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OPTIMALITY OF T-COERCIVITY FOR SCALAR INTERFACE PROBLEMS BETWEEN DIELECTRICS AND METAMATERIALS

Anne-Sophie Bonnet-Ben Dhia[1], Lucas Chesnel[2] and Patrick Ciarlet, Jr.[3]

Abstract. Some electromagnetic materials have, in a given frequency range, an effective dielectric permittivity and/or a magnetic permeability which are real-valued negative parameters when dissipation is neglected. They are usually called metamaterials. We study a scalar wave transmission problem between a classical dielectric material and a metamaterial, set in an open, bounded subset of $\mathbb{R}^d$, with $d = 2, 3$. Our aim is to characterize occurences where the problem is well-posed within the Fredholm (or coercive + compact) framework. For that, we build some criteria, based on the geometry of the interface between the dielectric and the metamaterial. The proofs rely on localization techniques, together with the theory of T-coercivity introduced by the first and third authors and co-worker. In particular, we establish the optimality of the criteria, when the dielectric permittivity is piecewise constant.

Introduction

In electromagnetism, one can model materials that, after homogenization, exhibit real-valued strictly negative electric permittivity and/or magnetic permeability, within given frequency ranges. These so-called metamaterials, or left-handed materials, raise unusual questions. Among others, in a domain Ω of $\mathbb{R}^d$ $(d = 2, 3)$, divided into a classical dielectric material and a metamaterial, proving the existence of electromagnetic fields, and computing them, is a challenging issue (see for instance [6, 14]). For example, let us consider a problem in a two-dimensional domain, set in the time-harmonic regime with pulsation $\omega > 0$. Then, the transmission problems in the Transverse Magnetic and Transverse Electric modes can be reduced to scalar problems like

$$\operatorname{div}(\sigma \nabla u) + \omega^2 \varsigma u = f \text{ in } \Omega,$$

with a source term f, and (σ, ς) equal to (ε^{-1}, μ) , (μ^{-1}, ε) or $(\varepsilon, 0)$, plus boundary conditions. Above, ε is the dielectric permittivity, and μ is the magnetic permeability.

Mathematically speaking, let $\sigma_k \in L^\infty(\Omega_k)$, $k = 1, 2$, be real-valued functions such that

$$\sigma_1 \geq c_1 > 0 \text{ a.e. in } \Omega_1 \text{ et } \sigma_2 \leq c_2 < 0 \text{ a.e. in } \Omega_2,$$

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with c_k, $k = 1, 2$, constant numbers. Define $\sigma \in L^\infty(\Omega)$ in the following way: $\sigma := \sigma_k$ in Ω_k, $k = 1, 2$. In other words, there is a dielectric material in Ω_1, and a metamaterial in Ω_2, and we have $\overline{\Omega} = \overline{\Omega_1} \cup \overline{\Omega_2}$. We assume that Ω, Ω_1 and Ω_2 are domains of $\mathbb{R}^d$ $(d = 2, 3)$. We recall that a domain is an open, bounded and connected subset of $\mathbb{R}^d$ $(d = 2, 3)$ with a Lipschitz boundary.
We supplement the PDE with a homogeneous Dirichlet boundary condition, which writes $u = 0$ on $\partial\Omega$. In this setting, the source term f belongs to $H^{-1}(\Omega)$, and solutions u are sought in $H^1_0(\Omega)$. As the imbedding of $H^1_0(\Omega)$ into $H^{-1}(\Omega)$ is compact, it is enough to study the principal part of the PDE $u \mapsto \mathrm{div}(\sigma\,\nabla u)$. Hence, we study the operator $A : u \mapsto -\mathrm{div}(\sigma\,\nabla u)$ of $\mathcal{L}(H^1_0(\Omega), H^{-1}(\Omega))$ (the set of linear continuous mappings from $H^1_0(\Omega)$ to $H^{-1}(\Omega)$), associated with the problem

$$(\mathscr{P}) \left| \begin{array}{ll} \text{Find } u \in H^1_0(\Omega) & \text{such that} \\ -\mathrm{div}(\sigma\,\nabla u) = f & \text{in } \Omega. \end{array} \right.$$

Classically, one proves that u is a solution to $(\mathscr{P})$ if, and only if, u solves "Find $u \in H^1_0(\Omega)$ such that $a(u, v) = l(v)$ for all $v \in H^1_0(\Omega)$", with respectively

$$a(u, v) = (\sigma\nabla u, \nabla v)_\Omega, \quad l(v) = {}_{H^{-1}(\Omega)}\langle f, v\rangle_{H^1_0(\Omega)}\ .$$

Above, $(\cdot,\cdot)_\Omega$ is the usual scalar product of $(L^2(\Omega))^d$, whereas ${}_{H^{-1}(\Omega)}\langle\cdot,\cdot\rangle_{H^1_0(\Omega)}$ denotes the duality product between $H^{-1}(\Omega)$ and $H^1_0(\Omega)$. Of course, because of the sign shift of σ across the interface Σ dividing Ω, the form a is not coercive over $H^1_0(\Omega) \times H^1_0(\Omega)$. In particular, one can not apply the Lax-Milgram theorem.

To overcome this difficulty, one can use the T-coercivity approach, introduced in [1]. Let us recall the main features of this method. If there exists an isomorphism T of $H^1_0(\Omega)$ such that the bilinear form $(u, v) \mapsto a(u, T\,v)$ is coercive, then the Lax-Milgram theorem now applies. Indeed, the problem "Find $u \in H^1_0(\Omega)$ such that $a(u, T\,v) = l(T\,v)$ for all $v \in H^1_0(\Omega)$" is well-posed. In addition, because T is an isomorphism of $H^1_0(\Omega)$, one solves in this way the original problem "Find $u \in H^1_0(\Omega)$ such that $a(u, v) = l(v)$ for all $v \in H^1_0(\Omega)$". Therefore, within this framework, one has to find suitable operators T. In [1, 16], it is shown that A is actually an isomorphism of $\mathcal{L}(H^1_0(\Omega), H^{-1}(\Omega))$ if $\max\,(\inf_{\Omega_1}\sigma_1/\sup_{\Omega_2}|\sigma_2|, \inf_{\Omega_2}|\sigma_2|/\sup_{\Omega_1}\sigma_1) > I_\Sigma \geq 1$, where I_Σ is a constant number, that depends only on the geometry of the interface Σ between the dielectric material and the metamaterial. However, the value of I_Σ is not explicitly provided: indeed, it is defined with the help of the norms of abstract operators. In this paper, we shall complement the results of [1] in two ways. First, we provide some explicit values of the constants. Second, we localize the derivation of the extrema to a neighborhood of the interface Σ. To achieve those aims, we prove that the problem $(\mathscr{P})$ is well-posed in the sense that the operator A is Fredholm, using simple, geometrically defined, operators T.

In the case where σ_1 and σ_2 are constant numbers, there exist in the literature at least two other approaches that allow one to tackle problem $(\mathscr{P})$. With the help of integral equations, it was first proven in [3] by Costabel-Stephan that, when the interface Σ is smooth (of $\mathscr{C}^2$-class), problem $(\mathscr{P})$ is well-posed in the Fredholm sense if, and only if, the contrast $\kappa_\sigma := \sigma_2/\sigma_1$ is different from -1. Second, the influence of corners over the interface was specifically studied in [2] (see also [4] and [13]). The authors proved that, when there is a right angle on the interface, problem $(\mathscr{P})$, with a right-hand side f in $L^2(\Omega)$, is not well-posed in the Fredholm sense if, and only if, $\kappa_\sigma \in [-3; -1/3]$ (similar results can be obtained for any value of the angle). Note that we recover those results within the framework we develop herefater, with the explicit operators T. In this sense, we shall refer to them as optimal results.

The outline is the following. After introducing some notations and proving a preliminary result, we first study elementary cases, in simple geometries of $\mathbb{R}^2$ $(d = 2)$. Then, we combine those results with a localization technique, to solve the problem $(\mathscr{P})$ in the Fredholm sense, in general geometries of $\mathbb{R}^2$, and provide some applications when σ_k, $k = 1, 2$, are smooth and/or constants. In particular, we prove that one can obtain a

criterion, based only on the values of the contrast on the interface. Finally, we discuss the optimality of the results we obtain in a domain of $\mathbb{R}^2$. Last, we provide elements of the theory in a domain of $\mathbb{R}^3$ ($d = 3$). We cover in particular the elementary cases, which can not always be reduced to 2D configurations: as an ilustrative example, we study the problem set in a domain like Fichera's corner.

1. Notations and a preliminary result

Before we proceed, let us introduce some notations.
Given $\mathcal{O}$ an open set of $\mathbb{R}^d$, $(\cdot,\cdot)_{\mathcal{O}}$ denotes the usual scalar products of $L^2(\mathcal{O})$ and $(L^2(\mathcal{O}))^d$, $\|\cdot\|_{\mathcal{O}}$ the associated norms, $\|\cdot\|_{L^p(\mathcal{O})}$ the norm of $L^p(\mathcal{O})$ or $(L^p(\mathcal{O}))^d$ ($p \in [1,\infty] \setminus \{2\}$), and finally $\|\cdot\|_{H^1_0(\mathcal{O})} = \|\nabla\cdot\|_{\mathcal{O}}$ the norm of $H^1_0(\mathcal{O})$ and $\|\cdot\|_{H^{-1}(\mathcal{O})}$ the norm of $H^{-1}(\mathcal{O})$.
The boundaries $\partial\Omega$ and $\partial\Omega_k$, $k = 1, 2$, are divided as follows: let $\Gamma_k := \partial\Omega \cap \partial\Omega_k$, for $k = 1, 2$. Obviously, the interface Σ is such that $\overline{\Sigma} = \overline{\Omega_1} \cap \overline{\Omega_2}$. L^p-norms ($p \in [1,\infty]$) over $\overline{\Sigma}$ are written as above, with $\overline{\Sigma}$ replacing $\mathcal{O}$.
Then, if v is measurable in Ω, we use the notations $v_k := v|_{\Omega_k}$, $k = 1, 2$. Next, we introduce[1]

$$\sigma_1^+ := \sup_{\Omega_1} \sigma_1\,, \quad \sigma_2^+ := \sup_{\Omega_2} |\sigma_2|\,, \quad \sigma_1^- := \inf_{\Omega_1} \sigma_1 \quad \text{and} \quad \sigma_2^- := \inf_{\Omega_2} |\sigma_2|.$$

Whenever applicable, the contrast $\kappa_\sigma := \sigma_2/\sigma_1$ will be defined over $\overline{\Sigma}$: for instance as a constant number when σ_k, $k = 1, 2$ are constant numbers, or as an element of $\mathscr{C}^0(\overline{\Sigma})$ when σ_k, $k = 1, 2$ are resp. continuous over $\overline{\Omega_k}$, $k = 1, 2$.
Last, we define the Sobolev spaces

$$H^1_{0,\,\Gamma_k}(\Omega_k) := \left\{ v|_{\Omega_k},\, v \in H^1_0(\Omega) \right\},\ k = 1, 2.$$

Let us now prove the result below.

Theorem 1.1. *Consider an operator $R_1 \in \mathcal{L}(H^1_{0,\,\Gamma_1}(\Omega_1), H^1_{0,\,\Gamma_2}(\Omega_2))$ with matching condition $(R_1 u_1)|_\Sigma = u_1|_\Sigma$ for all $u_1 \in H^1_{0,\,\Gamma_1}(\Omega_1)$, and define*

$$T_1\, u = \begin{cases} u_1 & \text{in } \Omega_1 \\ -u_2 + 2R_1\, u_1 & \text{in } \Omega_2 \end{cases}. \tag{1}$$

If $\sigma_1^-/\sigma_2^+ > \|R_1\|^2$, then the form a is T_1-coercive and $A : u \mapsto -div(\sigma\,\nabla u)$ is an isomorphism from $H^1_0(\Omega)$ to $H^{-1}(\Omega)$.
Consider an operator $R_2 \in \mathcal{L}(H^1_{0,\,\Gamma_2}(\Omega_2), H^1_{0,\,\Gamma_1}(\Omega_1))$ with matching condition $(R_2 u_2)|_\Sigma = u_2|_\Sigma$ for all $u_2 \in H^1_{0,\,\Gamma_2}(\Omega_2)$, and define

$$T_2\, u = \begin{cases} u_1 - 2R_2\, u_2 & \text{in } \Omega_1 \\ -u_2 & \text{in } \Omega_2 \end{cases}. \tag{2}$$

If $\sigma_2^-/\sigma_1^+ > \|R_2\|^2$, then the form a is T_2-coercive and $A : u \mapsto -div(\sigma\,\nabla u)$ is an isomorphism from $H^1_0(\Omega)$ to $H^{-1}(\Omega)$.

Proof. By construction, T_1 belongs to $\mathcal{L}(H^1_0(\Omega))$. In addition, one has $T_1 \circ T_1 = Id$. In particular, T_1 is an isomorphism of $H^1_0(\Omega)$. Let us compute now $a(u, T_1\, u)$, for $u \in H^1_0(\Omega)$. With the help of Young's inequality, one can write, for all $\eta > 0$,

$$\begin{aligned} a(u, T_1\, u) &= (\sigma_1\,\nabla u_1, \nabla u_1)_{\Omega_1} + (|\sigma_2|\,\nabla u_2, \nabla u_2)_{\Omega_2} - 2(|\sigma_2|\,\nabla u_2, \nabla(R_1\, u_1))_{\Omega_2} \\ &\geq (\sigma_1\,\nabla u_1, \nabla u_1)_{\Omega_1} + (|\sigma_2|\,\nabla u_2, \nabla u_2)_{\Omega_2} - \eta\,(|\sigma_2|\,\nabla u_2, \nabla u_2)_{\Omega_2} - 1/\eta\,(|\sigma_2|\,\nabla(R_1\, u_1), \nabla(R_1\, u_1))_{\Omega_2} \\ &\geq ((\sigma_1 - \|R_1\|^2\,\sigma_2^+/\eta)\,\nabla u_1, \nabla u_1)_{\Omega_1} + (|\sigma_2|\,(1-\eta)\,\nabla u_2, \nabla u_2)_{\Omega_2}. \end{aligned}$$

[1] Everywhere, we write sup for sup ess, respectively inf for inf ess.

