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Chow rings and decomposition theorems for families of $K3$ surfaces and Calabi-Yau hypersurfaces

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Abstract

The decomposition theorem for smooth projective morphisms $\pi : \mathcal{X} \to B$ says that $R\pi_*\mathbb{Q}$ decomposes as $\oplus R^i\pi_*\mathbb{Q}[-i]$. We describe simple examples where it is not possible to have such a decomposition compatible with cup-product, even after restriction to Zariski dense open sets of B. We prove however that this is always possible for families of $K3$ surfaces (after shrinking the base), and show how this result relates to a result by Beauville and the author ([2]) on the Chow ring of $K3$ surfaces S. We give two proofs of this result, the first one involving K-autocorrespondences of $K3$ surfaces, seen as analogues of isogenies of abelian varieties, the second one involving a certain decomposition of the small diagonal in S^3 obtained in [2]. We also prove an analogue of such a decomposition of the small diagonal in X^3 for Calabi-Yau hypersurfaces X in $\mathbb{P}^n$, which in turn provides strong restrictions on their Chow ring.

Let $\pi : \mathcal{X} \to B$ be a smooth projective morphism. The decomposition theorem, proved by Deligne in [4] as a consequence of the hard Lefschetz theorem, is the following statement:

Theorem 0.1. *(Deligne 1968) In the derived category of sheaves of $\mathbb{Q}$-vector spaces on B, there is a decomposition*

$$R\pi_*\mathbb{Q} = \oplus_i R^i\pi_*\mathbb{Q}[-i]. \tag{0.1}$$

This statement is equivalent, as explained by Deligne in *loc. cit.* to (a universal version of) the degeneracy at E_2 of the Leray spectral sequence of π. Deligne came back in [5] to the problem of constructing a *canonical* such decomposition, given the topological Chern class l of a relatively ample line bundle on $\mathcal{X}$ and imposing partial compatibilities with the morphism of cup-product with l.

Note that both sides of (0.1) carry a cup-product. On the right, we put the direct sum of the cup-product maps $\mu_{i,j} : R^i\pi_*\mathbb{Q} \otimes R^j\pi_*\mathbb{Q} \to R^{i+j}\pi_*\mathbb{Q}$. On the left, one needs to choose an explicit representation of $R\pi_*\mathbb{Q}$ by a complex C^*, together with an explicit morphism of complexes $\mu : C^* \otimes C^* \to C^*$ which induces the cup-product in cohomology. When passing to coefficients $\mathbb{R}$ or $\mathbb{C}$, one can take $C^* = \pi_*\mathcal{A}^*_{\mathcal{X}}$, where $\mathcal{A}^*_{\mathcal{X}}$ is the sheaf of $\mathcal{C}^\infty$ real or complex differential forms on $\mathcal{X}$ and for μ the wedge product of forms. For rational coefficients, the explicit construction of the cup-product at the level of complexes (for example Čech complexes) is more painful (see [8, 6.3]). The resulting cup-product morphism μ will be canonical only in the derived category.

The question we study in this paper is the following:

Question 0.2. *Given a family of smooth projective varieties $\pi : \mathcal{X} \to B$, does there exist a decomposition as above which is multiplicative, that is compatible with the morphism*

$$\mu : R\pi_*\mathbb{Q} \otimes R\pi_*\mathbb{Q} \to R\pi_*\mathbb{Q}$$

given by cup-product?

Let us give three examples: In the first one, which is the case of families of abelian varieties, the answer to Question 0.2 is affirmative. This was proved by Deninger and Murre in [6] as a consequence of a much more general "motivic" decomposition result.

Proposition 0.3. *For any family $\pi : \mathcal{A} \to B$ of abelian varieties (or complex tori), there is a multiplicative decomposition isomorphism $R\pi_*\mathbb{Q} = \oplus_i R^i\pi_*\mathbb{Q}[-i]$.*

In the next two examples, the answer to Question 0.2 is negative. The simplest example is that of projective bundles $\pi : \mathbb{P}(\mathcal{E}) \to B$, where $\mathcal{E}$ is a locally free sheaf on B.

Proposition 0.4. *Assume that $c_1^{top}(\mathcal{E}) = 0$ in $H^2(B, \mathbb{Q})$. Then, if there exists a multiplicative decomposition isomorphism for $\pi : \mathbb{P}(\mathcal{E}) \to B$, one has $c_i^{top}(\mathcal{E}) = 0$ in $H^{2i}(B, \mathbb{Q})$ for all $i > 0$.*

Proof. Let $h = c_1^{top}(\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)) \in H^2(\mathbb{P}(\mathcal{E}), \mathbb{Q})$. It is standard that

$$H^2(\mathbb{P}(\mathcal{E}), \mathbb{Q}) = \pi^* H^2(B, \mathbb{Q}) \oplus \mathbb{Q}h,$$

where $\pi^* H^2(B, \mathbb{Q})$ identifies canonically with the deepest term $H^2(B, R^0\pi_*\mathbb{Q})$ in the Leray filtration. A multiplicative decomposition isomorphism as in (0.1) induces by taking cohomology another decomposition of $H^2(\mathbb{P}(\mathcal{E}), \mathbb{Q})$ as $\pi^* H^2(B, \mathbb{Q}) \oplus \mathbb{Q}h'$, where $h' = h + \pi^*\alpha$, for some $\alpha \in H^2(B, \mathbb{Q})$. In this multiplicative decomposition, h' will generate a summand isomorphic to $H^0(B, R^2\pi_*\mathbb{Q})$. Let $r = rank\, \mathcal{E}$. As $c_1^{top}(\mathcal{E}) = 0$, one has $\pi_* h^r = 0$ in $H^2(B, \mathbb{Q})$. As $(h')^r = 0$ in $H^0(B, R^{2r}\pi_*\mathbb{Q})$, and $(h')^r$ belongs by multiplicativity to a direct summand naturally isomorphic (by restriction to fibers) to $H^0(B, R^{2r}\pi_*\mathbb{Q}) = 0$, one must also have $(h')^r = 0$ in $H^{2r}(\mathbb{P}(\mathcal{E}), \mathbb{Q})$. On the other hand $(h')^r = h^r + rh^{r-1}\pi^*\alpha + \ldots + \pi^*\alpha^r$, and it follows that

$$\pi_*(h')^r = 0 = \pi_* h^r + r\alpha \text{ in } H^2(B, \mathbb{Q}).$$

Thus $\alpha = 0$, $h' = h$, and $h^r = 0$ in $H^{2r}(\mathbb{P}(\mathcal{E}), \mathbb{Q})$. The definition of Chern classes and the fact that $h^r = 0$ show then that $c_i^{top}(\mathcal{E}) = 0$ for all $i > 0$. ■

In this example, the obstructions to the existence of a multiplicative decomposition isomorphism are given by cycle classes on B. These classes vanish locally on B for the Zariski topology and this suggests studying the following variant of Question 0.2:

Question 0.5. *Given a family of smooth projective varieties $\pi : \mathcal{X} \to B$, does there exist a Zariski dense open set B^0 of B, and a multiplicative decomposition isomorphism as in (0.1) for the restricted family $\mathcal{X}^0 \to B^0$?*

Our last example is given by families of curves and shows that already in this case, we can have a negative answer to this weakened question. We fix an abelian surface, choose a Lefschetz pencil of curves $C_t \subset A$, $t \in \mathbb{P}^1$, and let $B \subset \mathbb{P}^1$ be the open set parameterizing smooth fibers.

Proposition 0.6. *The family $\pi : \mathcal{C} \to B$ does not admit a multiplicative decomposition isomorphism over any non empty Zariski open set of B.*

Proof. Assume there is a multiplicative decomposition isomorphism for the restricted family $\pi : \mathcal{C}^0 \to B^0$ over some non-empty Zariski open set B^0 of B. Then we get by taking cohomology a decomposition

$$H^1(\mathcal{C}^0, \mathbb{Q}) \cong \pi^* H^1(B^0, \mathbb{Q}) \oplus K,$$

where $K \cong H^0(B^0, R^1\pi_*\mathbb{Q})$ has the property that the cup-product map:

$$\mu : K \otimes K \to H^2(\mathcal{C}^0, \mathbb{Q})$$

factors through the cup-product map

$$\mu : H^0(B, R^1\pi_*\mathbb{Q}) \otimes H^0(B, R^1\pi_*\mathbb{Q}) \to H^0(B, R^2\pi_*\mathbb{Q}).$$

Now let α, $\beta \in H^1(A, \mathbb{C})$ be the classes of two independent sections of Ω^1_A. Let us denote by $q : \mathcal{C} \to A$ the natural map. Then we can decompose

$$q^*\alpha = \alpha_K + \pi^*\alpha',\ q^*\beta = \beta_K + \pi^*\beta',$$

with α_K, $\beta_K \in K$ and α', $\beta' \in H^1(B^0, \mathbb{C})$. Taking their cup-product, and using the fact that the cup-product is trivial on the summand $\pi^*H^1(B^0, \mathbb{C})$, we get the equality

$$q^*(\alpha \cup \beta) = \alpha_K \cup \beta_K + \alpha_K \cup \pi^*\beta' + \pi^*\alpha' \cup \beta_K,$$

and the first term $\alpha_K \cup \beta_K$ vanishes because it vanishes in $H^0(B^0, R^2\pi_*\mathbb{C})$ (indeed, the classes α, β are of type $(1,0)$ and so are their restrictions to the fibers $\mathcal{C}_b$ which are 1-dimensional). The same arguments show that

$$q^*(\alpha \cup \beta) = q^*\alpha \cup \pi^*\beta' + \pi^*\alpha' \cup q^*\beta \ \text{ in } \ H^2(\mathcal{C}^0, \mathbb{C}).$$

