Minimum-weight perfect matching for non-intrinsic distances on the line

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MINIMUM-WEIGHT PERFECT MATCHING FOR NON-INTRINSIC DISTANCES ON THE LINE

JULIE DELON, JULIEN SALOMON, AND ANDREI SOBOLEVSKI

Abstract. We consider a minimum-weight perfect matching problem on the line and establish a "bottom-up" recursion relation for weights of partial minimum-weight matchings.

1. Introduction

We start with recalling a few notions from combinatorial optimization on graphs. A *matching* in an undirected graph is any set of its mutually disjoint edges: no two edges from such set can share a vertex. A matching is called *perfect* if it involves all vertices of the graph (the number of vertices is then necessarily even).

Depending on the structure of the graph, perfect matchings may be many. Suppose that edges of a graph are endowed with real *weights*; then it makes sense to look for a perfect matching composed from a set of edges with a minimum sum of weights. In this note we treat a particular case of this *minimum-weight perfect matching* problem where the graph is complete, all its vertices are located on a line, and edge weights are related to distances between the vertices along the line.

A bipartite version of this problem, in which vertices are divided into two equal classes and edges of the matching must connect vertices of different class, reduces to transport optimization. For a particular class of cost functions (those of "concave type") this problem has been thoroughly treated in the measure-theoretic setting in [10]. Similar problems have also long been considered in the algorithmics literature for the specific case of the distance $|x - y|$, assuming that the measures are discrete [1, 9, 11]. An algorithm for a general cost function of a concave type has been proposed recently in [4, 5].

In the present note we consider the minimum-weight perfect matching problem on a complete graph without assuming it to be bipartite. However, for the class of weight functions generated by distances, the non-bipartite problem turns out to be essentially equivalent to a bipartite problem with alternating points due to a "no-crossing" property of the optimal matching. This observation, which we owe to S. Nechaev, allows to employ the theory developed for the bipartite case in [4, 5] to the non-bipartite matching.

The main contribution of the present note is a specific "bottom-up" recursion relation (6) for partial minimum-weight matchings. This recursion follows from a "localization" property of minimum-cost perfect matchings for concave cost functions (Theorem 5) and provides a new perspective on the construction of [4, 5].

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The note is organized as follows. Section 2 is a review of the basic construction of metrics on the real line. In Section 3 we recall the problem of minimum-weight perfect matching and cite a few useful results about the structure of its solution. Section 4 contains the key technical result, a kind of localization principle for minimum-weight matching. Finally in Section 5 we derive the recursive relation for partial minimum-weight matchings. We also discuss the ensuing algorithm and compare it to modern variants of the Edmonds blossom algorithm for the minimum-weight perfect matching problem.

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2. Intrinsic and non-intrinsic distances on the line

Recall the usual axioms for a distance $d(\cdot,\cdot)$ on the real line $\mathbf{R}$: for all $x, y, z \in \mathbf{R}$,

(D1) $d(x, y) \geq 0$ with $d(x, y) = 0$ iff $x = y$;
(D2) $d(x, y) = d(y, x)$;
(D3) $d(x, y) + d(y, z) \geq d(x, z)$.

These axioms are satisfied by the distance $d(x, y) = |x-y|$ as well as by any distance of the form

$$d_g(x, y) = g(|x - y|), \tag{1}$$

where $g(\cdot)$ is a nonnegative concave function defined for all $x \geq 0$ such that $g(x) = 0$ iff $x = 0$. Here concavity means that $g(\lambda x + (1 - \lambda)y) \geq \lambda g(x) + (1 - \lambda)g(y)$ for all $x, y \geq 0$ and all $\lambda \in [0, 1]$. Note that the distance d_g is *homogeneous* with respect to shifts:

(D4) $d_g(x + t, y + t) = d_g(x, y)$ for all $x, y, t \in \mathbf{R}$.

Conversely, any distance satisfying the axioms (D1)–(D4) has the form (1) for a suitable nonnegative concave function g: indeed, take $g(x) = d(0, x)$ for $x \geq 0$ and check that concavity follows from (D3).

An example of this construction is the distance $|x - y|^\alpha$ with $0 \leq \alpha \leq 1$, where

$$|x - y|^0 = \begin{cases} 0, & x = y, \\ 1, & x \neq y \end{cases}$$

is the "discrete distance"; here we are mostly interested in the case $0 < \alpha < 1$.

An important property of a distance $d(\cdot,\cdot)$ is whether it is *intrinsic*. To recall the corresponding definition, take two distinct points $x, y \in \mathbf{R}$ and connect them with parameterized curves taking values $x(t)$ in $\mathbf{R}$, i.e., suppose that $0 \leq t \leq 1$, $x(0) = x$, and $x(1) = y$. By the triangle inequality,

$$d(x, y) \leq \inf \sum_{0 \leq i < N} d(x(t_i), d(x(t_{i+1})), \tag{2}$$

where the infimum is taken over all curves connecting x to y and all meshes $0 = t_0 < t_1 < t_2 < \cdots < t_N = 1$ with $N \geq 1$. The distance d is called intrinsic if (2) is an equality for all x, y (see, e.g., [2]).

