Estimates for Solutions of the ∂-Equation and Application to the Characterization of the Zero Varieties of the Functions of the Nevanlinna Class for Linearly Convex Domains of Finite Type

Philippe Charpentier, Yves Dupain, Modi Mounkaila,

▶ **To cite this version:**

Philippe Charpentier, Yves Dupain, Modi Mounkaila,. Estimates for Solutions of the ∂-Equation and Application to the Characterization of the Zero Varieties of the Functions of the Nevanlinna Class for Linearly Convex Domains of Finite Type. Journal of Geometric Analysis, 2014, 24 (4), pp.1860-1881. 10.1007/s12220-013-9398-5 . hal-00563932v2

HAL Id: hal-00563932
https://hal.science/hal-00563932v2

Submitted on 19 Feb 2013

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

ESTIMATES FOR SOLUTIONS OF THE $\bar\partial$-EQUATION AND APPLICATION TO THE CHARACTERIZATION OF THE ZERO VARIETIES OF THE FUNCTIONS OF THE NEVANLINNA CLASS FOR LINEALLY CONVEX DOMAINS OF FINITE TYPE

PHILIPPE CHARPENTIER, YVES DUPAIN & MODI MOUNKAILA

ABSTRACT. In the late ten years, the resolution of the equation $\bar\partial u = f$ with sharp estimates has been intensively studied for convex domains of finite type in $\mathbb{C}^n$ by many authors. Generally they used kernels constructed with holomorphic support function satisfying "good" global estimates. In this paper, we consider the case of lineally convex domains. Unfortunately, the method used to obtain global estimates for the support function cannot be carried out in that case. Then we use a kernel that does not gives directly a solution of the $\bar\partial$-equation but only a representation formula which allows us to end the resolution of the equation using Kohn's L^2 theory.

As an application we give the characterization of the zero sets of the functions of the Nevanlinna class for lineally convex domains of finite type.

1. INTRODUCTION

The general notion of extremal basis and the class of "geometrically separated" domains has been introduced in [CD08]. For such domains it is proved that if there exist "good" plurisubharmonic functions, in which case the domains are called "completely geometrically separated", then sharp estimates on the Bergman and Szegö projections and on the classical invariants metrics can be obtained.

Moreover, using the description of the complex geometry of lineally convex domains of finite type initiated in [Con02] and the construction of a local support function described in [DF03], it is shown, already in [CD08], that every lineally convex domain of finite type is completely geometrically separated.

The present paper is a continuation of the study of complex analysis in such domains. We are now interested in the problem of the characterization of the zero sets of functions in the Nevanlinna class. The main result obtained concerns the class of lineally convex domains of finite type and generalizes the results obtained in the case of convex domains ([BCD98, Cum01b, DM01]):

Theorem 1.1. *Let Ω be a bounded lineally convex domain of finite type in $\mathbb{C}^n$ with smooth boundary. Then a divisor in Ω can be defined by a function of the Nevanlinna class of Ω if and only if it satisfies the Blaschke condition.*

The general scheme of the proof is identical to the one used in the convex case and consists in three steps. First, for the general case of geometrically separated domains, we prove some "Malliavin conditions" on closed positive $(1,1)$-currents Θ and then we solve the equation $dw = \Theta$ with good estimates. The third step, which solves the $\bar\partial$-equation for $(0,1)$-form with L^1 estimates on the boundary, is only done in the case of lineally convex domains of finite type:

Theorem 1.2. *Let Ω be a bounded lineally convex domain of finite type in $\mathbb{C}^n$ with smooth boundary. Let f be a $(0,1)$-form in Ω whose coefficients are $\mathscr{C}^1(\overline{\Omega})$ functions and which is $\bar\partial$-closed. Then there exists a solution of the equation $\overline{\partial} u = f$, smooth on Ω and continuous on $\overline{\Omega}$ such that $\|u\|_{L^1(\partial\Omega)} \leq C|||f|||_k$ (see Section 2.1 formula* (2.3)*), the constant C depending only on Ω. In other words, there exists a solution of the equation $\overline{\partial_b} u = f$, in the sense of* [Sko76]*, in $L^1(\partial\Omega)$.*

2. SOLUTIONS FOR THE $\bar\partial$-EQUATION FOR LINEALLY CONVEX DOMAINS OF FINITE TYPE

First of all, we recall the definition of lineally convex domain:

Definition 2.1. A domain Ω in $\mathbb{C}^n$, with smooth boundary is said to be lineally convex at a point $p \in \partial\Omega$ if there exists a neighborhood W of p such that, for all point $z \in \partial\Omega \cap W$,

$$\left(z + T^{10}_z\right) \cap (D \cap W) = \emptyset,$$

where T^{10}_z is the holomorphic tangent space to $\partial\Omega$ at the point z.

Furthermore, we always suppose that $\partial\Omega$ is of finite type at every point of $\partial\Omega \cap W$. Shrinking W if necessary, we may assume that there exists a $\mathscr{C}^\infty$defining function ρ for Ω and a number $\eta_0 > 0$ such that $\nabla\rho(z) \neq 0$ at every point of W and the level sets $\{z \in W \text{ such that } \rho(z) = \eta\}$, $-\eta_0 \leq \eta \leq \eta_0$, are lineally convex of finite type.

As we want to obtain global results, we need these properties at every boundary point. Thus, in all our work, by "lineally convex domain" we mean a bounded smooth domain having a (global) defining function satisfying the previous hypothesis at every point of $\partial\Omega$.

In Section 2.1 we define a punctual anisotropic norm for forms, $\|.\|_k$, related to the geometry of the domain (formula (2.2)). With this notation, the main goal of this Section is to prove the following reformulation of Theorem 1.2 for $(0,q)$-forms:

2000 *Mathematics Subject Classification.* 32F17, 32T25, 32T40.

Key words and phrases. lineally convex, finite type, $\bar\partial$-equation, Nevanlinna class.

Theorem 2.1. *Let Ω be a smooth bounded lineally convex domain of finite type in $\mathbb{C}^n$. Then there exists a constant $C>0$ such that, for any smooth $\bar\partial$-closed $(0,q)$-form f, $1\le q\le n-1$, on $\overline{\Omega}$ there exists a solution u of the equation $\overline{\partial}u=f$, continuous on $\overline{\Omega}$, such that*

$$\int_{\partial\Omega}|u(z)|\,d\sigma(z)\le C\int_\Omega \|f(z)\|_k\,dV(z).$$

Except for the case of finite type domains in $\mathbb{C}^2$ where such an estimate was proved by D. C. Chang, A. Nagel and E. M. Stein ([CNS92]) for the $\bar\partial$-Neumann problem, this kind of result was always proved using explicit kernels solving the $\bar\partial$-equation. The first result was obtained, independently, by G. M. Henkin and H. Skoda for strictly pseudo-convex domains ([Hen75, Sko76]). Afterward, some generalizations to special pseudo-convex domains of finite type (in dimension $n\ge 3$) were obtained by several authors. For example, the case of complex ellipsoids was obtained by A. Bonami and Ph. Charpentier ([BC82]), and, probably the most notable result, the case of convex domains of finite type by J. Bruna, Ph. Charpentier & Y. Dupain, A. Cumenge and K. Diederich & E. Mazzilli ([BCD98, Cum01b, DM01]).

Here, we consider the more general case of lineally convex domains of finite type. Our starting point is similar to the one used in [Cum01b] and [DM01]. We try to construct a kernel solving the $\bar\partial$-equation following the method described in the classical paper of B. Berndtsson and M. Andersson [BA82]. Such kernel is constructed using two forms, $s(z,\zeta)$ and $Q(z,\zeta)$ satisfying some conditions. In particular $Q(z,\zeta)$ is supposed to be holomorphic in z. In many works using these constructions, the forms s and Q (or only Q) are constructed using a holomorphic support function for the domain. In the case of lineally convex domains of finite type such support functions have been constructed by K. Diederich and J. E. Fornaess in [DF03]. Let us denote by $S_0=\sum Q_i^0(z,\zeta)\,(z_i-\zeta_i)$ this support function. If we want to use S_0 to define s and/or Q, a problem appears immediately: some precise global estimates of S_0 are necessary since this function appears in the denominators of the kernels, and Diederich and Fornaess result gives only local estimates (i.e. when the two points z and ζ are close and close to the boundary of the domain). This problem has been noticed previously in the case of convex domains by W. Alexandre in [Ale01] where a modification of the support function is done. Unfortunately, this modification cannot apply for lineally convex domains, the convexity being strongly used to solve a division problem with estimates. Another way to construct a kernel without support function, introduced by A. Cumenge for convex domains in [Cum01b], is to use the Bergman kernel with the estimates obtained in [McN94]. These needed estimates on the Bergman kernel have been obtained for lineally convex domains in [CD08] but, once again, the method cannot be carried out for lineally convex domains for the same reason.

Thus we start with the method of Berndtsson and Andersson with Q constructed with S_0, but the form Q being holomorphic in z only when the two points z and ζ are close (and close to the boundary). Then the construction does not give a kernel solving the $\bar\partial$-equation but a representation formula of the following form: if f is a $(0,q)$-form smooth in $\bar\Omega$, there exist kernels $K(z,\zeta)$, $K_1(z,\zeta)$ and $P(z,\zeta)$ such that

$$f(z)=\bar\partial\left(\int_\Omega f(\zeta)\wedge K(z,\zeta)\right)+\int_\Omega \bar\partial f(\zeta)\wedge K_1(z,\zeta)+\int_\Omega f(\zeta)\wedge P(z,\zeta).$$

In this formula one important point is that, by construction, the kernel P is $\mathscr{C}^\infty\left(\bar\Omega\times\bar\Omega\right)$. If f is $\bar\partial$-closed then the form $g=\int_\Omega f(\zeta)\wedge P(z,\zeta)$ is also $\bar\partial$-closed, and, by the regularity of P (Lemma 2.2), for all integer r, the Sobolev norm $\|g\|_{W^r}$ of order r is controlled by $C_r\|f\|_{L^1(\Omega)}$. Then, using Kohn's theory ([Koh73]), it is possible, using a linear operator, to solve the equation $\bar\partial v=g$ with an estimate of the form $\|v\|_{W^r}\le C_r\|f\|_{L^1(\Omega)}$ (Lemma 2.3). By Sobolev Lemma, choosing r sufficiently large (depending only on the dimension), there exists a constant C such that, if $f\in L^1(\Omega)$, this solution v is continuous on $\overline{\Omega}$ and $\|v\|_{L^1(\partial\Omega)}\le C\|f\|_{L^1(\Omega)}$. Finally, to obtain a solution of the equation $\bar\partial u=f$, given by a linear operator, satisfying the desired estimate it suffices to estimate the integral $\int_\Omega f(\zeta)\wedge K(z,\zeta)$ which can be done, as we will see, using only the local estimates of the support function S_0 given in [DF03].

