbf{Z}^n, s \in \mathbf{Z}\}$ ; a full classification of the lattices in  $H$  is given in [Ausl77, I. 2].)

The Lebesgue measure on  $\mathbf{R}^{2n} \times \mathbf{R}$  is a Haar measure on  $H$  and induces an invariant measure  $m$  on the nilmanifold  $\Lambda \backslash H$ . (For the classification of  $\Gamma$ -invariant measures on  $\Lambda \backslash H$  for "large" groups  $\Gamma \subset \text{Aut}(\Lambda \backslash H)$ , see [Heu09].)

### Proof of Theorem 1.

Let  $\Gamma$  be a subgroup of  $\text{Aut}(\Lambda \backslash H)$ . Then  $\Gamma$  is a discrete subgroup of  $\text{Aut}(H)$ , for the topology of uniform convergence on compact subsets of  $H$ . Moreover, the subgroup of  $\Gamma$  consisting of the automorphisms fixing pointwise the centre of  $H$  has finite index in  $\Gamma$ . Indeed, the mapping

$$\text{Aut}(H)_0 \rightarrow \mathbf{R}^*, \quad \begin{pmatrix} rA & 0 \\ a^t & r^{2n} \end{pmatrix} \mapsto r$$

is a homomorphism and the image of  $\Gamma \cap \text{Aut}(H)_0$  is a discrete subgroup of  $\mathbf{R}^*$ .

It is clear that, if Theorem 1 is true for a subgroup of finite index in  $\Gamma$ , then it is true for  $\Gamma$ . So, we can (and will) assume that  $\Gamma$  is a subgroup of  $Sp_{2n}(\mathbf{R}) \ltimes \mathbf{R}^{2n}$ .

Since every  $\gamma \in \Gamma$  preserves the measure  $m$  on  $\Lambda \backslash H$ , we have an associated unitary representation  $U : \gamma \mapsto U_\gamma$  of  $\Gamma$  on  $L^2(\Lambda \backslash H, m)$ .

Let  $\rho_{\Lambda \backslash H}$  be the unitary representation of  $H$  on  $L^2(\Lambda \backslash H, m)$  given by right translation:

$$\rho_{\Lambda \backslash H}(h)\xi(x) = \xi(xh) \quad \text{for all } h \in H, \xi \in L^2(\Lambda \backslash H, m), x \in \Lambda \backslash H.$$



The representations  $U$  and  $\rho_{\Lambda \backslash H}$  are linked in the following way. For every  $\gamma \in \Gamma$ , we have:

$$(1) \quad U_\gamma \rho_{\Lambda \backslash H}(h) U_{\gamma^{-1}} = \rho_{\Lambda \backslash H}(\gamma(h)) \quad \text{for all } h \in H.$$

We have a decomposition of  $L^2(\Lambda \backslash H, m)$  into  $\rho_{\Lambda \backslash H}$ -invariant subspaces

$$L^2(\Lambda \backslash H, m) = \bigoplus_{m \in \mathbf{Z}} \mathcal{H}_m,$$

where

$$\mathcal{H}_m = \{\xi \in L^2(\Lambda \backslash H) : \rho_{\Lambda \backslash H}(0, 0, s)\xi = e^{2\pi i m s} \xi \quad \text{for all } t \in \mathbf{R}\}.$$

The space  $\mathcal{H}_0$  coincides with the space  $L^2(T)$ , where  $T \cong \mathbf{R}^{2n}/\mathbf{Z}^{2n}$  is the maximal torus factor associated to  $\Lambda \backslash H$ . Moreover, for every  $m \in \mathbf{Z} \setminus \{0\}$ , the subspace  $\mathcal{H}_m$  is an isotypical component for  $\rho_{\Lambda \backslash H}$  and is equivalent to a finite multiple of the irreducible representation  $\pi_m$  from above. (For a computation of the multiplicities, see [Toli78], [Moor65].)

Let  $m \in \mathbf{Z} \setminus \{0\}$ . Since  $\Gamma$  fixes pointwise  $Z$ , we see from (1) that

$$U_\gamma(\mathcal{H}_m) = \mathcal{H}_m \quad \text{for all } \gamma \in \Gamma.$$

Denote by  $U^{(m)}$  the restriction of  $U$  to  $\mathcal{H}_m$ .