The distance $|x - y|$ and its scalar multiples are the only homogeneous intrinsic distances in $\mathbf{R}$. For distances $|x - y|^\alpha$ with $0 < \alpha < 1$, connecting curves have infinite length unless $x = y$, and therefore these distances are not intrinsic.

The intuition behind this notion is that whereas the geometry of $\mathbf{R}$ equipped with an intrinsic distance is fully determined by the distance, the same line $\mathbf{R}$ with a non-intrinsic distance should be viewed as embedded in an auxiliary space of a larger dimension, and geometry on $\mathbf{R}$ is induced by that in the embedding space. According to the Assouad embedding theorem (see, e.g., [8]), for the distance $|x-y|^\alpha$ on $\mathbf{R}$ the embedding space has dimension of the order $1/\alpha$ for small α.

3. Minimum-weight perfect matchings

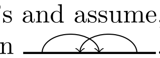
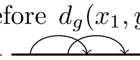
Consider an even number of points $x_1 < x_2 < \cdots < x_{2n}$ on the real line $\mathbf{R}$ equipped with a distance d and look for a *minimum-weight perfect matching* in a complete graph K_{2n} on these points, where the weight of an edge connecting x_i to x_j is $d(x_i, x_j)$. It is convenient to represent $\mathbf{R}$ with a horizontal interval and use arcs in the upper halfplane to show the edges of the graph K_{2n}.

The following lemma, adapted from McCann [10, Lemma 2.1], allows to describe the structure of minimum-weight matchings.

Lemma 1. *Suppose the distance has the form d_g (1) with a* strictly concave *function g (i.e., $g(\lambda x + (1-\lambda)y) > \lambda g(x) + (1-\lambda)g(y)$ for all $x, y \geq 0$, $x \neq y$, and $0 < \lambda < 1$). Then the inequality*

$$d_g(x_1, y_1) + d_g(x_2, y_2) \leq d_g(x_1, y_2) + d_g(x_2, y_1) \tag{3}$$

implies that the intervals connecting x_1 to y_1 and x_2 to y_2 are either disjoint or one of them is contained in the other.

Proof. Let arcs representing the matching be directed from x's to y's and assume, without loss of generality, that x_1 is leftmost. Then the configuration , where $x_1 < y_2 < y_1 < x_2$, cannot satisfy (3) because a strictly concave function g must be strictly growing and therefore $d_g(x_1, y_2) < d_g(x_1, y_1)$ and $d_g(x_2, y_1) < d_g(x_2, y_2)$. To rule out configuration , where $x_1 < x_2 < y_1 < y_2$, choose $0 < \lambda < 1$ such that

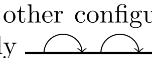
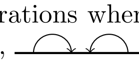
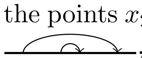
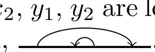
$$y_1 - x_1 = (1-\lambda)(y_2 - x_1) + \lambda(y_1 - x_2).$$

Then, since $(y_1 - x_1) + (y_2 - x_2) = (y_2 - x_1) + (y_1 - x_2)$, we obtain

$$y_2 - x_2 = \lambda(y_2 - x_1) + (1-\lambda)(y_1 - x_2)$$

and can further use the strict concavity of g to get

$$d_g(x_1, y_1) > (1-\lambda)d_g(x_1, y_2) + \lambda d_g(x_2, y_1),$$
$$d_g(x_2, y_2) > \lambda d_g(x_1, y_2) + (1-\lambda)d_g(x_2, y_1).$$

The sum of these inequalities contradicts (3). All the other configurations where the points x_2, y_1, y_2 are located to the right of x_1, namely , , , , are consistent with (3). □

It is possible to show that conversely, for an arbitrary bivariate function $d(x, y)$ the property of Lemma 1 together with homogeneity (D4) imply that d has the form (1) for a suitable concave function g [10, Lemma B4].

We call a matching *nested* if, for any two arcs (x_i, x_j) and $(x_{i'}, x_{j'})$ that are present in the matching, the corresponding intervals in $\mathbf{R}$ are either disjoint or one of them is contained in the other.

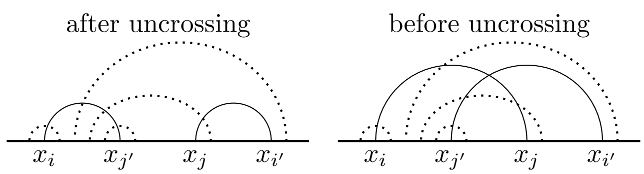
Theorem 2 ([1, 10])**.** *A minimum-weight matching is nested.*

Proof. This statement essentially follows from Lemma 1, because whenever two arcs (x_i, x_j) and $(x_{i'}, x_{j'})$ are crossed, the sum of distances corresponding to the "uncrossed" arcs $(x_i, x_{j'})$, $(x_{i'}, x_j)$ must be smaller according to (3).

However uncrossing may introduce new crossings with other arcs, and it remains to be checked that it does decrease the total number of crossings. This is done using an argument adapted from [1, Lemma 1].