As a consequence, if $\sigma_1^-/\sigma_2^+ > \|R_1\|^2$, then there exists $C > 0$ such that

$$C\ \|u\|^2_{H^1_0(\Omega)} \leq a(u, T_1\, u),\ \forall u \in H^1_0(\Omega).$$

In other words, a is T_1-coercive.
On the other hand, one has $T_2 \in \mathcal{L}(H^1_0(\Omega))$ and $T_2 \circ T_2 = Id$. Given $u \in H^1_0(\Omega)$, we find for all $\eta > 0$,

$$a(u, T_2\, u) \quad \geq \quad (\sigma_1(1-\eta)\nabla u_1, \nabla u_1)_{\Omega_1} + ((|\sigma_2| - \|R_2\|^2\, \sigma_1^+/\eta)\nabla u_2, \nabla u_2)_{\Omega_2}.$$

Therefore, if $\sigma_2^-/\sigma_1^+ > \|R_2\|^2$, then there exists $C > 0$ such that

$$C\ \|u\|^2_{H^1_0(\Omega)} \leq a(u, T_2\, u),\ \forall u \in H^1_0(\Omega),$$

i.e. a is T_2-coercive.
To conclude the proof, we know that there exists an isomorphism T of $H^1_0(\Omega)$, such that the continuous, bilinear form $(u, v) \mapsto \tilde{a}(u, v) = a(u, T\, v)$ is coercive over $H^1_0(\Omega) \times H^1_0(\Omega)$. Evidently, $v \mapsto \tilde{l}(v) = l(T\, v)$ is a continuous, linear form over $H^1_0(\Omega)$. According to Lax-Milgram's theorem, there exists one, and only one, $u \in H^1_0(\Omega)$ such that $\tilde{a}(u, v) = \tilde{l}(v)$ for all $v \in H^1_0(\Omega)$, with continuous dependency with respect to the data $\tilde{l}$. Recall that T is an isomorphism of $H^1_0(\Omega)$. So, there exists one, and only one, $u \in H^1_0(\Omega)$ such that $a(u, v) = l(v)$ for all $v \in H^1_0(\Omega)$, with continuous dependency with respect to the data l. We conclude that A is an isomorphism. $\square$

In the rest of the paper, R_1 denotes an operator of $\mathcal{L}(H^1_{0,\,\Gamma_1}(\Omega_1), H^1_{0,\,\Gamma_2}(\Omega_2))$, and R_2 denotes an operator of $\mathcal{L}(H^1_{0,\,\Gamma_2}(\Omega_2), H^1_{0,\,\Gamma_1}(\Omega_1))$. Also, T_1 and T_2 denote the operators of $\mathcal{L}(H^1_0(\Omega))$ respectively defined by (1) and (2), for operators R_1 and R_2 that fulfill the matching conditions.

2. A study of elementary cases: global conditions

Let us explicit operators that ensure T-coercivity, on a series of particular geometries. In a second step (see §3), we shall handle general geometries. The underlying idea is to provide a criterion, based on the values of σ, that allows one to prove that A is an isomorphism from $H^1_0(\Omega)$ to $H^{-1}(\Omega)$.

2.1. **Symmetric domain**

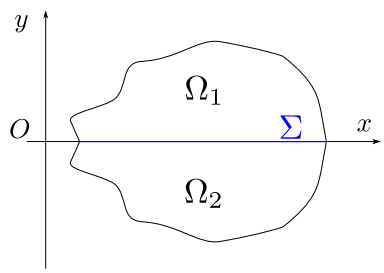

Figure 1. A symmetric geometry.

Let Ω be a *symmetric* domain, in the sense that Ω_1 and Ω_2 can be mapped from one to the other with the help of a reflection symmetry. Without loss of generality, we assume that the interface Σ is included in the line of equation $y = 0$ (see figure 1 for an example). In this case, we can prove the result below.

Theorem 2.1. (SYMMETRIC DOMAIN) *Assume that*

$$\max(\sigma_1^-/\sigma_2^+, \sigma_2^-/\sigma_1^+) > 1.$$

Then, there exists an isomorphism $T \in \mathcal{L}(H_0^1(\Omega))$ such that the form a is T-coercive and $A : u \mapsto -div(\sigma\,\nabla u)$ is an isomorphism from $H_0^1(\Omega)$ to $H^{-1}(\Omega)$.

Proof. Consider the operators R_1 and R_2 respectively defined by $(R_1\,u_1)(x,y) = u_1(x,-y)$ and $(R_2\,u_2)(x,y) = u_2(x,-y)$. Clearly, one has the matching conditions $(R_k u_k)|_\Sigma = u_k|_\Sigma$ for all $u_k \in H^1_{0,\,\Gamma_k}(\Omega_k)$, $k = 1,2$. Moreover, $\|R_k\| = 1$, for $k = 1,2$. The conclusion follows from theorem 1.1. □

Remark 2.2. *In the case where σ_1 and σ_2 are constant numbers, theorem 2.1 shows that A is an isomorphism as soon as the contrast $\kappa_\sigma = \sigma_2/\sigma_1$ is not equal to -1.*

2.2. **Interior vertex**

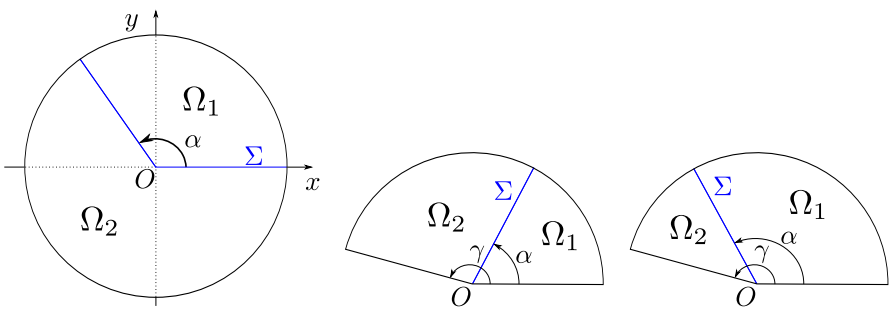

FIGURE 2. (Left) Geometry of an interior vertex. (Middle, right) Geometries of a boundary vertex.

Consider the geometry of figure 2–left. More precisely, let us denote by (r,θ) the polar coordinates centered at O with $\theta = 0$ on the half-line Ox (positive x). Given $R > 0$, let us define

$$\Omega_1 := \{(r\cos\theta, r\sin\theta)\,|\,0 < r < R,\ 0 < \theta < \alpha\}\ ;$$
$$\Omega_2 := \{(r\cos\theta, r\sin\theta)\,|\,0 < r < R,\ \alpha < \theta < 2\pi\}\,.$$

Theorem 2.3. (INTERIOR VERTEX) *Assume that*

$$\max(\sigma_1^-/\sigma_2^+, \sigma_2^-/\sigma_1^+) > I_\alpha,\ \textit{with}\ I_\alpha := \max(\frac{2\pi-\alpha}{\alpha}, \frac{\alpha}{2\pi-\alpha}).$$

Then, there exists an isomorphism $T \in \mathcal{L}(H_0^1(\Omega))$ such that the form a is T-coercive and $A : u \mapsto -div(\sigma\,\nabla u)$ is an isomorphism from $H_0^1(\Omega)$ to $H^{-1}(\Omega)$.

Proof. We keep the same notations for functions expressed either in cartesian coordinates or in polar coordinates. Consider the operators R_1 and R_2 respectively defined by $(R_1\,u_1)(\rho,\Theta) = u_1(\rho, \frac{\alpha}{\alpha-2\pi}\,(\Theta - 2\pi))$ and $(R_2\,u_2)(\rho,\Theta) = u_2(\rho, \frac{\alpha-2\pi}{\alpha}\,\Theta + 2\pi)$. By construction, one has the matching condition $(R_1\,u_1)(\rho,\alpha) = u_1(\rho,\alpha)$ and $(R_1\,u_1)(\rho,2\pi) = u_1(\rho,0)$, for all $u_1 \in H^1_{0,\,\Gamma_1}(\Omega_1)$. Let us now compute the norm of R_1. For that, let

$u_1 \in H^1_{0,\Gamma_1}(\Omega_1)$. Performing the change of variables $(r,\theta) = (\rho, \frac{\alpha}{\alpha-2\pi}(\Theta - 2\pi))$, we find successively

$$\begin{aligned}
\|\nabla(R_1\, u_1)\|^2_{\Omega_2} &= \int_{\Omega_2} \left(\frac{\partial(R_1\, u_1)}{\partial\rho}\right)^2 + \frac{1}{\rho^2}\left(\frac{\partial(R_1\, u_1)}{\partial\Theta}\right)^2 \rho d\rho d\Theta \\
&\leq \frac{2\pi-\alpha}{\alpha}\int_{\Omega_1}\left(\frac{\partial u_1}{\partial r}\right)^2 r dr d\theta + \frac{\alpha}{2\pi-\alpha}\int_{\Omega_1}\frac{1}{r^2}\left(\frac{\partial u_1}{\partial\theta}\right)^2 r dr d\theta \\
&\leq I_\alpha\, \|\nabla u_1\|^2_{\Omega_1}\ ; \\
\text{so } \|R_1\|^2 &\leq I_\alpha.
\end{aligned}$$

Similarly, the matching condition holds for R_2 on the interface, and $\|R_2\|^2 \leq I_\alpha$.
The conclusion follows thanks to theorem 1.1. □

Remark 2.4. *One has $-1 \in [-I_\alpha; -1/I_\alpha]$. Also, if $\alpha = \pi$ this interval reduces to $\{-1\}$, which is consistent with our analysis of symmetric domains (see §2.1).*

Remark 2.5. *When σ_1 are σ_2 are constant numbers, theorem 2.3 implies that A is an isomorphism if $\kappa_\sigma = \sigma_2/\sigma_1 \notin [-I_\alpha; -1/I_\alpha]$. For instance, if $\alpha = \pi/2$, there holds $[-I_\alpha; -1/I_\alpha] = [-3; -1/3]$. So, given $\kappa_\sigma \in]-\infty; -3[\cup]-1/3; 0[$, we know that A is an isomorphism.*

Remark 2.6. *More generally, one could consider an operator $R_1^\dagger$ defined by $(R_1^\dagger\, u_1)(\rho,\Theta) = u_1(\rho, g_1(\Theta))$ where g_1 is a $\mathscr{C}^1$ diffeomorphism from $[\alpha; 2\pi]$ to $[0;\alpha]$ such that $g_1(2\pi) = 0$ and $g_1(\alpha) = \alpha$. Then, one obtains $\|R_1^\dagger\|^2 = \max(\|g_1'\|_{L^\infty([\alpha;\pi])}, \|1/(g_1')\|_{L^\infty([\alpha;\pi])})$. According to the mean value theorem, one has $\|R_1^\dagger\|^2 \geq I_\alpha$, so our choice $g_1(\Theta) = \frac{\alpha}{\alpha-2\pi}(\Theta - 2\pi)$ is optimal in this configuration. That will not always be the case in 3D (see §6.4).*

2.3. Boundary vertex

Given $R > 0$ and $0 < \alpha < \gamma < 2\pi$, let us introduce, with (r,θ) the polar coordinates defined as before:

$$\begin{aligned}
\Omega_1 &:= \{(r\cos\theta, r\sin\theta)\,|\, 0 < r < R,\ 0 < \theta < \alpha\}\ ; \\
\Omega_2 &:= \{(r\cos\theta, r\sin\theta)\,|\, 0 < r < R,\ \alpha < \theta < \gamma\}\,.
\end{aligned}$$

Theorem 2.7. *(*BOUNDARY VERTEX*) Assume that*

$$\begin{cases}
\sigma_1^-/\sigma_2^+ > 1 \quad \text{or} \quad \sigma_2^-/\sigma_1^+ > \dfrac{\gamma-\alpha}{\alpha} & \text{if } \alpha \leq \gamma/2\,; \\
\sigma_2^-/\sigma_1^+ > 1 \quad \text{or} \quad \sigma_1^-/\sigma_2^+ > \dfrac{\gamma-\alpha}{\alpha} & \text{if } \alpha \geq \gamma/2.
\end{cases}$$

Then, there exists an isomorphism $T \in \mathcal{L}(H^1_0(\Omega))$ such that the form a is T-coercive and $A : u \mapsto -div(\sigma\,\nabla u)$ is an isomorphism from $H^1_0(\Omega)$ to $H^{-1}(\Omega)$.

Proof. Let us consider first that $\alpha \leq \gamma/2$ (figure 2–middle), with the operators R_1 and R_2, respectively defined by

$$(R_1\, u_1)(\rho,\Theta) = \begin{cases} u_1(\rho, 2\alpha - \Theta) & \text{if } \Theta \leq 2\alpha \\ 0 & \text{else} \end{cases}\ ;\ (R_2\, u_2)(\rho,\Theta) = u_2(\rho, \frac{\alpha-\gamma}{\alpha}\,\Theta + \gamma)\,.$$

One proves the results as before (see theorems 2.1 (for R_1) and 2.3 (for R_2)).
Similarly, one can handle the case where $\alpha \geq \gamma/2$ (figure 2–right). □

Remark 2.8. *If $\alpha = \gamma/2$, we recover the result on symmetric domains (see theorem 2.1).*

Remark 2.9. *Consider that σ_1 and σ_2 are constant numbers then, for instance with $\gamma = \pi$ and $\alpha = \pi/4$, the previous result indicates that A is an isomorphism, as soon as $\kappa_\sigma \in]-\infty; -3[\cup]-1; 0[$.*

2.4. **Interface of $\mathscr{C}^1$-class**

Let us conclude this overview of particular cases with a study of a smooth interface Σ. Let f be a real-valued

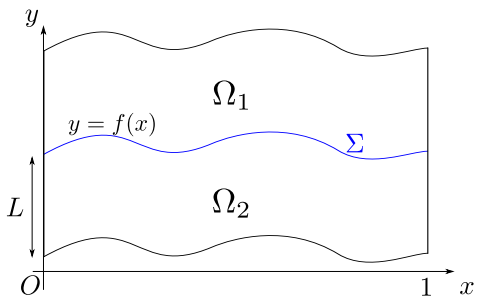

FIGURE 3. Geometry for an interface of $\mathscr{C}^1$-class.

function that belongs to $\mathscr{C}^1([0;1])$, and let $L > 0$. Let us introduce (see figure 3)

$$\begin{array}{lcl} \Omega & := & \{(x,y)\,|\,0 < x < 1,\ f(x) - L < y < f(x) + L\}\ ; \\ \Omega_1 & := & \{(x,y)\,|\,0 < x < 1,\ f(x) < y < f(x) + L\}\ ; \\ \Omega_2 & := & \{(x,y)\,|\,0 < x < 1,\ f(x) - L < y < f(x)\}\ . \end{array}$$

Theorem 2.10. *Assume that*

$$\max(\sigma_1^-/\sigma_2^+, \sigma_2^-/\sigma_1^+) > (1 + 2\,\|f'\|_{L^\infty(\overline{\Sigma})} + 4\,\|f'\|^2_{L^\infty(\overline{\Sigma})}).$$

Then, there exists an isomorphism $T \in \mathcal{L}(H^1_0(\Omega))$ such that the form a is T-coercive and $A : u \mapsto -div(\sigma\,\nabla u)$ is an isomorphism from $H^1_0(\Omega)$ to $H^{-1}(\Omega)$.