The contradiction comes from the fact that $q^*(\alpha \cup \beta)$ does not vanish in $H^2(\mathcal{C}^0, \mathbb{C})$ (because this is the restriction of the class of a nonzero $(2,0)$-form on a projective completion of $\mathcal{C}^0$, namely the blow-up of A at the base-points of the pencil) and has trivial residues along all fibers C_b, $b \in \mathbb{P}^1 \setminus B^0$, while the independence of the restrictions of the classes α, β to the fibers C_b, $b \in \mathbb{P}^1 \setminus B^0$ implies that the term on the right can have trivial residues along all fibers if and only if β' and α' have trivial residues at all points $b \in \mathbb{P}^1 \setminus B^0$, which implies $\beta' = 0$, $\alpha' = 0$. ■

Our main result in this paper is:

Theorem 0.7. *(i) For any smooth projective family $\pi : \mathcal{X} \to B$ of $K3$ surfaces, there exist a decomposition isomorphism as in (0.1) and a nonempty Zariski open subset B^0 of B, such that this decomposition becomes multiplicative for the restricted family $\pi : \mathcal{X}^0 \to B^0$.*

(ii) The class of the relative diagonal $[\Delta_{\mathcal{X}^0/B^0}] \in H^4(\mathcal{X}^0 \times_{B^0} \mathcal{X}^0, \mathbb{Q})$ belongs to the direct summand $H^0(B^0, R^4(\pi,\pi)_*\mathbb{Q})$ of $H^4(\mathcal{X}^0 \times_{B^0} \mathcal{X}^0, \mathbb{Q})$, for the induced decomposition of $R(\pi,\pi)_*\mathbb{Q}$.

(iii) For any algebraic line bundle $\mathcal{L}$ on $\mathcal{X}$, there is a dense Zariski open set B^0 of B such that the topological Chern class $c_1^{top}(\mathcal{L}) \in H^2(\mathcal{X}, \mathbb{Q})$ restricted to $\mathcal{X}^0$ belongs to the direct summand $H^0(B^0, R^2\pi_*\mathbb{Q})$ of $H^2(\mathcal{X}^0, \mathbb{Q})$ induced by this decomposition.

Statement (i) is definitely wrong if we do not restrict to a Zariski open set (cf. section 1.2 for an example). Statement (iii) is in fact implied by (i), according to Lemma 1.4.

We note that statements (i) and (iii) together imply that the decomposition above coincide locally over B in the Zariski topology with the first one defined by Deligne in [5]. This follows from the characterization of the latter given in [5, Prop. 2.7].

We will explain in section 1 how Theorem 0.7 is related to the results of [2], [1] (see also [13] for further developments) concerning the Chow ring of $K3$ surfaces. In fact the statement was motivated by the following result, which is an easy consequence of the results of [2], but can be seen as well as a consequence of Theorem 0.7 by Proposition 1.3.

Proposition 0.8. *Let $\pi : \mathcal{S} \to B$ be a family of $K3$ surfaces, $\mathcal{L}_i \in \operatorname{Pic}\mathcal{S}$ and n_{ij} be integers. Assume that the degree 4 cohomology class $c = \sum_{ij} n_{ij}\, c_1^{top}(\mathcal{L}_i)c_1^{top}(\mathcal{L}_j) \in H^4(\mathcal{S},\mathbb{Q})$ has trivial restriction on the fibers $\mathcal{S}_t$, $t \in B$ (or equivalently, has trivial restriction on one fiber $\mathcal{S}_t$, if B is connected). Then there exists a nonempty Zariski open subset B^0 of B such that c vanishes in $H^4(\mathcal{S}^0,\mathbb{Q})$, where $\mathcal{S}^0 := \pi^{-1}(B^0)$.*

In section 1, we prove Proposition 1.3, which says in particular that Proposition 0.8 is satisfied more generally by any family $\mathcal{X} \to B$ of varieties with trivial irregularity, admitting a multiplicative decomposition isomorphism, and for any fiberwise polynomial cohomological relation between Chern classes of line bundles on $\mathcal{X}$. This strongly relates the present work to the paper [1].

We will also use this proposition in section 1.1 to provide further examples of families of surfaces for which there is no multiplicative decomposition isomorphism over any dense Zariski open set of the base, although there is no variation of Hodge structures in the fibers.

Let us mention one consequence of Theorem 0.7. Let $\pi : \mathcal{X} \to B$ be a projective family of $K3$ surfaces, with B irreducible, and $\mathcal{L} \in \operatorname{Pic}\mathcal{X}$. Consider the 0-cycle $o_{\mathcal{X}} := \frac{1}{\deg_{\mathcal{X}_t}\mathcal{L}^2}\mathcal{L}^2 \in CH^2(\mathcal{X})_{\mathbb{Q}}$. By theorems 1.1 and 1.2, this 0-cycle is independent of $\mathcal{L}$, at least after restriction to $\mathcal{X}^0 = \pi^{-1}(B^0)$, for an adequate Zariski dense open set B^0 of B. We also have the relative diagonal $\Delta_{\mathcal{X}/B} \in CH^2(\mathcal{X} \times_B \mathcal{X})$. Let L_s, $s \in I$, be line bundles on $\mathcal{X}$. Set $\mathcal{X}^{m/B} := \mathcal{X} \times_B \ldots \times_B \mathcal{X}$, $\pi^m : \mathcal{X}^{m/B} \to B$, the m-th fibered product of $\mathcal{X}$ over B.

Corollary 0.9. *Consider a codimension $2r$ cycle Z with $\mathbb{Q}$-coefficients in $\mathcal{X}^{m/B}$ which is a polynomial in the cycles $pr_i^* o_{\mathcal{X}}$, $pr_j^* L_s$, $pr_{kl}^*\Delta_{\mathcal{X}/B}$, where $1 \le i,j,k,l \le m$. Assume that the restriction of Z to one (equivalently, any) fiber $\mathcal{X}_t^m$ is cohomologous to 0. Then there exists a dense Zariski open set B^0 of B such that Z is cohomologous to 0 in $(\mathcal{X}^0)^{m/B}$.*

Proof. Indeed, it follows from Theorem 0.7, (iii) that over a dense Zariski open set B^0, the classes $c_1^{top}(L_s) \in H^2(\mathcal{X}^0,\mathbb{Q})$ belong to the direct summand $H^0(B^0, R^2\pi_*\mathbb{Q})$ of $H^2(\mathcal{X}^0,\mathbb{Q})$ induced by the multiplicative decomposition isomorphism of Theorem 0.7. By multiplicativity, we have that the class $[o_{\mathcal{X}}]$ belongs to the direct summand $H^0(B^0, R^4\pi_*\mathbb{Q})$ of $H^4(\mathcal{X}^0,\mathbb{Q})$. Theorem 0.7, (ii) tells us that, over a Zariski open set B^0 of B, the class $[\Delta_{\mathcal{X}/B}]$ of the relative diagonal belongs to the direct summand $H^0(B^0, R^4(\pi,\pi)_*\mathbb{Q})$ of $H^4(\mathcal{X}^0 \times_{B^0} \mathcal{X}^0,\mathbb{Q})$. We thus conclude by multiplicativity that the class $[Z]$ belongs to the direct summand $H^0(B^0, R^{2r}(\pi^m)_*\mathbb{Q})$ of $H^{2r}(\mathcal{X}^0)^{m/B},\mathbb{Q})$. But by assumption, the class $[Z]$ projects to 0 in $H^0(B^0, R^{2r}(\pi^m)_*\mathbb{Q})$. We thus deduce that it is identically 0. ■

This corollary provides an evidence (of a rather speculative nature, in the same spirit as [9]) for the conjecture made in [13, Conj. 1.3] concerning the Chow ring of hyper-Kähler manifolds, at least for those of type $S^{[n]}$, where S is a $K3$ surface. Indeed, this conjecture states the following:

Conjecture 0.10. *Let Y be an algebraic hyper-Kähler variety. Then any polynomial cohomological relation $P([c_1(L_j)],[c_i(T_Y)]) = 0$ in $H^{2k}(Y,\mathbb{Q})$, $L_j \in \operatorname{Pic} Y$, already holds at the level of Chow groups : $P(c_1(L_j), c_i(T_Y)) = 0$ in $CH^k(Y)_{\mathbb{Q}}$.*

Indeed, we proved in [13, Prop. 2.5] that for $Y = S^{[n]}$, this conjecture is implied by the following conjecture:

Conjecture 0.11. *Let S be an algebraic $K3$ surface. For any integer m, let $P \in CH^p(S^m)_{\mathbb{Q}}$ be a weighted degree k polynomial expression in $pr_i^* c_1(L_s)$, $L_s \in \operatorname{Pic} S$, $pr_{jl}^*\Delta_S$: Then if $[P] = 0$ in $H^{2k}(S^m,\mathbb{Q})$, we have $P = 0$ in $CH^k(S^m)_{\mathbb{Q}}$.*

By the general principle 1.2, Conjecture 0.11 implies Corollary 0.9. In the other direction, we can say the following (which is rather speculative): In the situation of Conjecture 0.11, we can find a family $\mathcal{X} \to B$ of smooth projective $K3$ surfaces, endowed with line bundles $\mathcal{L}_s \in \operatorname{Pic} \mathcal{X}$, where everything is defined over $\mathbb{Q}$, such that S and the L_s's are the fiber over some $t \in B$ of $\mathcal{X}$ and the $\mathcal{L}_j$'s. Then we can construct using the same polynomial expression the cycle $\mathcal{P} \in CH^k(\mathcal{X}^{m/B})_{\mathbb{Q}}$ and Corollary 0.9 tells that the class of this cycle vanishes in $H^{2k}((\mathcal{X}^0)^{m/B}, \mathbb{Q})$. As $(\mathcal{X}^0)^{m/B}$ and $\mathcal{P}$ are defined over $\mathbb{Q}$, the Beilinson conjecture predicts that it is trivial if furthermore its Abel-Jacobi invariant vanishes, which is presumably provable by the same method used to get the vanishing of the cycle class.