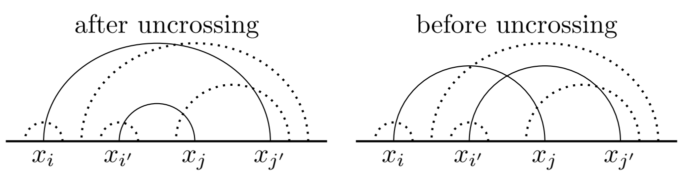
Let an arc (x_k, x_ℓ) cross either of the arcs $(x_i, x_{j'})$ or $(x_{i'}, x_j)$ obtained after uncrossing the arcs (x_i, x_j) and $(x_{i'}, x_{j'})$; we would like to prove that before uncrossing the arc (x_k, x_ℓ) had the same number of crossings with (x_i, x_j) and $(x_{i'}, x_{j'})$. It suffices to consider the following two cases.

Case $x_i < x_{j'} < x_j < x_{i'}$. Assume that (x_k, x_ℓ) crosses $(x_i, x_{j'})$. Then (x_k, x_ℓ), shown as a dotted arc, may be situated in one of the following ways:



We see that the number of crossings between (x_k, x_ℓ) and the other two arcs is the same before and after uncrossing. The case where (x_k, x_ℓ) crosses $(x_{i'}, x_j)$ is symmetrical.

Case $x_i < x_{i'} < x_j < x_{j'}$. The following situations of the dotted arc (x_k, x_ℓ) are possible:



Again the number of crossings between (x_k, x_ℓ) and the other two arcs is the same before and after uncrossing.

It follows that each uncrossing removes exactly one crossing from the matching, and therefore any possible sequence of uncrossings leads in a finite number of steps to a nested matching with a strictly smaller weight. In other words, it suffices to look for the minimum-weight matching only among the nested ones. □

This result implies that the minimum-weight perfect matching problem is essentially bipartite, and it is indeed the bipartite setting that is considered in [1, 4, 5, 9–11]

Corollary 3. *In a minimum-weight perfect matching, points with even numbers are matched to points with odd numbers.*

Proof. Suppose on the contrary that x_i is matched to x_j with i and j both even or both odd. The interval connecting x_i to x_j contains an odd number $|i - j| - 1$ of points, one of which has to be matched to a point outside and thus cause a crossing with (x_i, x_j). Since a minimum-weight matching is nested, the result follows. □

Observe finally the following simple form of the Bellman optimality principle.

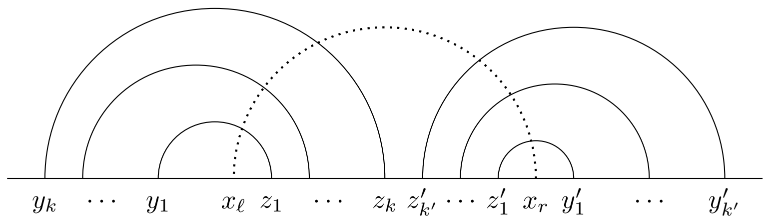


FIGURE 1. Notation used in the proof of Theorem 5. Note that in general $x_\ell \le z_1$ and $z'_1 \le x_r$, but in the figure these pairs of points are shown to be distinct.

Lemma 4. *Any subset of arcs in a minimum-weight matching is itself a minimum-weight perfect matching on the set of endpoints of the arcs that belong to this subset.*

Proof. Indeed, if one could rematch these points achieving a smaller total weight, then the full original matching itself would not be minimum-weight: rematching the corresponding subset of arcs (and possibly uncrossing any crossed arcs that might result from the rematch) would give a matching with a strictly smaller weight. □

4. Preservation of hidden arcs

Call an arc (x_i, x_j) in a nested matching *exposed* if it is not contained in any other arc, i.e., if there is no arc $(x_{i'}, x_{j'})$ with x_i, x_j contained between $x_{i'}$ and $x_{j'}$. We call all other arcs in a nested matching non-exposed or *hidden*. Intuitively, exposed arcs are those visible "from above" and hidden arcs are those covered with exposed ones.

Suppose $X = \{x_i\}_{1\le i\le 2n}$ with $x_1 < x_2 < \cdots < x_{2n}$ and $X' = \{x'_{i'}\}_{1\le i'\le 2n'}$ with $x'_1 < x'_2 < \cdots < x'_{2n'}$ be two sets such that $x_{2n} < x'_1$, i.e., X' lies to the right of X. We will refer to minimum-weight perfect matchings on X and X' as *partial matchings* and to the minimum-weight perfect matching on $X \cup X'$ as *joint matching*. The following result is closely related to properties of "local matching indicators" introduced and studied in [4, 5].

Theorem 5. *Whenever an arc (x_i, x_j) [respectively $(x'_{i'}, x'_{j'})$] is hidden in the partial matching on X [respectively on X'], it belongs to the joint optimal matching and is hidden there too.*

Observe that exposed arcs in partial matchings are generally not preserved in the joint matching: they may disappear altogether or become hidden.

Proof. By contradiction, assume that some of hidden arcs in the partial matching on X do not belong to the joint matching. Then there will be at least one exposed arc (x_ℓ, x_r) in the partial matching on X such that some points x_i with $x_\ell < x_i < x_r$ are connected in the joint matching to points outside (x_ℓ, x_r).