Proof. Define respectively the operators R_1 and R_2 by $(R_1\,u_1)(s,t) = u_1(s, 2f(s) - t)$ and $(R_2\,u_2)(s,t) = u_2(s, 2f(s) - t)$. We note that if $(s,t) \in \Sigma$, then $t = f(s)$ and accordingly $(R_1\,u_1)(s,t) = u_1(s, 2f(s) - t) = u_1(s,t)$, for all $u_1 \in H^1_{0,\,\Gamma_1}(\Omega_1)$. Next, let us bound the norm of R_1. Given $u_1 \in H^1_{0,\,\Gamma_1}(\Omega_1)$ and using the change of variables $(x,y) = (s, 2f(s) - t)$, we find

$$\begin{array}{rcl} \|\nabla(R_1\,u_1)\|^2_{\Omega_2} & = & \displaystyle\int_{\Omega_2} \left(\frac{\partial(R_1\,u_1)}{\partial s}\right)^2 + \left(\frac{\partial(R_1\,u_1)}{\partial t}\right)^2 dsdt \\ & \leq & \displaystyle\int_{\Omega_1} \left(\frac{\partial u_1}{\partial x} + 2f'(x)\frac{\partial u_1}{\partial y}\right)^2 + \left(\frac{\partial u_1}{\partial y}\right)^2 dxdy \\ & \leq & \displaystyle\int_{\Omega_1} \left(\frac{\partial u_1}{\partial x}\right)^2 + 4\,|f'(x)|\left|\frac{\partial u_1}{\partial x}\right|\left|\frac{\partial u_1}{\partial y}\right| + 4\,|f'(x)|^2\left(\frac{\partial u_1}{\partial y}\right)^2 + \left(\frac{\partial u_1}{\partial y}\right)^2 dxdy \\ & \leq & (1 + 2\,\|f'\|_{L^\infty(\overline{\Sigma})} + 4\,\|f'\|^2_{L^\infty(\overline{\Sigma})})\,\|\nabla u_1\|^2_{\Omega_1} \end{array}$$

It follows that $\|R_1\|^2 \leq (1 + 2\,\|f'\|_{L^\infty(\overline{\Sigma})} + 4\,\|f'\|^2_{L^\infty(\overline{\Sigma})})$.
Reversing the roles of Ω_1 and Ω_2, one recovers the matching condition for R_2, and moreover $\|R_2\|^2 \leq (1 + 2\,\|f'\|_{L^\infty(\overline{\Sigma})} + 4\,\|f'\|^2_{L^\infty(\overline{\Sigma})})$.
The conclusion follows from theorem 1.1. □

Remark 2.11. *In the special case where f' is uniformly equal to 0, the domain Ω is symmetric and the result is identical to the one of theorem 2.1.*

3. A study of general geometries via localization

The problem $(\mathscr{P})$ is said to be well-posed in the Fredholm sense when the operator $A \in \mathcal{L}(H_0^1(\Omega), H^{-1}(\Omega))$ is Fredholm of index 0. Let us recall the definition below (see for instance [15]).

Definition 3.1. *Let X and Y be two Banach spaces, and B an operator of $\mathcal{L}(X,Y)$. The operator B is Fredholm if*

i) $\dim \ker B < \infty$, $Im\, B$ is closed.
ii) $\dim \operatorname{coker} B < \infty$, where $\operatorname{coker} B := Y/Im\, B$.

When B is a Fredholm operator, its index is defined by $\operatorname{ind} B := \dim \ker B - \dim \operatorname{coker} B$.

3.1. Setting of the problem and additional notations

We recall that Ω is a domain of $\mathbb{R}^2$, that is an open, bounded and connected subset of $\mathbb{R}^2$ with a Lipschitz boundary. The domain Ω is divided into two open subsets Ω_1 and Ω_2 by an interface Σ, namely $\overline{\Omega_1} \cup \overline{\Omega_2} = \overline{\Omega}$, $\Omega_1 \cap \Omega_2 = \emptyset$ and $\overline{\Omega_1} \cap \overline{\Omega_2} = \overline{\Sigma}$. Let $\boldsymbol{n}$ be the unit normal vector to Σ, going from Ω_1 to Ω_2. Below, we make a number of regularity assumptions, focusing on the corners and endpoints of the interface:

- The subsets Ω_1 and Ω_2 have a Lipschitz boundary.
- The interface Σ is of $\mathscr{C}^1$-class, to the exception of a finite number of *interior vertices* $\mathcal{S}_{int} = \{\boldsymbol{x}^i,\ 1 \leq i \leq N_{int}\}$. And, for $1 \leq i \leq N_{int}$, the subsets Ω_1 and Ω_2 coincide with open cones in a neighborhood $\mathcal{V}^i$ of $\boldsymbol{x}^i$, locally in Ω:

$$\begin{aligned} &\Omega_1 \cap \mathcal{V}^i = \mathcal{K}_1(\boldsymbol{x}^i) \cap \mathcal{V}^i \ \text{ and } \ \Omega_2 \cap \mathcal{V}^i = \mathcal{K}_2(\boldsymbol{x}^i) \cap \mathcal{V}^i, \\ &\text{where } \mathcal{K}_1(\boldsymbol{x}^i) \text{ and } \mathcal{K}_2(\boldsymbol{x}^i) \text{ are open cones, centered at } \boldsymbol{x}^i. \end{aligned} \tag{3}$$

- There are either 0 or 2 endpoints, called *boundary vertices*: $\mathcal{S}_{ext} := \overline{\Sigma} \cap \partial\Omega = \{\boldsymbol{x}^i,\ N_{int}+1 \leq i \leq N_{int}+N_{ext}\}$, with $N_{ext} \in \{0,2\}$. And, for $N_{int}+1 \leq i \leq N_{int}+N_{ext}$, the subsets Ω_1 and Ω_2 coincide with open cones in a neighborhood $\mathcal{V}^i$ of $\boldsymbol{x}^i$, locally in Ω: i.e., (3) holds.

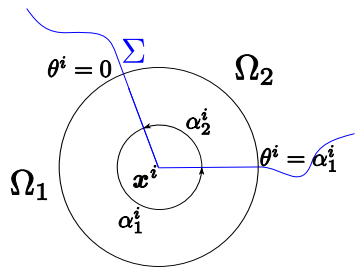

Figure 4. Notations for $\boldsymbol{x}^i \in \mathcal{S}_{int}$ - $\alpha^i = \alpha_2^i$.

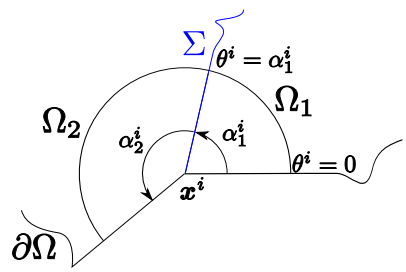

Figure 5. Notations for $\boldsymbol{x}^i \in \mathcal{S}_{ext}$ - $\alpha^i = \alpha_1^i$.

For each index i, we define the apertures $\alpha_k^i \in]0; 2\pi[$ of the cones $\mathcal{K}_k(\boldsymbol{x}^i)$, $k = 1, 2$. We introduce $\gamma^i := \alpha_1^i + \alpha_2^i$ and $\alpha^i := \min(\alpha_1^i, \alpha_2^i)$. Evidently, one has $\gamma^i = 2\pi$ for interior vertices, and $\gamma^i < 2\pi$ for boundary vertices. On the other hand, at an interior vertex $\boldsymbol{x}^i$, Σ is not of $\mathscr{C}^1$-class, so $0 < \alpha^i < \pi$.
We denote by (r^i, θ^i) the polar coordinates centered at $\boldsymbol{x}^i$ with the angle θ^i such that

$$\begin{aligned} &\mathcal{K}_1(\boldsymbol{x}^i) \text{ is isometric to } \left\{(r^i\cos\theta^i, r^i\sin\theta^i) \,|\, r^i > 0,\ 0 < \theta^i < \alpha_1^i\right\}; \\ &\mathcal{K}_2(\boldsymbol{x}^i) \text{ is isometric to } \left\{(r^i\cos\theta^i, r^i\sin\theta^i) \,|\, r^i > 0,\ \alpha_1^i < \theta^i < \gamma^i\right\}. \end{aligned}$$

We let $\mathcal{S}_{ext}^1 := \{\boldsymbol{x}^i \in \mathcal{S}_{ext} \,|\, \alpha_1^i \leq \alpha_2^i\}$, $\mathcal{S}_{ext}^2 := \{\boldsymbol{x}^i \in \mathcal{S}_{ext} \,|\, \alpha_2^i < \alpha_1^i\}$ and $\mathcal{S} := \mathcal{S}_{int} \cup \mathcal{S}_{ext}$. The cardinality of $\mathcal{S}$ is denoted by N.

Finally, we define

$$I_{\alpha^i} := \frac{\gamma^i - \alpha^i}{\alpha^i} \text{ for } 1 \leq i \leq N.$$

Remark 3.2. *Given any interior vertex, there holds $I_{\alpha^i} > 1$. The same is true for any boundary vertex of $\mathcal{S}^2_{ext}$. On the other hand, for a boundary vertex of $\mathcal{S}^1_{ext}$, one has only $I_{\alpha^i} \geq 1$ (it can happen that $I_{\alpha^i} = 1$).*

3.2. Statement of the result

In our setting, we shall prove that A is Fredholm, under some conditions on the geometry of the domain Ω and on σ.
Below, we let $B(\boldsymbol{x}, d)$ be the open ball centered at $\boldsymbol{x}$ with radius d.

Theorem 3.3. *Assume that either 1. or 2. below holds:*

1.
   - $\forall \boldsymbol{x} \in \Sigma \backslash \mathcal{S}$ *(smooth part of the interface):* $\exists d > 0, \inf\limits_{B(\boldsymbol{x},d)\cap\Omega_1} \sigma_1 > \sup\limits_{B(\boldsymbol{x},d)\cap\Omega_2} |\sigma_2|$,
   - $\forall \boldsymbol{x}^i \in \mathcal{S}_{int} \cup \mathcal{S}^2_{ext}$: $\exists d > 0, \inf\limits_{B(\boldsymbol{x}^i,d)\cap\Omega_1} \sigma_1 > I_{\alpha^i} \sup\limits_{B(\boldsymbol{x}^i,d)\cap\Omega_2} |\sigma_2|$,
   - $\forall \boldsymbol{x}^i \in \mathcal{S}^1_{ext}$: $\exists d > 0, \inf\limits_{B(\boldsymbol{x}^i,d)\cap\Omega_1} \sigma_1 > \sup\limits_{B(\boldsymbol{x}^i,d)\cap\Omega_2} |\sigma_2|$;
2.
   - $\forall \boldsymbol{x} \in \Sigma \backslash \mathcal{S}$ *(smooth part of the interface):* $\exists d > 0, \inf\limits_{B(\boldsymbol{x},d)\cap\Omega_2} |\sigma_2| > \sup\limits_{B(\boldsymbol{x},d)\cap\Omega_1} \sigma_1$,
   - $\forall \boldsymbol{x}^i \in \mathcal{S}_{int} \cup \mathcal{S}^1_{ext}$: $\exists d > 0, \inf\limits_{B(\boldsymbol{x}^i,d)\cap\Omega_2} |\sigma_2| > I_{\alpha^i} \sup\limits_{B(\boldsymbol{x}^i,d)\cap\Omega_1} \sigma_1$,
   - $\forall \boldsymbol{x}^i \in \mathcal{S}^2_{ext}$: $\exists d > 0, \inf\limits_{B(\boldsymbol{x}^i,d)\cap\Omega_2} |\sigma_2| > \sup\limits_{B(\boldsymbol{x}^i,d)\cap\Omega_1} \sigma_1$.

Then, the operator $A : u \mapsto -div(\sigma \nabla u)$ of $\mathcal{L}(H^1_0(\Omega), H^{-1}(\Omega))$ is Fredholm of index 0.

Remark 3.4. *Under the assumptions of theorem 3.3, if A is injective, then A is an* isomorphism *of $H^1_0(\Omega)$ into $H^{-1}(\Omega)$. On the other hand, it can happen that the dimension of $\ker A$ is finite and not equal to 0.*

The proof is divided in several steps, following §5, chapter 2 of Lions-Magenes [8], §6.3 of Kozlov-Maz'ya-Rossmann [7] or §4.1.2 of Nazarov-Plamenevsky [10]. First, we introduce a partition of unity, which fits the geometry of the domain (and of the interface). Then, we prove an *a priori* estimate for solutions to $(\mathscr{P})$, with the help of T-coercivity. To reach that goal, we use the T-coercivity framework that we developed previously on a series of elementary cases. Finally, a classical application of Peetre's lemma leads to the conclusion.

3.3. Construction of a partition of unity

Let $\boldsymbol{x}^i \in \mathcal{S}$. According to one of the two assumptions (case 1. or case 2.) of theorem 3.3, there exists $d^i > 0$ such that $(B(\boldsymbol{x}^i, d^i) \cap \Omega) \subset \mathcal{V}^i$, where $\mathcal{V}^i$ is the neighborhood of $\boldsymbol{x}^i$ that appears in (3), and

$$\left.\begin{array}{ll} \inf\limits_{B(\boldsymbol{x}^i,d^i)\cap\Omega_1} \sigma_1 > I_{\alpha^i} \sup\limits_{B(\boldsymbol{x}^i,d^i)\cap\Omega_2} |\sigma_2| & \text{if } \boldsymbol{x}^i \in \mathcal{S}_{int} \cup \mathcal{S}^2_{ext} \\ \inf\limits_{B(\boldsymbol{x}^i,d^i)\cap\Omega_1} \sigma_1 > \sup\limits_{B(\boldsymbol{x}^i,d^i)\cap\Omega_2} |\sigma_2| & \text{if } \boldsymbol{x}^i \in \mathcal{S}^1_{ext} \end{array}\right\} \text{ in case 1.};$$

$$\left.\begin{array}{ll} \inf\limits_{B(\boldsymbol{x}^i,d^i)\cap\Omega_2} |\sigma_2| > I_{\alpha^i} \sup\limits_{B(\boldsymbol{x}^i,d^i)\cap\Omega_1} \sigma_1 & \text{if } \boldsymbol{x}^i \in \mathcal{S}_{int} \cup \mathcal{S}^1_{ext} \\ \inf\limits_{B(\boldsymbol{x}^i,d^i)\cap\Omega_2} |\sigma_2| > \sup\limits_{B(\boldsymbol{x}^i,d^i)\cap\Omega_1} \sigma_1 & \text{if } \boldsymbol{x}^i \in \mathcal{S}^2_{ext} \end{array}\right\} \text{ in case 2..}$$

For $1 \leq i \leq N$, let $\zeta^i \in \mathscr{C}^\infty(\overline{\Omega})$ be a truncation function, equal to 1 in $\overline{B(\boldsymbol{x}^i, d^i/2)} \cap \Omega$, with support included in $(B(\boldsymbol{x}^i, d^i) \cap \Omega) \subset \mathcal{V}^i$, and such that ζ^i is a function of the radius r^i only, and $0 \leq \zeta^i \leq 1$.

