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# Generalized companion matrix for approximate GCD

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We study a variant of the univariate approximate GCD problem, where the coefficients of one polynomial  $f(x)$  are known exactly, whereas the coefficients of the second polynomial  $g(x)$  may be perturbed. Our approach relies on the properties of the matrix which describes the operator of multiplication by  $g$  in the quotient ring  $\mathbb{C}[x]/(f)$ . In particular, the structure of the null space of the multiplication matrix contains all the essential information about  $\text{GCD}(f, g)$ . Moreover, the multiplication matrix exhibits a displacement structure that allows us to design a fast algorithm for approximate GCD computation with quadratic complexity w.r.t. polynomial degrees.

## 1 Introduction

The approximate polynomial greatest common divisor (denoted as AGCD) is a central object of symbolic-numeric computation. The main difficulty of the problem comes from the fact that there is no universal notion of AGCD. One can find different approaches and different notions for AGCD. We will not give a review of all the existing work on this subject, but we will recall one of the most popular approaches to show how our work brings a different point of view on the problem.

The main approach to the computation of an AGCD consists in considering two univariate polynomials whose coefficients are known with uncertainty. This uncertainty can be the result of the fact that the polynomials have floating point coefficients coming from previous computation (and so are subject to round-off errors). The most frequently adopted formulation is related to semi-algebraic optimization : given  $\tilde{f}$  and  $\tilde{g}$  two approximate polynomials, find two polynomials  $f$  and  $g$  such that  $\|\tilde{f} - f\|$  and  $\|\tilde{g} - g\|$  are small (lower than a given tolerance for instance) and such that the degree of  $\text{gcd}(f, g)$  is maximal. That is, one looks

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for the most singular system close to the input  $(\tilde{f}, \tilde{g})$ . An  $\varepsilon$ -gcd is obtained if the conditions  $\|\tilde{f} - f\| < \varepsilon$  and  $\|\tilde{g} - g\| < \varepsilon$  are satisfied. One can try to compute the tolerance on the perturbation of the input polynomial thanks to direct computation (for instance from a jump on singular values of particular matrix for instance). This last approach has received a great interest following the work of Zeng using Sylvester like matrices ([14]).

Here, we consider a slightly different problem. One of the polynomials, say  $f$ , is known exactly (it is the result of an exact model) and the second one, say  $g$ , is an approximate polynomial (result of measures or previous approximation for instance). This case occurs in applications such as model checking (to compare results of an exact model and measures). There are many other instances of such a problem, such as simplification of fractions when one of the polynomial is known exactly but the other one is not.

We give an example of such a situation. When modeling an electromagnetic filter, one might want to parametrize its behavior with respect to the frequency. But one may need to do so even if there are singularities and to do so one may use Padé approximations of the electromagnetic signal at each point as a function of the frequency. In some cases of interest, one can know all the singularities and so compute an exact polynomial called characteristic. Padé approximations are computed independently for each point by a numerical process and denominators may have a non trivial gcd with the “characteristic” polynomial. The denominators are not known exactly. So, in order to identify unwanted common factors in denominators one has to compute approximate gcds between an exact and non exact polynomials.

This AGCD problem can also be interpreted as an optimization problem. Given  $f$  exactly and  $\tilde{g}$  approximately, compute a polynomial  $g$  close to  $\tilde{g}$  such that  $g$  has a maximal degree gcd with  $f$ . Our approach takes advantage of the asymmetry of the problem and of the structure of the quotient algebra  $\mathbb{C}[x]/(f(x))$  (more accurately, of the displacement rank of the multiplication operator in this algebra). So, we address the following problem :

**Problem 1** Let  $f(x) \in \mathbb{C}[x]$  a given polynomial and  $g(x)$  another polynomial. Find  $\tilde{g}(x)$  close to  $g(x)$  (in a sense that will be explained) such that  $f(x)$  and  $\tilde{g}(x)$  have a gcd of maximal degree.

This may be also an interesting approach when one has two polynomials, one known with high confidence and another with worse accuracy. This approach may take advantage of this asymmetry which would not be possible for classical framework based on Sylvester or Bézout matrices.

In this paper, we propose an approach and an algorithm to address this problem. The proposed algorithm is “fast” since the exponent of its complexity is better than the classical linear algebra exponent in the degree of the input polynomials.

Organisation of the paper: The second section is devoted to some basic result on algebra needed after, the third section gives an algebraic method for gcd based on linear algebra, the fourth section recalls the Barnett formula allowing

to compute the multiplication matrix without division, the fifth gives the displacement rank structure of the multiplication matrix, the sixth describes the final algorithm and experiments before finishing with conclusions and perspectives.

## 2 Euclidian structure and quotient algebra

In this section, we recall basic algebraic results allowing to understand the principle of our approach. All material in this section can be found (even in the non reduced case and in the multivariate setting) in [11].

Assume that  $\mathbb{K}$  is an algebraically closed field (here we think about  $\mathbb{C}$ ).

