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# Convexity of injectivity domains on the ellipsoid of revolution: The oblate case (addendum)

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**Convexity of injectivity domains on the ellipsoid of revolution: The oblate case (addendum)**

With the same notations as [1], convexity holds whenever ( $' = \partial/\partial p_\theta$ )

$$T(T + p_\theta T') + (X_0 - p_\theta^2)(2T'^2 - TT'') \geq 0, \quad p_\theta \in [0, \sqrt{X_0}].$$

The period  $T(p_\theta, \lambda)$  of the  $\varphi$  coordinate is computed using the quadrature in the form of the algebraic curve ( $X = \sin^2 \varphi$ )

$$\left[ \frac{\dot{X}(\lambda - X)}{\sqrt{\lambda}} \right]^2 = 4(X - p_\theta^2)(X - 1)(X - \lambda).$$

Setting  $y = 1 - p_\theta^2$  and  $x = \lambda - 1$ , the invariants are

$$g_2(x, y) = \frac{4}{3}(x^2 + xy + y^2), \quad g_3(x, y) = \frac{4}{27}(2x^3 + 3x^2y - 3xy^2 - 2y^3).$$

The period is  $T = 4\tau/(3\sqrt{x+1})$  with

$$\tau = (2x + y)\omega + 3\eta$$

where  $\omega$  is the real half-period of the Weierstraß function associated with  $(g_2, g_3)$ , and  $\eta = \zeta(\omega)$ . Differentiation with respect to  $x$  is obtained through the following rules,

$$\delta_x \frac{\partial \omega}{\partial x} = -A_x \omega - B_x \eta, \quad \delta_x \frac{\partial \eta}{\partial x} = C_x \omega + A_x \eta,$$

where

$$\delta_x = 18x(x + y), \quad A_x = 3(2x + y), \quad B_x = 9, \quad C_x = x^2 + xy + y^2.$$

Symmetrically,

$$\delta_y \frac{\partial \omega}{\partial y} = -A_y \omega - B_y \eta, \quad \delta_y \frac{\partial \eta}{\partial y} = C_y \omega + A_y \eta,$$

where

$$\delta_y = 18y(x + y), \quad A_y = 3(x + 2y), \quad B_y = -9, \quad C_y = -(x^2 + xy + y^2).$$

**Proposition 1.** *The first and second order derivatives of  $\tau$  with respect to (positive)  $p_\theta$  are*

$$\tau' = -\frac{\sqrt{1-y}}{y} [-(x-y)\omega + 3\eta],$$

$$\tau'' = -\frac{1}{y^2(x+y)} \{[-2x^2 + x(x-2)y + (2x+1)y^2]\omega + 3[2x - (x-1)y]\eta\}.$$

Define

$$\alpha(x, y) = \frac{1}{y^2} \left[ \chi(x, y) - \frac{x}{3} - \frac{y}{6} \right], \quad \chi = \frac{\eta}{\omega}.$$

So as to estimate the curvature sign, one essentially needs to compute directional limits of  $\alpha$  at the two degeneracies  $(x, y) = (0, 0)$  and  $(\infty, 0)$ .

**Lemma 1.** For positive  $x$  and  $y$ ,  $\alpha > -\frac{1}{2y}$ .

*Proof.* It is geometrically clear that the period  $T$  (hence  $\tau$ ) must be strictly decreasing with  $p_\theta > 0$  on an ellipsoid of revolution with prescribed oblateness ( $x$  is fixed). Then, according to Proposition 1,  $-(x - y) + 3\chi > 0$ , hence the result on  $\alpha$ .  $\square$

*Remark 1.* When  $x \rightarrow 0$  (flat ellipsoid),  $\chi$  degenerates to the rational value  $\lim_{x=0} 3g_3/(2g_2) = -y/3$  so one gets  $\alpha(0, y) = -1/(2y)$  for positive  $y$ .

**Lemma 2.** For positive  $x$ ,  $\alpha(x, 0) = -\frac{1}{16x}$ .

*Proof.* When  $y \rightarrow 0$  (equator),  $\chi$  degenerates to  $\lim_{y=0} 3g_3/(2g_2) = x/3$ . The differentiation rules imply that

$$\delta_y \frac{\partial \chi}{\partial y} = C_y + 2A_y \chi + B_y \chi^2$$

so, iterating, one obtains

$$\frac{\partial \chi}{\partial y}(x, 0) = \frac{1}{6}, \quad \frac{\partial^2 \chi}{\partial y^2}(x, 0) = -\frac{1}{8x},$$

whence the directional limit for  $\alpha$  (order two Taylor-Young).  $\square$

One can then devise a global coarse estimate of  $x\alpha$ , for instance the following.

**Corollary 1.** For positive  $x$  and  $y$ ,  $x\alpha > -1/15$ .

*Remark 2.* One actually has  $x\alpha > -1/16$  for positive  $x$  and  $y$ .