2.1. **Geometry and local support function.**

2.1.1. *Geometry of lineally convex domains of finite type.* Adapting the construction made by J. McNeal for convex domains ([McN94]) to the case of lineally convex domains of finite type, M. Conrad defined, in [Con02], the geometry of these domains and, in particular, the notion of extremal basis in this context (note that in his construction the basis are not maximal but minimal, see [Hef04, NPT09] for more details). Here we will only recall the results which are useful for our purpose. A more detailed exposition is given in [DF06].

For ζ close to $\partial\Omega$ and $\varepsilon\le\varepsilon_0$, ε_0 small, define, for all unitary vector v,

$$\tau(\zeta,v,\varepsilon)=\sup\{c \text{ such that } |\rho(\zeta+\lambda v)-\rho(\zeta)|<\varepsilon,\forall\lambda\in\mathbb{C},|\lambda|<c\}.$$

Note that, if v is tangent to the level set of ρ passing through ζ, $\tau(\zeta,v,\varepsilon)\gtrsim\varepsilon^{1/2}$ (with uniform constant in ζ, v and ε) and that, Ω being of finite type $\le 2m$, $\tau(\zeta,v,\varepsilon)\lesssim\varepsilon^{1/2m}$.

Let ζ and ε be fixed. Then, an orthonormal basis $(v_1,v_2,\dots,v_n)$ is called (ζ,ε)-*extremal* (or ε-*extremal*, or simply *extremal*) if v_1 is the complex normal (to ρ) at ζ, and, for $i>1$, v_i belongs to the orthogonal space of the vector space generated by $(v_1,\dots,v_{i-1})$ and minimizes $\tau(\zeta,v,\varepsilon)$ in that space. In association to this extremal basis, we denote

$$\tau(\zeta,v_i,\varepsilon)=\tau_i(\zeta,\varepsilon).$$

Note that there may exist many (ζ,ε)-extremal bases but they all give the same geometry we recall now.

With these notations, one defines polydiscs $AP_\varepsilon(\zeta)$ by

$$AP_\varepsilon(\zeta)=\left\{z=\zeta+\sum_{k=1}^n\lambda_k v_k \text{ such that } |\lambda_k|\le c_0A\tau_k(\zeta,\varepsilon)\right\},$$

c_0 depending on Ω, $P_\varepsilon(\zeta)$ being the corresponding polydisc with $A=1$ and we also define

$$d(\zeta,z)=\inf\{\varepsilon \text{ such that } z\in P_\varepsilon(\zeta)\}.$$

The fundamental result here is that d is a pseudo-distance which means that, $\forall\alpha>0$, there exist constants $c(\alpha)$ and $C(\alpha)$ such that

$$c(\alpha)P_\varepsilon(\zeta)\subset P_{\alpha\varepsilon}(\zeta)\subset C(\alpha)P_\varepsilon(\zeta) \text{ and } P_{c(\alpha)\varepsilon}(\zeta)\subset\alpha P_\varepsilon(\zeta)\subset P_{C(\alpha)\varepsilon}(\zeta). \tag{2.1}$$

We insist on the fact that this pseudodistance is well defined and is independent of the choice of the extremal bases.

We will make use of the following properties:

(1) Let $w=(w_1,\dots,w_n)$ be an orthonormal system of coordinates centered at ζ. Then
$$\left|\frac{\partial^{|\alpha+\beta|}\rho(\zeta)}{\partial w^\alpha\partial\bar w^\beta}\right|\lesssim\frac{\varepsilon}{\prod_i\tau(\zeta,w_i,\varepsilon)^{\alpha_i+\beta_i}},\ |\alpha+\beta|\ge1.$$

(2) Let ν be a unit vector. Let $a^\nu_{\alpha\beta}(\zeta)=\frac{\partial^{\alpha+\beta}\rho}{\partial\lambda^\alpha\partial\bar\lambda^\beta}(\zeta+\lambda\nu)_{|\lambda=0}$. Then
$$\sum_{1\le|\alpha+\beta|\le2m}\left|a^\nu_{\alpha\beta}(\zeta)\right|\tau(\zeta,\nu,\varepsilon)^{\alpha+\beta}\simeq\varepsilon.$$

(3) If $(v_1,\dots,v_n)$ is a (ζ,ε)-extremal basis and $\gamma=\sum_1^n a_jv_j\neq0$, then
$$\frac{1}{\tau(\zeta,\gamma,\varepsilon)}\simeq\sum_{j=1}^n\frac{|a_j|}{\tau_j(\zeta,\varepsilon)}.$$

(4) If ν is a unit vector then:
   (a) $z=\zeta+\lambda\nu\in P_\delta(\zeta)$ implies $|\lambda|\lesssim\tau(\zeta,\nu,\delta)$,
   (b) $z=\zeta+\lambda\nu$ with $|\lambda|\le\tau(\zeta,\nu,\delta)$ implies $z\in CP_\delta(\zeta)$.

(5) $\tau_1(\zeta,\varepsilon)=\varepsilon$, and, for $j>1$ and $\lambda\ge1$, $\lambda^{1/m}\tau_j(\zeta,\varepsilon)\lesssim\tau_j(\zeta,\lambda\varepsilon)\lesssim\lambda^{1/2}\tau_j(\zeta,\varepsilon)$.

Remark. Every lineally convex domain of finite type is completely geometrically separated and the pseudo-distance defined here is equivalent to the one defined in [CD08] using tangent complex vector fields (see Section 7.1 of [CD08] for some details).

With these notations, we define a punctual anisotropic norm $\|.\|_k$ for $(0,q)$-forms with functions coefficients f by

$$\|f(z)\|_k=\sup_{\|v_i\|=1}\frac{\left|\langle f;v_1,\dots,v_q\rangle(z)\right|}{\sum_{i=1}^q k(z,v_i)}, \tag{2.2}$$

where $k(z,v)=\frac{\delta_{\partial\Omega}(z)}{\tau(z,v,\delta_{\partial\Omega}(z))}$, $\delta_{\partial\Omega}(z)$ being the distance of z to the boundary of Ω. Note that this definition generalizes the definition given in [CD08] for $(0,1)$-forms. Moreover, in the coordinate system associated to an $(z,\delta(z))$-extremal basis, we have $\left\|d\bar z^I\right\|_k\simeq\min_{i\in I}\frac{\tau_i(z,\delta_{\partial\Omega}(z))}{\delta_{\partial\Omega}(z)}$, and, if $f=\sum_I a_Id\bar z^I$,

$$\|f\|_k\simeq\sup_I|a_I|\min_{i\in I}\frac{\tau_i(z,\delta_{\partial\Omega}(z))}{\delta_{\partial\Omega}(z)}.$$

If f is a $(0,q)$-form with continuous coefficients, $\|f(z)\|_k$ is also continuous and we define it's $|||.|||_k$ norm by

$$|||f|||_k=\int_\Omega\|f(z)\|_k\,dV(z). \tag{2.3}$$

2.1.2. *The holomorphic support function.* In [DF03] the following result is proved:

Theorem 2.2 (K. Diederich & J. E. Fornaess)**.** *Let Ω be a bounded lineally convex domain in $\mathbb{C}^n$ of finite type $2m$ with $\mathscr{C}^\infty$ boundary. Then there exist a neighborhood W_0 of the boundary of Ω and, for any $\varepsilon>0$ small enough a function $S_0(z,\zeta)\in\mathscr{C}^\infty(\mathbb{C}^n,W_0)$ which is a holomorphic polynomial of degree $2m$ in z for any $\zeta\in W_0$ fixed, such that $S_0(\zeta,\zeta)=0$, satisfying the following precise properties:*

Let M, $K>0$ be chosen sufficiently large and $\varepsilon>0$ sufficiently small. Choose l_ζ a family of affine unitary transformations on W_0 translating ζ to 0 and rotating the complex normal n_ζ to ρ at ζ to the vector $(1,0,\dots,0)$. Then there exists, on W_0, a family of holomorphic polynomials A_ζ, $A_\zeta(0)=0$, such that, if Φ_ζ is defined by $\Phi_\zeta^{-1}(z)_1=z_1\left(1-A_\zeta(z)\right)$, $\Phi_\zeta^{-1}(z)_k=z_k$, $k=2,\dots,n$, then

$$S_0\left(l_\zeta\circ\Phi_\zeta(\xi),\zeta\right)=\xi_1+K\xi_1^2-\varepsilon\sum_{j=2}^{2m}M^{2^j}\sigma_j\sum_{\substack{|\alpha|=j\\ \alpha=(0,\alpha_1,\dots,\alpha_n)}}\frac{1}{\alpha!}\frac{\partial\rho_\zeta(0)}{\partial\xi^\alpha}\xi^\alpha \tag{2.4}$$

where $\rho_\zeta(\xi)=\rho\left(l_\zeta\circ\Phi_\zeta(\xi)\right)-\rho(\zeta)$ and

$$\sigma_j=\begin{cases}1 & \text{for } j=0 \bmod 4\\ -1 & \text{for } j=2 \bmod 4\\ 0 & \text{otherwise}\end{cases}.$$

Moreover, there exist $d = d(\varepsilon) > 0$ and $c > 0$ such that, if n_ζ is the unit real exterior normal to ρ at ζ, for $(w_1, w_2) \in \mathbb{C}^2$ and t a unit vector in the holomorphic tangent space to ρ at ζ, for $|w| < d$, the following estimate holds

$$\mathfrak{Re} S_0\left(\zeta + w_1 n_\zeta + w_2 t, \zeta\right) \leq \left[\rho\left(\zeta + w_1 n_\zeta + w_2 t\right) - \rho(\zeta)\right] h\left(\zeta + w_1 n_\zeta + w_2 t\right) - \varepsilon c \sum_{j=2}^{n} \left\| P^j_{\zeta,t} \right\| |w_2|^j , \tag{2.5}$$

where h is a positive function bounded away from 0, $P^j_{\zeta,t}(w) = P^j_\zeta\left(\zeta + w_1 n_\zeta + w_2 t\right)$, with

$$P^j_\zeta(z) = \sum_{|\alpha|+|\beta|=j} \frac{1}{\alpha!\beta!} \frac{\partial^j \rho(\zeta)}{\partial z^\alpha \partial \bar{z}^\beta} (z-\zeta)^\alpha \left(\bar{z} - \bar{\zeta}\right)^\beta$$

and, for any polynomial $P = \sum a_{\alpha\beta} z^\alpha \bar{z}^\beta$, $\|P\| = \sum \left|a_{\alpha\beta}\right|$.