Next, define $\Sigma_r := \Sigma \backslash \bigcup_{i=1}^{N} \overline{B(\boldsymbol{x}^i, d^i/2)}$, and let $\boldsymbol{x} \in \overline{\Sigma_r}$. According to the assumption on the smooth part of Σ, there exists $d^{\boldsymbol{x}} > 0$ such that $B(\boldsymbol{x}, d^{\boldsymbol{x}}) \subset \Omega \backslash \mathcal{S}$, and

$$\inf_{B(\boldsymbol{x},d^{\boldsymbol{x}})\cap\Omega_1} \sigma_1 > \sup_{B(\boldsymbol{x},d^{\boldsymbol{x}})\cap\Omega_2} |\sigma_2| \quad \text{or} \quad \inf_{B(\boldsymbol{x},d^{\boldsymbol{x}})\cap\Omega_2} |\sigma_2| > \sup_{B(\boldsymbol{x},d^{\boldsymbol{x}})\cap\Omega_1} \sigma_1. \tag{4}$$

On the other hand, as Σ is of piecewise $\mathscr{C}^1$-class, it coincides locally with the graph of a function $f^{\boldsymbol{x}}$ of $\mathscr{C}^1(\mathbb{R})$ (see Annex C of [5]). Let $s_0 \in \mathbb{R}$ be such that $\boldsymbol{x} = (s_0, f^{\boldsymbol{x}}(s_0))$. Up to a rotation of the coordinates system, one can assume that $f^{\boldsymbol{x}\prime}(s_0) = 0$.
Consider next three real numbers $a^{\boldsymbol{x}}$, $b^{\boldsymbol{x}}$ and $\delta^{\boldsymbol{x}} > 0$ such that the set

$$\Omega^{\boldsymbol{x}} := \left\{(s,t) \in \mathbb{R}^2 \,|\, a^{\boldsymbol{x}} < s < b^{\boldsymbol{x}},\ f^{\boldsymbol{x}}(s) - \delta^{\boldsymbol{x}} < t < f^{\boldsymbol{x}}(s) + \delta^{\boldsymbol{x}}\right\} \tag{5}$$

is included in $B(\boldsymbol{x}, d^{\boldsymbol{x}})$, and such that $a^{\boldsymbol{x}} < s_0 < b^{\boldsymbol{x}}$ (so that $\boldsymbol{x}$ belongs to $\Omega^{\boldsymbol{x}}$). Choosing the direction of the coordinate axes, one can ensure that $\Omega^{\boldsymbol{x}} \cap \Omega_1$ and $\Omega^{\boldsymbol{x}} \cap \Omega_2$ coincide respectively with $\Omega_1^{\boldsymbol{x}}$ and $\Omega_2^{\boldsymbol{x}}$, that are defined by

$$\begin{aligned} \Omega_1^{\boldsymbol{x}} &:= \left\{(s,t) \in \mathbb{R}^2 \,|\, a^{\boldsymbol{x}} < s < b^{\boldsymbol{x}},\ f^{\boldsymbol{x}}(s) < t < f^{\boldsymbol{x}}(s) + \delta^{\boldsymbol{x}}\right\}; \\ \Omega_2^{\boldsymbol{x}} &:= \left\{(s,t) \in \mathbb{R}^2 \,|\, a^{\boldsymbol{x}} < s < b^{\boldsymbol{x}},\ f^{\boldsymbol{x}}(s) - \delta^{\boldsymbol{x}} < t < f^{\boldsymbol{x}}(s)\right\}. \end{aligned}$$

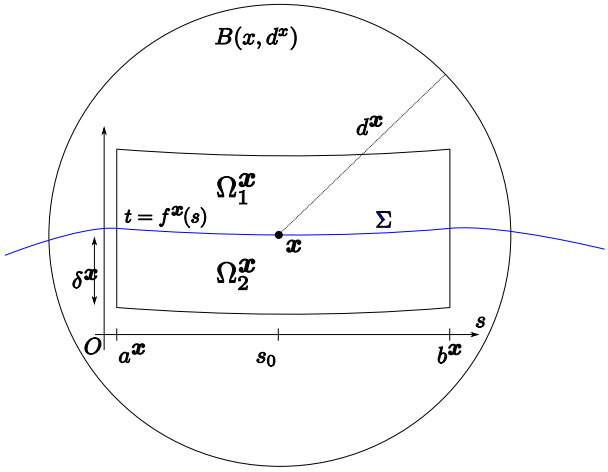

FIGURE 6. Situation in a neighborhood of $\boldsymbol{x}$.

But $f^{\boldsymbol{x}\prime}$ is continuous at $s = s_0$ and it vanishes there, so according to (4) one can take $a^{\boldsymbol{x}}$ and $b^{\boldsymbol{x}}$ close enough to s_0 so that

$$\begin{aligned} &\inf_{\Omega_1^{\boldsymbol{x}}} \sigma_1 > \sup_{\Omega_2^{\boldsymbol{x}}} |\sigma_2| \left(1 + 2\,\|f'\|_{L^\infty([a^{\boldsymbol{x}};b^{\boldsymbol{x}}])} + 4\,\|f'\|^2_{L^\infty([a^{\boldsymbol{x}};b^{\boldsymbol{x}}])}\right) \\ \text{or} \quad &\inf_{\Omega_2^{\boldsymbol{x}}} |\sigma_2| > \sup_{\Omega_1^{\boldsymbol{x}}} \sigma_1 \left(1 + 2\,\|f'\|_{L^\infty([a^{\boldsymbol{x}};b^{\boldsymbol{x}}])} + 4\,\|f'\|^2_{L^\infty([a^{\boldsymbol{x}};b^{\boldsymbol{x}}])}\right). \end{aligned} \tag{6}$$

Consider next

$$\tilde{\Omega}^{\boldsymbol{x}} := \left\{(s,t) \in \mathbb{R}^2 \,|\, a^{\boldsymbol{x}} + (s_0 - a^{\boldsymbol{x}})/2 < s < b^{\boldsymbol{x}} + (b^{\boldsymbol{x}} - s_0)/2,\ f^{\boldsymbol{x}}(s) - \delta^{\boldsymbol{x}}/2 < t < f^{\boldsymbol{x}}(s) + \delta^{\boldsymbol{x}}/2\right\}.$$

By construction, $\tilde{\Omega}^{\boldsymbol{x}}$ is a neighborhood of $\boldsymbol{x}$, and $\tilde{\Omega}^{\boldsymbol{x}} \subset \Omega^{\boldsymbol{x}}$.

The set $\overline{\Sigma_r}$ is compact, so one can extract from the set $(\tilde{\Omega}^{\boldsymbol{x}})_{\boldsymbol{x}\in\overline{\Sigma_r}}$ a finite collection, denoted by $(\tilde{\mathcal{O}}^i)_{i=1}^{N_\Sigma}$, whose union covers $\overline{\Sigma_r}$. Further, for $1 \le i \le N_\Sigma$, we let $\mathcal{O}^i$ denote the open set $\Omega^{\boldsymbol{x}}$ associated with $\tilde{\Omega}^{\boldsymbol{x}} = \tilde{\mathcal{O}}^i$. Thus, it is possible to introduce a bounded, open set $\mathcal{O}^0$ (of $\mathbb{R}^2$) which does not intersect Σ, and such that

$$\Omega \subset \left(\mathcal{O}^0 \cup (\bigcup_{i=1}^{N_\Sigma} \tilde{\mathcal{O}}^i) \cup (\bigcup_{i=1}^{N} B(\boldsymbol{x}^i, d^i/2)) \right).$$

Next, consider

- a function $\chi^0 \in \mathscr{C}^\infty(\overline{\Omega})$ whose support does not intersect $\overline{\Sigma}$, equal to 1 in $\mathcal{O}^0$ and such that $0 \le \chi^0 \le 1$;
- for $1 \le i \le N_\Sigma$, a function $\chi^i \in \mathscr{C}^\infty(\overline{\Omega})$, whose support is included in $\mathcal{O}^i$, equal to 1 in $\tilde{\mathcal{O}}^i$ and such that $0 \le \chi^i \le 1$.

It follows that, for all $\boldsymbol{x} \in \overline{\Omega}$,

$$\sum_{i=0}^{N_\Sigma} \chi^i(\boldsymbol{x}) + \sum_{i=1}^{N} \zeta^i(\boldsymbol{x}) > 1;\ \exists i_0 \text{ such that } \chi^{i_0}(\boldsymbol{x}) = 1 \text{ or } \zeta^{i_0}(\boldsymbol{x}) = 1.$$

3.4. *A priori* estimate for solutions to $(\mathscr{P})$

Given $u \in H_0^1(\Omega)$, let $f := A\,u = -\mathrm{div}(\sigma\,\nabla u) \in H^{-1}(\Omega)$. Let us prove there exists $C > 0$, independent of u, such that

$$\|u\|_{H_0^1(\Omega)} \le C \left(\|A\,u\|_{H^{-1}(\Omega)} + \|u\|_\Omega \right). \tag{7}$$

For $\chi \in \mathscr{C}^\infty(\overline{\Omega})$, define $supp^1\chi := \left\{ \boldsymbol{x} \in \overline{\Omega} \,|\, \chi(\boldsymbol{x}) = 1 \right\}$, so that one can write

$$\begin{aligned} \|u\|^2_{H_0^1(\Omega)} &\le \|u\|^2_{H_0^1(supp^1\chi^0)} + \sum_{i=1}^{N_\Sigma} \|u\|^2_{H_0^1(supp^1\chi^i)} + \sum_{i=1}^{N} \|u\|^2_{H_0^1(supp^1\zeta^i)} \\ &\le \left\|\chi^0 u\right\|^2_{H_0^1(\Omega)} + \sum_{i=1}^{N_\Sigma} \left\|\chi^i u\right\|^2_{H_0^1(\Omega)} + \sum_{i=1}^{N} \left\|\zeta^i u\right\|^2_{H_0^1(\Omega)}. \end{aligned} \tag{8}$$

Then, let us establish estimates for the three terms of the right-hand side of (8).
First,

$$\begin{aligned} \left\|\chi^0 u\right\|^2_{H_0^1(\Omega)} &\le C\,(|\sigma|\nabla(\chi^0 u), \nabla(\chi^0 u))_\Omega \\ &\le C \left(\left|(|\sigma|\, u\nabla\chi^0, \nabla(\chi^0 u))_\Omega\right| + \left|(|\sigma|\nabla u, \nabla((\chi^0)^2 u))_\Omega\right| + \left|(|\sigma|\nabla u, \chi^0 u \nabla\chi^0)_\Omega\right| \right) \\ &\le C \left(\|u\|_\Omega \left\|\chi^0 u\right\|_{H_0^1(\Omega)} + \|f\|_{H^{-1}(\Omega)} \left\|(\chi^0)^2 u\right\|_{H_0^1(\Omega)} + \|u\|_{H_0^1(\Omega)} \|u\|_\Omega \right) \\ &\le C \left(\|f\|_{H^{-1}(\Omega)} \|u\|_{H_0^1(\Omega)} + \|u\|_\Omega \|u\|_{H_0^1(\Omega)} \right). \end{aligned} \tag{9}$$

Using the operators T_1 or T_2 (see (1) or (2)) implicitly defined in the proof of theorem 2.10 over the domain $int(supp\,\chi^i)$ (with a continuation by 0 in $\Omega\backslash supp\,\chi^i$), one gets an operator of $\mathcal{L}(H_0^1(\Omega))$, denoted by T. Moreover, one finds

$$\begin{aligned} \left\|\chi^i u\right\|^2_{H_0^1(\Omega)} &\le C \left|(\sigma\nabla(\chi^i u), \nabla(T(\chi^i u)))_\Omega\right| \\ &\le C \left(\left|(\sigma\, u\nabla\chi^i, \nabla(T(\chi^i u)))_\Omega\right| + \left|(\sigma\,\nabla u, \nabla(\chi^i T(\chi^i u)))_\Omega\right| + \left|(\sigma\nabla u, T(\chi^i u)\nabla\chi^i)_\Omega\right| \right) \\ &\le C \left(\|f\|_{H^{-1}(\Omega)} \|u\|_{H_0^1(\Omega)} + \|u\|_\Omega \|u\|_{H_0^1(\Omega)} \right). \end{aligned} \tag{10}$$

Indeed, the operator T also belongs to $\mathcal{L}(L^2(\Omega))$.
Along the same lines, one obtains

$$\left\|\zeta^i u\right\|^2_{H^1_0(\Omega)} \quad \leq \quad C\left(\|f\|_{H^{-1}(\Omega)}\|u\|_{H^1_0(\Omega)} + \|u\|_\Omega \|u\|_{H^1_0(\Omega)}\right) \tag{11}$$

because the operators T_1 or T_2 (see (1) or (2)) implicitly used in the proofs of theorems 2.3 and 2.7 (with a continuation by 0 in $\Omega\backslash supp\,\zeta^i$), all belong to $\mathcal{L}(H^1_0(\Omega))$ and $\mathcal{L}(L^2(\Omega))$.

Finally, putting together the estimates (8), (9), (10) and (11), one concludes that the *a priori* estimate (7) holds.

3.5. **Concluding the proof of theorem 3.3**

Let us recall a classical result, due to J. Peetre [12] (see also lemma 5.1 in [8, Ch. 2], or lemma 3.4.1 in [7]).

Lemma 3.5. *Let X, Y and Z be three reflexive Banach spaces, such that X is compactly embedded into Z. Let $B \in \mathcal{L}(X,Y)$. Then the assertions below are equivalent:*

i) $\dim\ker B < \infty$, and $\mathrm{Im}\,B$ is closed in Y ;
ii) there exists $C > 0$ such that
$\|x\|_X \leq C\left(\|B\,x\|_Y + \|x\|_Z\right)$, $\forall x \in X$.

On the one hand, $H^1_0(\Omega)$ is compactly embedded into $L^2(\Omega)$, because Ω is a bounded subset of $\mathbb{R}^d$. On the other hand, $\mathrm{coker}\,A$ is isomorphic to $\ker A$. So theorem 3.3 follows from lemma 3.5, (7), and $\mathrm{ind}\,A = \dim\ker A - \dim\mathrm{coker}\,A = 0$.

4. Applications

4.1. **Case of smooth coefficients**

In the case where $\sigma_k \in \mathscr{C}^0(\overline{\Omega_k})$, $k = 1,2$, the statement of theorem 3.3 can be simplified. The contrast $\kappa_\sigma = \sigma_2/\sigma_1$ is considered here as an element of $\mathscr{C}^0(\overline{\Sigma})$.