**Lemma 3.** For positive  $y$ ,  $(x\alpha)(\infty, y) = -1/16$ .

*Proof.* Set  $\xi = 1/x$ . When  $\xi \rightarrow 0$  (round case<sup>1</sup>),  $\xi\chi$  degenerates to the limit at  $\xi = 0$  of

$$\xi \frac{3g_3}{2g_2}(1/\xi, y) = \frac{2 + 3y\xi - 3y^2\xi^2 - 2y^3\xi^3}{6(1 + y\xi + y^2\xi^2)}$$

so  $(\xi\chi)(0, y) = 1/3$ . Computing as in Lemma 2, one obtains

$$\frac{\partial(\xi\chi)}{\partial \xi}(0, y) = \frac{y}{6}, \quad \frac{\partial^2(\xi\chi)}{\partial \xi^2}(0, y) = -\frac{y^2}{8},$$

whence the directional limit for

$$\frac{\alpha}{\xi} = \frac{1}{\xi^2 y^2} \left[ \xi\chi - \frac{1}{3} - \frac{y}{6}\xi \right].$$

$\square$

A global coarse estimate of  $(x + 1)\alpha$  is for instance as follows.

**Corollary 2.** For positive  $x$  and  $y$ ,  $(x + 1)\alpha < -1/17 < 0$ .

<sup>1</sup>The degeneracy  $x \rightarrow \infty$  towards the round case is interpreted as follows: All geodesics tend to meridians, so the limit has to be independent of  $y = 1 - p_\theta^2$ , and the computation of  $\alpha(x, 0)$  in Lemma 2 for the equator already gives the result.

*Remark 3.* One actually has  $(x+1)\alpha < -1/16$  for positive  $x$  and  $y$ .

**Proposition 2.** *When  $x \in (0, 1/2)$ , the curvature for  $\varphi_0 = \pi/2$  changes sign.*

*Proof.* For  $X_0 = \sin^2 \varphi_0 = 1$ , up to some positive factor the curvature reads

$$\kappa = \tau(\tau + p_\theta \tau') + y(2\tau'^2 - \tau\tau'') \geq 0.$$

As

$$\tau + p_\theta \tau' = 3\omega[(x - \frac{1}{2}) + (1 - \alpha)y + 2\alpha y^2],$$

we see using Lemma 2 that this term has a negative limit as  $y \rightarrow 0$  since

$$\lim_{y=0} \omega = \lim_{y=0} \frac{\pi}{3} \sqrt{\frac{g_2}{2g_3}} = \frac{\pi}{2\sqrt{x}} > 0.$$

Moreover,

$$\tau' = -\frac{3\omega}{2} \sqrt{1-y}(1+2\alpha y)$$

and

$$\tau'' = -\frac{3\omega}{2(x+y)} [1 + (1+4\alpha)x + 2\alpha(1-x)y],$$

are both well defined for  $y = 0$  so  $\kappa$  has the sign of  $x - 1/2$  and is negative. Conversely, when  $y = 1$ ,  $\tau'$  vanishes and  $\kappa = \tau(\tau - \tau'')$  with

$$\tau - \tau''|_{y=1} = \omega[(x+2) + 6\chi] > 3\omega x > 0$$

by virtue of Lemma 1. Hence the change of sign.  $\square$

**Proposition 3.** *When  $x \geq 1/2$ ,  $\tau'' \leq 0$ .*

*Proof.* Write as in the previous proof

$$\tau'' = -\frac{3\omega}{2(x+y)} [1 + (1+4\alpha)x + 2\alpha(1-x)y],$$

and notice that, using Corollary 1, the term in the brackets is bounded from below according to

$$(1+x) + 2\alpha x \underbrace{[2 + y(\frac{1}{x} - 1)]}_{\geq 0} > (1+x) - \frac{2}{15} [2 + y(\frac{1}{x} - 1)] \geq \frac{11}{10}$$

for  $x \geq 1/2$ .  $\square$

**Proposition 4.** *When  $x \geq 1/2$ ,  $\tau + p_\theta \tau' \leq 0$ .*

*Proof.* Write as in the proof of Proposition 2

$$\tau + p_\theta \tau' = 3\omega[(x - \frac{1}{2}) + (1 - \alpha)y + 2\alpha y^2],$$

and notice that, using successively Lemma 1 and Corollary 2,

$$(1 - \alpha) + 2\alpha y \geq -\alpha > 0.$$

$\square$

**Theorem 1.** *Injectivity domains on the oblate ellipsoid are all convex if and only if the ratio between the minor and major axes is greater or equal to  $1/\sqrt{3}$ .*

*Proof.* When the ratio is less than  $1/\sqrt{3}$ , that is when  $x < 1/2$ , Proposition 2 shows that convexity does not hold for  $\varphi_0 = \pi/2$ . Conversely, when  $x \geq 1/2$ , as  $\tau'' \leq 0$  according to Proposition 3, nonnegativeness of

$$\tau(\tau + p_\theta\tau') + (X_0 - p_\theta^2)(2\tau'^2 - \tau\tau'')$$

holds as soon as  $\tau + p_\theta\tau' \geq 0$ , which is Proposition 4. □

## References

- [1] Bonnard, B.; Caillaud, J.-B.; Rifford, L. Convexity of injectivity domains on the ellipsoid of revolution: The oblate case. *C. R. Acad. Sci. Paris, Ser. I* **348** (2010), 1315–1318.