Indeed, if points inside every exposed arc (x_ℓ, x_r) would be matched in the joint matching only among themselves, then by Lemma 4 their matching would be exactly the same as in the partial matching on X, and therefore all hidden arcs between x_ℓ and x_r would be preserved in the joint matching.

Suppose (x_ℓ, x_r) is the leftmost arc of the above kind. Denote all the points in the segment $[x_\ell, x_r]$ that are connected in the joint matching to points on the left of x_ℓ by $z_1 < z_2 < \cdots < z_k$; denote the opposite endpoints of the corresponding arcs by $y_1 > y_2 > \cdots > y_k$, where the inequalities follow from the fact that the joint matching is nested. Likewise denote those points from $[x_\ell, x_r]$ that are connected in the joint matching to points on the right of x_r by $z'_1 > z'_2 > \cdots > z'_{k'}$ and their counterparts in the joint matching by $y'_1 < y'_2 < \cdots < y'_{k'}$ (fig. 1).

Although k or k' may be zero, the number $k + k'$ must be positive and even. Indeed, by Corollary 3 the segment $[x_\ell, x_r]$ contains an even number of points and all of them must be matched in a perfect matching; removing from the joint matching all arcs whose ends both lie in $[x_\ell, x_r]$, we are left with an even number of points that are matched outside this segment.

Let us now restrict our attention to the segment $[x_\ell, x_r]$ and consider a matching that consists of the following arcs: those arcs of the joint matching whose both ends belong to $[x_\ell, x_r]$; the arcs (z_1, z_2), …, $(z_{2\kappa-1}, z_{2\kappa})$, where[1] $\kappa = \lfloor k/2 \rfloor$; the arcs (z'_2, z'_1), …, $(z'_{2\kappa'}, z'_{2\kappa'-1})$, where $\kappa' = \lfloor k'/2 \rfloor$; and $(z_k, z'_{k'})$ if both k and k' are odd.

Denote by W' the weight of this matching. By Lemma 4, it cannot be smaller than the weight W'_0 of the restriction of the partial matching on X to $[x_\ell, x_r]$. For the total weight W of the joint matching on $X \cup X'$ we thus have

$$W \geq W - W' + W'_0. \tag{4}$$

The right-hand side of (4) is represented in fig. 2 (a) in the case when both k and k' are odd. It is a sum of positive terms corresponding to the arcs of the joint matching outside $[x_\ell, x_r]$ (not shown), the arcs of the partial matching on X inside $[x_\ell, x_r]$ (not shown, with exception of (x_ℓ, x_r) represented with a solid arc in the upper halfplane), the arcs of the joint matching having one end inside $[x_\ell, x_r]$ and the other end outside this segment (solid arcs in the upper halfplane), and negative terms that come from subtraction of W' and correspond to the arcs connecting the z points (solid arcs in the lower halfplane).

We now show that by a suitable sequence of uncrossings the right-hand side of (4) can be further reduced to a matching whose weight is strictly less than W.

STEP 1. Note that the arcs (z_1, y_1) and (x_ℓ, x_r), shown in fig. 2 (a) with thick lines, are crossing. Therefore

$$d(y_1, z_1) + d(x_\ell, x_r) > d(y_1, x_\ell) + d(z_1, x_r).$$

Uncrossing these arcs gives the matching represented in fig. 2 (b) and strictly reduces the right-hand side of (4):

$$W > W - W' + W'_0 - d(y_1, z_1) - d(x_\ell, x_r) + d(y_1, x_\ell) + d(z_1, x_r).$$

Now the arcs (y_2, z_2) and (z_1, x_r) are crossing, so

$$d(y_2, z_2) + d(z_1, x_r) - d(z_1, z_2) > d(y_2, x_r)$$

and therefore

$$W > W - W' + W'_0 - d(y_1, z_1) - d(y_2, z_2) - d(x_\ell, x_r) + d(y_1, x_\ell) + d(z_1, z_2) + d(y_2, x_r).$$

The right-hand side of this inequality is represented in fig. 2 (c).

[1] $\lfloor \xi \rfloor$ is the largest integer n such that $n \leq \xi$.