Theorem 4.1. (continuous coefficients) *Assume that*

$$\textit{either}\begin{cases} \forall \boldsymbol{x} \in \Sigma\backslash\mathcal{S} \ \textit{(smooth part of } \Sigma\textit{)},\ \kappa_\sigma(\boldsymbol{x}) < -1 \\ \forall \boldsymbol{x}^i \in \mathcal{S}_{int} \cup \mathcal{S}^2_{ext},\ \kappa_\sigma(\boldsymbol{x}) < -I_{\alpha^i}, \\ \forall \boldsymbol{x}^i \in \mathcal{S}^1_{ext},\ \kappa_\sigma(\boldsymbol{x}) < -1 \end{cases} \qquad \textit{or}\begin{cases} \forall \boldsymbol{x} \in \Sigma\backslash\mathcal{S} \ \textit{(smooth part of } \Sigma\textit{)},\ \kappa_\sigma(\boldsymbol{x}) > -1 \\ \forall \boldsymbol{x}^i \in \mathcal{S}_{int} \cup \mathcal{S}^1_{ext},\ \kappa_\sigma(\boldsymbol{x}) > -1/I_{\alpha^i}. \\ \forall \boldsymbol{x}^i \in \mathcal{S}^2_{ext},\ \kappa_\sigma(\boldsymbol{x}) > -1 \end{cases}$$

Then, the operator $A : u \mapsto -div(\sigma\,\nabla u)$ of $\mathcal{L}(H^1_0(\Omega), H^{-1}(\Omega))$ is Fredholm of index 0.

4.2. **Case of constant coefficients**

When in addition σ_k, $k = 1,2$, are constant numbers, define

$$\hat{R}_\Sigma := \max\left(\max_{\boldsymbol{x}^i \in \mathcal{S}_{int}\cup\mathcal{S}^1_{ext}} I_{\alpha^i}, 1\right), \quad \check{R}_\Sigma := \max\left(\max_{\boldsymbol{x}^i \in \mathcal{S}_{int}\cup\mathcal{S}^2_{ext}} I_{\alpha^i}, 1\right).$$

There holds the

Theorem 4.2. (constant coefficients) *Assume that $\sigma_2/\sigma_1 \in \mathbb{R}^*_-\backslash\left[-\hat{R}_\Sigma; -1/\check{R}_\Sigma\right]$. Then, the operator $A : u \mapsto -div(\sigma\,\nabla u)$ of $\mathcal{L}(H^1_0(\Omega), H^{-1}(\Omega))$ is Fredholm of index 0.*

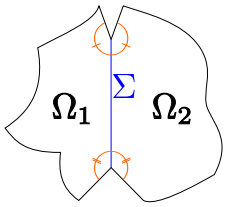

FIGURE 7. A is Fredholm of index 0 when $\sigma_2/\sigma_1 \in \mathbb{R}_-^* \setminus \{-1\}$.

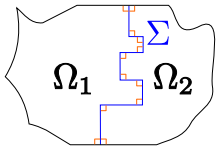

FIGURE 8. A is Fredholm of index 0 when $\sigma_2/\sigma_1 \in \mathbb{R}_-^* \setminus [-3; -1/3]$.

Remark 4.3. *With the help of Lax-Milgram's theorem, one proves easily that the operator A is an isomorphism of $\mathcal{L}(H_0^1(\Omega), H^{-1}(\Omega))$ when $\kappa_\sigma \in \mathbb{C}\backslash\mathbb{R}_-$. One concludes that, when $\sigma_2/\sigma_1 \in \mathbb{C}^*\backslash \left[-\hat{R}_\Sigma; -1/\check{R}_\Sigma\right]$, the operator A is Fredholm of index 0.*

We provide now some "practical" illustrations of these results in figures 7, 8, 9 and 10.

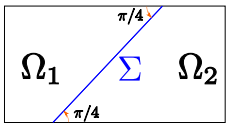

FIGURE 9. A is Fredholm of index 0 when $\sigma_2/\sigma_1 \in \mathbb{R}_-^* \backslash [-3; -1/3]$.

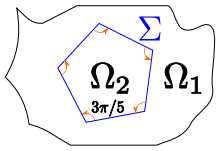

FIGURE 10. A is Fredholm of index 0 when $\sigma_2/\sigma_1 \in \mathbb{R}_-^* \backslash [-7/3; -3/7]$.

5. Optimality of the assumptions on σ

In this section, we establish some complementary results on the operator $A : u \mapsto -\mathrm{div}(\sigma \nabla u)$ of $\mathcal{L}(H_0^1(\Omega), H^{-1}(\Omega))$, in the case where σ does not fulfill all the assumptions of theorem 3.3. We use the contrast $\kappa_\sigma = \sigma_2/\sigma_1$ when σ_k, $k = 1, 2$ are constant numbers.

5.1. Case of the symmetric domain

Below, Ω is a *symmetric* domain.

Theorem 5.1. (SYMMETRIC DOMAIN & CONSTANT COEFFICIENTS) *Assume that*

- $\kappa_\sigma \neq -1$*: then A is an isomorphism;*
- $\kappa_\sigma = -1$*: then A is not a Fredholm operator ($\dim \ker A = \infty$).*

Proof. We consider without loss of generality that the interface Σ is included in the line of Eq. $y = 0$ (see figure 1).
Theorem 2.1 proves the result when $\kappa_\sigma \neq -1$.
Next, consider that $\kappa_\sigma = -1$. In this case, we prove that $\ker A$ is an infinite dimensional vector space. To that aim, let $g \in H_{00}^{1/2}(\Sigma)$, i.e. an element of $H^{1/2}(\Sigma)$ such that its continuation by 0 to the whole line of Eq. $y = 0$ belongs to $H^{1/2}(\mathbb{R})$. For $k = 1, 2$, consider then $u_k \in H^1_{0,\Gamma_k}(\Omega_k)$ such that

$$\left\{ \begin{array}{lcll} \Delta u_k & = & 0 & \text{in } \Omega_k \\ u_k & = & 0 & \text{on } \Gamma_k \\ u_k & = & g & \text{on } \Sigma \end{array} \right. .$$

By the uniqueness of the solution, we find that $u_2(x,y) = u_1(x,-y)$ a.e. in Ω_2, and it follows that

$$\sigma_1\, \partial_{\boldsymbol{n}} u_1 - \sigma_2\, \partial_{\boldsymbol{n}} u_2 = -\sigma_1(\, \partial_y u_1 + \partial_y u_2) = 0 \text{ a.e. on } \Sigma.$$

Summing up, the element u of $H^1_0(\Omega)$ defined by $u_{|\Omega_k} = u_k$ for $k = 1,2$ satisfies $\operatorname{div}(\sigma\, \nabla u) = 0$ in Ω, and as consequence $A\, u = 0$. As $H^{1/2}_{00}(\Sigma)$ is an infinite dimensional vector space, the same is true for $\ker A$. □

5.2. **Locally straight interface and contrast equal to -1**

Here, Ω is a domain of $\mathbb{R}^2$ which fulfills the assumptions of §3.1.

Theorem 5.2. (LOCALLY STRAIGHT INTERFACE & CONSTANT COEFFICIENTS) *Assume that $\kappa_\sigma = -1$, and that there is an open part of Σ which is straight. Then A is not a Fredholm operator.*

Remark 5.3. *The result remains true, assuming only that σ_1 and σ_2 are locally constant, and take opposite values, in a neighborhood of the straight part of Σ.*

Proof. According to lemma 3.5, if A is a Fredholm operator, then there exists $C > 0$ such that

$$\|u\|_{H^1_0(\Omega)} \leq C \left(\|A\, u\|_{H^{-1}(\Omega)} + \|u\|_\Omega \right), \forall u \in H^1_0(\Omega). \tag{12}$$

Following Hadamard's example, which allows one to prove that the Cauchy problem in the half-plane is not well-posed, we contradict (12).

Let $\boldsymbol{x}_0$ be a point on the (open) straight part of Σ. Up to a rotation of the coordinates system, we can assume that Σ is locally included in the line of Eq. $s = 0$, around $\boldsymbol{x}_0$. Next, let $b > 0$ be sufficiently small, so that $D :=]-b;b[\times]-b;b[\subset \Omega$. For $n \in \mathbb{N}$, define

$$u_n(s,t) := \begin{cases} \dfrac{\sinh n(b+s)\sin nt}{e^{nb}} & \text{in } [-b;0] \times [-b;b]\ ; \\ \dfrac{\sinh n(b-s)\sin nt}{e^{nb}} & \text{in } [0;b] \times [-b;b]\,. \end{cases} \tag{13}$$

Let $\chi_0 \in \mathscr{C}^\infty_0(\mathbb{R})$ be an even truncation function, equal to 1 in a neighborhood of 0, with support included in $]-b;b[$, and $0 \leq \chi_0 \leq 1$. Now, let $\chi(s,t) := \chi_0(s)\,\chi_0(t)$. Then, the continuation of $\chi\, u_n$ by 0 to Ω, still denoted by $\chi\, u_n$, belongs to $H^1_0(\Omega)$. We prove now the estimate below, with C independent of n:

$$\|A\,(\chi u_n)\|_{H^{-1}(\Omega)} \leq C\,(\|A\, u_n\|_{H^{-1}(D)} + \|u_n\|_D)\,. \tag{14}$$

Recall that

$$\|A\,(\chi u_n)\|_{H^{-1}(\Omega)} = \sup_{v \in H^1_0(\Omega),\, \|v\|_{H^1_0(\Omega)}=1} (\sigma\, \nabla(\chi u_n), \nabla v)_\Omega.$$

On the other hand, given $v \in H^1_0(\Omega)$, one has

$$(\sigma\, \nabla(\chi u_n), \nabla v)_\Omega = (\sigma\, \nabla u_n, \nabla(\chi\, v))_\Omega + (\sigma\, u_n\, \nabla\chi, \nabla v)_\Omega - (\nabla u_n, \sigma\, v\, \nabla\chi)_\Omega\,. \tag{15}$$

Consider next each term of the right-hand side of (15) separately.

- First term:
$$|(\sigma\, \nabla u_n, \nabla(\chi\, v))_\Omega| \leq C\, \|\operatorname{div}(\sigma\, \nabla u_n)\|_{H^{-1}(D)}\, \|v\|_{H^1_0(\Omega)}\,. \tag{16}$$
- Second term, using Cauchy-Schwarz inequality:
$$|(\sigma\, u_n\, \nabla\chi, \nabla v)_\Omega| \leq C\, \|u_n\|_D\, \|v\|_{H^1_0(\Omega)}\,. \tag{17}$$

- Third term, integrated by parts:

$$(\nabla u_n, \sigma\, v\, \nabla\chi)_\Omega = (u_n, \mathrm{div}(\sigma\, v\, \nabla\chi))_D. \tag{18}$$

Note that $\mathrm{div}(\sigma\, v\, \nabla\chi)$ belongs to $L^2(\Omega)$ (and so to $L^2(D)$), because one has $\sigma\, v\, \nabla\chi|_{\Omega_1} \in H^1(\Omega_1)$, $\sigma\, v\, \nabla\chi|_{\Omega_2} \in H^1(\Omega_2)$, and finally $\partial_{\boldsymbol{n}}\chi = 0$ on Σ. In addition, $\|\mathrm{div}(\sigma\, v\, \nabla\chi)\|_D \leq C\, \|v\|_{H^1_0(\Omega)}$. Therefore, (18) yields

$$|(\nabla u_n, \sigma\, v\, \nabla\chi)_\Omega| \quad \leq \quad C\, \|u_n\|_D\, \|v\|_{H^1_0(\Omega)}\,. \tag{19}$$

Adding up (16), (17) and (19) to bound the left-hand side of (15) leads to (14).

On the other hand, one can check by direct inspection that $A\, u_n = 0$ in D. Indeed, on $]-b;0[\times]-b;b[$ and respectively on $]0;b[\times]-b;b[$, there holds $\Delta u_n = 0$. Also, on the straight part of the interface, the trace of u_n matches. Then, as u_n is symmetric with respect to the interface and as the contrast is equal to -1, this implies that the flux $\sigma\, \partial_{\boldsymbol{n}} u_n$ also matches.
Next, $\|u_n\|_D \leq 2b\, \|u_n\|_{L^\infty(D)} < C$, with C independent of n. Consequently, according to (14), $(A\,(\chi u_n))_{n\in\mathbb{N}^*}$ is bounded in $H^{-1}(\Omega)$. But one can check, again by direct inspection (cf. lemma 7.1), that

$$\|\chi u_n\|_{H^1_0(\Omega)} \underset{n\to+\infty}{\longrightarrow} +\infty.$$

This contradicts (12), which ends the proof. □

5.3. **Criterion at boundary vertices**

Here, Ω is a domain of $\mathbb{R}^2$ which fulfills the assumptions of §3.1. Note that for $]-I_{\alpha^i};-1[$ and $]-1;-1/I_{\alpha^i}[$ to be non-empty intervals, one must have $\alpha_1^i \neq \alpha_2^i$.