Let  $f(x)$  and  $g(x) \in \mathbb{K}[x]$  and assume that  $f(x) = f_d * \prod_{i=1}^d (x - \zeta_i)$  and that  $\zeta_i \neq \zeta_j$  for all  $i \neq j$  in  $\{1, \dots, d\}$ . Let  $\mathbb{A} = \mathbb{K}[x]/(f)$  and  $\pi : \mathbb{K}[x] \rightarrow \mathbb{A}$  be the natural projection. For  $i \in \{1, \dots, d\}$ , we define  $L_i(x) = \frac{\prod_{j \neq i} (x - \zeta_j)}{\prod_{j \neq i} (\zeta_i - \zeta_j)}$ , the  $i^{\text{th}}$

Lagrange polynomial associated to  $\{\zeta_1, \dots, \zeta_d\}$ . Clearly, since  $\deg(L_i) < \deg(f)$  we have  $\pi(L_i) = L_i$ , for all  $i \in \{1, \dots, d\}$ . Let  $\mathbb{A}^* = \text{Hom}_{\mathbb{K}}(\mathbb{A}, \mathbb{K})$  be the usual dual space of  $\mathbb{A}$ . For all  $i \in \{1, \dots, d\}$ , we define  $\mathbf{1}_{\zeta_i} : \mathbb{A} \rightarrow \mathbb{K}$  by  $\mathbf{1}_{\zeta_i}(p) = p(\zeta_i)$  for all  $p \in \mathbb{A}$ . The following lemma is obvious from the definition of the

polynomials  $L_i$  that for  $i$  and  $j \in \{1, \dots, d\}$ , we have  $L_i(\zeta_j) = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{else} \end{cases}$ .

This implies that the set  $\{L_1, \dots, L_d\}$  is a basis of  $\mathbb{A}$ . A well known fact is that the set  $\{\mathbf{1}_{\zeta_1}, \dots, \mathbf{1}_{\zeta_d}\}$  form a basis  $\mathbb{A}^*$  dual of the basis  $\{L_1, \dots, L_d\}$  of  $\mathbb{A}$ . As a corollary, we have the Lagrange interpolation formula : Each  $p \in \mathbb{A}$  can

be written  $p(x) = \sum_{i=1}^d \mathbf{1}_{\zeta_i}(p) * L_i(x)$ . A funny consequence is that if we choose  $\{L_1, \dots, L_d\}$  as a basis of  $\mathbb{A}$ , for all  $g \in \mathbb{K}[x]$ , the remainder  $\pi(g)$  of the euclidian division of  $g$  by  $f$  is given by  $(g(\zeta_1), \dots, g(\zeta_d))$  in the basis  $\{L_1, \dots, L_d\}$ , i.e.  $r = \sum_{i=1}^d g(\zeta_i) L_i(x)$ . In other word, divide  $g$  by  $f$  is equivalent to evaluate  $g$  at the roots of  $f$ .

The general philosophy of this last proposition will allows us to make a lot of proof in a very simple way. For example, it is very easy to see the different operation in  $\mathbb{A}$  using this representation. Let  $g$  and  $h$  be to elements in  $\mathbb{A}$ , then

we have  $g + h = \sum_{i=1}^d (g(\zeta_i) + h(\zeta_i)) * L_i(x)$  and  $g * h = \sum_{i=1}^d (g(\zeta_i) * h(\zeta_i)) L_i(x)$  in  $\mathbb{A}$ .

This allows us to avoid the use of the section  $\sigma$ . In fact, the Lagrange polynomials  $L_1, \dots, L_d$  reveal a deeper structure on the algebra  $\mathbb{A}$  : The polynomials

$L_1, \dots, L_d$  are the idempotents of  $\mathbb{A}$ , i.e.  $L_i * L_j = \begin{cases} L_i & \text{if } i = j \\ 0 & \text{else} \end{cases}$ .

Thanks to this description of the quotient algebra, it is easy to derive algorithms for both polynomial solving and gcd computation even though the problems are of very different nature.

Remark that we have expressed everything in the monomial basis since it is the most widely used basis to express polynomials but we could use other bases. A particular basis is the Chebyshev basis where all results are exactly the same since it is a graduated basis.

### 3 An algebraic algorithm for gcd computation

To first give an idea on how to exploit the section above in order to design algorithm for gcd, We recall a classical method for polynomial solving (see [4] for instance). Proofs are given for the sake of completeness and because very similar ideas will lead us to the AGCD computation.

#### 3.1 Roots via eigenvalues

Let  $f(x) = \sum_{i=0}^d f_i x^i \in \mathbb{C}[x]$  be a polynomial of degree  $d$ . Then we consider the matrix of the multiplication by  $x$  in  $\mathbb{C}[x]/(f)$ . Its matrix in the monomial basis  $1, \dots, x^{d-1}$  is the following:

$$\text{Frob}(f) = \begin{pmatrix} 1 \\ x \\ x^2 \\ \vdots \\ x^{d-1} \end{pmatrix} \begin{pmatrix} (1 & x & x^2 & \dots & x^{d-1}) \\ 0 & 0 & 0 & \dots & -\frac{f_0}{f_d} \\ 1 & 0 & 0 & \dots & -\frac{f_1}{f_d} \\ 0 & 1 & 0 & \dots & -\frac{f_2}{f_d} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & -\frac{f_{d-1}}{f_d} \end{pmatrix}$$

well known as the Frobenius companion matrix associated to  $f$ .