Since  $\mathcal{H}_m$  is equivalent to a finite multiple of the irreducible representation  $\pi_m$ , we can assume that  $\mathcal{H}_m$  is the tensor product

$$\mathcal{H}_m = \mathcal{K}_m \otimes \mathcal{L}_m$$

of the Hilbert space  $\mathcal{K}_m$  of  $\pi_m$  with a finite dimensional Hilbert space  $\mathcal{L}_m$ , in such a way that

$$(2) \quad \rho_{\Lambda \backslash H}(h)|_{\mathcal{H}_m} = \pi_m(h) \otimes I_{\mathcal{L}_m} \quad \text{for all } h \in H.$$

Let  $\gamma \in \Gamma$ . By (1) and (2) above, we have

$$(3) \quad U_\gamma^{(m)}(\pi_m(h) \otimes I_{\mathcal{L}_m}) U_{\gamma^{-1}}^{(m)} = \pi_m(\gamma(h)) \otimes I_{\mathcal{L}_m} \quad \text{for all } x \in H.$$

Let  $\omega_m$  be Weil's representation from above. Recall that  $\omega_m$  is a projective representation of  $Sp_{2n}(\mathbf{R}) \ltimes \mathbf{R}^{2n}$  defined on  $\mathcal{K}_m = L^2(\mathbf{R}^n)$  and that  $\omega_m$  extends  $\pi_m$ . We have

$$(4) \quad \omega_m(\gamma) \pi_m(h) \omega_m(\gamma)^{-1} = \pi_m(\gamma(h)) \quad \text{for all } h \in H.$$

It follows from (3) and (4) that, on  $\mathcal{H}_m$ , the operator  $(\omega_m(\gamma)^{-1} \otimes I_{\mathcal{L}_m}) U_\gamma^{(m)}$  commutes with  $(\pi_m(h) \otimes I_{\mathcal{L}_m})$  for all  $h \in H$ . Hence, since  $\pi_m$  is irreducible, there exists a bounded operator  $V_\gamma^{(m)}$  on  $\mathcal{L}_m$  such that

$$(\omega_m(\gamma)^{-1} \otimes I_{\mathcal{L}_m}) U_\gamma^{(m)} = I_{\mathcal{K}_m} \otimes V_\gamma^{(m)},$$

that is,

$$(5) \quad U_\gamma^{(m)} = \omega_m(\gamma) \otimes V_\gamma^{(m)}.$$

Since  $U^{(m)}$  is a unitary representation, it is clear that  $\gamma \mapsto V_\gamma^{(m)}$  is a projective unitary representation of  $\Gamma$ .

Let  $\xi, \eta \in \mathcal{S}(\mathbf{R}^n)$  be Schwartz functions on  $\mathbf{R}^n$ . By [HoMo79, Proposition 6.4], for every  $\varepsilon > 0$ , the matrix coefficient

$$C_{\xi, \eta}^{\omega_m} : g \mapsto \langle \omega_m(g)\xi, \eta \rangle$$

of the metaplectic representation  $\omega_m$  belongs to  $L^{4n+2+\varepsilon}(Sp_{2n}(\mathbf{R}) \times \mathbf{R}^{2n})$ . Set  $G = Sp_{2n}(\mathbf{R}) \times \mathbf{R}^{2n}$ ; observe that  $\Gamma$  is a discrete and hence closed subgroup of  $G$ . Choosing a Borel subset  $X \subset G$  which is a fundamental domain for the quotient space  $\Gamma \backslash G$ , we can write (compare with the proof of Proposition 6.4 in [Howe82])

$$\begin{aligned} \int_G |C_{\xi, \eta}^{\omega_m}(g)|^{4n+2+\varepsilon} dg &= \int_G |\langle \omega_m(g)\xi, \eta \rangle|^{4n+2+\varepsilon} dg \\ &= \int_X \left( \sum_{\gamma \in \Gamma} |\langle \omega_m(\gamma g)\xi, \eta \rangle|^{4n+2+\varepsilon} \right) dg \\ &< \infty. \end{aligned}$$

Therefore, by Fubini's theorem, for almost every  $g \in X$ , we have

$$\sum_{\gamma \in \Gamma} |\langle \omega_m(\gamma)\omega_m(g)\xi, \eta \rangle|^{4n+2+\varepsilon} < \infty,$$

that is,  $C_{\omega_m(g)\xi, \eta}^{\omega_m|_\Gamma} \in \ell^{4n+2+\varepsilon}(\Gamma)$ .