Theorem 0.7 will be proved in section 2. In fact, we will give two proofs of it. In the first one, we use the existence of non trivial self K-correspondences (see [12]), whose action on cohomology allows to split the cohomology in different pieces, in a way which is compatible with the cup-product. This is very similar to the proof given in the abelian case (Proposition 0.3), for which one uses homotheties. The second proof is formal, and uses a curious decomposition of the small diagonal $\Delta \subset S^3$ of a $K3$ surface S, obtained in [2, Prop. 3.2] (see Theorem 2.17).

In section 3, we will investigate the case of Calabi-Yau hypersurfaces X in projective space $\mathbb{P}^n$ and establish for them the following analogue of this decomposition of the small diagonal. We denote by $\Delta \cong X \subset X^3$ the small diagonal of X and $\Delta_{ij} \cong X \times X \subset X^3$ the inverse image in X^3 of the diagonal of $X \times X$ by the projection onto the product of the i-th and j-th factors. There is a natural 0-cycle $o := \frac{c_1(\mathcal{O}_X(1))^{n-1}}{n+1} \in CH_0(X)$.

Theorem 0.12. *(cf. Theorem 3.1) The following relation holds in $CH^{2n-2}(X \times X \times X)_{\mathbb{Q}}$ (in the following equation, "+(perm.)" means that we symmetrize in the indices the considered expression):*

$$\Delta = \Delta_{12} \cdot o_3 + (perm.) + Z + \Gamma' \text{ in } CH^{2n-2}(X \times X \times X)_{\mathbb{Q}}, \tag{0.2}$$

where Z is the restriction to $X \times X \times X$ of a cycle of $\mathbb{P}^n \times \mathbb{P}^n \times \mathbb{P}^n$, and Γ' is a multiple of the following effective cycle of dimension $n-1$:

$$\Gamma := \cup_{t \in F(X)} \mathbb{P}^1_t \times \mathbb{P}^1_t \times \mathbb{P}^1_t,$$

where $F(X)$ is the variety of lines contained in X.

As a consequence, we get the following result concerning the Chow ring of a Calabi-Yau hypersurface X in $\mathbb{P}^n$, which generalizes Theorem 1 of [2] (see Theorem 1.1):

Theorem 0.13. *Let X be as above and let Z_i, Z'_i be cycles of codimension > 0 on X such that* $\operatorname{codim} Z_i + \operatorname{codim} Z'_i = n-1$. *Then if we have a cohomological relation*

$$\sum_i n_i \deg (Z_i \cdot Z'_i) = 0,$$

this relation already holds at the level of Chow groups:

$$\sum_i n_i Z_i \cdot Z'_i = 0 \text{ in } CH_0(X)_{\mathbb{Q}}.$$

We conjecture that the cycle Γ also comes from a cycle on $\mathbb{P}^n \times \mathbb{P}^n \times \mathbb{P}^n$. This would imply the analogue of Theorem 0.7 for families of Calabi-Yau hypersurfaces.

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1 Link with the results of [1], [2]

In this section, we first show how to deduce Proposition 0.8 from the following Theorem proved in [2] :

Theorem 1.1. *(Beauville-Voisin 2004) Let S be a $K3$ surface, $D_i \in CH^1(S)$ be divisors on S and n_{ij} be integers. Then if the 0-cycle $\sum_{i,j} n_{ij} D_i D_j \in CH_0(S)$ is cohomologous to 0 on S, it is equal to 0 in $CH_0(S)$.*

We will use here and many times later on in the paper the following "general principle" (cf. [3], [14, Theorem 10.19], [15, Corollary 3.1.6]:

Theorem 1.2. *Let $\pi : X \to B$ be a morphism with X, B smooth, and $Z \in CH^k(X)$ such that $Z_{|X_t} = 0$ in $CH^k(X_t)$ for any $t \in B$. Then there exists a dense Zariski open set $B^0 \subset B$ such that*

$$[Z] = 0 \text{ in } H^{2k}(X^0, \mathbb{Q}), \tag{1.3}$$

where $X^0 := \pi^{-1}(B^0)$.

Proof of Proposition 0.8. Indeed, under the assumption that the intersection number $\sum_{i,j} n_{ij} c_1^{top}(\mathcal{L}_{i,b}) c_1^{top}(\mathcal{L}_{j,b}) = 0$ vanishes in $H^4(\mathcal{S}_b, \mathbb{Q}) = \mathbb{Q}$ for all $b \in B$, Theorem 1.1 says that the codimension 2 cycle $\sum_{i,j} n_{ij} c_1(\mathcal{L}_i) c_1(\mathcal{L}_j) \in CH^2(\mathcal{S})$ has trivial restriction on each fiber $\mathcal{S}_b$. The general principle 1.2 then implies that there is a Zariski dense open set B^0 of B such that the class $\sum_{i,j} n_{ij} c_1^{top}(\mathcal{L}_i) c_1^{top}(\mathcal{L}_j)$ vanishes in $H^4(\mathcal{S}^0, \mathbb{Q})$. ■

We next prove the following Proposition 1.3, which provides a conclusion similar as above, under the assumption that the family has a multiplicative decomposition isomorphism over a Zariski open set.

Let $\pi : \mathcal{X} \to B$ be a projective family of smooth complex varieties such that $H^1(\mathcal{X}_b, \mathcal{O}_{\mathcal{X}_b}) = 0$ for any $b \in B$, parameterized by a connected complex quasi-projective variety B. Let $\mathcal{L}_i$, $i = 1, \ldots, m$ be line bundles on $\mathcal{X}$ and $l_i := c_1^{top}(\mathcal{L}_i) \in H^2(\mathcal{X}, \mathbb{Q})$. We will say that a cohomology class $\beta \in H^*(\mathcal{X}, \mathbb{Q})$ is Zariski locally trivial over B if B is covered by Zariski open sets $B^0 \subset B$, such that $\beta_{|\mathcal{X}^0} = 0$ in $H^*(\mathcal{X}^0, \mathbb{Q})$, where $\mathcal{X}^0 = \pi^{-1}(B^0)$.

Proposition 1.3. *Assume that there is a multiplicative decomposition isomorphism*

$$R\pi_* \mathbb{Q} = \oplus_i R^i \pi_* \mathbb{Q}[-i]. \tag{1.4}$$

Let P be a homogeneous polynomial of degree r in m variables with rational coefficients and let $\alpha := P(l_i) \in H^{2r}(\mathcal{X}, \mathbb{Q})$. Then, if $\alpha_{|\mathcal{X}_b} = 0$ in $H^{2r}(\mathcal{X}_b, \mathbb{Q})$ for some $b \in B$, the class α is Zariski locally trivial over B.

Proof. We will assume for simplicity that B is smooth although a closer look at the proof shows that this assumption is not necessary. The multiplicative decomposition isomorphism induces, by taking cohomology and using the fact that the fibers have no degree 1 rational cohomology, a decomposition

$$H^2(\mathcal{X}, \mathbb{Q}) = H^0(B, R^2\pi_* \mathbb{Q}) \oplus \pi^* H^2(B, \mathbb{Q}), \tag{1.5}$$

which is compatible with cup-product, so that the cup-product map on the first term factors through the map induced by cup-product:

$$\mu_r : H^0(B, R^2\pi_* \mathbb{Q})^{\otimes r} \to H^0(B, R^{2r}\pi_* \mathbb{Q}).$$

We write in this decomposition $l_i = l'_i + \pi^* k_i$, where $k_i \in H^2(B, R^0\pi_*\mathbb{Q}) = H^2(B,\mathbb{Q}) \overset{\pi^*}{\cong} \pi^* H^2(B,\mathbb{Q})$. We now have:

Lemma 1.4. *The assumptions being as in Proposition 1.3, the classes k_i are divisor classes on B. Thus B is covered by Zariski open sets B^0 such that the divisor classes l_i restricted to $\mathcal{X}^0$ belong to the direct summand $H^0(B^0, R^2\pi_*\mathbb{Q})$.*