Remark. In the above Theorem the function S_0 is globally defined for $\zeta \in W_0$, and (2.4) is independent of the choice of l_ζ. In particular, as it is stated in [DF03], if we restrict ζ to a small open set in W_0, the functions l_ζ, h and A_ζ can be chosen $\mathscr{C}^\infty$ in that set (with respect to the two variables ζ and z).

2.2. **Koppelman formulas.** With the notations used for the holomorphic support function S_0, we choose $R < d$ such that $\left|A_\zeta(z)\right| < 10^{-1}$ if $|z-\zeta| < R$ and, reducing η_0 if necessary, we may suppose that $\delta_{\partial\Omega}(\zeta) < \eta_0$ implies $\zeta \in W_0$.

Let us define two $\mathscr{C}^\infty$ functions $\chi_1(z,\zeta) = \hat{\chi}\left(|z-\zeta|\right)$ and $\chi_2(z) = \tilde{\chi}\left(\delta_{\partial\Omega}(z)\right)$ (where $\delta_{\partial\Omega}$ denotes the distance to the boundary of Ω) where χ and $\tilde{\chi}$ are $\mathscr{C}^\infty$ functions, $0 \leq \hat{\chi}, \tilde{\chi} \leq 1$, such that $\hat{\chi} \equiv 1$ on $[0, R/2]$ and $\hat{\chi} \equiv 0$ on $[R, +\infty[$ and $\tilde{\chi} \equiv 1$ on $[0, \eta_0/2]$ and $\tilde{\chi} \equiv 0$ on $[\eta_0, +\infty[$. Then we define

$$\chi(z,\zeta) = \chi_1(z,\zeta)\chi_2(\zeta)$$

and

$$S(z,\zeta) = \chi(z,\zeta) S_0(z,\zeta) - (1 - \chi(z,\zeta))|z-\zeta|^2 = \sum_{i=1}^{n} Q_i(z,\zeta)(z_i - \zeta_i).$$

Now we define the two forms s and Q used in [BA82] in the construction of the Koppelman formula by

$$s(z,\zeta) = \sum_{i=1}^{n} \left(\overline{\zeta_i} - \overline{z_i}\right) d(\zeta_i - z_i)$$

and

$$Q(z,\zeta) = \frac{1}{K_0 \rho(\zeta)} \sum_{i=1}^{n} Q_i(z,\zeta) d(\zeta_i - z_i),$$

where K_0 is a large constant chosen so that

$$\mathfrak{Re}\left(\rho(\zeta) + \frac{1}{K_0} S(z,\zeta)\right) < \frac{\rho(\zeta)}{2}. \tag{2.6}$$

Notice that $\mathfrak{Re} S \leq \chi \mathfrak{Re} S_0 \leq -C\rho(\zeta)$, by (2.5), so it suffices to take $K_0 \geq 2C$. Remark also that, if $\zeta \in \partial\Omega$, (2.5) implies $\mathfrak{Re} S(z,\zeta) < 0$ for $z \in \Omega$.

We point out also that Q is not holomorphic in z and that s satisfies

$$|z-\zeta|^2 = |\langle s, z-\zeta\rangle| \leq C|z-\zeta|, \, z, \zeta \in \Omega.$$

Following the construction done in [BA82], with $G(\xi) = \frac{1}{\xi}$, we obtain two kernels

$$K(z,\zeta) = C_n \sum_{k=0}^{n-1} \frac{(n-1)!}{k!} G^{(k)}\left(\frac{1}{K_0\rho(\zeta)} S(z,\zeta) + 1\right) \frac{s(z,\zeta) \wedge (dQ)^k \wedge (ds)^{n-k-1}}{|\zeta - z|^{2(n-k)}} \tag{2.7}$$

and

$$P(z,\zeta) = C'_n G^{(n)}\left(\frac{1}{K_0\rho(\zeta)} S(z,\zeta) + 1\right) (dQ)^n \tag{2.8}$$

giving the following Koppelman formula:

For all $(0,q)$-form f, with $\mathscr{C}^1(\bar{\Omega})$ coefficients, we have, for $z \in \Omega$,

$$f(z) = \int_{\partial\Omega} f(\zeta) \wedge K^0(z,\zeta) + (-1)^{q+1} \overline{\partial_z} \int_\Omega f(\zeta) \wedge K^1(z,\zeta) + (-1)^q \int_\Omega \bar{\partial} f(\zeta) \wedge K^2(z,\zeta) - \int_\Omega f(\zeta) \wedge P(z,\zeta) \tag{2.9}$$

where K^0 (resp. K^1, resp. K^2, resp. P) is the component of K of be-degree $(0,q)$ in z and $(n, n-q-1)$ in ζ (resp. $(0,q-1)$ in z and $(n,n-q)$ in ζ, resp. $(0,q)$ in z and $(n,n-q-1)$ in ζ, resp. $(0,q)$ in z and $(n,n-q)$ in ζ).

Moreover, by definition of S, $G^{(k)}(S(z,\zeta)+1) = \frac{c_k \rho(\zeta)^{k+1}}{\left[\frac{1}{K_0} S(z,\zeta) + \rho(\zeta)\right]^{k+1}}$, and, for $\zeta \in \partial\Omega$, $K^0(z,\zeta) = 0$ so that the first integral in the Koppelman formula disappears, and if f is $\bar{\partial}$-closed (2.9) becomes

$$f(z) = (-1)^{q+1} \overline{\partial_z} \int_\Omega f(\zeta) \wedge K^1(z,\zeta) - g, \tag{2.10}$$

with

$$g=\int_\Omega f(\zeta)\wedge P(z,\zeta)$$

and g is $\bar\partial$-closed.

To be able to estimate the kernels K^1 and P, we need a fundamental estimate for $\left|\rho(\zeta)+\frac{1}{K_0}S(z,\zeta)\right|$.

Lemma 2.1. *There exists K_0 such that, for $\zeta\in P^i_\varepsilon(z)=P_{2^{-i}\varepsilon}(z)\setminus P_{2^{-i-1}\varepsilon}(z)$, we have:*

(1) $\left|\rho(\zeta)+\frac{1}{K_0}S(z,\zeta)\right|\gtrsim 2^{-i}\varepsilon$, $(z,\zeta)\in\bar\Omega\times\bar\Omega$;

(2) $\left|\frac{1}{K_0}S(\zeta,z)\right|\gtrsim 2^{-i}\varepsilon$, $(z,\zeta)\in\partial\Omega\times\bar\Omega$.

Proof. Clearly (2) is a special case of (1) (because $d(z,\zeta)\simeq d(\zeta,z)$). Let us prove (1).

First note that, by (2.5), if $z=\zeta+w_1n_\zeta+w_2t$,

$$\mathfrak{Re}\left(\rho(\zeta)+\frac{1}{K_0}S(z,\zeta)\right)\le\chi(z,\zeta)\left[c_1\rho(z)+\frac{1}{2}\rho(\zeta)\right]-(1-\chi(z,\zeta))\,|z,\zeta|^2-c_2\sum_{j=2}^n\left\|P^j_{\zeta,t}\right\||w_2|^j,$$

and then, if $|z-\zeta|>\varepsilon_0$, $\mathfrak{Re}\left(\rho(\zeta)+\frac{1}{K_0}S(z,\zeta)\right)\lesssim -c_0(\varepsilon_0)$, and it is enough to prove the Lemma for $\varepsilon<\varepsilon_0$ small enough.

Let us denote $\varepsilon'=2^{-i-1}\varepsilon$ and let $(w_1,\dots,w_n)$ be an ε'-extremal basis at ζ. Let us write $z=\zeta+\sum_{i=1}^n\lambda_iw_i=\zeta+\lambda_1w_1+v_1=\zeta+\lambda_1w_1+\|v_1\|v$. Remark that, by (2.6), the result is trivial if $|\rho(\zeta)|\gtrsim\varepsilon'$. Let us suppose $|\rho(\zeta)|\ll\varepsilon'$.

Let $0<k_0\ll 1$ be a real number which will be fixed later.