**Proposition 1** *Let  $f(x) \in \mathbb{C}[x]$  be polynomial of degree  $d$  with  $d$  distinct roots  $\mathcal{Z}(f) = \{z_1, \dots, z_d\}$ , then the eigenvalues of  $\text{Frob}(f)$  are the roots of  $f(x)$ , i.e.  $\text{Spec}(\text{Frob}(f)) = \{z_1, \dots, z_d\}$ .*

**Proof** It follows directly from the fact that  $\text{Frob}(f)$  is the matrix of the multiplication by  $x$  in  $\mathbb{C}[x]/(f)$ . But here we propose to give a direct proof by induction. In fact, we prove by induction that the characteristic polynomial of  $\text{Frob}(f)$  is  $f(x)$  itself (up to a sign and a scalar factor  $1/f_d$ ), i.e.:

$$\begin{vmatrix} -x & 1 & 0 & \dots & -\frac{f_0}{f_d} \\ 0 & -x & 1 & \dots & -\frac{f_1}{f_d} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & -x - \frac{f_{d-1}}{f_d} \end{vmatrix} = -f(x).$$

since we have:

$$\begin{vmatrix} -x & 1 & 0 & \cdots & -\frac{f_0}{f_d} \\ 0 & -x & 1 & \cdots & -\frac{f_1}{f_d} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & -x - \frac{f_{d-1}}{f_d} \end{vmatrix} = -x^* \begin{vmatrix} -x & 1 & 0 & \cdots & -\frac{f_1}{f_d} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & -x - \frac{f_{d-1}}{f_d} \end{vmatrix} - \frac{f_0}{f_d} = -(x^* \tilde{f}(x) + \frac{f_0}{f_d})$$

and by assumption  $\tilde{f}(x) = \frac{f(x)-f_0}{f_d * x}$  and finally we have the wanted result. Remark that this proof allows to avoid the condition that all the roots of  $f(x)$  are distinct.  $\square$

Then, to compute the roots of  $f(x)$  one can compute the eigenvalues of its Froebenius companion matrix. This is the object of the method proposed (reintroduced) by Edelman and Murakami [5] and revisited by Fortune [6] and many others trying to use the displacement structure of the companion matrix. In fact, often, the author realized that the monomial basis of the quotient algebra is not the most suitable one and proposed to express the matrix of the same linear application but in other basis. In the case of the Chebyshev basis this algorithm was already known by Barnett [2] and Cardinal later [4].

In the next section, we will also take advantage of the structure of the quotient algebra to design an algorithm for gcd computation mainly using linear algebra (eigenvalues are used in theory and never computed).

### 3.2 Structure of quotient and gcd

Let  $f(x)$  and  $g(x) \in \mathbb{K}[x]$  such that they are both monic. As above, we denote  $\mathbb{A} = \mathbb{K}[x]/(f)$  and  $d = \deg(f)$ . We denote  $\{\zeta_1, \dots, \zeta_d\}$  the set of roots of  $f(x)$  and we assume that  $f(x)$  is squarefree, i.e.  $\zeta_i \neq \zeta_j$  if  $i \neq j$ . We define  $\mathcal{M}_g : \begin{cases} \mathbb{A} \longrightarrow \mathbb{A} \\ h \longmapsto \pi(gh) \end{cases}$  where  $\pi(p) \in \mathbb{A}$  denote the remainder of  $p(x) \in \mathbb{K}[x]$  by division by  $f(x)$ . We denote  $M_g$  the matrix of  $\mathcal{M}_g$  in the monomial basis  $1, x, \dots, x^{d-1}$  of  $\mathbb{A}$  but other bases can be used. A matrix representing the map  $\mathcal{M}_g$  is called an extended companion matrix.

**Proposition 2** *The eigenvalues of  $\mathcal{M}_g$  are  $\{g(\zeta_1), \dots, g(\zeta_d)\}$ .*

**Proof** It is a direct corollary of the proposition ?? since if we write the matrix of this linear map in the Lagrange basis associated to  $\{\zeta_1, \dots, \zeta_d\}$  is

$$\begin{pmatrix} g(\zeta_1) & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & g(\zeta_d) \end{pmatrix}$$

and gives the wanted result.  $\square$

Trivially, we have:

**Corollary 1** *We have  $\text{corank}(\mathcal{M}_g) = \deg(f) - \text{rank}(\mathcal{M}_g) = \deg(\gcd(f, g))$ .*

The column of index  $i$  of  $M_g$  is the column vector of the coefficients of  $x^{i-1} * g(x)$ .

