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► **To cite this version:**

Jean-Baptiste Caillau. Convexity of injectivity domains on the ellipsoid of revolution: The oblate case (addendum). 2011. hal-00564457

HAL Id: hal-00564457

<https://hal.science/hal-00564457v1>

Submitted on 9 Feb 2011

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Convexity of injectivity domains on the ellipsoid of revolution: The oblate case (addendum)

With the same notations as [1], convexity holds whenever ($' = \partial/\partial p_\theta$)

$$T(T + p_\theta T') + (X_0 - p_\theta^2)(2T'^2 - TT'') \geq 0, \quad p_\theta \in [0, \sqrt{X_0}].$$

The period $T(p_\theta, \lambda)$ of the φ coordinate is computed using the quadrature in the form of the algebraic curve ($X = \sin^2 \varphi$)

$$\left[\frac{\dot{X}(\lambda - X)}{\sqrt{\lambda}} \right]^2 = 4(X - p_\theta^2)(X - 1)(X - \lambda).$$

Setting $y = 1 - p_\theta^2$ and $x = \lambda - 1$, the invariants are

$$g_2(x, y) = \frac{4}{3}(x^2 + xy + y^2), \quad g_3(x, y) = \frac{4}{27}(2x^3 + 3x^2y - 3xy^2 - 2y^3).$$

The period is $T = 4\tau/(3\sqrt{x+1})$ with

$$\tau = (2x + y)\omega + 3\eta$$

where ω is the real half-period of the Weierstraß function associated with (g_2, g_3) , and $\eta = \zeta(\omega)$. Differentiation with respect to x is obtained through the following rules,

$$\delta_x \frac{\partial \omega}{\partial x} = -A_x \omega - B_x \eta, \quad \delta_x \frac{\partial \eta}{\partial x} = C_x \omega + A_x \eta,$$

where

$$\delta_x = 18x(x + y), \quad A_x = 3(2x + y), \quad B_x = 9, \quad C_x = x^2 + xy + y^2.$$

Symmetrically,

$$\delta_y \frac{\partial \omega}{\partial y} = -A_y \omega - B_y \eta, \quad \delta_y \frac{\partial \eta}{\partial y} = C_y \omega + A_y \eta,$$

where

$$\delta_y = 18y(x + y), \quad A_y = 3(x + 2y), \quad B_y = -9, \quad C_y = -(x^2 + xy + y^2).$$

Proposition 1. *The first and second order derivatives of τ with respect to (positive) p_θ are*

$$\tau' = -\frac{\sqrt{1-y}}{y} [-(x-y)\omega + 3\eta],$$

$$\tau'' = -\frac{1}{y^2(x+y)} \{[-2x^2 + x(x-2)y + (2x+1)y^2]\omega + 3[2x - (x-1)y]\eta\}.$$

Define

$$\alpha(x, y) = \frac{1}{y^2} \left[\chi(x, y) - \frac{x}{3} - \frac{y}{6} \right], \quad \chi = \frac{\eta}{\omega}.$$

So as to estimate the curvature sign, one essentially needs to compute directional limits of α at the two degeneracies $(x, y) = (0, 0)$ and $(\infty, 0)$.

Lemma 1. For positive x and y , $\alpha > -\frac{1}{2y}$.

Proof. It is geometrically clear that the period T (hence τ) must be strictly decreasing with $p_\theta > 0$ on an ellipsoid of revolution with prescribed oblateness (x is fixed). Then, according to Proposition 1, $-(x - y) + 3\chi > 0$, hence the result on α . \square

Remark 1. When $x \rightarrow 0$ (flat ellipsoid), χ degenerates to the rational value $\lim_{x=0} 3g_3/(2g_2) = -y/3$ so one gets $\alpha(0, y) = -1/(2y)$ for positive y .

Lemma 2. For positive x , $\alpha(x, 0) = -\frac{1}{16x}$.

Proof. When $y \rightarrow 0$ (equator), χ degenerates to $\lim_{y=0} 3g_3/(2g_2) = x/3$. The differentiation rules imply that

$$\delta_y \frac{\partial \chi}{\partial y} = C_y + 2A_y \chi + B_y \chi^2$$

so, iterating, one obtains

$$\frac{\partial \chi}{\partial y}(x, 0) = \frac{1}{6}, \quad \frac{\partial^2 \chi}{\partial y^2}(x, 0) = -\frac{1}{8x},$$

whence the directional limit for α (order two Taylor-Young). \square

One can then devise a global coarse estimate of $x\alpha$, for instance the following.

Corollary 1. For positive x and y , $x\alpha > -1/15$.

Remark 2. One actually has $x\alpha > -1/16$ for positive x and y .

Lemma 3. For positive y , $(x\alpha)(\infty, y) = -1/16$.