Suppose first that, $\forall i\ge 2$, $|\lambda_i|<k_0\tau_i(\zeta,\varepsilon')$. Then $|\lambda_1|\gtrsim\varepsilon'$ (property (4) of Section 2.1.1), and, as

$$\sum_{j=2}^m M^{2^j}\sigma_j\sum_{\substack{|\alpha|=j\\ \alpha_1=0}}\frac{\partial\rho_\zeta(0)}{\partial\xi^\alpha}\xi^\alpha=\sum_{j=2}^m M^{2^j}\sigma_j\sum_{\substack{|\alpha|=j\\ \alpha_1=0}}\frac{\partial\rho(z)}{\partial w^\alpha}_{|w=0}(\lambda w)^\alpha,$$

by (1) of Section 2.1.1, we have

$$\left|\sum_{j=2}^m M^{2^j}\sigma_j\sum_{\substack{|\alpha|=j\\ \alpha_1=0}}\frac{\partial\rho(z)}{\partial w^\alpha}_{|w=0}(\lambda w)^\alpha\right|\lesssim k_0^2\varepsilon'.$$

Using formula (2.4), this gives

$$\begin{aligned}\left|\rho(\zeta)+\frac{1}{K_0}S_0(z,\zeta)\right| &\ge \left|\rho(\zeta)+\frac{\lambda_1\left(1-A_\zeta(z)\right)}{K_0}+\frac{K\lambda_1^2\left(1-A_\zeta(z)\right)^2}{K_0}\right|-C_1k_0^2\varepsilon'\\ &\ge \left|\rho(\zeta)+\frac{\lambda_1\left(1-A_\zeta(z)\right)}{K_0}\right|-C_2\varepsilon'\left(k_0^2+\varepsilon'\right),\end{aligned}$$

the last inequality following the fact that $|\lambda_1|\lesssim\varepsilon'$ (property (4) of Section 2.1.1). As $\left|A_\zeta(z)\right|<10^{-1}$, it follows that

$$\left|\rho(\zeta)+\frac{1}{K_0}S_0(z,\zeta)\right|\gtrsim\varepsilon',$$

if k_0 and ε_0 are chosen sufficiently small.

We now fix ε_0. As $\mathfrak{Re}\left(\rho(\zeta)+\frac{1}{K_0}S_0(z,\zeta)\right)<0$, we have

$$\left|\rho(\zeta)+\frac{1}{K_0}S(z,\zeta)\right|\gtrsim\chi(z,\zeta)\left|\rho(\zeta)+\frac{1}{K_0}S_0(z,\zeta)\right|+(1-\chi(z,\zeta))\,|z-\zeta|^2,$$

and the inequality $|z-\zeta|^2\gtrsim\|v_1\|^2\gtrsim\tau(\zeta,v,\varepsilon)^2\gtrsim\varepsilon'$ (properties (3) and (5) of Section 2.1.1) proves the Lemma in that case.

Suppose then there exists $i\ge 2$ such that $|\lambda_i|\ge k_0\tau_i(\zeta,\varepsilon')$. By (4) of Section 2.1.1, we have $\|v_1\|\simeq\tau(\zeta,v,\varepsilon')$, and by (2) of Section 2.1.1, we get (v being tangent to $\rho=\rho(\zeta)$ at ζ)

$$\sum_{2\le|\alpha+\beta|\le m}\left|a^v_{\alpha\beta}(\zeta)\right|\tau(\zeta,v,\varepsilon)^{\alpha+\beta}=\sum_{1\le|\alpha+\beta|\le m}\left|a^v_{\alpha\beta}(\zeta)\right|\tau(\zeta,v,\varepsilon)^{\alpha+\beta}\simeq\varepsilon',$$

and (2.5) implies

$$\mathfrak{Re}S_0(z,\zeta)\lesssim C\rho(\zeta)-c_1\varepsilon',$$

and we conclude by the same argument on $|z-\zeta|^2$. □

Lemma 2.2. *If $q\ge 1$, all the derivatives, in z, of $P(z,\zeta)$ are uniformly bounded in $\bar\Omega\times\bar\Omega$.*

Proof. Recall $P(z,\zeta)=c_n\frac{\rho(\zeta)^{n+1}}{\left(\frac{1}{K_0}S(z,\zeta)+\rho(\zeta)\right)^{n+1}}(dQ)^n$. If $\delta_{\partial\Omega}(\zeta)>\eta_0/2$, this is clear by (2.6). Suppose $\delta_{\partial\Omega}(\zeta)\leq\eta_0/2$. If $|z-\zeta|<R/2$ then Q is holomorphic in z and the component of $(dQ)^n$ of be-degree $(0,q)$ in z (recall we suppose $q\geq 1$) is identically 0 which implies $P=0$. If $|z-\zeta|\geq R/2$, the preceding Lemmas, give $\left|\frac{1}{K_0}S(z,\zeta)+\rho(\zeta)\right|\gtrsim(R)^{2m}$ (because, for unitary v, $\tau(z,v,\varepsilon)\lesssim\varepsilon^{1/2m}$), and the Lemma follows easily. □

Lemma 2.3. *For each positive integer s there exists a linear operator $\mathscr{L}_s$ and a constant $C(s)$ such that, if $g=\int_\Omega f(\zeta)\wedge P(z,\zeta)$ is the $\bar\partial$-closed form of* (2.10) *there exists a solution $v_s=\mathscr{L}_s(g)$ to the equation $\bar\partial v_s=g$ satisfying the estimate*

$$\|v_s\|_s\leq C(s)\,\|f\|_{L^1(\Omega)},$$

where $\|.\|_s$ denotes the Sobolev norm of index s in Ω.

Proof. This is an immediate consequence of the preceding Lemma and the results on the weighted $\bar\partial$-Neumann problem obtained by J. J. Kohn in [Koh73]. □

As explained at the end of the introduction of Section 2 this last Lemma shows that to obtain sharp estimates for solution of the $\bar\partial$-equation in lineally convex domains of finite type, like the one stated in Theorem 2.1, it is enough to prove that the form $z\mapsto\int_\Omega f(\zeta)\wedge K^1(z,\zeta)$ (c.f.(2.10)) is continuous on $\overline{\Omega}$ and that

$$\left\|\int_\Omega f(\zeta)\wedge K^1(z,\zeta)\right\|_{L^1(\partial\Omega)}\leq C|||f|||_k.$$

This will be done in the next section.

2.3. **Proof of Theorem 2.1.** By formulas (2.7) and (2.9), we have

$$K^1(z,\zeta)=\sum_{k=n-q}^{n-1}C'_k\frac{\rho(\zeta)^{k+1}s\wedge\left(\partial_{\bar\zeta}Q\right)^{n-q}\wedge(\partial_{\bar z}Q)^{k+q-n}\wedge(\partial_{\bar z}s)^{n-k-1}}{|z-\zeta|^{2(n-k)}\left(\frac{1}{K_0}S(z,\zeta)+\rho(\zeta)\right)^{k+1}}.$$

A priory, formula (2.9) is only valid for $z\in\Omega$. But, as noted by H. Skoda in [Sko76], the form

$$\int_\Omega f(\zeta)\wedge K^1(z,\zeta)$$

is continuous in $\bar\Omega$ if the kernel K^1 satisfies a condition of uniform integrability:

$$\int_{\Omega\cap P_\varepsilon(z)}\left|K^1(z,\zeta)\right|dV(\zeta)=\mathrm{O}\left(\varepsilon^{1/m}\right),\tag{2.11}$$

uniformly for z satisfying $\delta_{\partial\Omega}(z)\leq\varepsilon$ and ε small enough.

Under these hypothesis ($\zeta\in P_\varepsilon(z)$), $Q(z,\zeta)=\frac{1}{K_0\rho(\zeta)}\sum_{i=1}^n Q_i(z,\zeta)d(\zeta_i-z_i)$ is holomorphic in z and K^1 is reduced to

$$K^1(z,\zeta)=c\frac{\rho(\zeta)^{n-q+1}s\wedge\left(\partial_{\bar\zeta}Q\right)^{n-q}\wedge(\partial_{\bar z}s)^{q-1}}{|z-\zeta|^{2q}\left(\frac{1}{K_0}S(z,\zeta)+\rho(\zeta)\right)^{n-q+1}}.$$

To prove (2.11), we use the coordinate system $(\zeta_1,\dots,\zeta_n)$ associated to the $(z,\delta_{\partial\Omega}(z))$-extremal basis and the following estimates:

Lemma 2.4. *For z close to $\partial\Omega$, ε small and $\zeta\in P_\varepsilon(z)$, we have:*

(1) $\left|\frac{\partial\rho}{\partial\zeta_i}(\zeta)\right|\lesssim\frac{\varepsilon}{\tau_i(z,\varepsilon)}$ *((1) of Section 2.1.1);*
(2) $|Q_i(z,\zeta)|+|Q_i(\zeta,z)|\lesssim\frac{\varepsilon}{\tau_i(z,\varepsilon)}$ *(see* [DF06]*);*
(3) $\left|\frac{\partial Q_i(z,\zeta)}{\partial\overline{\zeta_j}}\lesssim\frac{\varepsilon}{\tau_i(z,\varepsilon)\tau_j(z,\varepsilon)}\right|$ *(see* [DF06]*).*

A straightforward calculus shows that

$$\begin{aligned}\left(\partial_{\bar\zeta}Q\right)^{n-q} &= \left(\frac{1}{\rho(\zeta)}\right)^{n-q}\sum_{\substack{I=(i_1,\dots,i_{n-q})\\ J=(j_1,\dots,j_{n-q})}}{}'\prod_{k=1}^{n-q}\frac{\partial Q_{i_k}(z,\zeta)}{\partial\overline{\zeta_{j_k}}}\bigwedge_{i\in I}d\zeta_i\bigwedge_{j\in J}d\overline{\zeta_j}\pm\\ &\pm\left(\frac{1}{\rho(\zeta)}\right)^{n-q+1}\sum_{\substack{I=(i_1,\dots,i_{n-q})\\ J=(j_1,\dots,j_{n-q})}}{}'\sum_{k_0\in\{1,\dots,n-q\}}\frac{\partial\rho(\zeta)}{\partial\overline{\zeta_{j_{k_0}}}}Q_{i_{k_0}}(z,\zeta)\prod_{\substack{1\leq k\leq n-q\\ k\neq k_0}}\frac{\partial Q_{i_k}(z,\zeta)}{\partial\overline{\zeta_{j_k}}}\bigwedge_{i\in I}d\zeta_i\bigwedge_{j\in J}d\overline{\zeta_j},\end{aligned}$$

where $\sum'$ means that the i_r (resp. j_r) are all different.