Let  $p_1, \dots, p_l$  be a basis of  $\text{Ker}(M_g)$  and let  $P_1(x), \dots, P_l(x)$  be the corresponding polynomials. First remark that  $\text{Ann}_{\mathbb{A}}(g) = \{P(x) \in \mathbb{A} \mid P(x) * g(x) = 0\}$  is an ideal of  $\mathbb{A}$ .

**Lemma 1** *The ideal  $\text{Ann}_{\mathbb{A}}(g)$  is a principal ideal.*

**Proof** Let us define

$$s(x) = \prod_{\zeta \in \mathcal{Z}(f) \setminus (\mathcal{Z}(f) \cap \mathcal{Z}(g))} (x - \zeta).$$

For all  $h \in \text{Ann}_{\mathbb{A}}(g)$  it is clear that  $\mathcal{Z}(h) \supset \mathcal{Z}(f) \setminus (\mathcal{Z}(f) \cap \mathcal{Z}(g))$  and then  $s$  divide  $h$ . Furthermore  $s \in \text{Ann}_{\mathbb{A}}(g)$  since in the Lagrange basis

$$s(x) * g(x) = \sum_{i=1}^d s(\zeta_i) * g(\zeta_i) L_i(x) = 0.$$

This shows that  $\text{Ann}_{\mathbb{A}}(g) = (s)$ . □

To compute  $s(x)$ , we built the matrix with columns formed by  $p_1, \dots, p_l$  and we make a triangulation operating only on the columns. This way we obtain the polynomial of minimal degree linear combination of  $P_1(x), \dots, P_l(x)$  and it is easily seen that this  $s(x)$  up to a multiplicative scalar factor.

**Lemma 2** *The first column of a column echelon form of the matrix  $K_g$  built from a basis of  $\text{Ker}(M_g)$  is the generator of  $\text{Ann}_{\mathbb{A}}(g)$ , i.e. it is the vector of the coefficients of  $s(x)$  up to a scalar multiplication.*

**Proof** Since the columns of a column echelon form of the matrix  $K_g$  are linearly independent, they form a basis of  $\text{Ann}_{\mathbb{A}}(g)$  as  $\mathbb{K}$ -vector space. So  $s(x)$  is a linear combination of the polynomials associated to those columns. The polynomial associated to the column echelon form of  $K_g$  have all different degree (because it is an echelon form) and so  $s(x)$  is a linear combination of those polynomial. Because  $s(x)$  as the lowest degree possible, it is a scalar multiple of the polynomial associated to the first column. □

**Proposition 3**  $f(x) \wedge g(x) = \frac{f(x)}{s(x)}$ .

**Proof** By construction, we have  $s(x) * g(x) = 0 \pmod{f(x)}$  and so  $s(x)$  divide  $f(x)$ . We also have  $\gcd(\frac{f(x)}{s(x)}, g(x)) = \gcd(f(x), g(x))$  since the roots of  $\frac{f(x)}{s(x)}$  are the root of  $f(x)$  where  $g(x)$  vanishes. Since  $\deg(\frac{f(x)}{s(x)}) = \deg(\gcd(f(x), g(x)))$  we have the wanted result. □

In all this section, we did not care if the polynomials are known in monomial or Chebyshev basis for instance. In fact, in order to have an algebraic algorithm, we only need to be able to perform euclidian division and this is always the case if the polynomial basis is graduated (as for monomial, Chebyshev, most of the orthogonal bases).

## 4 Bezoutian and Barnett's formula

A classical matricial formulation of resultant is given by the Bézout matrix. In this part, we recall the construction of the Bézout matrix and a special factorization of the multiplication matrix expressed in the monomial basis. This factorization is called Barnett formula (see [2]). The Barnett's formula allows to build the classical extended companion matrix without using euclidian division and only stable numerical computations. Furthermore, this factorization reveals that the extended companion matrix has a special rank structure and we will use this fact later to design a fast algorithm to compute AGCD.

**Definition 1** Let  $f$  and  $g \in \mathbb{C}[x]$  of degree  $m$  and  $n$  respectively (with  $n \geq m$ ), we denote  $\Theta_{f,g}(x, y) = \frac{f(x)g(y) - f(y)g(x)}{x-y} = \sum_{i,j} \theta_{i,j} x^i y^j = \sum_{j=0}^{m-1} \kappa_{f,g,j}(x) y^j$ . The Bézout matrix associated with  $f$  and  $g$  is  $B_{f,g} = (\theta_{j,j})_{i,j \in \{0, \dots, m-1\}}$ .

Remark that since  $\Theta_{f,g}(x, y) = \Theta_{f,g}(y, x)$  the matrix  $B_{f,g}$  is symmetric. The polynomials  $\kappa_{f,g,j}(x)$  are univariate polynomials of degree at most  $m-1$ . One particular case of interest is when  $f = 1$ . In this case the Bézout matrix has a Hankel structure, i.e.  $\theta_{i,j} = \theta_{i-1, j+1}$ . In this case we denote  $H_{g,i}(x) = \kappa_{1,g,i}(x)$  for  $i \in \{0, \dots, m-1\}$  which are called the Horner polynomials.