Proof. Set $\xi = 1/x$. When $\xi \rightarrow 0$ (round case¹), $\xi\chi$ degenerates to the limit at $\xi = 0$ of

$$\xi \frac{3g_3}{2g_2}(1/\xi, y) = \frac{2 + 3y\xi - 3y^2\xi^2 - 2y^3\xi^3}{6(1 + y\xi + y^2\xi^2)}$$

so $(\xi\chi)(0, y) = 1/3$. Computing as in Lemma 2, one obtains

$$\frac{\partial(\xi\chi)}{\partial \xi}(0, y) = \frac{y}{6}, \quad \frac{\partial^2(\xi\chi)}{\partial \xi^2}(0, y) = -\frac{y^2}{8},$$

whence the directional limit for

$$\frac{\alpha}{\xi} = \frac{1}{\xi^2 y^2} \left[\xi\chi - \frac{1}{3} - \frac{y}{6}\xi \right].$$

\square

A global coarse estimate of $(x + 1)\alpha$ is for instance as follows.

Corollary 2. For positive x and y , $(x + 1)\alpha < -1/17 < 0$.

¹The degeneracy $x \rightarrow \infty$ towards the round case is interpreted as follows: All geodesics tend to meridians, so the limit has to be independent of $y = 1 - p_\theta^2$, and the computation of $\alpha(x, 0)$ in Lemma 2 for the equator already gives the result.

Remark 3. One actually has $(x+1)\alpha < -1/16$ for positive x and y .

Proposition 2. *When $x \in (0, 1/2)$, the curvature for $\varphi_0 = \pi/2$ changes sign.*

Proof. For $X_0 = \sin^2 \varphi_0 = 1$, up to some positive factor the curvature reads

$$\kappa = \tau(\tau + p_\theta \tau') + y(2\tau'^2 - \tau\tau'') \geq 0.$$

As

$$\tau + p_\theta \tau' = 3\omega[(x - \frac{1}{2}) + (1 - \alpha)y + 2\alpha y^2],$$

we see using Lemma 2 that this term has a negative limit as $y \rightarrow 0$ since

$$\lim_{y=0} \omega = \lim_{y=0} \frac{\pi}{3} \sqrt{\frac{g_2}{2g_3}} = \frac{\pi}{2\sqrt{x}} > 0.$$

Moreover,

$$\tau' = -\frac{3\omega}{2} \sqrt{1-y}(1+2\alpha y)$$

and

$$\tau'' = -\frac{3\omega}{2(x+y)} [1 + (1+4\alpha)x + 2\alpha(1-x)y],$$

are both well defined for $y = 0$ so κ has the sign of $x - 1/2$ and is negative. Conversely, when $y = 1$, τ' vanishes and $\kappa = \tau(\tau - \tau'')$ with

$$\tau - \tau''|_{y=1} = \omega[(x+2) + 6\chi] > 3\omega x > 0$$

by virtue of Lemma 1. Hence the change of sign. \square

Proposition 3. *When $x \geq 1/2$, $\tau'' \leq 0$.*

Proof. Write as in the previous proof

$$\tau'' = -\frac{3\omega}{2(x+y)} [1 + (1+4\alpha)x + 2\alpha(1-x)y],$$

and notice that, using Corollary 1, the term in the brackets is bounded from below according to

$$(1+x) + 2\alpha x \underbrace{[2 + y(\frac{1}{x} - 1)]}_{\geq 0} > (1+x) - \frac{2}{15} [2 + y(\frac{1}{x} - 1)] \geq \frac{11}{10}$$

for $x \geq 1/2$. \square

Proposition 4. *When $x \geq 1/2$, $\tau + p_\theta \tau' \leq 0$.*

Proof. Write as in the proof of Proposition 2

$$\tau + p_\theta \tau' = 3\omega[(x - \frac{1}{2}) + (1 - \alpha)y + 2\alpha y^2],$$

and notice that, using successively Lemma 1 and Corollary 2,

$$(1 - \alpha) + 2\alpha y \geq -\alpha > 0.$$

\square

Theorem 1. *Injectivity domains on the oblate ellipsoid are all convex if and only if the ratio between the minor and major axes is greater or equal to $1/\sqrt{3}$.*

Proof. When the ratio is less than $1/\sqrt{3}$, that is when $x < 1/2$, Proposition 2 shows that convexity does not hold for $\varphi_0 = \pi/2$. Conversely, when $x \geq 1/2$, as $\tau'' \leq 0$ according to Proposition 3, nonnegativeness of

$$\tau(\tau + p_\theta \tau') + (X_0 - p_\theta^2)(2\tau'^2 - \tau\tau'')$$

holds as soon as $\tau + p_\theta \tau' \geq 0$, which is Proposition 4. □

References

- [1] Bonnard, B.; Caillaud, J.-B.; Rifford, L. Convexity of injectivity domains on the ellipsoid of revolution: The oblate case. *C. R. Acad. Sci. Paris, Ser. I* **348** (2010), 1315–1318.