Suppose $\zeta \in P^0_\varepsilon = P_\varepsilon(z) \setminus P_{\varepsilon/2}(z)$. Then, by Lemma 2.1, $|z-\zeta|^{2q}\left(\frac{1}{K_0}S(z,\zeta)+\rho(\zeta)\right)^{n-q+1} \gtrsim \varepsilon^{n-q+1}|z-\zeta|^{2q}$, and, by Lemma 2.4,

$$\begin{aligned}\left|K^1(z,\zeta)\right| &\lesssim \sum'_{\substack{I=(i_1,\dots,i_{n-q})\\ J=(j_1,\dots,j_{n-q})}} \frac{(\rho(\zeta)+\varepsilon)\,\varepsilon^{n-q}}{\prod_{k=1}^{n-q}\tau_{i_k}\tau_{j_k}\varepsilon^{n-q+1}|z-\zeta|^{2q-1}}\\ &\lesssim \sum'_{\substack{I=(i_1,\dots,i_{n-q})\\ J=(j_1,\dots,j_{n-q})}} \frac{1}{\prod_{k=1}^{n-q}\tau_{i_k}\tau_{j_k}|z-\zeta|^{2q-1}},\end{aligned}$$

because $\delta_{\partial\Omega}(z)<\varepsilon$ and using (1) of Section 2.1.1.

The τ_i being supposed ordered increasingly, it is easy to see that

$$\int_{P^0_\varepsilon}\frac{1}{|z-\zeta|^{2q-1}} \lesssim \int_{P^0_\varepsilon}\frac{1}{|z-\zeta|^{2q-1}+\varepsilon^{2q-1}} \lesssim \tau_{n-q+1}(z,\varepsilon)\prod_{i=1}^{n-q}\tau_i^2(z,\varepsilon),$$

and we obtain

$$\int_{P^0_\varepsilon}\left|K^1(z,\zeta)\right|\,dV(\zeta) \lesssim \tau_{n-q+1}(z,\varepsilon) = \mathrm{O}\left(\varepsilon^{1/m}\right),$$

and, if we denote $P^i_\varepsilon = P^i_\varepsilon(z) = P\left(z,2^{-i}\varepsilon\right)\setminus P\left(z,2^{-(i+1)}\varepsilon\right)$,

$$\int_{P_\varepsilon}\left|K^1(z,\zeta)\right|\,dV(\zeta) = \sum_{i=0}^{\infty}\int_{P^i_\varepsilon}\left|K^1(z,\zeta)\right|\,dV(\zeta) \lesssim \sum_{i=0}^{\infty}\mathrm{O}\left(\left(\varepsilon 2^{-i}\right)^{1/m}\right) = \mathrm{O}\left(\varepsilon^{1/m}\right),$$

which ends the proof of (2.11).

To finish the proof of Theorem 2.1, by Fubini's Theorem, we have to prove that

$$\int_{\partial\Omega}\left|K^1(z,\zeta)\wedge f(\zeta)\right|\,d\sigma(z) \lesssim \|f(\zeta)\|_k,$$

and, by Lemma 2.1, it is enough to see that, for ζ near the boundary and η small enough (to have the reduced form of K^1),

$$\int_{\partial\Omega\cap P_\eta(\zeta)}\left|K^1(z,\zeta)\wedge f(\zeta)\right|\,d\sigma(z) \lesssim \|f(\zeta)\|_k.$$

To see it, let us denote $\varepsilon=\delta_{\partial\Omega}(\zeta)$, $Q^0_\varepsilon(\zeta)=P_\varepsilon(\zeta)$ and $Q^i_\varepsilon(\zeta)=P_{2^i\varepsilon}(\zeta)\setminus P_{2^{i-1}\varepsilon}(\zeta)$, $i\geq 1$, and let us estimate

$$\int_{\partial\Omega\cap Q^i_\varepsilon(\zeta)}\left|K^1(z,\zeta)\wedge f(\zeta)\right|\,d\sigma(z)$$

using the coordinate system associated to the $\left(\zeta,2^i\varepsilon\right)$-extremal basis. By the properties of the norm $\|.\|_k$ it is enough to prove the estimate when the form f is (in the extremal coordinate system) $f=\left(d\overline{\zeta}\right)^I$, with $|I|=q$. Then, denoting $I=(i_1,\dots,i_q)$ and $J=(j_1,\dots,j_{n-q})$, $I\cup J=\{1,\dots,n\}$, $K^1(z,\zeta)\wedge f(\zeta)$ is a sum of expressions of the form $\frac{W_i}{D}$, $i=1,2$, with

$$D(\zeta,z)=|z-\zeta|^{2q}\left(\frac{1}{K_0}S(z,\zeta)+\rho(\zeta)\right)^{n-q+1}$$

and

$$W_1=(\zeta_m-z_m)\rho(\alpha)\prod_{k=1}^{n-q}\frac{\partial Q_{i_k}(z,\zeta)}{\partial\overline{\zeta_{j_k}}}\bigwedge_{i=1}^{n}\left(d\zeta_i\wedge d\overline{\zeta_i}\right)\bigwedge_{\substack{l\in L\\ |L|=q-1}}d\overline{z_l}$$

and, if $1\notin I$,

$$W_2=(\zeta_m-z_m)\frac{\partial\rho(\zeta)}{\partial\overline{\zeta_1}}Q_{i_{k_0}}(\zeta,z)\prod_{\substack{1\leq k\leq n-q\\ k\neq k_0}}\frac{\partial Q_{i_k}(z,\zeta)}{\partial\overline{\zeta_{j_k}}}\bigwedge_{i=1}^{n}\left(d\zeta_i\wedge d\overline{\zeta_i}\right)\bigwedge_{\substack{l\in L\\ |L|=q-1}}d\overline{z_l}.$$

Now, remarking that

$$\int_{Q^i_\varepsilon(\zeta)\cap\partial\Omega}\frac{d\sigma(z)}{|\zeta-z|^{2q-1}} \lesssim 2^i\varepsilon\tau_{n-q+1}\left(\zeta,2^i\varepsilon\right)\prod_{i=2}^{n-q}\tau_i^2\left(\zeta,2^i\varepsilon\right),$$

we get, using property (5) of Section 2.1.1,

$$\int_{Q^i_\varepsilon(\zeta)\cap\partial\Omega}\left|\frac{W_1(\zeta,z)}{D(\zeta,z)}\wedge f(\zeta)\right|\,d\sigma(z) \lesssim \frac{\rho(\zeta)}{2^i\varepsilon}\min_{j\in I}\frac{\tau_j\left(\zeta,2^i\varepsilon\right)}{2^i\varepsilon} \lesssim \frac{\rho(\zeta)}{2^i\varepsilon}\min_{j\in I}\frac{\tau_j(\zeta,\varepsilon)}{\varepsilon} \lesssim \min_{j\in I}\frac{\tau_j\left(\zeta,\delta_{\partial\Omega}(\zeta)\right)}{2^i\delta_{\partial\Omega}(\zeta)}$$

and, because $1\notin I$,

$$\int_{Q^i_\varepsilon(\zeta)\cap\partial\Omega}\left|\frac{W_2(\zeta,z)}{D(\zeta,z)}\wedge f(\zeta)\right|\,d\sigma(z) \lesssim \min_{j\in I}\frac{\tau_j\left(\zeta,2^i\varepsilon\right)}{2^i\varepsilon} \lesssim \min_{j\in I}\frac{\tau_j(\zeta,\delta_{\partial\Omega}(\zeta))}{2^{i/2}\delta_{\partial\Omega}(\zeta)},$$

and the proof is complete.

3. Nonisotropic estimates of closed positive currents on geometrically separated domains

In this Section, Ω is a pseudo-convex domain in $\mathbb{C}^n$ with $\mathscr{C}^\infty$ boundary which is geometrically separated at a point $p_0 \in \partial\Omega$. To state the main result of this Section we fix some notations.

We denote by ρ a defining function of Ω and $N = \frac{1}{|\nabla\rho|^2}\sum_i \frac{\partial\rho}{\partial\overline{z_i}} dz_i$ the unit normal defined in a neighborhood U of $\partial\Omega$. Recall (see [CD08]) that the hypothesis made on Ω means that there exists a constant $K > 0$, a neighborhood $V = V(p_0) \subset U$ of p_0 and a $(n-1)$-dimensional complex vector space E^0 of $(1,0)$-vector fields $\mathscr{C}^\infty$ in V, tangent to ρ (i.e. $L(\rho) \equiv 0$ in V for $L \in E^0$) such that, at every point p of $V \cap \overline{\Omega}$ and for every $0 < \delta < \delta_0$ there exists a (K, p, δ))-extremal (or (p,δ)-extremal) basis whose elements belong to E^0. We will denote by E^1 the complex vector space generated by E^0 and N.

For $L \in E^1$, $0 < \varepsilon < \delta_0$ and $z \in V \cap \overline{\Omega}$, in [CD08] we defined the weight $F(L,z,\varepsilon)$ as follows:

If $L = L_\tau + a_n N$, the weight is defined by

$$F(L,z,\varepsilon) = \sum_{\mathscr{L} \in \mathscr{L}_M(L)} \left| \frac{\mathscr{L}(\partial\rho)(z)}{\varepsilon} \right|^{2/|\mathscr{L}|} + \frac{|a_n|^2}{\varepsilon^2},$$

where M is an integer larger than the type, $\mathscr{L}_M(L)$ denotes the set of lists $\left(L^1, \dots L^k\right)$, of length $k \leq M$, $L^j \in \left\{L_\tau, \overline{L_\tau}\right\}$, and

$$\mathscr{L}(\partial\rho)(z) = L^1 \dots L^{k-2}\left(\left\langle \partial\rho, \left[L^{k-1}, L^k\right]\right\rangle\right)(z).$$

Remark. To keep the same notations as in [CD08], the normal is denoted by L_n in opposite of the preceding Section where it was denoted by L_1.

Now we denote $\tau(L,z,\varepsilon) = F(L,z,\varepsilon)^{-1/2}$. If v is any non zero vector in $\mathbb{C}^n$, and $z \in V \cap \overline{\Omega}$, there exists a unique vector field $L = L_z$ in E^1 such that $L(z) = v$. Then we denote

$$\tau(z,v,\varepsilon) = \tau(L,z,\varepsilon) \text{ and } k(z,v,\varepsilon) = \frac{\delta_{\partial\Omega}(z)}{\tau(z,v,\varepsilon)},$$

where $\delta_{\partial\Omega}(z)$ is the distance from z to the boundary of Ω. When $\varepsilon = \delta_{\partial\Omega}(z)$ we denote $\tau(z,v) = \tau(z,v,\delta_{\partial\Omega}(z))$ and $k(z,v) = k(z,v,\delta_{\partial\Omega}(z))$.