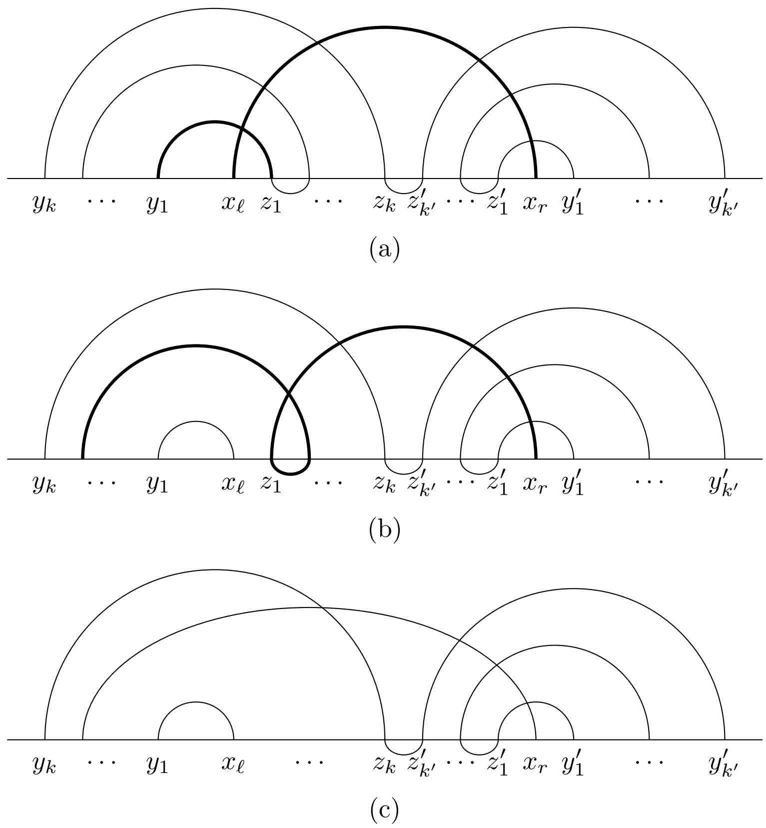


FIGURE 2. Step 1 of the proof (see explanation in the text).

Repeating this step $\kappa = \lfloor k/2 \rfloor$ times gives the inequality

$$W > W - W' + W'_0 - d(x_\ell, x_r) - \sum_{1 \le i \le 2\kappa} d(y_i, z_i) + \sum_{1 \le i \le \kappa} d(z_{2i-1}, z_{2i}) + \sum_{1 \le i \le \kappa} d(y_{2i-1}, y_{2i-2}) + d(y_{2\kappa}, x_r),$$

where in the rightmost sum y_0 is defined to be x_ℓ. Note that at this stage all arcs coming to points $z_1, z_2, \ldots$ from outside $[x_\ell, x_r]$ are eliminated from the matching, except possibly (y_k, z_k) if k is odd.

STEP 2. It is now clear by symmetry that a similar reduction step can be performed on arcs going from $z'_1, z'_2, \ldots$ to the right. Repeating this $\kappa' = \lfloor k'/2 \rfloor$

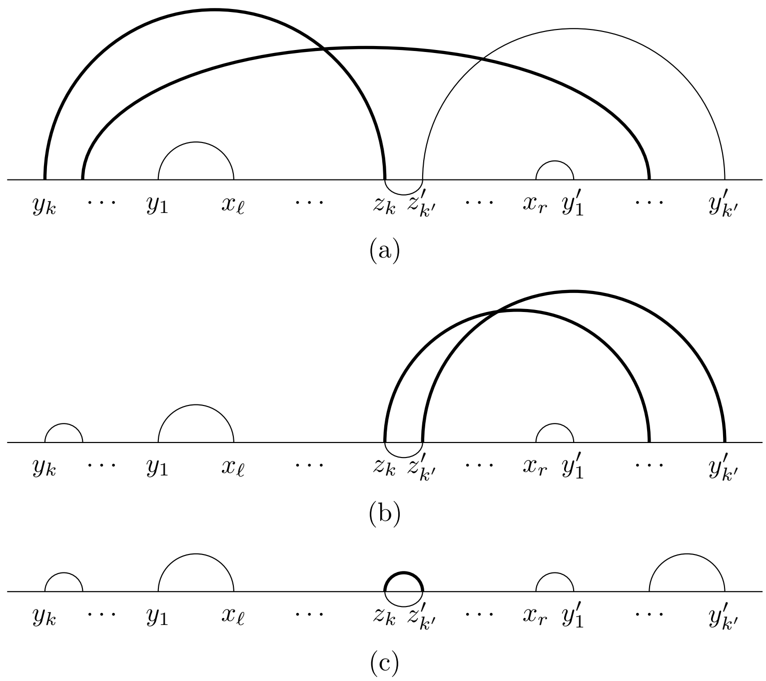


FIGURE 3. Step 3 of the proof. Note that in stage (c) the arc $(z_k, z'_{k'})$ gives two contributions with positive and negative signs, which cancel out each other.

times gives the inequality

$$\begin{aligned} W > W - W' + W'_0 - d(x_\ell, x_r) - \sum_{1\le i\le 2\kappa} d(y_i, z_i) - \sum_{1\le i'\le 2\kappa'} d(z'_{i'}, y'_{i'}) \\ + \sum_{1\le i\le \kappa} d(z_{2i-1}, z_{2i}) + \sum_{1\le i\le \kappa} d(y_{2i-1}, y_{2i-2}) \\ + \sum_{1\le i'\le \kappa'} d(z'_{2i'}, z'_{2i'-1}) + \sum_{1\le i'\le \kappa'} d(y'_{2i'-2}, y'_{2i'-1}) + d(y_{2\kappa}, y'_{2\kappa'}), \end{aligned}$$

where $y'_0 = x_r$.