Proof. Indeed, take any line bundle $\mathcal{L}$ on $\mathcal{X}$. Let $l = c_1^{top}(\mathcal{L}) \in H^2(\mathcal{X},\mathbb{Q})$ and decompose as above $l = l' + \pi^* k$, where l' has the same image as l in $H^0(B, R^2\pi_*\mathbb{Q})$ and k belongs to $H^2(B,\mathbb{Q})$. Denoting by n the dimension of the fibers, we get:

$$l^n l_i = (\sum_p \binom{n}{p} l'^p \pi^* k^{n-p})(l'_i + \pi^* k_i) = \sum_p \binom{n}{p} l'^p l'_i \pi^* k^{n-p} + \sum_p \binom{n}{p} l'^p \pi^*(k^{n-p} k_i). \quad (1.6)$$

Recall now that the decomposition is multiplicative. The class $l'^n l'_i$ thus belongs to the direct summand of $H^{2n+2}(\mathcal{X},\mathbb{Q})$ isomorphic to $H^0(B, R^{2n+2}\pi_*\mathbb{Q})$ deduced from the decomposition (1.4). As $R^{2n+2}\pi_*\mathbb{Q} = 0$, we conclude that $l'^n l'_i = 0$. Applying $\pi_* : H^{2n+2}(\mathcal{X},\mathbb{Q}) \to H^2(B,\mathbb{Q})$ to (1.6), we then get:

$$\pi_*(l^n l_i) = n\deg_{\mathcal{X}_b}(l'^{n-1} l'_i)k + \deg_{\mathcal{X}_b}(l'^n)k_i = n\deg_{\mathcal{X}_b}(l^{n-1} l_i)k + \deg_{\mathcal{X}_b}(l^n)k_i. \quad (1.7)$$

Observe that the term on the left is a divisor class on B. If the fiberwise self-intersection $\deg_{\mathcal{X}_b}({l_i}^n)$ is non zero, we can take $\mathcal{L} = \mathcal{L}_i$ and (1.7) gives:

$$\pi_*(l_i^{n+1}) = (n+1)\deg_{\mathcal{X}_b}({l_i}^n)k_i.$$

This shows that k_i is a divisor class on B and proves the lemma in this case. If $\deg_{\mathcal{X}_b}({l_i}^n)$ is equal to 0, choose a line bundle $\mathcal{L}$ on $\mathcal{X}$ such that both intersection numbers $\deg_{\mathcal{X}_b}(l^{n-1} l_i)$ and $\deg_{\mathcal{X}_b}(l^n)$ are nonzero (such an $\mathcal{L}$ exists because the morphism π is projective). Then, in the formula

$$\pi_*(l^n l_i) = n\deg_{\mathcal{X}_b}(l^{n-1} l_i)k + \deg_{\mathcal{X}_b}(l^n)k_i,$$

the left hand side is a divisor class on B and, as we just proved, the first term in the right hand side is also a divisor class on B. It thus follows that $\deg_{\mathcal{X}_b}(l^n)k_i$ is a divisor class on B. The lemma is thus proved. ■

Coming back to the proof of Proposition 1.3, Lemma 1.4 tells us that B is covered by Zariski open sets B^0 on which l_i belongs to the first summand $H^0(B^0, R^2\pi_*\mathbb{Q})$ in (1.5). It then follows by multiplicativity that any polynomial expression $P(l_i)_{|\mathcal{X}^0}$ belongs to a direct summand of $H^{2r}(\mathcal{X}^0,\mathbb{Q})$ isomorphic by the natural projection to $H^0(B^0, R^{2r}\pi_*\mathbb{Q})$. Consider now our fiberwise cohomological polynomial relation $\alpha_{|\mathcal{X}_b} = 0$ in $H^{2r}(\mathcal{X}_b,\mathbb{Q})$, for some $b \in B$. Since B is connected, it says equivalently that α vanishes in $H^0(B^0, R^{2r}\pi_*\mathbb{Q})$. It follows then from the previous statement that it vanishes in $H^{2r}(\mathcal{X}^0,\mathbb{Q})$. ■

1.1 Application

We can use Proposition 1.3 to exhibit very simple families of smooth projective surfaces, with no variation of Hodge structure, but for which there is no multiplicative decomposition isomorphism on any nonempty Zariski open set of the base.

We consider a smooth projective surface S, and set

$$X = (\widetilde{S \times S})_\Delta,\ B = S,\ \pi = pr_2 \circ \tau,$$

where $\tau : (\widetilde{S \times S})_\Delta \to S \times S$ is the blow-up of the diagonal.

Proposition 1.5. *Assume that $h^{1,0}(S) = 0$, $h^{2,0}(S) \neq 0$. Then there is no multiplicative decomposition isomorphism for $\pi : X \to B$ over any Zariski dense open set of $B = S$.*

Proof. Let H be an ample line bundle on S, and $d := \deg c_1(H)^2$. On X, we have then two line bundles, namely $L := \tau^*(pr_1^* H)$ and $L' = \mathcal{O}_X(E)$ where E is the exceptional divisor of τ. On the fibers of π, we have the relation

$$\deg c_1(L)^2 = -d \deg c_1(L')^2.$$

If there existed a multiplicative decomposition isomorphism over a Zariski dense open set of $B = S$, we would have by Proposition 1.3, using the fact that the fibers of π are regular, a Zariski dense open set $U \subset S$ such that the relation

$$c_1^{top}(L)^2 = -d\, c_1^{top}(L')^2 \tag{1.8}$$

holds in $H^4(X_U, \mathbb{Q})$. If we apply $\tau_* : H^4(X_U, \mathbb{Q}) \to H^4(S \times U, \mathbb{Q})$ to this relation, we now get:

$$pr_1^* c_1^{top}(H)^2 = -d[\Delta] \tag{1.9}$$

in $H^4(S \times U, \mathbb{Q})$.

This relation implies that the class $pr_1^* c_1^{top}(H)^2 + d[\Delta] \in H^4(S \times S, \mathbb{Q})$ comes from a class $\gamma \in H^2(S \times \widetilde{D}, \mathbb{Q})$, where $D := S \setminus U$ and $\widetilde{D}$ is a desingularization of D. Denoting by $\tilde{j} : \widetilde{D} \to S$ the natural map, we then conclude that for any class $\alpha \in H^2(S, \mathbb{Q})$,

$$d\alpha \in H^2(S, \mathbb{Q}) = \tilde{j}_*(\gamma_* \alpha)$$

is supported on D. This contradicts the assumption $h^{2,0}(S) \neq 0$. ■

1.2 Example where Theorem 0.7, (i) is not satisfied globally on B

Let us apply the same arguments as in the proof of Proposition 1.3 to exhibit simple families of smooth projective $K3$ surfaces for which a multiplicative decomposition isomorphism does not exist on the whole base.

We take $B = \mathbb{P}^1$ and $\mathcal{S} \subset \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^2$ a generic hypersurface of multidegree $(d, 2, 3)$. We put $\pi := pr_1$. This is not a smooth family of $K3$ surfaces because of the nodal fibers, but we can take a finite cover of $\mathbb{P}^1$ and introduce a simultaneous resolution of the pulled-back family to get a family of smooth $K3$ surfaces parameterized by a complete curve. (Note that the simultaneous resolution does not hold in the projective category, so the morphism $\pi' : \mathcal{S}' \to B'$ obtained this way is usually not projective : this is a minor point.) By the Grothendieck-Lefschetz theorem, one has $\mathrm{NS}\,\mathcal{S} = \mathbb{Z}^3 = \mathrm{NS}\,(\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^2)$, and the same is true for $\mathcal{S}'$ if one assumes that the general fiber of $\mathcal{S}$ over B has $\mathrm{Pic}\,\mathcal{S}_b = \mathrm{Pic}\,(\mathbb{P}^1 \times \mathbb{P}^2) = \mathbb{Z}^2$.

We prove now :

Lemma 1.6. *The family $\pi' : \mathcal{S}' \to B'$ does not admit a multiplicative decomposition isomorphism over B'.*

Proof. As the hypersurface $\mathcal{S}$ is generic, the family $\mathcal{S} \to \mathbb{P}^1$ is not locally isotrivial. It follows that $H^2(\mathcal{S}', \mathcal{O}_{\mathcal{S}'}) = 0$, and thus

$$H^2(\mathcal{S}', \mathbb{Q}) = NS(\mathcal{S}') \otimes \mathbb{Q}^{\cdot}$$

As already mentioned, the right hand side is isomorphic to $\mathbb{Q}^3$, generated by the pull-back to $\mathcal{S}'$ of the natural classes h_1, h_2, h_3 on $\mathrm{Pic}\,(\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^2)$. The first class h_1 belongs to the natural summand $\pi'^* H^2(B, \mathbb{Q}) = H^2(B, R^0\pi'_*\mathbb{Q})$ and, as explained above, the existence of a multiplicative decomposition isomorphism would imply the existence of a decomposition

$$\mathrm{NS}\,(\mathcal{S}') \otimes \mathbb{Q} = H^2(\mathcal{S}', \mathbb{Q}) = \pi'^* H^2(B', \mathbb{Q}) \oplus H,\ H \cong H^0(B, R^2\pi'_*\mathbb{Q})$$

such that the cup-product map on H factors through the map given by cup-product

$$\mu : H^0(B', R^2\pi'_*\mathbb{Q}) \otimes H^0(B', R^2\pi'_*\mathbb{Q}) \to H^0(B', R^4\pi'_*\mathbb{Q}) = \mathbb{Q}.$$

Let us show that such a decomposition does not exist. As all classes are obtained by pull-back from $\mathcal{S}$ and the pull-back map preserves the cup-product, we can make the computation on $\mathcal{S}$. Let $h'_2 = h_2 - \alpha h_1$, $h'_3 = h_3 - \beta h_1$ be generators for H. The class h'_2 has self-intersection 0 on the fibers $\mathcal{S}_b$, and it follows that we must have $h'^2_2 = 0$ in $H^4(\mathcal{S}, \mathbb{Q})$. As $h_2^2 = 0$ and $h'^2_2 = h_2^2 - 2\alpha h_1 h_2$, with $h_1 h_2 \neq 0$ in $H^4(\mathcal{S}, \mathbb{Q})$, we conclude that $\alpha = 0$ and $h_2 = h'_2$. Next, the class h_3^2 (hence also the class h'^2_3) has degree 2 on the fibers $\mathcal{S}_b$; furthermore the intersection number $h_2 h_3$ of the classes h_2 and h_3 on the fibers $\mathcal{S}_b$ is equal to 3 (thus we get as well that the intersection number $h_2 h'_3 = h'_2 h'_3$ on the fibers $\mathcal{S}_b$ is equal to 3).