Theorem 5.4. (BOUNDARY VERTEX & CONSTANT COEFFICIENTS)

1. *If there exists $\boldsymbol{x}^i \in \mathcal{S}^1_{ext}$ such that $\kappa_\sigma \in]-I_{\alpha^i};-1[$, then the operator $A : u \mapsto -div(\sigma\, \nabla u)$ of $\mathcal{L}(H^1_0(\Omega), H^{-1}(\Omega))$ is not Fredholm.*
2. *If there exists $\boldsymbol{x}^i \in \mathcal{S}^2_{ext}$ such that $\kappa_\sigma \in]-1;-1/I_{\alpha^i}[$, then the operator $A : u \mapsto -div(\sigma\, \nabla u)$ of $\mathcal{L}(H^1_0(\Omega), H^{-1}(\Omega))$ is not Fredholm.*

Remark 5.5. *In the cases $\kappa_\sigma = -I_{\alpha^i}$ (case 1. of theorem 5.4) or $\kappa_\sigma = -1/I_{\alpha^i}$ (case 2. of theorem 5.4), a logarithmic singularity appears. Consequently, the operator may not be Fredholm in those cases.*

Proof. Before we proceed, let us remark that, if we replace A by $-A$, then the roles of Ω_1 (where $\sigma > 0$ a.e.) and Ω_2 (where $\sigma < 0$ a.e.) are reversed: in particular, $\kappa_\sigma(-A) = 1/\kappa_\sigma(A)$. Also, to $\boldsymbol{x}^i \in \mathcal{S}^2_{ext}$ for the operator A corresponds $\boldsymbol{x}^i \in \mathcal{S}^1_{ext}$ for the operator $-A$, while $I_{\alpha^i}(-A) = I_{\alpha^i}(A)$. Now, assume that case 1. is true, and apply it to operator $-A$: if there exists $\boldsymbol{x}^i \in \mathcal{S}^2_{ext}$ such that $\kappa_\sigma(-A) \in]-I_{\alpha^i};-1[$, then the operator $-A : u \mapsto -\mathrm{div}(\sigma\, \nabla u)$ of $\mathcal{L}(H^1_0(\Omega), H^{-1}(\Omega))$ is not Fredholm. This is precisely case 2. for the operator A.
Let us now focus on the proof of case 1.. In the rest of the proof, we omit the index ^{i}. Assume that $\kappa_\sigma \in]-I_\alpha;-1[$: we prove in this case that (12) cannot hold, using a classical idea in the theory of elliptic operators in non-smooth domains (see for instance part V of the proof of theorem 1.2 of [10, page 104] or lemma 6.3.3 of [7]). For a value of the contrast lying in the interval $]-I_\alpha;-1[$, one can show that (follow §7.3.3 of [13]) there exists a singular function $S(r,\theta) = r^{i\,\eta}\varphi(\theta)$, with $\eta \in \mathbb{R}^*$ and φ piecewise smooth[2], such that $\mathrm{div}(\sigma\, \nabla S) = 0$ in a

[2]More precisely, one finds $\varphi|_{\Omega_1} = c_1\, \sinh(\eta\, \theta)$ and $\varphi|_{\Omega_2} = c_2\, \sinh(\eta\, (\theta - \gamma))$, where the constants c_1, c_2 are chosen to ensure matching traces and fluxes on the interface Σ.

neighborhood of the vertex. This singular function belongs to $L^2(\Omega)$, but not to $H^1(\Omega)$. Introduce next two truncation functions χ, $\zeta \in \mathscr{C}_0^\infty(\mathbb{R})$, such that

$$\chi(t) = \begin{cases} 1 & \text{if } t < -1 \\ 0 & \text{if } t > 0 \end{cases} \qquad \zeta(t) = \begin{cases} 1 & \text{if } t < d/2 \\ 0 & \text{if } t > d \end{cases} \qquad \text{with } d = d^i \text{ of §3.3.} \tag{20}$$

Define finally $u_n(r,\theta) := \chi_n(r,\theta)\,\zeta(r)\,S(r,\theta)$, with $\chi_n(r,\theta) := \chi(|\ln r| - n)$, $n \in \mathbb{N}^*$: in particular, χ_n is equal to 0 for $r \in [0; e^{-n}]$, and to 1 for $r \in [e^{-n+1}; 1]$. By construction (cf. lemma 7.3), for $n \in \mathbb{N}^*$, u_n belongs to $H_0^1(\Omega)$, and

$$\exists C > 0,\ \forall n,\ \|u_n\|_{L^\infty(\Omega)} < C \text{ and } \|u_n\|_{H_0^1(\Omega)} \underset{n\to+\infty}{\longrightarrow} +\infty. \tag{21}$$

To contradict (12), there remains to prove that the sequence $(\mathrm{div}(\sigma\,\nabla u_n))_{n\in\mathbb{N}^*}$ is bounded in $H^{-1}(\Omega)$, which is the more involved part of the proof.
As in the proof of theorem 5.2, we write

$$\|\mathrm{div}(\sigma\,\nabla u_n)\|_{H^{-1}(\Omega)} = \sup_{v\in H_0^1(\Omega),\ \|v\|_{H_0^1(\Omega)}=1} (\sigma\,\nabla u_n, \nabla v)_\Omega.$$

And, as before,

$$(\sigma\,\nabla u_n, \nabla v)_\Omega = (\sigma\,\nabla(\zeta\,S), \nabla(\chi_n\,v))_\Omega + (\sigma\,(\zeta\,S)\,\nabla\chi_n, \nabla v)_\Omega - (\nabla(\zeta\,S), \sigma\,v\,\nabla\chi_n)_\Omega. \tag{22}$$

Let us now study the right-hand side of (22).
For the first term, one has

$$|(\sigma\,\nabla(\zeta\,S), \nabla(\chi_n\,v))_\Omega| \le C\ \|\mathrm{div}(\sigma\,\nabla(\zeta\,S))\|_{H^{-1}(\Omega)}\,\|\chi_n\,v\|_{H_0^1(\Omega)}\,.$$

On the other hand, one has

$$\begin{aligned} \|\chi_n\,v\|_{H_0^1(\Omega)} &\le \|\chi_n\,\nabla v\|_\Omega + \|v\,\nabla\chi_n\|_\Omega \\ &\le \|\chi_n\|_{L^\infty(\Omega)}\,\|v\|_{H_0^1(\Omega)} + \left\|r^{-1}\,v\right\|_\Omega \|r\,\nabla\chi_n\|_{L^\infty(\Omega)}\,. \end{aligned}$$

Classically, one can prove the Hardy inequality below (see the paragraph before theorem 1.3.1 in [9])

$$\left\|r^{-1}\,v\right\|_\Omega \le C\ \|v\|_{H_0^1(\Omega)}\,. \tag{23}$$

By direct computations (cf. lemma 7.2.), one checks that $(r\,\nabla\chi_n)_{n\in\mathbb{N}^*}$ is bounded in $(L^\infty(\Omega))^2$. It follows that

$$|(\sigma\,\nabla(\zeta\,S), \nabla(\chi_n\,v))_\Omega| \le C\ \|\mathrm{div}(\sigma\,\nabla(\zeta\,S))\|_{H^{-1}(\Omega)}\,\|v\|_{H_0^1(\Omega)}\,. \tag{24}$$

One can check (cf. lemma 7.2) that $(\chi_n)_{n\in\mathbb{N}^*}$ is a bounded sequence in $H_0^1(\Omega)$. So, for the second term of (22), one writes simply

$$|(\sigma\,(\zeta\,S)\,\nabla\chi_n, \nabla v)_\Omega| \le C\ \|(\zeta\,S)\|_{L^\infty(\Omega)}\,\|v\|_{H_0^1(\Omega)}\,. \tag{25}$$

As far as the third term is concerned, let us integrate by parts, to find

$$(\nabla(\zeta\,S), \sigma\,v\,\nabla\chi_n)_\Omega = -((\zeta\,S), \mathrm{div}(\sigma\,v\,\nabla\chi_n))_\Omega.$$

Indeed, $\mathrm{div}(\sigma\,v\,\nabla\chi_n)$ belongs to $L^2(\Omega)$ (and so to $L^1(\Omega)$), because one has $\sigma\,v\,\nabla\chi_n|_{\Omega_1} \in H^1(\Omega_1)$, $\sigma\,v\,\nabla\chi_n|_{\Omega_2} \in H^1(\Omega_2)$ and finally $\partial_{\boldsymbol{n}}\chi_n = 0$ on Σ. It follows that

$$|(\nabla(\zeta\,S), \sigma\,v\,\nabla\chi_n)_\Omega| \le \|\zeta\,S\|_{L^\infty(\Omega)}\,\|\mathrm{div}(\sigma\,v\,\nabla\chi_n)\|_{L^1(\Omega)}\,.$$

Next, one gets the bound

$$\begin{array}{rcl}\|\mathrm{div}(\sigma\, v\, \nabla\chi_n)\|_{L^1(\Omega)} & \leq & C\,(\|\mathrm{div}(v\,\nabla\chi_n)\|_{L^1(\Omega_1)} + \|\mathrm{div}(v\,\nabla\chi_n)\|_{L^1(\Omega_2)}) \\ & \leq & C\,(\|\nabla v\cdot\nabla\chi_n\|_{L^1(\Omega)} + \|v\,\Delta\chi_n\|_{L^1(\Omega)}) \\ & \leq & C\,(\|v\|_{H_0^1(\Omega)}\,\|\chi_n\|_{H_0^1(\Omega)} + \left\|r^{-1}\,v\right\|_{\Omega}\,\|r\,\Delta\chi_n\|_{\Omega}) \\ & \leq & C\,\|v\|_{H_0^1(\Omega)}\ ,\end{array}$$

because $(r\,\Delta\chi_n)_{n\in\mathbb{N}^*}$ is a bounded sequence in $L^2(\Omega)$ (cf. lemma 7.2). For the third term, we conclude that

$$|(\nabla(\zeta\,S), \sigma\, v\,\nabla\chi_n)_\Omega| \leq C\ \|\zeta\,S\|_{L^\infty(\Omega)}\,\|v\|_{H_0^1(\Omega)}\,. \tag{26}$$

With the help of (24), (25) and (26), we can bound the right-hand side of (22) as follows:

$$|(\sigma\,\nabla(\chi_n\,(\zeta\,S)), \nabla v)_\Omega| \leq C\,(\|\mathrm{div}(\sigma\,\nabla(\zeta\,S))\|_{H^{-1}(\Omega)} + \|\zeta\,S\|_{L^\infty(\Omega)})\,\|v\|_{H_0^1(\Omega)}\,.$$

This leads to

$$\|\mathrm{div}(\sigma\,\nabla u_n)\|_{H^{-1}(\Omega)} \leq C\,(\|\mathrm{div}(\sigma\,\nabla(\zeta\,S))\|_{H^{-1}(\Omega)} + \|\zeta\,S\|_{L^\infty(\Omega)}). \tag{27}$$

Now, we recall that $(u_n)_{n\in\mathbb{N}^*}$ is bounded in $L^\infty(\Omega)$ (and so in $L^2(\Omega)$). As a consequence, the limit (21) and Ineq. (27), together with lemma 3.5, prove that A is not a Fredholm operator in the case where $\kappa_\sigma \in\,]-I_\alpha; -1[$. □

5.4. **Criterion at interior vertices**

Here, Ω is a domain of $\mathbb{R}^2$ which fulfills the assumptions of §3.1.

Theorem 5.6. (INTERIOR VERTEX & CONSTANT COEFFICIENTS) *If there exists $\boldsymbol{x}^i \in \mathcal{S}^1_{int}$ such that*

$$\begin{array}{llll} \textit{either:} & \alpha_1^i \leq \alpha_2^i & \textit{and} & \kappa_\sigma \in\,]-I_{\alpha^i}; -1]\,, \\ \textit{or:} & \alpha_1^i \geq \alpha_2^i & \textit{and} & \kappa_\sigma \in\, [-1; -1/I_{\alpha^i}[\,, \end{array} \tag{28}$$

then the operator $A : u \mapsto -div(\sigma\,\nabla u)$ of $\mathcal{L}(H_0^1(\Omega), H^{-1}(\Omega))$ is not Fredholm.

To prove this result, we would like to mimic the proof of theorem 5.4. However, Hardy's inequality (23) relies critically on the fact that the function v vanishes at the boundary. Assume now that one of the two conditions (28) is fulfilled, and consider the problem with an exterior vertex, obtained by restricting locally the geometry to the half-plane delimited by the bisector of the angles. Then, starting from the exterior singularity of theorem 5.4, one can build a continuation by using an odd reflection, which is actually a singularity for the problem around the interior vertex[3]. This singularity acts only on the skewsymmetric part of test-functions, so one can proceed as before to prove theorem 5.6.

5.5. **Comments on the optimality**

Loosely speaking, on a straight part of the interface for which $\kappa_\sigma = -1$, we established that the operator A is not Fredholm, because of a *linear singularity distribution.* Indeed, we proved that, at any point $\boldsymbol{x}_0$ of the (open) straight part of Σ, one can build a sequence of functions $(u_n)_n$ that prevents A from being a Fredholm operator (see theorem 5.2). On the other hand, if $\kappa_\sigma \neq -1$, the operator A is not Fredholm if there exist *pointwise singularities*, located at interior and/or boundary vertices of the interface. This situation happens for values of the contrast lying in an interval (see theorems 5.4 and 5.6).

Let us conclude by two cases not covered by theorem 3.3.
First, a domain $\Omega :=]-1;1[\times]-1;1[$, with subsets $\Omega_1 :=]-1;0[\times]-1;1[$ and $\Omega_2 :=]0;1[\times]-1;1[$ (see figure

[3]On the other hand, in either of the two cases $\kappa_\sigma \in\,]-1; -1/I_{\alpha^i}[$ and $\alpha_1^i < \alpha_2^i$, or $\kappa_\sigma \in\,]-I_{\alpha^i}; -1[$ and $\alpha_2^i < \alpha_1^i$, this does not work because the exterior singularity of theorem 5.4 is not available.

11–left). Assume that $\sigma = 1$ in Ω_1, $\sigma = -2$ in $]0;1[\times]0;1[$ in $\sigma = \beta \in \mathbb{R}^*_-$ in $]0;1[\times]-1;0[$. Given $\beta > -1$, there holds, for all $d > 0$,

$$\inf_{B(O,d)\cap\Omega_1} \sigma_1 < \sup_{B(O,d)\cap\Omega_2} |\sigma_2| \quad \text{and} \quad \inf_{B(O,d)\cap\Omega_2} |\sigma_2| < \sup_{B(O,d)\cap\Omega_1} \sigma_1.$$

So, the assumptions of theorem 3.3 are not fulfilled and, as a consequence, one can not conclude that the operator A of $\mathcal{L}(H^1_0(\Omega), H^{-1}(\Omega))$ is Fredholm.

Remark 5.7. *However, one can easily build by hand, for this simple configuration, an ad hoc operator T that allows one to prove T-coercivity directly for some $\beta > -1$. For that, the operator T is built using some line symmetries. For $u \in H^1_0(\Omega)$, the action of T is defined by*

$$(T\,u)(x,y) := \begin{cases} u_a(x,y) - 2u_d(-x,y) & \text{in } \Omega_a :=]-1;0[\times]0;1[\\ u_b(x,y) - 2u_d(-x,-y) & \text{in } \Omega_b :=]-1;0[^2 \\ -2u_a(-x,-y) + 2u_b(-x,y) - u_c(x,y) & \text{in } \Omega_c :=]0;1[\times]-1;0[\\ -u_d(x,y) & \text{in } \Omega_d :=]0;1[^2 \end{cases},$$

with $u_k := u|_{\Omega_k}$, for $k = a, b, c, d$.

On the other hand, if $\beta < -1$, then

$$\inf_{B(O,d)\cap\Omega_2} |\sigma_2| > \sup_{B(O,d)\cap\Omega_1} \sigma_1,$$

and theorem 3.3 allows to conclude that A is Fredholm (of index 0).
Second, a domain $\Omega :=]-1;1[\times]-1;1[$, with subsets $\Omega_1 :=]-1;0[\times]0;1[\cup]0;1[\times]-1;0[$ and $\Omega_2 :=]-1;0[\times]-1;0[\cup]0;1[\times]0;1[$ (see figure 11–right). Here, one cannot use theorem 3.3, because the boundaries $\partial\Omega_1$ and $\partial\Omega_2$ are not Lipschitz (see [1, Corrigendum]).