**Proposition 4** Let  $i \in \{0, \dots, m-1\}$ , the polynomial  $H_{g,i}(x) = c_{1,m-i} + \dots + c_{1,m} x^i$  has degree  $i$  and since they have different degree, they form a basis of  $\mathbb{C}[x]/(g)$ . Furthermore,  $\Theta_{1,g}(x, y) = \sum_{i=0}^{m-1} H_{g,m-i}(x) y^i$ .

**Corollary 2** The matrix  $B_{1,g}$  is the basis conversion from the Horner basis  $H_0, \dots, H_{m-1}$  to the monomial basis  $1, x, \dots, x^{m-1}$  of  $\mathbb{C}[x]/(g)$ .

This leads us to the following theorem, known as Barnett formula (see [2]):

**Theorem 1** Let  $M_f$  be the multiplication matrix associated to  $f$  in  $\mathbb{C}[x]/(g)$  in the monomial basis, we have:

$$M_f = B_{f,g} B_{1,g}^{-1}$$

**Proof** We have  $\Theta_{f,g}(x, y) = f(x) \frac{g(y) - g(x)}{x-y} + g(x) \frac{f(x) - f(y)}{x-y}$  and so  $f(x) \frac{g(x) - g(y)}{x-y} \equiv \Theta_{f,g}(x, y)$  in  $\mathbb{C}[x, y]/(g(x))$ . So, for each  $i \in \{0, \dots, m-1\}$ , we have  $\Theta_{f,g,i}(x) \equiv$



$f(x)\Theta_{1,g,i}(x)$ . This last equality means that  $B_{f,g}$  is the matrix of the multiplication by  $f(x)$  in  $\mathbb{C}[x]/(g)$ . The result follows directly from this fact.  $\square$

The Barnett's formula reveals the rank structure of the multiplication matrix. Furthermore, this formula is already known if we choose Chebyshev basis instead of monomial basis to express the polynomials and the matrices have exactly the same nature.

## 5 Structured matrices and asymptotically fast algorithms

In this section, we briefly recall some basics on displacement structured matrices and related algorithms.

### 5.1 Displacement structure

Given an integer  $n$  and a complex number  $\vartheta$  with  $|\vartheta| = 1$ , define the circulant matrix

$$Z_n^\vartheta = \begin{pmatrix} 0 & & & \vartheta \\ 1 & 0 & & \\ & 1 & \ddots & \\ & & \ddots & \ddots \\ & & & 1 & 0 \end{pmatrix} \in \mathbb{C}^{n \times n}.$$

Next, define the *Toeplitz-like displacement operator* as the linear operator

$$\nabla_T : \mathbb{C}^{m \times n} \longrightarrow \mathbb{C}^{m \times n}$$

$$\nabla_T(A) = Z_m^1 A - A Z_n^\vartheta.$$

A matrix  $A \in \mathbb{C}^{m \times n}$  is said to be *Toeplitz-like* if  $\nabla_T(A)$  is a small rank matrix (where “small” means small with respect to the matrix size). The number  $\alpha = \text{rank}(\nabla(A))$  is called the *displacement rank* of  $A$ . If  $A$  is Toeplitz-like, then there exist (non-unique) *displacement generators*  $G \in \mathbb{C}^{m \times \alpha}$  and  $H \in \mathbb{C}^{\alpha \times n}$  such that

$$\nabla(A) = GH.$$

Toeplitz matrices and their inverses are examples of Toeplitz-like matrices. Another useful example is the multiplication matrix  $M_f$ , which has Toeplitz-like displacement rank equal to 2, regardless of its size.

A similar definition holds for *Cauchy-like* structure; here the relevant displacement operator is

$$\begin{aligned} \nabla_C : \mathbb{C}^{m \times n} &\longrightarrow \mathbb{C}^{m \times n} \\ \nabla_C(A) &= D_1 A - A D_2, \end{aligned}$$

where  $D_1$  and  $D_2$  are diagonal matrices of appropriate size with disjoint spectra. See [10] for a detailed description of displacement structure.

## 5.2 Fast solution of displacement structured linear systems

Gaussian elimination with partial pivoting (GEPP) is a well-known and reliable algorithm that computes the solution of a linear system. Its arithmetic complexity for an  $n \times n$  matrix is asymptotically  $\mathcal{O}(n^3)$ . But if the system matrix exhibits displacement structure, it is possible to apply a variant of GEPP with complexity  $\mathcal{O}(n^2)$ . The main idea consists in operating on displacement generators rather than on the whole matrix; see [7] for details.

Strictly speaking, the GKO algorithm performs GEPP (or, equivalently, computes the PLU factorization) for Cauchy-like matrices. However, several authors have pointed out (see [7], [9], [12]) that Toeplitz-like matrices can be stably and cheaply transformed into Cauchy-like matrices; the same is true for displacement generators.