Since  $\mathcal{S}(\mathbf{R}^n)$  contains a countable set which is dense in  $L^2(\mathbf{R}^n)$ , it follows that there exist dense subspaces  $D_1^{(m)}$  and  $D_2^{(m)}$  of  $L^2(\mathbf{R}^n)$  such that

$$C_{\xi, \eta}^{\omega_m|_\Gamma} \in \ell^{4n+2+\varepsilon}(\Gamma)$$

for all  $\xi \in D_1^{(m)}$  and  $\eta \in D_2^{(m)}$ .

Since  $U_\gamma^{(m)} = \omega_m(\gamma) \otimes V_\gamma^{(m)}$  and since matrix coefficients of projective unitary representations are bounded, the matrix coefficients  $C_{\xi \otimes \xi', \eta \otimes \eta'}^{U^{(m)}}$  of  $U^{(m)}$  belong to  $\ell^{4n+2+\varepsilon}(\Gamma)$  for  $\xi \in D_1^{(m)}, \eta \in D_2^{(m)}$  and  $\xi' \in \mathcal{L}_m, \eta' \in \mathcal{L}_m$ .

Let now  $D_1, D_2$  be the linear subspaces of  $\mathcal{H}$  generated respectively by

$$\{\xi \otimes \xi' : \xi \in D_1^{(m)}, \xi' \in \mathcal{L}_m, m \in \mathbf{Z} \setminus \{0\}\}$$

and

$$\{\eta \otimes \eta' : \eta \in D_2^{(m)}, \eta' \in \mathcal{L}_m, m \in \mathbf{Z} \setminus \{0\}\}.$$

Then  $D_1$  and  $D_2$  are dense in  $\mathcal{H}$  and the matrix coefficients  $C_{v,w}^U$  belong to  $\ell^{4n+2+\varepsilon}(\Gamma)$  for  $v \in D_1$  and  $w \in D_2$ . ■

### Proof of Theorem 3

Let  $\mu$  be a probability measure on  $\text{Aut}(\Lambda \setminus H)$ . Denote by  $\Gamma$  the subgroup of  $\text{Aut}(\Lambda \setminus H)$  generated by the support of  $\mu$ .

Let  $\lambda_\Gamma$  be the left regular representation of  $\Gamma$  on  $\ell^2(\Gamma)$ . Let  $U^0$  and  $V^0$  be the corresponding unitary representations of  $\Gamma$  on  $L_0^2(\Lambda \setminus H)$  and  $L_0^2(T)$ , respectively. We claim that

$$(6) \quad \|U^0(\mu)\| \leq \max\{\|V^0(\mu)\|, \|\lambda_\Gamma(\mu)\|^{1/k}\}$$

for  $k = 2n + 2$ .

Denoting by  $U^\mathcal{H}$  the restriction of  $U$  to  $\mathcal{H}$ , it suffices to show that

$$(7) \quad \|U^\mathcal{H}(\mu)\| \leq \|\lambda_\Gamma(\mu)\|^{1/k}.$$

By Theorem 1, the matrix coefficients  $C_{v,w}^{U^\mathcal{H}}$  of  $U^\mathcal{H}$  are in  $\ell^{2k}(\Gamma)$  for  $v$  and  $w$  in dense subspaces  $D_1$  and  $D_2$  of  $\mathcal{H}$ . It follows that the  $k$ -fold tensor power  $(U^\mathcal{H})^{\otimes k}$  of  $U^\mathcal{H}$  is unitarily equivalent to a subrepresentation of an infinite multiple  $\infty\lambda_\Gamma$  of  $\lambda_\Gamma$  (see [HoTa92, Chapter V, 1.2.4]). Hence,

$$\|(U^\mathcal{H})^{\otimes k}(\mu)\| \leq \|\infty\lambda_\Gamma(\mu)\| = \|\lambda_\Gamma(\mu)\|$$

and Inequality (7) will be proved if we show that

$$(8) \quad \|U^\mathcal{H}(\mu)\| \leq \|(U^\mathcal{H})^{\otimes k}(\mu)\|^{1/k}.$$

To show Inequality (8), we use the following argument from [Nevo98]. Denote by  $\check{\mu}$  the probability measure on  $\Gamma$  defined by  $\check{\mu}(\gamma) = \mu(\gamma^{-1})$ . For every vector  $v \in \mathcal{H}$ , using Jensen's inequality, we have

$$\begin{aligned}
\|U(\mu)v\|^{2k} &= |\langle U(\check{\mu} * \mu)v, v \rangle|^k \\
&= \left| \sum_{\gamma \in \Gamma} \langle U(\gamma)v, v \rangle (\check{\mu} * \mu)(\gamma) \right|^k \\
&\leq \sum_{\gamma \in \Gamma} |\langle U(\gamma)v, v \rangle|^k (\check{\mu} * \mu)(\gamma) \\
&= \sum_{\gamma \in \Gamma} \langle U(\gamma)^{\otimes k} v^{\otimes k}, v^{\otimes k} \rangle (\check{\mu} * \mu)(\gamma) \\
&= |\langle U^{\otimes k}(\check{\mu} * \mu)v^{\otimes k}, v^{\otimes k} \rangle| \\
&= \|U^{\otimes k}(\mu)v^{\otimes k}\|^2.
\end{aligned}$$