If our multiplicative decomposition exists, we conclude that we must have the following relation in $H^4(\mathcal{S}, \mathbb{Q})$:

$$3h'^2_3 - 2h_2 h'_3 = 0. \tag{1.10}$$

Equivalently, as the class of $\mathcal{S}$ in $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^2$ is an ample class equal to $dh_1 + 2h_2 + 3h_3$, we should have:

$$(dh_1 + 2h_2 + 3h_3)(3(h_3^2 - 2\beta h_3 h_1) - 2h_2(h_3 - \beta h_1)) = 0 \text{ in } H^6(\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^2, \mathbb{Q}). \tag{1.11}$$

However this class is equal to $(3d - 18\beta)h_1 h_3^2 + (-2d - 6\beta)h_1 h_2 h_3$, where the two classes $h_1 h_3^2$, $h_1 h_2 h_3$ are independent in $H^6(\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^2, \mathbb{Q})$. We conclude that for the equation (1.11) to hold, one needs

$$3d - 18\beta = 0,\ -2d - 6\beta = 0,$$

which has no solution for $d \neq 0$. Hence the relation (1.10) is not satisfied for any choice of h'_3.

■

2 Proof of Theorem 0.7

2.1 A criterion for the existence of a decomposition

Our proofs will be based on the following easy and presumably standard lemma, applied to the category of sheaves of $\mathbb{Q}$-vector spaces on B.

Let A be a $\mathbb{Q}$-linear abelian category, and let $\mathcal{D}(A)$ be the corresponding derived category of left bounded complexes. Let $M \in \mathcal{D}(A)$ be an object with bounded cohomology such that $End\, M$ is finite dimensional. Assume M admits a morphism $\phi : M \to M$ such that

$$H^i(\phi) : H^i(M) \to H^i(M)$$

is equal to $\lambda_i Id_{H^i(M)}$, where all the $\lambda_i \in \mathbb{Q}$ are distinct.

Lemma 2.1. *The morphism ϕ induces a canonical decomposition*

$$M \cong \oplus_i H^i(M)[-i], \tag{2.12}$$

characterized by the properties :
1) The induced map on cohomology is the identity map.
2) One has

$$\phi \circ \pi_i = \lambda_i \pi_i : M \to M. \tag{2.13}$$

where π_i corresponds via the isomorphism (2.12) to the i-th projector pr_i.

Proof. We first prove using the arguments of [4] that M is decomposed, namely there is an isomorphism

$$f : M \cong \oplus_i H^i(M)[-i].$$

For this, given an object $K \in Ob\, A$, we consider the left exact functor T from A to the category of $\mathbb{Q}$-vector spaces defined by $T(N) = Hom_A(K, N)$, and for any integer i the induced functor, denoted by T_i, $N \mapsto Hom_{\mathcal{D}(A)}(K[-i], N)$ on $\mathcal{D}(A)$. For any $N \in \mathcal{D}(A)$, there is the hypercohomology spectral sequence with E_2-term

$$E_2^{p,q} = R^p T_i(H^q(N)) = Ext_A^{p+i}(K, H^q(N)) \Rightarrow \mathbb{R}^{p+q} T_i(N).$$

Under our assumptions, this spectral sequence for $N = M$ degenerates at E_2. Indeed, the morphism ϕ acts then on the above spectral sequence starting from E_2. The differential $d_2 : E_2^{p,q} \to E_2^{p+2,q-1}$

$$Ext_A^{p+i}(K, H^q(M)) \Rightarrow Ext_A^{p+2+i}(K, H^{q-1}(M)) \tag{2.14}$$

commutes with the action of ϕ. On the other hand, ϕ acts as $\lambda_q Id$ on the left hand side and as $\lambda_{q-1} Id$ on the right hand side of (2.14). Thus we conclude that $d_2 = 0$ and similarly that all d_r, $r \geq 2$ are 0.

We take now $K = H^i(M)$. We conclude from the degeneracy at E_2 of the above spectral sequence that the map

$$Hom_{\mathcal{D}(A)}(H^i(M)[-i], M) \to Hom_A(H^i(M), H^i(M)) = E_2^{-i,i}$$

is surjective, so that there is a morphism

$$f_i : H^i(M)[-i] \to M$$

inducing the identity on degree i cohomology. The direct sum $f = \sum f_i$ is a quasi-isomorphism which gives the desired splitting.

The morphism ϕ can thus be seen as a morphism of the split object $\oplus_i H^i(M)[-i]$. Such a morphism is given by a block-uppertriangular matrix

$$\phi_{j,i} \in Ext_A^{i-j}(H^i(M), H^j(M)),\ i \geq j,$$

with $\lambda_i Id$ on the i-th diagonal block. Let ψ be the endomorphism of $End\, M$ given by left multiplication by ϕ. We have by the above description of ϕ:

$$\prod_{i, H^i(M) \neq 0} (\psi - \lambda_i Id_{End\, M}) = 0, \tag{2.15}$$

which shows that the endomorphism ψ is diagonalizable. More precisely, as ψ is block-uppertriangular in an adequately ordered decomposition

$$End\, M = \oplus_{i\geq j} Ext_A^{i-j}(H^i(M), H^j(M)),$$

with term $\lambda_j Id$ on the block diagonals $Ext_A^{i-j}(H^i(M), H^j(M))$, hence in particular on $End_A\, H^j(M)$, we conclude that there exists $\pi_i' \in End\, M$ such that π_i' acts as the identity on $H^i(M)$, and $\phi \circ \pi_i' = \lambda_i \pi_i'$.

Let $\rho_i := \pi_i' \circ f_i : H^i(M)[-i] \to M$. Then $\rho := \sum \rho_i$ gives another decomposition $\oplus_i H^i(M)[-i] \cong M$ and we have $\phi \circ \rho_i = \lambda_i \rho_i$, which gives $\phi \circ \pi_i = \lambda_i \pi_i$, where $\pi_i = \rho \circ pr_i \circ \rho^{-1}$.

The uniqueness of the π_i's satisfying properties 1) and 2) is obvious, since these properties force the equality $\pi_i = \frac{\prod_{j\neq i}(\phi - \lambda_j Id_M)}{\prod_{j\neq i} \lambda_i - \lambda_j}$.

■

The following result is proved in [6], by similar but somehow more complicated arguments (indeed they use Fourier-Mukai transforms, which exist only in the projective case):

Corollary 2.2. *(Deninger-Murre 1991) For any family $\pi : \mathcal{A} \to B$ of abelian varieties or complex tori, there is a multiplicative decomposition isomorphism $R\pi_*\mathbb{Q} = \oplus_i R^i\pi_*\mathbb{Q}[-i]$.*

Proof. Choose an integer $n \neq \pm 1$ and consider the multiplication map

$$\mu_n : \mathcal{A} \to \mathcal{A}, \;\; a \mapsto na.$$

We then get morphisms $\mu_n^* : R\pi_*\mathbb{Q} \to R\pi_*\mathbb{Q}$ with the property that the induced morphisms on each $R^i\pi_*\mathbb{Q} = H^i(R\pi_*\mathbb{Q})$ is multiplication by n^i. We use now Lemma 2.1 to deduce from such a morphism a canonical splitting

$$R\pi_*\mathbb{Q} \cong \oplus_i R^i\pi_*\mathbb{Q}[-i], \tag{2.16}$$

characterized by the properties that the induced map on cohomology is the identity map, and

$$\mu_n^* \circ \pi_i = n^i \pi_i : \; R\pi_*\mathbb{Q} \to R\pi_*\mathbb{Q}. \tag{2.17}$$

where π_i is the endomorphism of $R\pi_*\mathbb{Q}$ which identifies to the i-th projector via the isomorphism (2.16). On the other hand, the morphism $\mu : R\pi_*\mathbb{Q} \otimes R\pi_*\mathbb{Q} \to R\pi_*\mathbb{Q}$ given by cup-product is compatible with μ_n^*, in the sense that

$$\mu \circ (\mu_n^* \otimes \mu_n^*) = \mu_n^* \circ \mu : R\pi_*\mathbb{Q} \otimes R\pi_*\mathbb{Q} \to R\pi_*\mathbb{Q}.$$

Combining this last equation with (2.17), we find that

$$\mu \circ (\mu_n^* \otimes \mu_n^*) \circ (\pi_i \otimes \pi_j) = n^{i+j} \mu \circ (\pi_i \otimes \pi_j)$$

$$= \mu_n^* \circ \mu \circ (\pi_i \otimes \pi_j) : R\pi_*\mathbb{Q} \otimes R\pi_*\mathbb{Q} \to R\pi_*\mathbb{Q}$$

from which it follows applying again (2.17) that $\mu \circ \pi_i \otimes \pi_j$ factors through $R^{i+j}\pi_*[-i-j]$, or equivalently that in the splitting (2.16), the cup-product morphism μ maps $R^i\pi_*\mathbb{Q}[-i] \otimes R^j\pi_*\mathbb{Q}[-j])$ to the summand $R^{i+j}\pi_*[-i-j]$. ■

2.2 K-autocorrespondences

K-correspondences were introduced in [12] in order to study intrinsic volume forms on complex manifolds.