STEP 3. If k and k' are odd, we perform two more uncrossings shown in fig. 3. The final estimate for W has the form

$$\begin{aligned} W > W - W' + W'_0 - d(x_\ell, x_r) - \sum_{1\le i\le k} d(y_i, z_i) - \sum_{1\le i'\le k'} d(z'_{i'}, y'_{i'}) \\ + \sum_{1\le i\le \kappa} d(z_{2i-1}, z_{2i}) + \sum_{1\le i'\le \kappa'} d(z'_{2i'}, z'_{2i'-1}) + d(z_k, z'_{k'}) \cdot [k,\ k' \text{ are odd}] \\ + \sum_{1\le i\le \kappa} d(y_{2i-1}, y_{2i-2}) + \sum_{1\le i'\le \kappa'} d(y'_{2i'-2}, y'_{2i'-1}) + d(y_k, y'_{k'}) \cdot [k,\ k' \text{ are even}], \end{aligned} \tag{5}$$

where notation such as [k, k' are odd] means 1 if k, k' are odd and 0 otherwise.

The right-hand side of (5) contains four groups of terms: first,

$$W - \sum_{1 \le i \le k} d(y_i, z_i) - \sum_{1 \le i' \le k'} d(z'_{i'}, y'_{i'}),$$

corresponding to the joint matching without the arcs connecting points inside $[x_\ell, x_r]$ to points outside this segment; second,

$$W' - \sum_{1 \le i \le \kappa} d(z_{2i-1}, z_{2i}) - \sum_{1 \le i' \le \kappa'} d(z'_{2i'}, z'_{2i'-1}) - d(z_k, z'_{k'}) \cdot [k,\, k' \text{ are odd}],$$

which comes with a negative sign and corresponds to the arcs of the joint matching with both ends inside $[x_\ell, x_r]$, and cancels them from the total; third,

$$W'_0 - d(x_\ell, x_r),$$

with positive sign, which corresponds to the hidden arcs of the partial matching on X inside the exposed arc (x_ℓ, x_r), not including the latter; and finally the terms in the last line of (5), corresponding to the arcs matching x_ℓ, x_r, and points $y_1, \ldots, y_k, y'_1, \ldots, y'_{k'}$, i.e., those points outside $[x_\ell, x_r]$ that were connected in the joint matching to points inside this segment.

Gathering together contributions of these four groups of terms, we observe that all negative terms cancel out and what is left corresponds to a perfect matching with a weight strictly smaller than W, in which all arcs hidden by (x_ℓ, x_r) in the partial matching on X are restored. There may still be some crossings caused by terms of the fourth group and *not* involving the hidden arcs in $[x_\ell, x_r]$; uncrossing them if necessary gives a nested perfect matching whose weight is strictly less than that of the joint matching. This contradicts the assumption that the latter is the minimum-weight matching on $X \cup X'$. Therefore all hidden arcs in the partial matching on X (and, by symmetry, those in the partial matching on X') belong to the joint matching. □

5. Recursion for minimum weights

In this section we show how to apply Theorem 5 to compute the minimum-weight perfect matching algorithmically. Let $x_1 < x_2 < \cdots < x_{2n}$ be a set of points on the real line $\mathbf{R}$ equipped with a homogeneous distance d of the form (1). For indices i, j of opposite parity and such that $i < j$, let $W_{i,j}$ be the weight of the minimum-weight perfect matching on the $j - i + 1$ points $x_i < x_{i+1} < \cdots < x_j$. It is convenient to organize weights $W_{i,j}$ with $i < j$ into a pyramidal table:

$$\begin{array}{cccccccc}
 & & & & W_{1,2n} & & & \\
 & & & & \cdots\cdots\cdots\cdots & & & \\
 & & W_{1,6} & W_{2,7} & W_{3,8} & W_{4,9} & \ldots & \\
 & W_{1,4} & W_{2,5} & W_{3,6} & W_{4,7} & W_{5,8} & \ldots & W_{2n-3,2n} \\
W_{1,2} & W_{2,3} & W_{3,4} & W_{4,5} & W_{5,6} & W_{6,7} & \ldots & W_{2n-2,2n-1} \quad W_{2n-1,2n}
\end{array}$$

Theorem 6. *For all indices i, j of opposite parity with $1 \le i < j \le 2n$, weights $W_{i,j}$ satisfy the following second-order recursion*

$$W_{i,j} = \min\left[d(x_i, x_j) + W_{i+1,j-1},\; W_{i,j-2} + W_{i+2,j} - W_{i+2,j-2}\right] \tag{6}$$

with "initial conditions"

$$W_{i,i-1} = 0, \quad W_{i+2,i-1} = -d(x_i, x_{i+1}). \tag{7}$$

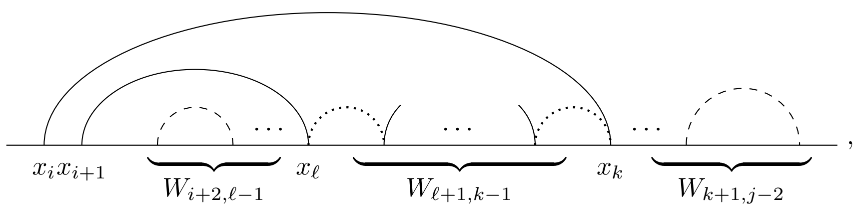
Proof. By an abuse of notation, we will refer to the minimum-weight perfect matching on points $x_r < x_{r+1} < \cdots < x_s$ as the "matching $W_{r,s}$."