Consider, for instance, the case  $\vartheta = 1$  and let  $A$  be an  $n \times n$  Toeplitz-like matrix with generators  $G$  and  $H$ . Denote by  $D_0$  the matrix  $\text{diag}(1, e^{\pi i/n}, \dots, e^{(n-1)\pi i/n})$  and let  $F$  be the Fourier matrix of size  $n \times n$ . Then the matrix  $FAD_0^{-1}F^H$  is Cauchy-like, of the same displacement rank as  $A$ , with respect to the displacement operator defined by  $D_1 = \text{diag}(1, e^{2\pi i/n}, \dots, e^{2\pi i(n-1)/n})$  and  $D_2 = \text{diag}(e^{\pi i/n}, e^{3\pi i/n}, \dots, e^{(2n-1)\pi i/n})$ . Its Cauchy-like generators can be computed as  $\hat{G} = FG$  and  $\hat{H}^H = FD_0H^H$ .

Generalization to the case of  $m \times n$  rectangular matrices is possible. In this case, the parameter  $\vartheta$  should be chosen so that the spectra of  $D_1$  and  $D_2$  are well separated (see [1] and [3]).

We also point out that the GKO algorithm can be adapted to pivoting techniques other than partial pivoting ([8], [13]). This is especially useful in case of instability due to internal growth of generator entries. A Matlab implementation of the GKO algorithm that takes into account several pivoting strategies is found in the package DRsolve described in [1]. In our implementation, we use the pivoting strategy proposed in [8].

## 6 A structured approach to AGCD computation

We propose here an algorithm that exploits the algebraic and displacement structure of the multiplication matrix to compute the AGCD of two given polynomials with real coefficients (as defined in section 1).

### 6.1 Rank estimation

It has been pointed out in Section 3 that the rank deficiency of the multiplication matrix equals the AGCD degree. Here we use the structured pivoted LU decomposition to estimate the approximate rank of the multiplication matrix. Recall that  $M_g$  has a Toeplitz-like structure with displacement rank 2; it can then be transformed into a Cauchy-like matrix  $\hat{M}_g$  as described in Section 5.2. Fast pivoted Gauss elimination yields a factorization  $\hat{M}_g = P_1LUP_2$ , where  $L$

is a square, nonsingular, lower triangular matrix with diagonal entries equal to 1,  $U$  is upper triangular and  $P_{1,2}$  are permutation matrices. Inspection of the diagonal entries (or of the row norms) of  $U$  allows to estimate the approximate rank of  $\hat{M}_g$  and, therefore, of  $M_g$ .

## 6.2 Minimization of a quadratic functional

Let us suppose that:

- the polynomial  $f(x) = \sum_{j=0}^n f_j x^j$  is exactly known,
- the polynomial  $g(x) = \sum_{j=0}^m g_j x^j$  is approximately known and may be perturbed, so that we consider its coefficients as variables,
- the AGCD degree is known.

Then we can reformulate the problem of AGCD computation as the minimization of a quadratic functional. Indeed, recall that the cofactor  $v(x)$  with respect to  $f(x)$  is defined by the “shortest” vector (i.e., the vector with the maximum number of trailing zeros) that belongs to the null space of  $M_g$ . We assume  $v(x)$  to be monic; we denote its degree as  $k$  and we have

$$M_g v = M_g \cdot \begin{pmatrix} v_0 \\ \vdots \\ v_{k-1} \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ 0 \end{pmatrix}.$$

Also observe that the entries of  $M_g$  are linear functions of the coefficients of  $g(x)$ . Then the equation  $M_g v = 0$  can be rewritten as  $\mathcal{F}(g, v) = 0$ , where the functional  $\mathcal{F}$  is defined as

$$\begin{aligned} \mathcal{F} : \mathbb{C}^{m+1} \times \mathbb{C}^k &\longrightarrow \mathbb{R}_+ \\ \mathcal{F}(g, v) &= \|M_g v\|_2^2. \end{aligned}$$

For a preliminary study of the problem, we have chosen to solve the equation  $\mathcal{F}(g, v) = 0$  by means of Newton’s method, applied so as to exploit structure. Denote by  $z = [g_0, \dots, g_m, v_0, \dots, v_{k-1}]^T$  the vector of unknowns; then each Newton step has the form

$$z^{(j+1)} = z^{(j)} - J(g^{(j)}, v^{(j)})^\dagger M_{g^{(j)}} v^{(j)}.$$

In particular, notice that the Jacobian matrix associated with  $\mathcal{F}$  is an  $n \times (m + k + 1)$  Toeplitz-like matrix of displacement rank 3. This property allows to

compute a solution of the linear system  $J(g^{(j)}, v^{(j)})y = M_{g^{(j)}}v^{(j)}$  in a fast way; therefore, the arithmetic complexity of each iteration is quadratic w.r.t. the degree of the input polynomials.

We propose in the future to take into consideration other optimization methods in the quasi-Newton family, such as BFGS.