To simplify the exposition, we will also use the following terminology. For each point $z \in V \cap \overline{\Omega}$ and each $0 < \delta \leq \delta_0$, if $(v_i)_{1\leq i\leq n}$ is a basis of $\mathbb{C}^n$ such that $v_n = N(z)$ and $(L_i)_{1\leq i\leq n-1}$ ($L_i \in E^0$) is a (z,δ)-extremal basis such that $L_i(z) = v_i$ we will say that $(v_i)_{1\leq i\leq n}$ is a (z,δ)-extremal basis.

Suppose now that Θ is a $\mathscr{C}^\infty$ $(1,1)$-form defined in $V \cap \overline{\Omega}$. Then, following [BCD98], we define, for $z \in V \cap \overline{\Omega}$,

$$\|\Theta(z)\|_k = \|\Theta(z)\|_k^{p_0} = \sup_{u,v\in\mathbb{C}^*} \frac{|\Theta(z)(u,v)|}{k(z,u)k(z,v)}$$

and

$$\|\Theta(z)\|_E = \sup_{u,v\in\mathbb{C}^*} \frac{|\Theta(z)(u,v)|}{\|u\|\,\|v\|},$$

where $\|.\|$ denotes the euclidean norm.

Similarly, with the above notations, we extend the notations defined in formulas (2.2) and (2.3) for $(0,q)$-forms to forms defined in a geometrically separated domain.

The aim of the Section is to prove the following result:

Theorem 3.1. *Let Ω be a pseudo-convex domain in $\mathbb{C}^n$ which is geometrically separated at a boundary point p_0. Then there exist two neighborhoods V and $W \subset V$ of p_0 and a constant $C > 0$ such that, if Θ is a smooth positive closed $(1,1)$-form defined in V then*

$$\int_W \delta_{\partial\Omega}(z)\|\Theta(z)\|_k\, dV(z) \leq C\int_V \delta_{\partial\Omega}(z)\|\Theta(z)\|_E\, dV(z).$$

If Ω is a bounded pseudo-convex domain which is geometrically separated at every point of it's boundary, we choose a finite number of points p_i, $1 \leq i \leq N$, in $\partial\Omega$ such that the set of the neighborhoods $V_i = V(p_i)$ is a covering $\mathscr{V}$ of $\partial\Omega$, $V_0 = \Omega \setminus \bigcup_{i\geq 1} V_i$, and we define

$$\|\Theta(z)\|_k^{\mathscr{V}} = \max_{z\in V_i} \|\Theta(z)\|_k^i,$$

where, $\|\Theta(z)\|_k^i = \|\Theta(z)\|_k^{p_i}$ if $i \geq 1$ and $\|\Theta(z)\|_k^0 = \|\Theta(z)\|_E$.

With this notation, the local result immediately implies a global one:

Theorem 3.2. *If Ω is a pseudo-convex domain in $\mathbb{C}^n$ which is geometrically separated at every point of it's boundary, a covering $\mathscr{V}$ of $\partial\Omega$ being chosen, there exists a constant $C > 0$ such that*

$$\int_\Omega \delta_{\partial\Omega}(z)\|\Theta(z)\|_k^{\mathscr{V}}\, dV(z) \leq C\int_\Omega \delta_{\partial\Omega}(z)\|\Theta(z)\|_E\, dV(z)$$

for all smooth closed positive $(1,1)$-form Θ in Ω.

In fact the two statements are equivalent. In [CD08] it is proved that if Ω is geometrically separated at $p_0 \in \partial\Omega$ then there exists a bounded pseudo-convex domain D with $\mathscr{C}^\infty$ boundary contained in Ω which is geometrically separated at every point of it's boundary and whose boundary contains a neighborhood of p_0 in the boundary of Ω. Thus, Theorem 3.1 for Ω follows immediately Theorem 3.2.

We now prove the Theorems. As it is very similar to the proof of the same result for convex domains given in Section 2 of [BCD98], we will refer to that paper for many details. Precisely, we will only give the mains articulations and the proof of Lemmas where the differences due to the fact that the weights F are defined with vector fields instead of coordinates systems are relevant.

We still use the previous notations for the defining function ρ of Ω, the point $p_0 \in \partial\Omega$, the neighborhood V of p_0 and W a neighborhood of p_0 relatively compact in V.

In Section 3.3 of [CD08] it is shown that there exists a (z,ε)-adapted coordinate system $(\xi_i)_i$ used, in particular, to define "polydiscs" centered at the point z by

$$P_\varepsilon(z) = P(z,\varepsilon) = \{q = (\xi)_i \text{ such that } |\xi_i| \le c\tau(z,v_i,\varepsilon)\},$$

where $v_i = L_i(z)$, $1 \le i \le n$, $(L_i)_{1\le i\le n}$ being the (z,ε)-extremal basis, and c a sufficiently small constant (depending on Ω). Moreover, the set of these polydiscs are associated to a pseudo-distance. With these notations we have

Lemma 3.1. *For $z \in V \cap \overline{\Omega}$ and $v \in \mathbb{C}^n$, $v \neq 0$, if $w \in P_\varepsilon(z)$ we have $\tau(w,v,\varepsilon) \simeq \tau(z,v,\varepsilon)$.*

Proof. Let L_z (resp. L_w) the vector field belonging to E_1 such that $L_z(z) = v$ (resp. $L_w(w) = v$). Let $(L_i)_{1\le i\le n}$ be the (z,ε)-extremal basis and let us write $L_z = \sum_{i=1}^n b_i L_i$ and $L_w = \sum_{i=1}^n b_i' L_i$, $b_i, b_i' \in \mathbb{C}$. If $(\xi_i)_{1\le i\le n}$ is the (z,ε)-adapted coordinate system let us write $L_j(.) = \sum_i a_j^i(.)\frac{\partial}{\partial\xi_i}$, with $a_j^i(z) = \delta_i^j$ so that

$$L_z(z) = \sum_{i=1}^n b_i \frac{\partial}{\partial\xi_i} \text{ and } L_w(w) = \sum_{i=1}^n \left(\sum_{j=1}^n b_j' a_j^i(w)\right)\frac{\partial}{\partial\xi_i}.$$

In [CD08] we proved (Proposition 3.6) that,if $w \in P_\varepsilon(z)$,

$$\left|a_j^i(w)\right| \lesssim F_j^{1/2}(z,\varepsilon)F_i^{-1/2}(z,\varepsilon).$$

The basis (L_i) being (z,ε)-extremal we have immediately that $F(L_z,z,\varepsilon) \simeq F(L_w,z,\varepsilon)$, and, using another time Proposition 3.6 of [CD08], that $F(L_w,z,\varepsilon) \simeq F(L_w,w,\varepsilon)$ finishing the proof of the Lemma. □

The proof of the Theorems is done in three steps.

First step: definition of a family of polydiscs. As the polydiscs $P_\varepsilon(z)$ are associated to a pseudodistance, there exists a constant M such that, for δ and ε sufficiently small, there exists points z_i, $1 \le i \le n(\delta,\varepsilon)$ belonging to the set $\{\rho = -\delta\}$ such that, if $S_{\delta,\varepsilon}(z) = S_\varepsilon(z) = P_\varepsilon(z) \cap \{\rho = -\delta\}$ and $S^*_{\delta,\varepsilon}(z) = S^*_\varepsilon(z) = 2P_\varepsilon(z) \cap \{\rho = -\delta\}$ then

(1) $\{S_\varepsilon(z_i)\}_i$ is a covering of $W \cap \{\rho = -\delta\}$;
(2) for all i, $S^*_\varepsilon(z_i)$ is contained in $V \cap \{\rho = -\delta\}$;
(3) for all $z \in \{\rho = -\delta\}$, there exist at most M index i such that $r \in S^*_\varepsilon(z_i)$.

Let $\alpha \in]0,1[$ be a parameter that will be fixed later. For $\rho = \alpha^k$, $k \ge k_0$, let $\left(z_i^k\right)_{1\le i\le n_k}$ be a family of points satisfying these properties for $\delta = \alpha^k$ and $\varepsilon = \alpha^k/2$. The family $\bigcup_{k\ge k_0}\left\{z_i^k, 1 \le i \le n_k\right\}$ will be denoted by $(z_i)_i$.

For z close enough to $\partial\Omega$ and $\delta > 0$ sufficiently small, let $\pi_\delta(z)$ denote the point where the integral curve of $\nabla\rho$ passing through z meets the set $\{\rho = -\delta\}$. V being small enough, π_δ is $\mathscr{C}^\infty$ in V and any finite number of derivatives of π_δ is bounded independently of δ.

For all z_j belonging to $(z_i)_i$, if $-\rho(z_j) = \alpha^k$ we set $S_j = S_{\alpha^k/2}(z_j)$, $S_j^* = S^*_{\alpha^k/2}(z_j)$,

$$Q_j = \left\{w \in \Omega \text{ such that } \alpha^k \ge -\rho(w) \ge \alpha^{k+1}, \pi_\delta(w) \in S_j\right\},$$

and

$$Q_j^* = \left\{w \in \Omega \text{ such that } \alpha^{k-1} \ge -\rho(w) \ge \alpha^{k+2}, \pi_\delta(w) \in S_j^*\right\}.$$

We suppose k_0 chosen sufficiently large so that $\bigcup_j Q_j^* \subset V$. Moreover, note that there exist at most $4M$ index j such that $z \in Q_j^*$.

Second step: some estimates related to the radius τ_i.