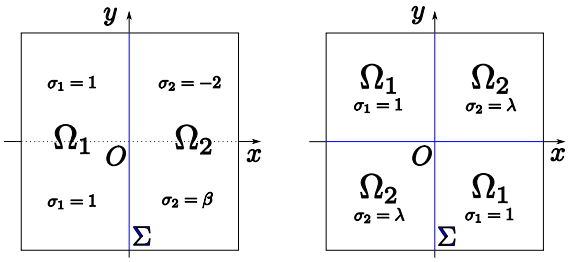

FIGURE 11. Two situations not covered by theorem 3.3: $\beta > -1$ on the left; $\lambda \in \mathbb{R}^*_-$ on the right.

6. Domains of $\mathbb{R}^3$

Generally speaking, one can use the same lines of thought to tackle the problem $(\mathscr{P})$ in a domain Ω of $\mathbb{R}^3$. Provided one can establish T-coercivity locally (cf. §2), one can prove that the operator $A \in \mathcal{L}(H^1_0(\Omega), H^{-1}(\Omega))$ is Fredholm. The main difference is that one has to deal with a larger number of elementary cases, and among them some can not be reduced to their lower-dimensional counterparts. Notations used previously are kept here.

We begin the study by elementary cases. We provide proofs only in the most illustrative cases.

6.1. Symmetric domain of $\mathbb{R}^3$

One obtains easily the same results as the ones stated in theorem 2.1.

6.2. **Prismatic edges**

Introduce the cylindrical coordinates (r, θ, z) centered on the edge, so that the cartesian coordinates are mapped as $(x, y, z) = (r\,\cos\,\theta, r\,\sin\,\theta, z)$. Let $H > 0$ denote the height of the cylinder, $R > 0$ its radius.

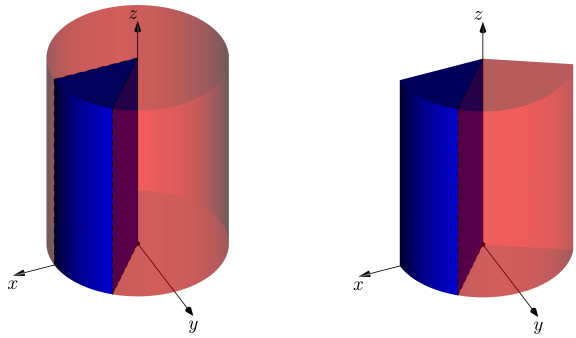

FIGURE 12. Geometry of prismatic edges: (left) interior edge. (Right) boundary edge.

6.2.1. *Interior edge*

Consider the geometry of figure 12–left. Given $0 < \alpha < 2\pi$, define

$$\Omega_1 := \{(r\,\cos\,\theta, r\,\sin\,\theta, z)\,|\,0 < r < R,\, 0 < \theta < \alpha,\, 0 < z < H\}\ ;$$
$$\Omega_2 := \{(r\,\cos\,\theta, r\,\sin\,\theta, z)\,|\,0 < r < R,\, \alpha < \theta < 2\pi,\, 0 < z < H\}\ .$$

Theorem 6.1. *(*INTERIOR EDGE IN 3D*) Assume that*

$$\max(\sigma_1^-/\sigma_2^+, \sigma_2^-/\sigma_1^+) > I_\alpha,\ \textit{with}\ I_\alpha := \max(\frac{\alpha}{2\pi - \alpha}, \frac{2\pi - \alpha}{\alpha}).$$

Then, there exists an isomorphism $T \in \mathcal{L}(H_0^1(\Omega))$ such that the form a is T-coercive and $A : u \mapsto -div(\sigma\,\nabla u)$ is an isomorphism from $H_0^1(\Omega)$ to $H^{-1}(\Omega)$.

Proof. Define the two operators R_1 and R_2 respectively by $(R_1\,u_1)(\rho, \Theta, Z) = u_1(\rho, \frac{\alpha}{\alpha - 2\pi}\,(\Theta - 2\pi), Z)$ and $(R_2\,u_2)(\rho, \Theta, Z) = u_2(\rho, \frac{\alpha - 2\pi}{\alpha}\,\Theta + 2\pi, Z)$. As before, the matching condition holds for R_1. We find as in theorem 2.3 that $\|R_1\|^2 \leq I_\alpha$. Similarly, the matching condition holds for R_2 and $\|R_2\|^2 \leq I_\alpha$. We conclude the proof as usual (see theorem 1.1). □

6.2.2. *Boundary edge*

Consider the geometry of figure 12–right. Given $0 < \alpha < \gamma < 2\pi$, define

$$\Omega_1 := \{(r\,\cos\,\theta, r\,\sin\,\theta, z)\,|\,0 < r < R,\, 0 < \theta < \alpha,\, 0 < z < H\}\ ;$$
$$\Omega_2 := \{(r\,\cos\,\theta, r\,\sin\,\theta, z)\,|\,0 < r < R,\, \alpha < \theta < \gamma,\, 0 < z < H\}\,.$$

One obtains the same results as the ones of theorem 2.7.

6.3. **Axisymmetric edges**

We refer to the geometry of figure 13, with toroidal coordinates (r, θ, φ) such that cartesian coordinates are mapped as $(x, y, z) = (\cos\,\theta\,(R + r\,\cos\,\varphi), \sin\,\theta\,(R + r\,\cos\,\varphi), r\,\sin\,\varphi)$. Here, $R > 0$ denotes the radius of the

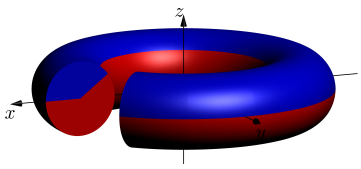

FIGURE 13. Geometry of an interior axisymmetric edge.

torus. Given $0 < d < R$ and $0 < \alpha < 2\pi$, define

$$\Omega_1 := \{(\cos\theta\,(R + r\,\cos\varphi), \sin\theta\,(R + r\,\cos\varphi), r\,\sin\varphi)\,|\,0 < r < d,\ 0 \le \theta < 2\pi,\ 0 < \varphi < \alpha\}\ ;$$
$$\Omega_2 := \{(\cos\theta\,(R + r\,\cos\varphi), \sin\theta\,(R + r\,\cos\varphi), r\,\sin\varphi)\,|\,0 < r < d,\ 0 \le \theta < 2\pi,\ \alpha < \varphi < 2\pi\}\ .$$

Theorem 6.2. *(*AXISYMMETRIC INTERIOR EDGE IN 3D*) Assume that*

$$\max(\sigma_1^-/\sigma_2^+, \sigma_2^-/\sigma_1^+) > \frac{1 + d/R}{1 - d/R} I_\alpha,\ \textit{with } I_\alpha := \max(\frac{\alpha}{2\pi - \alpha}, \frac{2\pi - \alpha}{\alpha}).$$

Then, there exists an isomorphism $T \in \mathcal{L}(H_0^1(\Omega))$ such that the form a is T-coercive and $A : u \mapsto -div(\sigma\,\nabla u)$ is an isomorphism from $H_0^1(\Omega)$ to $H^{-1}(\Omega)$.

Proof. Introduce the operators R_1 and R_2 respectively defined by $(R_1\,u_1)(\rho, \Theta, \Phi) = u_1(\rho, \Theta, \frac{\alpha}{\alpha - 2\pi}(\Phi - 2\pi))$ and $(R_2\,u_2)(\rho, \Theta, \Phi) = u_2(\rho, \Theta, \frac{\alpha - 2\pi}{\alpha}\,\Phi + 2\pi)$. The matching conditions hold for R_1 and R_2.
To compute the norm of R_1, let $u_1 \in H^1_{0,\,\Gamma_1}(\Omega_1)$. With the help of the change of (toroidal) variables $(r, \theta, \varphi) = (\rho, \Theta, \frac{\alpha}{\alpha - 2\pi}(\Phi - 2\pi))$, we find[4]

$$\begin{aligned}
\|\nabla(R_1\,u_1)\|^2_{\Omega_2} &= \int_{\Omega_2} \left(\left(\frac{\partial(R_1\,u_1)}{\partial\rho} \right)^2 + \frac{1}{(R + \rho\,\cos\Phi)^2} \left(\frac{\partial(R_1\,u_1)}{\partial\Theta} \right)^2 \right) \rho\,(R + \rho\,\cos\Phi)\,d\rho d\Phi d\Theta \\
&\quad + \int_{\Omega_2} \frac{1}{\rho^2} \left(\frac{\partial(R_1\,u_1)}{\partial\Phi} \right)^2 \rho\,(R + \rho\,\cos\Phi)\,d\rho d\Phi d\Theta \\
&\le \frac{2\pi - \alpha}{\alpha} \int_{\Omega_1} \left(\frac{\partial u_1}{\partial r} \right)^2 r\,(R + r\,\cos(\frac{2\pi - \alpha}{\alpha}\,\varphi))\,dr d\varphi d\theta \\
&\quad + \frac{2\pi - \alpha}{\alpha} \int_{\Omega_1} \frac{1}{(R + r\,\cos(\frac{2\pi - \alpha}{\alpha}\,\varphi))^2} \left(\frac{\partial u_1}{\partial\theta} \right)^2 r\,(R + r\,\cos(\frac{2\pi - \alpha}{\alpha}\,\varphi))\,dr d\varphi d\theta \\
&\quad + \frac{\alpha}{2\pi - \alpha} \int_{\Omega_1} \frac{1}{r^2} \left(\frac{\partial u_1}{\partial\varphi} \right)^2 r\,(R + r\,\cos(\frac{2\pi - \alpha}{\alpha}\,\varphi))\,dr d\varphi d\theta.
\end{aligned}$$

By direct inspection, one finds

$$\frac{R + r\,\cos(\frac{2\pi - \alpha}{\alpha}\,\theta)}{R + r\,\cos\theta} \le \frac{1 + d/R}{1 - d/R} \quad \text{and} \quad \frac{R + r\cos\theta}{R + r\cos(\frac{2\pi - \alpha}{\alpha}\,\theta)} \le \frac{1 + d/R}{1 - d/R}, \quad \forall r \in]0; d[,\ \forall\theta \in]0; \alpha[,$$

so one obtains $\|R_1\|^2 \le \frac{1 + d/R}{1 - d/R}\,I_\alpha$. Similarly, $\|R_2\|^2 \le \frac{1 + d/R}{1 - d/R}\,I_\alpha$. We conclude as in the proof of theorem 1.1. □

Remark 6.3. *If $\max(\sigma_1^-/\sigma_2^+, \sigma_2^-/\sigma_1^+) > I_\alpha$, then according to theorem 6.2, $A : u \mapsto -div(\sigma\,\nabla u)$ is an isomorphism from $H_0^1(\Omega)$ to $H^{-1}(\Omega)$ for d/R small enough.*

[4]See §7.3 for complementary computations.

Remark 6.4. *We focused here on the case of an interior axisymmetric edge. Boundary axisymmetric edges can be handled as before, with a final result like theorem 2.7.*

6.4. **Conical vertex**

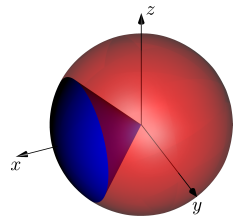

FIGURE 14. Geometry of an interior conical vertex.

Consider the geometry of figure 14, and the associated spherical coordinates (r, θ, φ) centered at the origin. The cartesian coordinates are now mapped as $(x, y, z) = (r \cos\theta, r \sin\theta \cos\varphi, r \sin\theta \sin\varphi)$. Let $R > 0$ and $0 < \alpha < \pi$, and define

$$\Omega_1 := \{(r \cos\theta, r \sin\theta \cos\varphi, r \sin\theta \sin\varphi),\ 0 < r < R,\ 0 \le \theta < \alpha,\ 0 \le \varphi < 2\pi\}\ ;$$
$$\Omega_2 := \{(r \cos\theta, r \sin\theta \cos\varphi, r \sin\theta \sin\varphi),\ 0 < r < R,\ \alpha < \theta \le \pi,\ 0 \le \varphi < 2\pi\}\ .$$

Theorem 6.5. *(*CONICAL INTERIOR VERTEX IN 3D*) Assume that*

$$\begin{cases} \sigma_1^-/\sigma_2^+ > I_\alpha^2 & \text{or} \quad \sigma_2^-/\sigma_1^+ > I_\alpha \quad \text{if } \alpha \le \pi/2 \\ \sigma_2^-/\sigma_1^+ > I_\alpha^2 & \text{or} \quad \sigma_1^-/\sigma_2^+ > I_\alpha \quad \text{if } \alpha \ge \pi/2 \end{cases}, \quad \text{with } I_\alpha := \max\left(\frac{\alpha}{\pi-\alpha}, \frac{\pi-\alpha}{\alpha}\right).$$

Then, there exists an isomorphism $T \in \mathcal{L}(H_0^1(\Omega))$ such that the form a is T-coercive and $A : u \mapsto -div(\sigma\,\nabla u)$ is an isomorphism from $H_0^1(\Omega)$ to $H^{-1}(\Omega)$.