### 6.3 Computation of displacement generators

In order to perform fast factorization of the multiplication matrix  $M_g$ , we need to compute Toeplitz-like displacement generators. It turns out that the range of  $\nabla(M_g)$  is spanned by the first and last column of the displaced matrix, and the columns of indices from 2 to  $n - 1$  are multiples of the first one. Therefore, it suffices to compute a few rows and columns of  $M_g$  in order to obtain displacement generators. This can be done in a fast and stable way by using Barnett's formula. If we denote as  $e_j$  the  $j$ th vector of the canonical basis of  $\mathbb{C}^n$ , then the computation of the  $j$ -th column of  $M_g$  can be seen as

$$M_g(:, j) = B(f, g) \cdot (B(1, f)^{-1}e_j),$$

that is, it consists in solving a triangular Hankel linear system and computing a matrix-vector product. For row computation, recall that the Bezoutian is a symmetric matrix; we have analogously:

$$M_g(j, :) = e_j^T \cdot B(f, g) \cdot B(1, f)^{-1} = (B(1, f)^{-1}B(f, g)e_j^T)^T,$$

so that the computation of a row of  $M_g$  amounts to performing a matrix-vector product and solving a Hankel triangular system.

A similar approach holds for computation of displacement generators of the Jacobian matrix  $J(g, v)$  associated with the functional  $\mathcal{F}(g, v)$ .

### 6.4 Description of the algorithm

Input: coefficients of polynomials  $f(x)$  and  $g(x)$ .

Output: a perturbed polynomial  $\tilde{g}(x)$  such that  $f$  and  $\tilde{g}$  have a nontrivial common factor.

1. Estimate the approximate rank  $k$  of  $M_g$  by computing a fast pivoted LU decomposition of the associated Cauchy-like matrix.
2. Again by using fast LU, compute a vector  $v = [v_0, v_1, \dots, v_{k-1}, 1, 0, \dots, 0]^T$  in the approximate null space of  $M_g$ .
3. Apply structured Newton with initial guess  $(g, v)$  and compute polynomials  $\tilde{g}$  and  $\tilde{v}$  such that  $f$  and  $\tilde{g}$  have a common factor of degree  $\deg f - k$  and  $\tilde{v}$  is the monic cofactor for  $f$ .

### 6.5 Numerical experiments and computational issues

We have written a preliminary implementation of the proposed method in Matlab (available at the URL [http://www.unilim.fr/pages\\_perso/paola.boito/MMgcd.m](http://www.unilim.fr/pages_perso/paola.boito/MMgcd.m)).

The results of a few numerical experiments are shown below. The polynomials  $f$  and  $g$  are monic and have random coefficients uniformly distributed over  $[-1, 1]$ . They have an exact GCD of prescribed degree. A perturbation is then added to  $g$ . The perturbation vector has random entries uniformly distributed over  $[-\eta, \eta]$  and its norm is of the order of magnitude of  $\eta$ . We show:

- the residual  $\mathcal{F}(\tilde{g}, \tilde{v})$ ,
- the 2-norm distance between the exact and the computed cofactor  $v$ ,
- the 2-norm distance between the exact and the computed perturbed polynomial  $g$  (which is expected to be roughly of the same order of magnitude as  $\eta$ ).

In the following table we have taken  $\eta = 1e-5$ .

$n, m, \deg f$	$\mathcal{F}(\tilde{g}, \tilde{v})$	$\ v - \tilde{v}\ _2$	$\ g - \tilde{g}\ _2$
8, 7, 3	1.02e-15	1.19e-15	1.40e-5
15, 14, 5	1.51e-15	2.26e-15	1.35e-4
22, 22, 7	2.07e-13	1.40e-13	2.20e-4
36, 36, 11	1.19e-12	5.07e-14	0.0012

Here are results for  $\eta = 1e-8$ :

$n, m, \deg f$	$\mathcal{F}(\tilde{g}, \tilde{v})$	$\ v - \tilde{v}\ _2$	$\ g - \tilde{g}\ _2$
8, 7, 3	5.49e-15	1.63e-15	5.85e-8
28, 27, 13	7.90e-14	8.98e-14	6.50e-7
38, 37, 13	4.88e-12	4.26e-12	2.30e-5
58, 57, 23	2.03e-12	4.40e-12	2.54e-4

There are several issues in our approach that deserve further investigation. Let us mention in particular:

- The choice of a threshold (or a more refined technique) for estimating approximate rank.
- Normalization of polynomials: here we mostly work with monic polynomials, but other normalizations may be considered.
- The structured implementation of the optimization step (minimizing  $\mathcal{F}(g, v)$ ). We have used for now a heuristic structured version of the Gauss-Newton algorithm. Observe that each step of classical Gauss-Newton applied to our problem has the form  $z^{(j+1)} = z^{(j)} - y^{(j)}$ , where  $z^{(j)}$  is the vector containing the coefficients of the  $j$ -th iterate polynomials  $g^{(j)}$  and  $v^{(j)}$ , and  $y^{(j)}$  is the least-norm solution to the underdetermined system  $J(g^{(j)}, v^{(j)})y^{(j)} = M_{g^{(j)}} v^{(j)}$ . Computing this least-norm solution in a structured and fast way is a difficult point that will require more work. Our implementation gives a solution which is not, in general, the least-norm one, even though it is typically quite close. Further work will also include a study of other possible optimization methods that lend themselves well to a structured approach.