Hence,  $\|U^{\mathcal{H}}(\mu)\| \leq \| (U^{\mathcal{H}})^{\otimes k}(\mu) \|^{1/k}$ , as claimed.

Assume that  $\mu$  is aperiodic and that the  $\Gamma$  action on  $T$  has a spectral gap. Then, as mentioned in the Introduction,  $\Gamma$  is not amenable. Hence,  $\|\lambda_{\Gamma}(\mu)\| < 1$  (see [BeHV08, G.4.2]) and therefore  $\|U^0(\mu)\| < 1$ , that is, the  $\Gamma$  action on  $\Lambda \backslash H$  has a spectral gap. ■

### Proof of Corollary 5

Let  $H = H_3(\mathbf{R})$  be the 3-dimensional Heisenberg group and  $\Lambda$  a lattice in  $H$ . The unitary representation  $V$  of  $\text{Aut}(\Lambda \backslash H)$  on  $L^2(T)$  factors through  $p : \text{Aut}(\Lambda \backslash H) \rightarrow \text{Aut}(T) \cong GL_2(\mathbf{Z})$  to the standard representation of  $GL_2(\mathbf{Z})$  on  $L^2(T) = L^2(\mathbf{R}^2/\mathbf{Z}^2)$ . By Fourier duality, this last representation is unitarily equivalent to the representation of  $GL_2(\mathbf{Z})$  on  $\ell^2(\mathbf{Z}^2)$  obtained from the dual action of  $GL_2(\mathbf{Z})$  on  $\mathbf{Z}^2$ . We have an orthogonal decomposition into  $GL_2(\mathbf{Z})$ -invariant subspaces

$$\ell^2(\mathbf{Z}^2) = \mathbf{C}\delta_0 \bigoplus \bigoplus_{t \in T} \ell^2(GL_2(\mathbf{Z})/\Gamma_t),$$

where  $T$  is a set of representatives for the  $GL_2(\mathbf{Z})$ -orbits in  $\mathbf{Z}^2 \setminus \{0\}$  and  $\Gamma_t$  is the stabilizer of  $t$  in  $GL_2(\mathbf{Z})$ . Since every  $\Gamma_t$  is solvable (and hence amenable), it follows that  $V^0$  is weakly contained in  $\lambda_{GL_2(\mathbf{Z})} \circ p$  (see [BeHV08, Appendix F]). Hence, for every probability measure  $\mu$  on  $\text{Aut}(\Lambda \backslash H)$ , we have

$$\|V^0(\mu)\| \leq \|(\lambda_{GL_2(\mathbf{Z})} \circ p)(\mu)\|.$$

Since

$$\ker p \subset \left\{ \begin{pmatrix} I_2 & 0 \\ a^t & 1 \end{pmatrix} : a \in \mathbf{R}^2 \right\} \cong \mathbf{R}^2,$$

$\ker p$  is amenable and it follows that

$$\|(\lambda_{GL_2(\mathbf{Z})} \circ p)(\mu)\| \leq \|\lambda_{\text{Aut}(\Lambda \setminus H)}(\mu)\|.$$

Therefore, we have

$$\|V^0(\mu)\| \leq \|\lambda_{\text{Aut}(\Lambda \setminus H)}(\mu)\|.$$

Denote by  $\Gamma$  the subgroup generated by the support of  $\mu$ . Since

$$\|\lambda_\Gamma(\mu)\| = \|\lambda_{\text{Aut}(\Lambda \setminus H)}(\mu)\|,$$

it follows from Theorem 3 that

$$\|U^0(\mu)\| \leq \max\{\|V^0(\mu)\|, \|\lambda_\Gamma(\mu)\|^{1/4}\} = \|\lambda_\Gamma(\mu)\|^{1/4}.$$