Definition 2.3. *(Voisin 2004) A K-isocorrespondence between two projective complex manifolds X and Y of dimension n is a n-dimensional closed algebraic subvariety $\Sigma \subset X \times Y$, such that each irreducible component of Σ dominates X and Y by the natural projections, and satisfying the following condition : Let $\widetilde{\Sigma} \xrightarrow{\tau} \Sigma$ be a desingularization, and let $f := pr_1 \circ \tau : \widetilde{\Sigma} \to X$, $g := pr_2 \circ \tau : \widetilde{\Sigma} \to Y$. Then we have the equality*

$$R_f = R_g \tag{2.18}$$

of the ramification divisors of f and g on $\widetilde{\Sigma}$.

A K-autocorrespondence of X is a K-isocorrespondence between X and itself.

We will be interested in K-autocorrespondences $\Sigma \subset X \times X$, where X is a smooth complex projective variety with trivial canonical bundle. In fact, we are not interested in this paper in the equality (2.18) of ramification divisors, but in the proportionality of pulled-back top holomorphic forms, which is an equivalent property by the following lemma:

Lemma 2.4. *Let X be a smooth complex compact manifold with trivial canonical bundle, and let $\Sigma \subset X \times X$ be an irreducible self-correspondence, with desingularization $\tau : \widetilde{\Sigma} \to \Sigma$. Then Σ is a K-autocorrespondence if and only if for some coefficient $\lambda \in \mathbb{C}^*$, one has*

$$0 \neq f^*\eta = \lambda g^*\eta \text{ in } H^0(\widetilde{\Sigma}, K_{\widetilde{\Sigma}}) \tag{2.19}$$

for any nonzero holomorphic section η of K_X, where as before $f = pr_1 \circ \tau$, $g = pr_2 \circ \tau$.

Proof. Indeed, as $f^*\eta$ and $g^*\eta$ are not identically 0, the maps f and g are dominating and thus generically finite. As K_X is trivial, R_f and R_g are respectively the divisors of the pulled-back forms $f^*\eta$, $g^*\eta \in H^0(\widetilde{\Sigma}, K_{\widetilde{\Sigma}})$. As $\widetilde{\Sigma}$ is irreducible, these two forms are thus proportional if and only if $R_f = R_g$. ■

The simplest way to construct such a K-autocorrespondence is by studying rational equivalence of points on X: We recall for the convenience of the reader the proof of the following statement, which can be found in [12, Sec. 2]: Let X be a complex projective n-fold with trivial canonical bundle, and $z_0 \in CH_0(X)$ be a fixed 0-cycle. Let m_1, m_2 be non zero integers.

Proposition 2.5. *Let $\Sigma \subset X \times X$ be a n-dimensional subvariety which dominates X by both projections, and such that, for any $(x, y) \in \Sigma$, $m_1 x + m_2 y = z_0$ in $CH_0(X)$. Then Σ is a K-autocorrespondence of X. More precisely, we have the equality $m_1 f^*\eta = -m_2 g^*\eta$ in $H^0(\widetilde{\Sigma}, K_{\widetilde{\Sigma}})$ for any holomorphic n-form η on X.*

Proof. Let $\tau : \widetilde{\Sigma} \to \Sigma$ be a desingularization of Σ and let as above $f := pr_1 \circ \tau$, $g = pr_2 \circ \tau$. We apply Mumford's theorem [10] or its generalization [14, Proposition 10.24] to the cycle

$$\Gamma = m_1 \mathrm{Graph}(f) + m_2 \mathrm{Graph}(g) \in CH^n(\widetilde{\Sigma} \times X)$$

which has the property that $\mathrm{Im}(\Gamma_* : CH_0(\widetilde{\Sigma})_{hom} \to CH_0(X))$ is supported on $\mathrm{Supp}\, z_0$. It follows that for any holomorphic form η of degree > 0 on X, $\Gamma^*\eta = 0$ on $\widetilde{\Sigma}$. But we have

$$\Gamma^*\eta = m_1 f^*\eta + m_2 g^*\eta \text{ in } H^0(\widetilde{\Sigma}, \Omega^l_{\widetilde{\Sigma}}).$$

For $l = n$, we get the desired equality $m_1 f^*\eta = -m_2 g^*\eta$ in $H^0(\widetilde{\Sigma}, K_{\widetilde{\Sigma}})$.

■

Let S be an algebraic $K3$ surface, and L an ample line bundle on S of self-intersection $c_1(L)^2 = 2d$. We assume that $\operatorname{Pic} S$ has rank 1, generated by a class proportional to L. There is a 1-dimensional family of singular elliptic curves in $|L|$ which sweep-out S. They may be not irreducible, and have in particular fixed rational components, but as $(\operatorname{Pic} S) \otimes \mathbb{Q}$ is generated by L, the classes of all irreducible components are proportional to $c_1(L)$. Changing L if necessary, we may then assume the general fibers of this 1-dimensional family of elliptic curves are irreducible. Starting from this one dimensional family of irreducible elliptic curves $\Sigma_1 := \bigcup_{b \in \Gamma_1} \Sigma'_b$, we get by desingularizing Σ_1 and Γ_1 the following data: A smooth projective surface Σ, and two morphisms

$$\phi : \Sigma \to S,\ p : \Sigma \to \Gamma,$$

where p is surjective with elliptic fibers Σ_b such that $\phi_*(\Sigma_b) \in |L|$, Γ is a smooth curve, and ϕ is generically finite.

Choose an integer $m \equiv 1$ mod. $2d$, and write $m = 2kd + 1$. For a general point $x \in \Sigma$, the fiber $\Sigma_x := p^{-1}(p(x))$ is a smooth elliptic curve, and there is an unique $y \in \Sigma_x$ such that

$$mx = y + kL_{|\Sigma_x} \text{ in } \operatorname{Pic} \Sigma_x.$$

This determines a rational map $\psi : \Sigma \dashrightarrow \Sigma$, $x \mapsto y$ which is of degree m^2. Let $\tau : \widetilde{\Sigma} \to \Sigma$ be a birational morphism such that $\psi \circ \tau$ is a morphism, and let

$$f := \phi \circ \tau : \widetilde{\Sigma} \to S,\ g := \phi \circ \psi \circ \tau : \widetilde{\Sigma} \to S.$$

Remark 2.6. The degree of f is equal to the degree of ϕ, hence independent of m.

Lemma 2.7. *The image $\Sigma_m := (f, g)(\widetilde{\Sigma})$ is a K-autocorrespondence of S, which satisfies the following numerical properties:*

1) For any $\eta \in H^{2,0}(S)$, $g^*\eta = mf^*\eta$.

2) $f_*g^*L = \lambda_m L$ in $\operatorname{Pic} S$, where

$$m\lambda_m \notin \{0,\ m^2 \deg f,\ m \deg f,\ \deg f\}$$

for m large enough.

Proof. By construction, we have for $\sigma \in \Sigma$

$$g(\sigma) = mf(\sigma) - kL^2 \text{ in } CH_0(S). \tag{2.20}$$

Thus Σ_m is a K-correspondence and 1) is satisfied by Proposition 2.5.

As $(\operatorname{Pic} S) \otimes\ = \mathbb{Q}L$, we certainly have a formula $f_*g^*L = \lambda_m L$ in $\operatorname{Pic} S$ and it only remains to show that $m\lambda_m \notin \{0,\ \deg f,\ m \deg f,\ m^2 \deg f\}$ for m large. This is however obvious, as the degree of f is independent of m according to Remark 2.6, while the intersection number $g^*L \cdot \Sigma_b$ is equal to $2m^2d$, which implies that the intersection number $f_*\Sigma_b \cdot f_*g^*L = L \cdot f_*g^*L$ is $\geq 2m^2d$, so that $\lambda_m \geq m^2$.