Remark 6.6. *In this case, the question of finding the optimal criteria remains open. For instance, one can improve the bound on $\|R_1\|^2$ by using in the proof below a non-linear map.*

Proof. Below, we use the general formalism introduced in remark 2.6, and we consider the case $\alpha \le \pi/2$.
Define the operators R_1 and R_2 by $(R_1\, u_1)(\rho, \Theta, \Phi) = u_1(\rho, g_1(\Theta), \Phi)$, and by $(R_2\, u_2)(\rho, \Theta, \Phi) = u_2(\rho, g_2(\Theta), \Phi)$. Here, g_1 is a $\mathscr{C}^1$ diffeomorphism from $[\alpha; \pi]$ to $[0; \alpha]$ such that $g_1(\pi) = 0$ and $g_1(\alpha) = \alpha$ whereas g_2 is a $\mathscr{C}^1$ diffeomorphism from $[0; \alpha]$ to $[\alpha; \pi]$ such that $g_2(0) = \pi$ and $g_2(\alpha) = \alpha$. We denote h_1 (resp. h_2) the inverse of g_1 (resp. g_2).
The matching conditions hold. We evaluate the norm of R_1. Let $u_1 \in H^1_{0,\Gamma_1}(\Omega_1)$. Performing the change of variables $(r, \theta, \varphi) = (\rho, g_1(\Theta), \Phi)$, we find successively

$$\begin{aligned} \|\nabla(R_1\, u_1)\|^2_{\Omega_2} &= \int_{\Omega_2} \left(\frac{\partial(R_1\, u_1)}{\partial\rho}\right)^2 + \frac{1}{\rho^2}\left(\frac{\partial(R_1\, u_1)}{\partial\Theta}\right)^2 + \frac{1}{(\rho \sin\Theta)^2}\left(\frac{\partial(R_1\, u_1)}{\partial\Phi}\right)^2 \rho^2\, d\rho \sin\Theta\, d\Theta d\Phi \\ &\le \int_{\Omega_1} \left(\frac{\partial u_1}{\partial r}\right)^2 r^2\, dr \sin(h_1(\theta))\, |h_1'(\theta)|\, d\theta d\varphi + \int_{\Omega_1} \frac{1}{r^2\, |h_1'(\theta)|^2}\left(\frac{\partial u_1}{\partial\theta}\right)^2 r^2\, dr \sin(h_1(\theta))\, |h_1'(\theta)|\, d\theta d\varphi \\ &\quad + \int_{\Omega_1} \frac{1}{(r \sin h_1(\theta))^2}\left(\frac{\partial u_1}{\partial\varphi}\right)^2 r^2\, dr \sin(h_1(\theta))\, |h_1'(\theta)|\, d\theta d\varphi. \end{aligned}$$

Let us take the linear map $g_1(\Theta) = \frac{\alpha}{\alpha-\pi}\,(\Theta-\pi)$ (so $h_1(\theta) = \frac{\alpha-\pi}{\alpha}\,\theta+\pi$). One finds

$$\frac{|h_1'(\theta)|\,\sin(h_1(\theta))}{\sin\theta} \le I_\alpha^2, \quad \frac{\sin(h_1(\theta))}{|h_1'(\theta)|\,\sin\theta} \le 1, \quad \frac{|h_1'(\theta)|\,\sin\theta}{\sin(h_1(\theta))} \le I_\alpha, \quad \forall\theta \in]0;\alpha[.$$

Thus, there holds $\|\nabla(R_1\,u_1)\|_{\Omega_2}^2 \le I_\alpha^2\,\|\nabla u_1\|_{\Omega_1}^2$ and $\|R_1\|^2 \le I_\alpha^2$. On the other hand, using $g_2(\Theta) = \frac{\alpha-\pi}{\alpha}\,\Theta+\pi$ (so $h_2(\Theta) = \frac{\alpha}{\alpha-\pi}\,(\Theta-\pi)$), one finds $\|R_2\|^2 \le I_\alpha$. We conclude as in the proof of theorem 1.1.
One proceeds similarly to deal with the case $\pi/2 < \alpha < \pi$. □

Remark 6.7. *We focused here on the case of an interior conical vertex. Boundary conical vertices can be handled as above.*

6.5. **Fichera's corner**

In a domain of $\mathbb{R}^3$, it can happen that edges and vertices interact with one another, in ways which are not covered by the theory we developed before for domains of $\mathbb{R}^2$. To illustrate this situation, we consider a "famous" example, the so-called Fichera's corner. More precisely, let us define $\Omega :=]-1;1[^3$, with $\Omega_1 :=]0;1[^3$, and Ω_2 the open set such that $\overline{\Omega_2} := \overline{\Omega}\backslash\overline{\Omega_1}$.

Theorem 6.8. *(*Fichera's corner*) Assume that*

$$\max(\sigma_1^-/\sigma_2^+, \sigma_2^-/\sigma_1^+) > 7.$$

Then, there exists an isomorphism $T \in \mathcal{L}(H_0^1(\Omega))$ such that the form a is T-coercive and $A : u \mapsto -div(\sigma\,\nabla u)$ is an isomorphism from $H_0^1(\Omega)$ to $H^{-1}(\Omega)$.

Proof. With the help of reflection symmetries[5], we define the operator R_1 by

$$(R_1\,u_1)(x,y,z) = \begin{cases} u_1(-x,y,z) & \text{in} \quad \Omega_2^1 :=]-1;0[\times]0;1[^2 \\ u_1(x,-y,z) & \text{in} \quad \Omega_2^2 :=]0;1[\times]-1;0[\times]0;1[\\ u_1(x,y,-z) & \text{in} \quad \Omega_2^3 :=]0;1[^2\times]-1;0[\\ u_1(-x,-y,z) & \text{in} \quad \Omega_2^4 :=]-1;0[^2\times]0;1[\\ u_1(-x,y,-z) & \text{in} \quad \Omega_2^5 :=]-1;0[\times]0;1[\times]-1;0[\\ u_1(x,-y,-z) & \text{in} \quad \Omega_2^6 :=]0;1[\times]-1;0[^2 \\ u_1(-x,-y,-z) & \text{in} \quad \Omega_2^7 :=]-1;0[^3 \end{cases}.$$

Next, we define R_2 by

$$\begin{aligned}(R_2\,u_2)(x,y,z) \;=\; & u_2^1(-x,y,z) + u_2^2(x,-y,z) + u_2^3(x,y,-z) \\ & -u_2^4(-x,-y,z) - u_2^5(-x,y,-z) - u_2^6(x,-y,-z) \\ & +u_2^7(-x,-y,-z).\end{aligned}$$

Above, $(u_2^\ell)_{\ell=1,7}$ respectively denote the restriction of u_2 to $(\Omega_2^\ell)_{\ell=1,7}$.
The matching conditions hold. Then, one obtains easily that for all $u_1 \in H_{0,\Gamma_1}^1(\Omega_1)$, $\|\nabla(R_1\,u_1)\|_{\Omega_2}^2 = 7\,\|\nabla u_1\|_{\Omega_1}^2$. On the other hand, for all $u_2 \in H_{0,\Gamma_2}^1(\Omega_2)$, $\|\nabla(R_2\,u_2)\|_{\Omega_1}^2 \le 7\,\|\nabla u_2\|_{\Omega_2}^2$. Indeed, there holds classically $(\sum_{k=1}^7 a_k)^2 \le 7\,\sum_{k=1}^7 {a_k}^2$, for all $(a_1,\ldots,a_7) \in \mathbb{R}^7$. We conclude as in the proof of theorem 1.1. □

6.6. **General geometries in $\mathbb{R}^3$**

To establish that the operator A is Fredholm, in the case of general geometries in $\mathbb{R}^3$, one can proceed by localization, as in §3 (cf. theorem 3.3) and §4. Also, one can prove optimality results, in the same spirit of §5.

[5] A similar approach has been recently used by Nicaise *et al* in [11] in a geometry of $\mathbb{R}^2$, with $\Omega :=]-1;1[^2$ and $\Omega_1 :=]0;1[^2$.

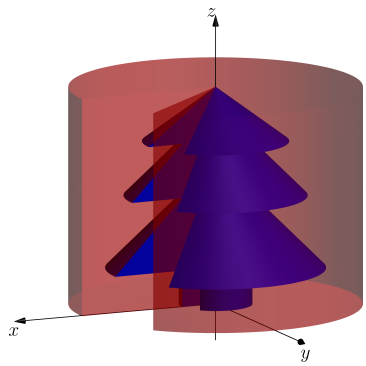

FIGURE 15. Joyeux Noël, aka. Merry Christmas!

We do not provide the details here, but instead comment on the case of Fichera's corner. For simplicity, let us consider constant coefficients σ_1 and σ_2, and a situation in which the contrast $\kappa_\sigma = \sigma_2/\sigma_1$ lies within the critical interval $[-7; -1/7]$, i.e. the case not covered by theorem 6.8. Loosely speaking, one finds that

- If $\kappa_\sigma = -1$ then there exists a *surface singularity distribution*. Indeed, at each point standing on one of the three (open) faces of the interface, one can build a sequence of functions that prevents A from being Fredholm. To achieve this result, one extends the construction given in the proof of theorem 5.2.
- If $\kappa_\sigma \in]-3; -1[$ then there exists a *linear singularity distribution*: at each point standing on one of the three (open) lines of the interface, one can build a sequence of functions that prevents A from being Fredholm, using the pointwise singularities exhibited in theorem 5.6.
- If $\kappa_\sigma \in]-7; -1[$ then there exists a *pointwise singularity*, which can be build in the same spirit as those of theorem 5.6.

7. Missing computations

7.1. Computations for theorem 5.2

Let $b > 0$ such that $[-b; b] \times [-b; b] \subset \Omega$. We define $(u_n)_n$ as in (13), and a truncation function $\chi \in \mathscr{C}_0^\infty(\mathbb{R}^2)$, equal to 1 in $[-b/2; b/2] \times [-b/2; b/2]$.

Lemma 7.1. *There holds* $\|\chi u_n\|_{H_0^1(\Omega)} \underset{n\to+\infty}{\longrightarrow} +\infty$.

Proof. Introduce $\tilde{D} := [-b/2; b/2] \times [-b/2; b/2]$, and write

$$\begin{aligned}
\|\chi u_n\|^2_{H_0^1(\Omega)} &\geq \|\nabla u_n\|^2_{\tilde{D}} \geq \|\partial_t u_n\|^2_{\tilde{D}} \\
&\geq 2\int_{-b/2}^{b/2}\int_0^{b/2} n^2 \cos^2 nt \, \frac{\sinh^2 n(b-s)}{e^{2nb}} \, ds\, dt \\
&\geq 2\, n^2 \int_{-b/2}^{b/2} \cos^2 nt \, dt \int_0^{b/2} \frac{\sinh^2 n(b-s)}{e^{2nb}} \, ds \\
&\geq 2\, n^2 \left[\frac{b}{2} + \frac{\sin nb}{2n}\right] \int_0^{b/2} \frac{\sinh^2 n(b-s)}{e^{2nb}} \, ds.
\end{aligned}$$

But one has

$$\begin{aligned} 4\int_0^{b/2} \frac{\sinh^2 n(b-s)}{e^{2nb}}\,ds &= \int_0^{b/2} e^{-2ns} - 2e^{-2nb} + e^{2ns-4nb}\,ds \\ &= (\frac{1}{2n} - \frac{e^{-nb}}{2n}) - (b\,e^{-2nb}) + e^{-4nb}(\frac{e^{nb}}{2n} - \frac{1}{2n}) \sim \frac{1}{2n}. \end{aligned}$$

Hence, there exists $C > 0$, such that for large n, one has $\|\chi u_n\|^2_{H^1_0(\Omega)} > C\,n$. □

7.2. **Computations for theorem 5.4**

Define $u_n(r,\theta) := \chi_n(r,\theta)\,\zeta(r)\,S(r,\theta)$, where χ and ζ are two truncation functions of $\mathscr{C}^\infty_0(\mathbb{R})$ with properties described in (20), $\chi_n(r,\theta) := \chi(-\ln r - n)$ and $S(r,\theta) := r^{i\,\eta}\varphi(\theta)$. We suppose that $d < 1$: χ_n is equal to 0 in $[0; e^{-n}]$, and to 1 in $[e^{-n+1}; 1]$. Obviously, $(u_n)_{n\in\mathbb{N}}$ is bounded in $L^\infty(\Omega)$, and also in $L^2(\Omega)$.

Lemma 7.2. *The sequence $(\chi_n)_{n\in\mathbb{N}^*}$ (respectively $(r\nabla\chi_n)_{n\in\mathbb{N}}$, $(r\Delta\chi_n)_{n\in\mathbb{N}}$) is bounded in $H^1_0(\Omega)$ (resp. $(L^\infty(\Omega))^2$, $L^2(\Omega)$).*

Proof. The sequence $(\chi_n)_{n\in\mathbb{N}^*}$ is bounded in $L^\infty(\Omega)$, and also in $L^2(\Omega)$. On the other hand, for d small enough and n large enough, as χ_n is independent of θ, one finds successively

$$\begin{aligned} \|\nabla\chi_n\|^2_\Omega &= \int_0^d\int_0^{2\pi} |\partial_r\chi_n|^2\,rd\theta dr = 2\pi\int_0^d |\partial_r\chi_n|^2\,rdr = 2\pi\int_{e^{-n}}^{e^{-n+1}} |\partial_r\chi_n|^2\,rdr \\ &= 2\pi\int_{e^{-n}}^{e^{-n+1}} r^{-1}|\chi'(-\ln r - n)|^2\,dr \\ &\le C\int_{e^{-n}}^{e^{-n+1}} r^{-1}\,dr = C\,[-n+1-(-n)] = C. \end{aligned}$$

Concerning the boundedness of $(r\nabla\chi_n)_{n\in\mathbb{N}}$ in $(L^\infty(\Omega))^2$, the result follows simply from $\partial_r\chi_n = r^{-1}\chi'(-\ln r - n)$, $\partial_\theta\chi_n = 0$.
Finally, to prove that the sequence $(r\Delta\chi_n)_{n\in\mathbb{N}}$ is bounded in $L^2(\Omega)$, one computes

$$\begin{aligned} \|r\Delta\chi_n\|^2_\Omega &\le C\int_0^d\int_0^{2\pi} r^2(|r^{-2}(r\partial_r)^2\chi_n|^2)\,rd\theta dr \\ &\le C\int_0^d r^{-1}|(r\partial_r)^2\chi_n|^2\,dr \le C\int_{e^{-n}}^{e^{-n+1}} r^{-1}|(r\partial_r)^2\chi_n|^2\,dr \\ &\le C\int_{e^{-n}}^{e^{-n+1}} r^{-1}\,dr = C\,[(1-n)-(-n)] = C. \end{aligned}$$

□

Lemma 7.3. *There holds $\|u_n\|_{H^1_0(\Omega)} \underset{n\to+\infty}{\longrightarrow} +\infty$.*

Proof. One writes

$$\begin{aligned} \|u_n\|^2_{H^1_0(\Omega)} &\ge \int_0^{d/2}\int_0^{2\pi} r^{-2}|\chi_n|^2|\partial_\theta S|^2\,rd\theta dr \\ &\ge \int_0^{d/2}\int_0^{2\pi} r^{-2}|\chi_n|^2|r^{i\eta}\partial_\theta\varphi\,r^{-i\eta}\partial_\theta\overline{\varphi}|^2\,rd\theta dr \\ &\ge C\int_0^{d/2} r^{-1}|\chi_n|^2\,dr \\ &\ge C\int_{e^{-n+1}}^{d/2} r^{-1}\,dr = C\,(n-1+d/2) \underset{n\to\infty}{\longrightarrow} +\infty. \end{aligned}$$

□

7.3. **Toroidal coordinates**

Considering the geometry of figure 13, introduce the change of variables $(x, y, z) = (\cos\,\theta\,(R{+}r\,\cos\,\varphi), \sin\,\theta\,(R{+}r\,\cos\,\varphi), r\,\sin\,\varphi)$, for $R > 0$. The jacobian associated with this change of variables is

$$\begin{pmatrix} \cos\theta\cos\varphi & -\sin\theta\,(R + r\cos\varphi) & -r\cos\theta\sin\varphi \\ \sin\theta\cos\varphi & \cos\theta\,(R + r\cos\varphi) & -r\sin\theta\sin\varphi \\ \sin\varphi & 0 & r\cos\varphi \end{pmatrix}.$$

The elementary volume in toroidal coordinates is then $r\,(R + r\,\cos\varphi)\,dr d\varphi d\theta$.
Also, the gradient in toroidal coordinates writes

$$\nabla u = \begin{pmatrix} \dfrac{\partial u}{\partial r} \\ \dfrac{1}{R + r\,\cos\varphi}\dfrac{\partial u}{\partial \theta} \\ \dfrac{1}{r}\dfrac{\partial u}{\partial \varphi} \end{pmatrix}.$$

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