Lemma 3.2. *Let $z \in V \cap \overline{\Omega}$, $v \in \mathbb{C}^n$, $v \neq 0$, and m an integer larger than the type of p_0.*

(1) *If $\varepsilon_1 \ge \varepsilon_2$, $\left(\frac{\varepsilon_1}{\varepsilon_2}\right)^{1/2}\tau(z,v,\varepsilon_2) \gtrsim \tau(z,v,\varepsilon_1) \gtrsim \left(\frac{\varepsilon_1}{\varepsilon_2}\right)^{1/m}\tau(z,v,\varepsilon_2)$.*
(2) *If $v = \sum_{i=1}^n w_i L_i(z)$, where $(L_i, 1 \le i \le n)$ is the (z,ε)-extremal basis, then*

$$\sum_{i=1}^n |w_i|\, k(z,v_i,\varepsilon) \simeq k(z,v,\varepsilon).$$

Lemma 3.3. *If $z = \pi_\delta(w)$, then*

(1) *if $\delta_{\partial\Omega}(w) \le \delta_{\partial\Omega}(z)$, $\tau(z,v)\left(\frac{\delta_{\partial\Omega}(w)}{\delta_{\partial\Omega}(z)}\right)^{1/2} \lesssim \tau(w,v) \lesssim \tau(z,v)\left(\frac{\delta_{\partial\Omega}(w)}{\delta_{\partial\Omega}(z)}\right)^{1/m}$,*

(2) *if* $\delta_{\partial\Omega}(w) \geq \delta_{\partial\Omega}(z)$, $\tau(z,v)\left(\frac{\delta_{\partial\Omega}(w)}{\delta_{\partial\Omega}(z)}\right)^{1/m} \lesssim \tau(w,v) \lesssim \tau(z,v)\left(\frac{\delta_{\partial\Omega}(w)}{\delta_{\partial\Omega}(z)}\right)^{1/2}$.

Proof. We only need to prove 1., and, by Lemma 3.2 (1), it is enough to prove that

$$\tau\left(z,v,\delta_{\partial\Omega}(z)/2\right) \simeq \tau\left(w,v,\delta_{\partial\Omega}(z)/2\right).$$

This follows the fact that, if $L_z = \sum_i a_i L_i^0$ and $L_w = \sum_i b_i L_i^0$ are so that $L_z(z) = L_w(w) = v$, then $L_z = L_w + \mathrm{O}\left(\delta_{\partial\Omega}(z)\right)$, and, thus $F\left(L_z,z,\delta_{\partial\Omega}(z)/2\right) = F\left(L_w,z,\delta_{\partial\Omega}(z)/2\right) + \mathrm{O}(1)$ and $F\left(L_w,w,\delta_{\partial\Omega}(z)/2\right) = F\left(L_w,z,\delta_{\partial\Omega}(z)/2\right) + \mathrm{O}(1)$. □

Lemma 3.4. *If* $(v_i)_{1\leq i\leq n}$ *is a* $(z_j,\delta_{\partial\Omega}(z_j)/2)$*-extremal basis then, for* $w \in S_j^*$, $\sup_{u,v}\frac{|\Theta(w)(u,v)|}{k(z_j,u)k(z_j,v)} \simeq \sum_{l=1}^n \frac{\Theta(w)(v_l,v_l)}{k(z_j,v_l)^2}$.

Proof. The proof is exactly the same as in [BCD98]. It follows Lemma 3.2 (2) and Cauchy-Schwarz inequality. □

Lemma 3.5. *Let* w *be a point in* Q_j^* *and* $(\tilde{w}_i)_i$ *be the coordinates of* $\pi_{-\rho(z_j)}(w)$ *in the* $(z_j,\delta_{\partial\Omega}(z_j)/2)$*-adapted coordinate system* $\left(z_j^i\right)_i$. *Then*

$$\left|\frac{\partial\tilde{w}_r}{\partial z_j^l}(w)\right| \lesssim \frac{\tau(z_j,v_r)}{\tau(z_j v_l)},\ r \leq n-1,\ l \leq n,$$

where $(v_i)_{1\leq i\leq n}$ *is* $(z_j,\delta_{\partial\Omega}(z_j)/2)$*-extremal, and*

$$\left|\frac{\partial\Im\mathrm{m}(\tilde{w}_n)}{\partial z_j^l}\right| \lesssim \frac{\delta_{\partial\Omega}(z_j)}{\tau(z_j,v_l)},$$

the constants depending on Ω *and* α.

Proof. As in [BCD98], the proof is reduced to the case where $\rho(w) = \rho(z_j)$, then one consider a $(w,\delta_{\partial\Omega}(z_j)/2)$-extremal basis and uses Lemma 3.1. □

Third step: proof of Theorem 4.1 and Theorem 3.2. Let φ_0 be a $\mathscr{C}^\infty$ function supported in $[-2,2]$ and identically equal to 1 in $[-1,1]$, $0 \leq \varphi_0 \leq 1$. Let $\Psi = \Psi_\alpha$ be a $\mathscr{C}^1$ function supported in $\left[\alpha^2,\alpha^{-1}\right]$ equal to 1 on $[\alpha,1]$, $0 \leq \Psi \leq 1$.

With the notations of Lemma 3.5 we define, for $w \in Q_j^*$,

$$\varphi(w) = \Psi_\alpha\left(\frac{\delta_{\partial\Omega}(w)}{\delta_{\partial\Omega}(z_j)}\right)\varphi_0\left(\frac{|\Im\mathrm{m}(\tilde{w}_n)|}{\tau(z_j,v_n)}\right)\prod_{i=2}^n\varphi_0\left(\frac{|\tilde{w}_i|}{\tau(z_j,v_i)}\right).$$

Applying Stokes's formula to the form $(-\rho)^{2/m}\varphi\Theta\wedge\eta_l$, where $\eta_l = i^{n-1}dz_j^n\bigwedge_{\substack{r=1\\r\neq l}}^{n-1}dz_j^r\wedge d\overline{z_j^r}$, on Ω, using the previous Lemmas and a convenient function Ψ_α (see [BCD98, CD97]), for all $\alpha > 0$, the calculus made p. 404-408 in [BCD98] leads to the following estimate

$$\begin{aligned}\int_{Q_j}\delta_{\partial\Omega}(w)\left\|\Theta(w)\right\|_k dV(w) \quad &\leq \quad \left[\beta(\alpha) + \mathrm{O}(\mathrm{diam}(Q_j)/\alpha + \varepsilon C(\alpha)\right]\int_{Q_j^*}\delta_{\partial\Omega}(w)\left\|\Theta(w)\right\|_k dV(w)\\ &\quad + C(\varepsilon)C(\alpha)\int_{Q_j^*}\delta_{\partial\Omega}(w)\left\|\Theta(w)\right\|_E dV(w),\end{aligned}$$

with $\lim_{\alpha\to 0}\beta(\alpha) = 0$.

To finish the proof of the Theorems choose points p_i, $0 \leq i \leq N$ such that the neighborhood $V_0(p_i)$ is a covering of $\partial\Omega$. Then choosing α small enough so that $\beta(\alpha)MN$ is small, then ε small enough so that $\varepsilon C(\alpha)MN$ is small and finally k_0 large enough so that $\mathrm{O}(\mathrm{diam}(Q_j)/\alpha$ is small, we get a $\delta_0 > 0$ such that

$$\int_{W\cap\{\delta_{\partial\Omega}(w)\leq\delta_0\}}\delta_{\partial\Omega}(w)\left\|\Theta(w)\right\|_k dV(w) \lesssim \int_V \delta_{\partial\Omega}(w)\left\|\Theta(w)\right\|_E dV(w)$$

and

$$\int_{\Omega\cap\{\delta_{\partial\Omega}(w)\leq\delta_0\}}\delta_{\partial\Omega}(w)\left\|\Theta(w)\right\|_k dV(w) \lesssim \int_\Omega \delta_{\partial\Omega}(w)\left\|\Theta(w)\right\|_E dV(w)$$

which proves the Theorems. □

4. Non isotropic estimates for the d-operator for geometrically separated domains

Theorem 4.1. *Let* Ω *be a pseudo-convex domain in* $\mathbb{C}^n$ *which is geometrically separated at a boundary point* p_0. *Then there exist two neighborhoods of* p_0, $W \Subset V$ *such that there exists a constant* $C > 0$ *such that, for any smooth closed* $(1,1)$*-form* Θ *in* V, *there exists a smooth solution* w *of the equation* $dw = \Theta$ *satisfying the following estimate*

$$\int_W \|w\|_k\, dV \leq C\int_W \delta_{\partial\Omega}(z)\left\|\Theta(z)\right\|_k dV(z).$$

The proof of this result is quite standard and we will just adapt the proof made in Section 3 of [BCD98].

Proof. We choose for V a neighborhood contained in the neighborhood $V(p_0)$ defined in Section 3 and sufficiently small so that there exists a neighborhood $W \Subset V$ of p_0 and a point $P \in V$ such that there exists an open set $U \subset \Omega$ containing $(W \cap \Omega) \cup \{P\}$ which is starshaped with respect to P. Solving the equation $dw = \Theta$ in U, we can assume, without loss of generality, that the support of Θ does not contain P. By translation, we can also assume that P is the origin in $\mathbb{C}^n$.

Now, for $z \in W \cap \Omega$, $M_t : z \mapsto tz$ being the homotopy between 0 and z, Poincaré's formula

$$w = \int_0^1 M_t^* \left(i_{z_t}(\Theta)\right) dt,$$

where $i_{z_t}(\Theta)$ denotes the inner contraction of Θ with the field $Z_t = \frac{z}{t}$ and M_t^* the pull-back operator, gives a solution w of the equation $dw = \Theta$ in $W \cap \Omega$.