Consider first the matching that consists of the arc (x_i, x_j) and all arcs of the matching $W_{i+1,j-1}$, and observe that by Lemma 4 its weight $d(x_i, x_j) + W_{i+1,j-1}$ is minimal among all matchings that contain (x_i, x_j).

We now examine the meaning of the expression $W_{i,j-2} + W_{i+2,j} - W_{i+2,j-2}$. Denote the point connected in the matching $W_{i,j-2}$ to x_i by x_k and the point connected to x_{i+1} by x_ℓ. By Corollary 3, the pairs of indices i, k and $i+1, \ell$ both have opposite parity. Assume first that

$$x_{i+1} < x_\ell < x_k \le x_{j-2}. \tag{8}$$

Applying Theorem 5 to the sets $X = \{x_i, x_{i+1}\}$ and $X' = \{x_{i+2}, \ldots, x_{j-2}\}$ and taking into account parity of k and ℓ, we see that x_k and x_ℓ (as well as their neighbors x_{k+1} and $x_{\ell-1}$ if they are contained in $[x_{i+2}, x_{j-2}]$) belong to exposed arcs of the matching $W_{i+2,j-2}$. Thus the matching $W_{i,j-2}$ has the following structure:



where dashed (resp., dotted) arcs correspond to those exposed arcs of the matching $W_{i+2,j-2}$ that belong (resp., do not belong) to $W_{i,j-2}$.

Since points $x_{\ell-1}$ and x_{k+1} belong to exposed arcs in the matching $W_{i+2,j-2}$, by Lemma 4 we see that the (possibly empty) parts of this matching that correspond to points $x_{i+2} < \cdots < x_{\ell-1}$ and $x_{k+1} < \cdots < x_{j-2}$ coincide with the (possibly empty) matchings $W_{i+2,\ell-1}$ and $W_{k+1,j-2}$. For the same reason the (possibly empty) part of the matching $W_{i,j-2}$ supported on $x_{\ell+1} < \cdots < x_{k-1}$ coincides with $W_{\ell+1,k-1}$. Therefore

$$W_{i,j-2} = d(x_i, x_k) + d(x_{i+1}, x_\ell) + W_{i+2,\ell-1} + W_{\ell+1,k-1} + W_{k+1,j-2}. \tag{9}$$

Taking into account (7), observe that in the case $k = i+1$ and $\ell = i$, which was left out in (8), this expression still gives the correct formula $W_{i,j-2} = d(x_i, x_{i+1}) + W_{i+2,j-2}$.

Now assume that in the matching $W_{i+1,j}$ the point x_j is connected to $x_{\ell'}$ and the point x_{j-1} to $x_{k'}$. A similar argument gives

$$W_{i+2,j} = W_{i+2,\ell'-1} + W_{\ell'+1,k'-1} + W_{k'+1,j-2} + d(x_{\ell'}, x_j) + d(x_{k'}, x_{j-1}); \tag{10}$$

in particular, if $\ell' = j-1$ and $k' = j$, then $W_{i+2,j} = W_{i+2,j-2} + d(x_{j-1}, x_j)$.

Suppose that $x_k < x_{\ell'}$. Using again Lemma 4 and taking into account that x_k, x_{k+1}, $x_{\ell'-1}$, and $x_{\ell'}$ all belong to exposed arcs in $W_{i+2,j-2}$, we can write

$$W_{k+1,j-2} = W_{k+1,\ell'-1} + W_{\ell',j-2}, \quad W_{i+2,\ell'-1} = W_{i+2,k} + W_{k+1,\ell'-1} \tag{11}$$

and

$$W_{i+2,j-2} = W_{i+2,k} + W_{k+1,\ell'-1} + W_{\ell',j-2}. \tag{12}$$

Substituting (11) into (9) and (10) and taking into account (12), we obtain

$$W_{i,j-2} + W_{i+2,j} - W_{i+2,j-2} = d(x_i, x_k) + d(x_{i+1}, x_\ell) + W_{i+2,\ell-1} + W_{\ell+1,k-1} + W_{k+1,\ell'-1} + d(x_{\ell'}, x_j) + W_{\ell'+1,k'-1} + d(x_{k'}, x_{j-1}) + W_{k'+1,j-2}.$$

The right-hand side of this expression corresponds to a matching that coincides with $W_{i,j-2}$ on $[x_i, x_k]$, with $W_{i+2,j-2}$ on $[x_{k+1}, x_{\ell'-1}]$, and with $W_{i+1,j}$ on $[x_{\ell'}, x_j]$. By Lemma 4 this matching cannot be improved on any of these three segments and is therefore optimal among all matchings in which x_i and x_j belong to different exposed arcs.

It follows that under the assumption that $x_k < x_{\ell'}$ the expression in the right-hand side of (6) gives the minimum weight of all matchings on $x_i < x_{i+1} < \cdots < x_j$. Moreover, the only possible candidates for the optimal matching are those constructed above: one that corresponds to $d(x_i, x_j) + W_{i+1,j-1}$ and one given by the right-hand side of the latter formula.