■

Corollary 2.8. *For a very general pair (S, L) as above, we have*

$$mf^* = g^* : H^2(S, \mathbb{Q})^{\perp c_1(L)} \to H^2(\widetilde{\Sigma}, \mathbb{Q}).$$

Proof. Indeed the morphism of Hodge structures $mf^* - g^* : H^2(S, \mathbb{Q})^{\perp c_1(L)} \to H^2(\widetilde{\Sigma}, \mathbb{Q})$ vanishes on $H^{2,0}(S)$ by Lemma 2.7. Its kernel K is thus a Hodge substructure of $H^2(S, \mathbb{Q})^{\perp c_1(L)}$ which contains both $H^{2,0}(S)$ and its complex conjugate $H^{0,2}(S)$. The orthogonal complement of K in $H^2(S, \mathbb{Q})^{\perp c_1(L)}$ is thus contained in $\operatorname{NS}(S) \otimes \mathbb{Q}$ and orthogonal to $c_1(L)$, hence is 0 because for a very general pair (S, L), we have $\operatorname{NS}(S) \otimes \mathbb{Q} = \mathbb{Q}c_1(L)$. ■

Corollary 2.9. *The eigenvalues of f_*g^* acting on $H^*(S,\mathbb{Q})$ are*

$$\deg f,\ m\deg f,\ \lambda_m,\ m^2\deg f.$$

Proof. Indeed, f_*g^* acts as $\deg f\, Id$ on $H^0(S,\mathbb{Q})$. Corollary 2.8 and Proposition 2.7, 2) show that the eigenvalues of f_*g^* on $H^2(S,\mathbb{Q})$ are $m\deg f$ and λ_m, and finally f_*g^* acts as $\deg g\, Id$ on $H^4(S,\mathbb{Q})$. But $\deg g = m^2\deg f$ because for any non zero holomorphic 2-form η on S, we have $g^*\eta = mf^*\eta$ and thus

$$\int_{\widetilde{\Sigma}} g^*\eta\wedge g^*\overline{\eta} = \deg g\int_S \eta\wedge\overline{\eta} = m^2\int_{\widetilde{\Sigma}} f^*\eta\wedge f^*\overline{\eta} = m^2\deg f\int_S \eta\wedge\overline{\eta},$$

where the integral $\int_S \eta\wedge\overline{\eta}$ is non zero. ■

We are going to use now the above constructions to prove Theorem 0.7, (i) for families of $K3$ surfaces with generic Picard number 1.

Proof of Theorem 0.7, (i). We start with our family $\pi:\mathcal{S}\to B$ of $K3$ surfaces, which has the property that the very general fibers $\mathcal{S}_b$ have Picard number 1. Let $\mathcal{L}$ be a relatively ample line bundle on $\mathcal{S}$ of self-intersection $2d$. The construction mentioned previously of a 1-dimensional family of irreducible elliptic curves with smooth total space works in family, at least over a Zariski open set of B. Hence, replacing B by a Zariski open set, $\mathcal{S}$ by its inverse image under π, and $\mathcal{L}$ by a rational multiple of $\mathcal{L}$ if necessary, we can assume that there are a family of smooth surfaces $p:\mathcal{T}\to B$ and two morphisms

$$f,\ g:\mathcal{T}\to\mathcal{S} \tag{2.21}$$

whose fibers over $b\in B$ satisfy the conclusions of Lemma 2.7 and Corollary 2.8.

The relative cycle

$$\Gamma := (f,g)_*(\mathcal{T}) + (\frac{m\deg f-\lambda_m}{2d})pr_1^*c_1(\mathcal{L})\cdot pr_2^*c_1(\mathcal{L})\in CH^2(\mathcal{S}\times_B\mathcal{S})_{\mathbb{Q}} \tag{2.22}$$

induces a morphism

$$\Gamma^*: R\pi_*\mathbb{Q}\to R\pi_*\mathbb{Q},$$

which acts by Corollary 2.9 with respective eigenvalues

$$\lambda_0=\deg f,\ \lambda_2 = m\deg f,\ \lambda_4 = m^2\deg f$$

on $R^0\pi_*\mathbb{Q}$, $R^2\pi_*\mathbb{Q}$, $R^4\pi_*\mathbb{Q}$.

These three eigenvalues being distinct, we can apply Lemma 2.1 to the morphism Γ^* acting on the object $R\pi_*\mathbb{Q}$ of the bounded derived category of sheaves of $\mathbb{Q}$-vector spaces on B. We thus get a decomposition

$$R\pi_*\mathbb{Q} = R^0\pi_*\mathbb{Q}\oplus R^2\pi_*\mathbb{Q}[-2]\oplus R^4\pi_*\mathbb{Q}[-4], \tag{2.23}$$

which is preserved by Γ^*. Note furthermore that $R^2\pi_*\mathbb{Q}[-2]$ is canonically the direct sum $\mathbb{Q}L[-2]\oplus R^2\pi_*\mathbb{Q}^{\perp L}[-2]$, which provides us with the two direct summands

$$\mathbb{Q}L[-2],\ R^2\pi_*\mathbb{Q}^{\perp L}[-2] \tag{2.24}$$

of $R\pi_*\mathbb{Q}$.

The proof of Theorem 0.7 then concludes with the following:

Proposition 2.10. *The decomposition (2.23) is multiplicative on a nonempty Zariski open set of B.*

■

It remains to prove Proposition 2.10. The proof will use the following lemma: Let f, $g : \Sigma \to S$ be two morphisms from a smooth surface Σ to a $K3$ surface S equipped with a line bundle L with non zero self-intersection.

Lemma 2.11. *Assume that for some integers m_1, m_2, and for some fixed 0-cycle z_0 of S, the relation*

$$m_1 f(\sigma) + m_2 g(\sigma) = z_0 \tag{2.25}$$

holds in $CH_0(S)$ for every $\sigma \in \Sigma$. Then we have

$$f_* g^*(c_1(L)^2) = \deg g \, c_1(L)^2 \tag{2.26}$$

in $CH_0(S)$.

Proof. We just have to show that $f_* g^*(c_1(L)^2)$ is proportional to $c_1(L)^2$ in $CH_0(S)$, since $CH_0(S)$ has no torsion and the degrees of both sides in (2.26) are equal. There are various criteria for a point x of S to be proportional to $c_1(L)^2$ in $CH_0(S)$. The one used in [2] is that it is enough that x belongs to some (singular) rational curve in S. The following criterion is a weaker characterization:

Sublemma 2.12. *Let S be a $K3$ surface and L be a line bundle on S such that $\deg c_1(L)^2 \neq 0$. Let $j : C \to S$ be a non constant morphism from an irreducible curve C to S, such that $j_* : CH_0(C) \to CH_0(S)$ has for image $\mathbb{Z}$ (that is all points $j(c)$, $c \in C$, are rationally equivalent in S). Then for any $c \in C$, $j(c)$ is proportional to $c_1(L)^2$ in $CH_0(S)$.*

Proof. Let H be an ample line bundle on S. As all points $j(c)$, $c \in C$ are rationally equivalent in S, they are proportional in $CH_0(S)$ to the cycle $j_* j^* H = j_* C \cdot H$, because the latter has a non zero degree. But it follows from Theorem 1.1 that $j_* C \cdot H$ and $c_1(L)^2$ are proportional in $CH_0(S)$. ■

Coming back to our situation, we start from a singular rational curve $D \subset S$ in some ample linear system $\mid H \mid$. Then we know by [2, Thm 1] that any point x of D is proportional to $c_1(L)^2$ in $CH_0(S)$. On the other hand, the curve $g^{-1}(D)$ is connected and $f(g^{-1}(D))$ is not reduced to a point, because $f_* g^* H \neq 0$ in $\mathrm{NS}(S)$. Let C be a component of $g^{-1}(D)$ which is not contracted to a point by f. We now apply Sublemma 2.12 to the morphism f restricted to C. Indeed, as $g_*(c)$ is constant in $CH_0(S)$ because $g(C)$ is rational, it follows from (2.25) that $f_*(c)$ is also constant in $CH_0(S)$. Hence $f_*(c)$ is proportional to $c_1(L)^2$ in $CH_0(S)$ by Sublemma 2.12. As $g^{-1}(D)$ is connected, the same conclusion also holds for the components C of $g^{-1}(D)$ which are contracted by f. As this is true for any $c \in C$, we get a fortiori that denoting by g_C the restriction of g to C, $f_* g_C^* x$ is proportional to $c_1(L)^2$ in $CH_0(S)$. Summing over all components C of $g^{-1}(D)$, and recalling that x is proportional to $c_1(L)^2$ in $CH_0(S)$ concludes the proof of Lemma 2.11. ■

Corollary 2.13. *Over a nonempty Zariski open set of B, we have*

$$\Gamma^*(c_1^{top}(\mathcal{L})^2) = m^2 \deg f \, c_1^{top}(\mathcal{L})^2, \tag{2.27}$$

where Γ is as in (2.22).

The morphism $c_1^{top}(\mathcal{L})^2 \cup : \mathbb{Q}[-4] \to R\pi_* \mathbb{Q}$ factors through the direct summand $R^4 \pi_* \mathbb{Q}[-4]$.

Proof. The second statement is an immediate consequence of the first by definition of the decomposition.

Next, for any point $b \in B$, Γ_b^* acts as f_*g^* on $CH_0(\mathcal{S}_b)$. Furthermore, the pair (f, g) satisfies the condition that

$$mf(\sigma) = g(\sigma) + kc_1(L)^2 \text{ in } CH_0(\mathcal{S}_b)$$

for any $\sigma \in \mathcal{T}_b$. As $\deg g = m^2 \deg f$, Lemma 2.11 tells us that

$$\Gamma_b^*(c_1(\mathcal{L}_b)^2) = f_*g^*(c_1(\mathcal{L}_b)^2) = m^2 \deg f\, c_1(\mathcal{L}_b)^2$$

in $CH_0(\mathcal{S}_b)$.