Assume that  $\mu$  is aperiodic. If  $\Gamma$  is non-amenable, then  $\|\lambda_\Gamma(\mu)\| < 1$  and hence  $\|U^0(\mu)\| < 1$ . If  $\Gamma$  is amenable, then  $\Gamma$  has no spectral gap in  $\Lambda \setminus H$ . ■

### 3 Some further applications

Let  $G$  be a locally compact group acting by measure preserving transformations on a probability space  $(X, m)$ . Let  $U$  denote the associated unitary representation of  $G$  on  $L^2(X, m)$ . The action of  $G$  on  $X$  is weakly mixing if  $L^2_0(X, m)$  contains no non-zero finite dimensional  $U(G)$ -invariant subspace (equivalently: if the diagonal action of  $G$  on  $X \times X$  is ergodic; see [BeMa00, Chapter I, 2.17]). The action is strongly mixing if, for all  $\xi, \eta \in L^2_0(X, m)$ , the matrix coefficient  $g \mapsto \langle U_g \xi, \eta \rangle$  belongs to  $C_0(G)$ .

With the notation of Theorem 1, all matrix coefficients  $C_{v,w}^U$  are in  $c_0(\Gamma)$  for  $v \in D_1$  and  $w \in D_2$ . By density of  $D_1$  and  $D_2$  in  $\mathcal{H}$ , the same is true for all  $v, w \in \mathcal{H}$ . It follows that  $\mathcal{H}$  contains no non-zero finite dimensional  $U(G)$ -invariant subspace if  $\Gamma$  is infinite (see [BeMa00, Chapter I, 2.15.iii]). Therefore, we immediately obtain the following corollary.

**Corollary 6** *Let  $\Gamma$  be a group of automorphisms of the compact Heisenberg nilmanifold  $\Lambda \setminus H$  and  $T$  the maximal  $T$  torus factor associated to  $\Lambda \setminus H$ . The following properties are equivalent.*

(i) *The action of  $\Gamma$  on  $\Lambda \backslash H$  is ergodic (weakly mixing or strongly mixing, respectively).*

(ii) *The action of  $\Gamma$  on  $T$  is ergodic (weakly mixing or strongly mixing, respectively).*

**Remark 7** In the case where  $\Gamma$  is generated by a single automorphism (or even an affine transformation) of an arbitrary compact nilmanifold, the previous corollary was obtained by W. Parry (see [Parr69], [Parr70]). The result concerning ergodicity has been generalized by J.-P. Conze ([Conz09]) to arbitrary groups of affine transformations of a general compact nilmanifold  $\Lambda \backslash H$ . Moreover, [Conz09] gives an example of an ergodic group  $\Gamma$  of automorphisms of the standard 7-dimensional Heisenberg nilmanifold  $\Lambda \backslash H_7(\mathbf{R})$  such that no element  $\gamma \in \Gamma$  acts ergodically on  $\Lambda \backslash H_7(\mathbf{R})$ .

Let  $G$  be a locally compact group acting by measure preserving transformations on a probability space  $(X, m)$  and  $U$  the associated unitary representation of  $G$  on  $L^2(X, m)$ . Assume that  $\|(U^0 \otimes U^0)(\mu)\| < 1$ . This condition, which is formally stronger than the spectral gap condition  $\|U^0(\mu)\| < 1$ , plays an important role in [FuSh99]. Indeed, it is shown in Theorem 1.4 there that, with the notation as in the Introduction, for every  $\xi, \eta \in L^2(X, m)$ , the correlation coefficient  $\langle U(S_n^\omega)\xi, \eta \rangle$  converge almost surely to  $\int_X \xi dm \int_X \eta dm$ , with exponentially fast speed.

However, we can see that both conditions are equivalent in our situation.

**Corollary 8** *With the notation as in Theorem 3, the following properties are equivalent.*

(i)  $\|(U^0 \otimes U^0)(\mu)\| < 1$ ;

(ii)  $\|U^0(\mu)\| < 1$ ;

(iii)  $\|V^0(\mu)\| < 1$ .

Indeed, a proof similar to the one of Theorem 3 shows that the following inequality holds:

$$\|(U^0 \otimes U^0)(\mu)\| \leq \max\{\|(V^0 \otimes V^0)(\mu)\|, \|\lambda_\Gamma(\mu)\|^{1/k}\}.$$

On the other hand, as was shown in [FuSh99, Theorem 6.4], the condition  $\|V^0(\mu)\| < 1$  is equivalent to the condition  $\|(V^0 \otimes V^0)(\mu)\| < 1$ . This shows the equivalence of Conditions (i), (ii), and (iii).

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