## 7 Conclusions

We have proposed and implemented a fast structured matrix-based approach to a variant of the AGCD problem, namely, the problem of computing an approximate greatest common divisor of two univariate polynomials, one of which is known to be exact. To our knowledge, this variant has been so far neglected in the existing literature. It may be also interesting when one polynomial is known with high accuracy and the other is not.

Our approach is based on the structure of the multiplication matrix and on the subsequent reformulation of the problem as the minimization of a suitably defined functional. Our choice of the multiplication matrix  $M_g$  over other resultant matrices (e.g., Sylvester, Bézout...) is motivated by

- the smaller size of  $M_g$ , with respect e.g. to the Sylvester matrix,
- the strong link between the null space of  $M_g$  and the gcd, and in particular the fact that the null space of  $M_g$  immediately yields a gcd cofactor,
- the displacement structure of  $M_g$ ,
- the possibility of computing selected rows and columns of  $M_g$  in a stable and cheap way, thanks to Barnett's formula.

This is, however, a preliminary study. Further work will include generalizations of the proposed problem and a more thorough analysis of the optimization part of the algorithm. Furthermore, this approach can be generalized in several interesting way:

- using better bases than the monomial one,
- it can be extended to some multivariate setting to compute the co-factor of a polynomial  $g$  in  $\mathbb{C}[x_1, \dots, x_n]/(f_1, \dots, f_n)$  when  $f_1, \dots, f_n$  define a complete intersection since Barnett formula still holds,
- to compute the AGCD of  $f$  with  $g_1, \dots, g_k$  where  $f$  is known with accuracy but  $g_1, \dots, g_k$  are inaccurate, one can take  $g$  as a linear combination of  $g_1, \dots, g_k$  with our method and succeed with a high probability.

## References

- [1] A. Aricò, G. Rodriguez. *A fast solver for linear systems with displacement structure*. Numer. Algorithms, 2010. DOI: 10.1007/s11075-010-9421-x.
- [2] S. Barnett, *Polynomials and linear control systems*, Monographs and Textbooks in Pure and Applied Mathematics, 77, Marcel Dekker, Inc., New York, 1983. xi+452 pp. ISBN: 0-8247-1898-4.

- [3] D. A. Bini, P. Boito, *A fast algorithm for approximate polynomial gcd based on structured matrix computations*, in Numerical Methods for Structured Matrices and Applications: Georg Heinig memorial volume, Operator Theory: Advances and Applications, vol. 199, Birkhäuser (2010), 155-173.
- [4] J. P. Cardinal, *On two iterative methods for approximating roots of a polynomial*, In J. Renegar, M. Shub and S. Smale editors, Proc. SIAM-AMS Summer Seminar on Math. of Numerical Analysis, Vol. 32 of Lecture Notes in Applied Math., AMS Press (1996), 165-188.
- [5] A. Edelman, H. Murakami, *Polynomial roots from companion matrix eigenvalues*, Math. Comp. 64 (1995), no. 210, 763-776.
- [6] S. Fortune, *An iterated eigenvalue algorithm for approximating roots of univariate polynomials*, Journal of Symbolic Computation, 33 (2002), no. 5, 627-646.
- [7] I. Gohberg, T. Kailath, and V. Olshevsky, *Fast Gaussian elimination with partial pivoting for matrices with displacement structure*, Math. Comp. 64 (212), 1557-1576 (1995).
- [8] M. Gu, *Stable and Efficient Algorithms for Structured Systems of Linear Equations*, SIAM J. Matrix Anal. Appl. 19, 279-306 (1998).
- [9] G. Heinig, *Inversion of generalized Cauchy matrices and other classes of structured matrices*, Linear Algebra in Signal Processing, IMA volumes in Mathematics and its Applications 69, 95-114 (1994).
- [10] T. Kailath, A. H. Sayed, *Displacement Structure: Theory and Applications*, SIAM Review 37(3), 297-386 (1995).
- [11] B. Mourrain, O. Ruatta, *Relations between roots and coefficients, interpolation and application to system solving*, Journal of Symbolic Computation, 33 (2002), no. 5, 679-699.
- [12] V. Y. Pan, *On computations with dense structured matrices*, Math. of Comput. 55(191), 179-190 (1990).
- [13] M. Stewart, *Stable Pivoting for the Fast Factorization of Cauchy-Like Matrices*, preprint (1997).
- [14] Z. Zeng, *The approximate GCD of inexact polynomials Part I: a univariate algorithm*, <http://www.neiu.edu/~zzeng/uvgcd.pdf>.