To finish the proof, we have to estimate this solution, and, for this, following the calculus made in [BCD98] p. 409, we only have to verify the following Lemma:

Lemma. *There exists a constant $\alpha > 0$ such that, for $0 \le t \le 1$ and $v \in \mathbb{C}^*$,*

$$\tau\left(tz, tv, \delta_{\partial\Omega}(tz)/2\right) \ge \alpha \left(\frac{\delta_{\partial\Omega}(tz)}{\delta_{\partial\Omega}(z)}\right)^{1/m} \tau\left(z, v, \delta_{\partial\Omega}(z)/2\right).$$

Proof of the Lemma. By definition of the polydiscs, there exists a constant K (depending only on c) such that $z \in P_{K\delta_{\partial\Omega}(tz)}(tz)$. Then, by Lemma 3.1, $\tau\left(tz, tv, K\delta_{\partial\Omega}(tz)\right) \simeq \tau\left(z, tv, K\delta_{\partial\Omega}(tz)\right)$, and thus $\tau\left(tz, tv, \delta_{\partial\Omega}(tz)/2\right) \simeq_K \tau\left(z, tv, \delta_{\partial\Omega}(tz)\right) \ge \tau\left(z, v, \delta_{\partial\Omega}(tz)\right)$, because $t \le 1$, and the Lemma follows Lemma 3.2. □

□

From Theorem 4.1 the following global result is easily deduced:

Theorem 4.2. *Let Ω be a bounded pseudo-convex domain in $\mathbb{C}^n$ which is geometrically separated at every point of it's boundary. There exists a constant $C > 0$ such that, for any smooth closed $(1,1)$-form Θ in Ω whose cohomology class in $H^2(\Omega, \mathbb{C})$ is 0, there exists a smooth solution of the equation $dw = \Theta$ in Ω satisfying*

$$|||w|||_k \le C |||\Theta|||_k.$$

Here, the $|||.|||_k$ norm of a smooth $(1,1)$-form Θ is defined using the covering V_i defined before the statement of Theorem 3.2:

$$|||\Theta|||_k = \sum_{i=0}^{N} \int_{V_i} \left(\sup_{u,v \in \mathbb{C}^*} \frac{|\langle \Theta; u, v\rangle|}{k_i(z,u) k_i(z,v)} \right) \delta_{\partial\Omega}(z) dV(z),$$

with $k_0(z,u) = k_0(z,v) = 1$.

5. Characterization of the zero-sets of the functions of the Nevanlinna class for lineally convex domains of finite type

It is a well-known fact that if X is the zero-set of a function f of the Nevanlinna class $N(\Omega)$, then it satisfies the Blaschke condition: if X_k are the irreducible components of X and n_k the multiplicity of f on X_k, then

$$\sum_k n_k \int_{X_k} \delta_{\partial\Omega} \, d\mu_k < +\infty,$$

where μ_k is the Euclidean measure on the regular part of X_k. Classically the data $\{X_k, n_k\}$ is called a divisor.

If Θ is the $(1,1)$-positive current classically associated to the divisor $\{X_k, n_k\}$, this condition is

$$\int_\Omega \delta_{\partial\Omega} d\left(\sum_{j,k} \left|\Theta_{j,k}\right|\right) < +\infty.$$

For more details we refer to [BCD98].

To prove Theorem 1.1, Ω satisfying automatically some topological condition, by a standard regularization process, it is sufficient to prove that there exists a constant $C > 0$ such that if Θ is a $(1,1)$-closed positive current, $\mathscr{C}^\infty$ in $\bar\Omega$, there exists a function u in $\bar\Omega$ solution of the equation $i\partial\bar\partial u = \Theta$ satisfying the estimate

$$\int_{\partial\Omega} |u| \, d\sigma \le C \int_\Omega \delta_{\partial\Omega} \|\Theta\|_E .$$

For details see [BCD98] and [Sko76].

As usual, the main part of the proof is done in two steps: resolution of the d-equation and, then, resolution of the $\bar\partial$-equation. The second step is done in Section 2. For the first one we use the results of Sections 3 and 4. As this two Sections are written for general geometrically separated domains and in a local context, we give some precisions (see also [Con02]).

5.1. **Non isotropic estimates on closed positive currents in lineally convex domains.** Ω being lineally convex of finite type it is completely geometrically separated ([CD08]). On the other hand, the proof of the Corollary at the beginning of Section 7.1 of [CD08] shows that the radius $\tau(z,v,\varepsilon)$ defined for general geometrically separated domains are equivalents to the ones defined by M. Conrad in [Con02] (and used in Section 2)

Then, the norm $\|\Theta(z)\|_k^v$ used in Theorem 3.2 is equivalent to the norm $\|\Theta(z)\|_k$ defined with the radius τ_i of [Con02] (and is independent of v). Thus, Theorem 3.2 means that there exists a constant C_1 such that

$$\int_\Omega \delta_{\partial\Omega}(z)\,\|\Theta(z)\|_k\,dV \le C_1 \int_\Omega \delta_{\partial\Omega}(z)\,\|\Theta(z)\|_E\,dV.$$

5.2. **Resolution of the d equation in lineally convex domains.** As Ω is lineally convex we have $H^2(\Omega,\mathbb{C})=0$ (see for example [Con02]). Thus, by Theorem 4.2 there exists a smooth form w such that $dw=\Theta$ in Ω and $\int_\Omega \|w(z)\|_k \le C_2 \int_\Omega \delta_{\partial\Omega}(z)\,\|\Theta(z)\|_k\,dV$.

6. Remarks

The method of resolution of the equation $\bar\partial u = f$ presented in Section 2 can be used to obtain various other estimates.

For example, the estimates obtained for convex domains of finite type in [Cum01a, DFF99, Fis04, Hef04, Ale05, Ale06] can be proved for lineally convex domains using our method (see also [DF06]).

References

[Ale01] W. Alexandre, *Construction d'une fonction de support à la Diederich-Fornaess*, Pub. IRMA, Lille **54** (2001), no. III.

[Ale05] ———, *$\mathscr{C}^k$-estimates for the $\bar\partial_b$-equation on convex domains of finite type,*, Michigan Math. J. **2** (2005), 357–382.

[Ale06] ———, *$\mathscr{C}^k$ estimates for $\bar\partial$ on convex domain of finite type*, Math. Z. **252** (2006), 473–496.

[BA82] B. Berndtsson and M. Andersson, *Henkin-Ramirez formulas with weight factors*, Ann. Inst. Fourier **32** (1982), no. 2, 91–110.

[BC82] A. Bonami and P Charpentier, *Solutions de l'équation $\bar\partial$ et zéros de la classe de Nevanlinna dans certains domains faiblement pseudo-convexes*, Ann. Inst. Fourier **XXXII** (1982), no. 4, 53–89.

[BCD98] J. Bruna, Ph. Charpentier, and Y. Dupain, *Zeros varieties for the Nevanlinna class in convex domains of finite type in $\mathbb{C}^n$*, Ann. of Math. **147** (1998), 391–415.

[CD97] P. Charpentier and Y. Dupain, *Pseudodistances et courants positifs dans les domaines de $\mathbb{C}^3$*, Ann. Scuola Norm. Pisa **XXIV** (1997), no. 2, 299–350.

[CD08] ———, *Extremal basis, geometrically separated domains and applications*, arXiv:0810.1889 (2008).

[CNS92] D. C. Chang, A. Nagel, and E. Stein, *Estimates for the $\bar\partial$-Neumann problem for pseudoconvex domains in $\mathbb{C}^2$ of finite type*, Acta Math. **169** (1992), 153–228.

[Con02] M. Conrad, *Anisotrope optimale Pseudometriken für lineal konvex Gebeite von endlichem Typ (mit Anwendungen)*, PhD thesis, Berg.Universität-GHS Wuppertal (2002).

[Cum01a] A. Cumenge, *Sharp estimates for $\bar\partial$ on convex domains of finite type*, Ark. Math. **39** (2001), no. 1, 1–25.

[Cum01b] ———, *Zero sets of functions in the Nevanlinna or the Nevanlinna-Djrbachan classes*, Pacific J. Math. **199** (2001), no. 1, 79–92.

[DF03] K Diederich and J. E. Fornaess, *Lineally convex domains of finite type: holomorphic support functions*, Manuscripta Math. **112** (2003), 403–431.

[DF06] K. Diederich and B. Fischer, *Hölder estimates on lineally convex domains of finite type*, Michigan Math. J. **54** (2006), no. 2, 341–452.

[DFF99] K. Diederich, B. Fischer, and J. E. Fornaess, *Hölder estimates on convex domains of finite type*, Math. Z. **232** (1999), 43–61.

[DM01] K. Diederich and E. Mazzilli, *Zero varieties for the Nevanlinna class on all convex domains of finite type*, Nagoya Math. Journal **163** (2001), 215–227.

[Fis04] B. Fischer, *Nonisotropic Hôlder Extimates on Convex Domains of Finite Type*, Michigan Math. J. **52** (2004), 219–239.

[Hef04] T. Hefer, *Extremal bases and Hölder Estimates for $\bar\partial$ on Convex Domains of Finite Type*, Michigan Math. J. **52** (2004), 573–602.

[Hen75] G. M. Henkin, *Solutions with bounds for the equations of H. Lewy and Poincaré-Lelong. Construction of functions of Nevanlinna class with given zeros in a strongly pseudoconvex domain*, Dokl. Akad. Nauk SSSR **224** (1975), no. 4, 771–774.

[Koh73] J. J. Kohn, *Global regularity for $\bar\partial$ on weakly pseudo-convex manifolds*, Trans. Amer. Math. Soc. **181** (1973), 273–292.

[McN94] J. McNeal, *Estimates on Bergman Kernels of Convex Domains*, Advances in Math (1994), 108–139.

[NPT09] N. Nikolov, P. Pflug, and P. J. Thomas, *On different extremal bases for $\mathbb{C}$-convex domains*, arXiv 0912.4828v2 (2009).

[Sko76] H. Skoda, *Valeurs au bord pour les solutions de l'opérateur $\bar\partial$, et caractérisation des zéros des fonctions de la classe de Nevanlinna*, Bull. Soc. Math. France **104** (1976), 225–299.

P. Charpentier & Y. Dupain, Université Bordeaux I, Institut de Mathématiques de Bordeaux, 351, Cours de la Libération, 33405, Talence, France

M. Mounkaila, Université Abdou Moumouni, Faculté des Sciences, B.P. 10662, Niamey, Niger

E-mail address: P. Charpentier: philippe.charpentier@math.u-bordeaux1.fr

E-mail address: Y. Dupain: yves.dupain@math.u-bordeaux1.fr

E-mail address: M. Mounkaila: modi.mounkaila@yahoo.fr