It remains to consider the case $x_k \geq x_{\ell'}$. Since $x_k \neq x_{\ell'}$ for parity reasons, it follows that $x_k > x_{\ell'}$; now a construction similar to the above yields a matching which corresponds to $W_{i,j-2}+W_{i+2,j}-W_{i+2,j-2}$ and in which the arcs (x_i, x_k) and $(x_{\ell'}, x_j)$ are crossed. Uncrossing them leads to a matching with strictly smaller weight, which contains the arc (x_i, x_j) and therefore cannot be better than $d(x_i, x_j) + W_{i+1,j-1}$. This means that (6) holds in this case too with $W_{i,j} = d(x_i, x_j) + W_{i+1,j-1}$. □

Obviously, recursion (6) can be solved for all $1 \leq i < j \leq 2n$ in $O(n^2)$ operations, resulting in computation of weights W_{ij} of all partial optimal matchings. This process is carried out in a "bottom to top" fashion: in the pyramid, weights $W_{i,j}$ with smaller values of $j - i$ are computed first.

To determine the optimal matching on all the points $x_1, x_2, \ldots, x_{2n}$, one should keep track of those pairs (i, j) for which minimum in (7) is attained at the first alternative. Indeed, if the minimum is never attained at the first alternative, then it is easy to see that the optimal perfect matching is $(x_1, x_2), (x_3, x_4), \ldots, (x_{2n-1}, x_{2n})$. Suppose now that the first alternative provides minimum for for some $W_{i_0 j_0}$. Then according to Theorem 5 one can retain the matching for the points $x_{i_0+1}, \ldots, x_{j_0-1}$ that has been computed by this moment and consider a new, smaller minimum-weight perfect matching problem on the points $x_1, x_2, \ldots, x_{i_0}, x_{j_0}, x_{j_0+1}, \ldots, x_{2n}$. More precisely, it suffices to remove from the pyramidal table quantities W_{ij} with i, j satisfying at least one of the conditions $i_0 < i < j_0$ or $i_0 < j < j_0$, replace $W_{i_0 j_0}$ with $d(x_{i_0}, x_{j_0})$, stack the cells with either i or j outside (i_0, j_0) into a smaller pyramidal table, and continue solving the recursion.

This reduction step is illustrated in fig. 4. The elements to be removed from the table are shown in light gray in the top pane. Assuming that the rows are scanned left to right, at the moment when it is found that the element $W_{i_0,j_0} = W_{3,6}$ (in brackets) involves the first alternative in (6), the elements shown in parentheses have not yet been computed. Those of them that are to be kept in the table (the "black" ones) are then stuck with the already computed elements to form a smaller pyramid shown in the bottom, and the recursion resumes with the element $W_{3,6}$ replaced with $d(x_3, x_6)$.

At the reduction step illustrated in fig. 4, the matching is updated with one arc (x_4, x_5), which is guaranteed to belong to the optimal matching by Theorem 5. Generally, every time a reduction step is performed on an element W_{i_0,j_0} and results in removal of the points $x_{k_1} < x_{k_2} < \cdots < x_{k_{2m}}$ that have been "covered" with

$$
\begin{array}{ccccccccc}
 & & & & (W_{1,10}) & & & & \\
 & & & (W_{1,8}) & (W_{2,9}) & (W_{3,10}) & & & \\
 & & (W_{1,6}) & (W_{2,7}) & (W_{3,8}) & (W_{4,9}) & (W_{5,10}) & & \\
 & W_{1,4} & W_{2,5} & [W_{3,6}] & (W_{4,7}) & (W_{5,8}) & (W_{6,9}) & (W_{7,10}) & \\
W_{1,2} & W_{2,3} & W_{3,4} & W_{4,5} & W_{5,6} & W_{6,7} & W_{7,8} & W_{8,9} & W_{9,10}
\end{array}
$$

$$\downarrow$$

$$
\begin{array}{ccccccc}
 & & & (W_{1,10}) & & & \\
 & & (W_{1,8}) & (W_{2,9}) & (W_{3,10}) & & \\
 & (W_{1,6}) & (W_{2,7}) & (W_{3,8}) & (W_{6,9}) & (W_{7,10}) & \\
W_{1,2} & W_{2,3} & d(x_3, x_6) & W_{6,7} & W_{7,8} & W_{8,9} & W_{9,10}
\end{array}
$$

FIGURE 4. Reduction of the pyramidal table ($i_0 = 3$, $j_0 = 6$, $n = 5$).

the arc (i_0, j_0), the matching is updated with arcs $(x_{k_1}, x_{k_2}), \dots, (x_{k_{2m-1}}, x_{k_{2m}})$. When the process reaches the top element of the pyramid, a similar matching of consecutive pairs is performed on the remaining elements.

A formal description of this procedure is given in [4, 5] in somewhat different terms (emphasizing "local matching indicators" instead of the quantities W_{ij}). Since this algorithm requires computing all the weights W_{ij} in the worst case, it takes at most $O(n^2)$ operations for the matching on $2n$ points.

It is interesting to compare performance of our algorithm to the celebrated Edmonds algorithm for minimum-weight perfect matching in a general graph with N vertices and M edges [6, 7]. Note that we effectively consider a complete graph on $N = 2n$ vertices, i.e., $M = n(2n-1)$. In this case recent improvements of the Edmonds algorithm run in $O(n^3)$ time (see, e.g., the survey [3]) whereas our specialized algorithm is at most quadratic in n.

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