The general principle 1.2 then tells us that, for a nonempty Zariski open set B^0 of B,

$$\Gamma^*(c_1^{top}(\mathcal{L})^2) = m^2 \deg f\, c_1^{top}(\mathcal{L})^2 \text{ in } H^4(\mathcal{S}^0, \mathbb{Q}).$$

■

Corollary 2.14. *The two morphisms Γ^* and f_*g^* agree, over a nonempty Zariski open set of B, on the direct summand $R^4\pi_*\mathbb{Q}[-4]$ of the decomposition (2.23). More precisely, they both act by multiplication by $m^2 \deg f$ on this direct summand.*

Proof. Indeed, this direct summand is equal by Corollary 2.13 to the image of the morphism

$$\mathbb{Q}[-4] \to R\pi_*\mathbb{Q} \tag{2.28}$$

given by the class $c_1(\mathcal{L})^2$. The difference $f_*g^* - \Gamma^*$ is the morphism given by the class

$$\frac{m \deg f - \lambda_m}{2d} pr_1^* c_1^{top}(\mathcal{L}) \cdot pr_2^* c_1^{top}(\mathcal{L}) \in H^4(\mathcal{S} \times_B \mathcal{S}, \mathbb{Q}),$$

hence is given up to a coefficient by the formula:

$$pr_{1*} \circ (pr_1^* c_1^{top}(\mathcal{L}) \cup pr_2^* c_1^{top}(\mathcal{L}) \cup) \circ pr_2^* : R\pi_*\mathbb{Q} \to R\pi_*\mathbb{Q}. \tag{2.29}$$

But the composition of the morphism (2.28) with the morphism (2.29) obviously vanishes over a Zariski open set of B because the class $c_1^{top}(\mathcal{L})^3 \in H^6(\mathcal{S}, \mathbb{Q})$ is the class of an algebraic cycle of codimension 3. ■

We will also need the following easy lemma:

Lemma 2.15. *1) The morphisms Γ_* and f_*g^*, restricted to the direct summand $R^2\pi_*\mathbb{Q}^{\perp L}[-2]$ (see (2.24)), are equal.*

2) The summand $\mathbb{Q}L[-2]$ of $R^2\pi_*\mathbb{Q}[-2] \subset R\pi_*\mathbb{Q}$ introduced in (2.24) is locally over B in the Zariski topology generated by the class $c_1^{top}(\mathcal{L})$, that is, is the image of the morphism

$$c_1(\mathcal{L}) \cup : \mathbb{Q}[-2] \to R\pi_*\mathbb{Q}. \tag{2.30}$$

Proof. 1) Indeed their difference is up to a coefficient the morphism given by formula (2.29). But this morphism obviously vanishes on $R^2\pi_*\mathbb{Q}^{\perp L}[-2]$, by the projection formula and because for degree reasons it factors through the morphism of local systems

$$R^2\pi_*\mathbb{Q} \overset{\cup c_1^{top}(\mathcal{L})}{\to} R^4\pi_*\mathbb{Q} \overset{\pi_*}{\to} \mathbb{Q}$$

which by definition vanishes on $R^2\pi_*\mathbb{Q}^{\perp L}$.

2) Indeed, we have locally over B in the Zariski topology

$$\Gamma^* c_1^{top}(\mathcal{L}) = m \deg f\, c_1^{top}(\mathcal{L}).$$

By definition of the decomposition, this implies that locally over B, the morphism (2.30) takes value in the direct summand $R^2\pi_*\mathbb{Q}[-2]$ of the decomposition. It then follows obviously that it locally belongs in fact to the direct summand $\mathbb{Q}L[-2]$. ■

Proof of Proposition 2.10. We have the data of the family of smooth surfaces $p : \mathcal{T} \to B$ and of the morphisms f, $g : \mathcal{T} \to \mathcal{S}$ as in (2.21). The induced morphisms

$$f^* : R\pi_*\mathbb{Q} \to Rp_*\mathbb{Q},\ g^* : R\pi_*\mathbb{Q} \to Rp_*\mathbb{Q},$$

are multiplicative, i.e. compatible with cup-products on both sides.

Consider now our decomposition

$$R\pi_*\mathbb{Q} \cong \oplus_i R^i\pi_*\mathbb{Q}[-i], \tag{2.31}$$

together with the orthogonal decomposition of the local system $R^2\pi_*\mathbb{Q}$

$$R^2\pi_*\mathbb{Q} = R^2\pi_*\mathbb{Q}^{\perp L} \oplus \mathbb{Q}L.$$

The decomposition (2.31) is by definition preserved by Γ^* and Γ^* acts with eigenvalues

$$\deg f,\ m\deg f,\ m^2\deg f$$

on the respective summands.

Lemma 2.16. *1) Over a nonempty Zariski open set of B, we have the equality*

$$g^* = mf^* : R^2\pi_*\mathbb{Q}^{\perp L}[-2] \to Rp_*\mathbb{Q}. \tag{2.32}$$

2) The morphism $f_*g^* : R\pi_*\mathbb{Q} \to R\pi_*\mathbb{Q}$ preserves the direct summand $R^2\pi_*\mathbb{Q}^{\perp L}$ and acts by multiplication by $m\deg f$ on it.

Proof. 2) follows from 1) by applying f_* to both sides of (2.32).

To prove 1), note that the morphisms f^*, g^* are induced by the classes of the codimension 2 cycles $\Gamma_f := \mathrm{Graph}\, f$, $\Gamma_g := \mathrm{Graph}\, g$ in $\mathcal{T} \times_B \mathcal{S}$. For any $b \in B$, consider the cycle

$$\Gamma_b := m\Gamma_{f,b} - \Gamma_{g,b} - k\, pr_2^* c_1(\mathcal{L}_b)^2 \in CH^2(\mathcal{T}_b \times \mathcal{S}_b).$$

By construction, the induced map $\Gamma_{b*} : CH_0(\mathcal{T}_b) \to CH_0(\mathcal{S}_b)$ is equal to 0. It follows by applying the general principle 1.2 that, after passing to rational coefficients and modulo rational equivalence, Γ_b is supported on $D_b \times \mathcal{S}_b$ for some curve $D_b \subset \mathcal{T}_b$. However, as $\mathrm{Pic}^0(\mathcal{S}_b) = 0$, denoting $\widetilde{D}_b$ the desingularization of D_b, we have $\mathrm{Pic}\,(\widetilde{D}_b \times \mathcal{S}_b) = \mathrm{Pic}\,\widetilde{D}_b \oplus \mathrm{Pic}\,\mathcal{S}_b$. We thus conclude that

$$m\Gamma_{f,b} - \Gamma_{g,b} - kpr_2^* c_1(\mathcal{L}_b)^2 = pr_1^* Z_b + pr_1^* Z_b' \cdot pr_2^* Z_b'' \text{ in } CH^2(\mathcal{T}_b \times \mathcal{S}_b)_{\mathbb{Q}}, \tag{2.33}$$

for some zero cycle $Z_b \in CH^2(\mathcal{T}_b)$ and 1-cycles Z_b' on $\mathcal{T}_b$, Z_b'' on $\mathcal{S}_b$. Note that the cycle Z_b'' has to be proportional to $c_1(\mathcal{L}_b)$, since the point b is general in B.

Applying again the general principle 1.2, the pointwise equality (2.33) in the Chow groups of the fibers produces the following equality of cohomology classes over a Zariski open subset B^0:

$$m[\Gamma_f] - [\Gamma_g] - kpr_2^* c_1^{top}(\mathcal{L})^2 = pr_1^*[\mathcal{Z}] + pr_1^*[\mathcal{Z}'] \cup pr_2^*[\mathcal{Z}''] \text{ in } H^4(\mathcal{T}^0 \times_{B^0} \mathcal{S}^0, \mathbb{Q}) \tag{2.34}$$

for some codimension 2 cycles $\mathcal{Z} \in CH^2(\mathcal{T}^0)_{\mathbb{Q}}$, and codimension 1 cycles $\mathcal{Z}' \in CH^1(\mathcal{T}^0)_{\mathbb{Q}}$, $\mathcal{Z}'' \in CH^1(\mathcal{S}^0)_{\mathbb{Q}}$, where we may assume furthermore $[\mathcal{Z}''] = c_1^{top}(\mathcal{L})$ by shrinking B^0 if necessary. We thus get over B^0 the following equality of associated morphisms:

$$mf^* = g^* + k(pr_2^* c_1^{top}(\mathcal{L})^2)^* + (pr_1^*[\mathcal{Z}]^* + pr_1^*[\mathcal{Z}'] \cup pr_2^* c_1^{top}(\mathcal{L}))^* : R\pi_*\mathbb{Q} \to Rp_*\mathbb{Q}. \tag{2.35}$$

The morphism

$$(pr_1^*[\mathcal{Z}])^* = pr_{1*} \circ (pr_1^*[\mathcal{Z}]\cup) \circ pr_2^* : R\pi_*\mathbb{Q} \to Rp_*\mathbb{Q}$$

induced by the cycle class $pr_1^*[\mathcal{Z}]$ vanishes on $R^0\pi_*\mathbb{Q} \oplus R^2\pi_*\mathbb{Q}[-2]$, by the projection formula and because for degree reasons

$$pr_{1*} \circ pr_2^* = 0,\ R^0\pi_*\mathbb{Q} \oplus R^2\pi_*\mathbb{Q}[-2] \to Rp_*\mathbb{Q}[-4]$$

vanishes.

Similarly, the morphism

$$pr_{1*} \circ (pr_1^*[\mathcal{Z}'] \cup pr_2^* c_1^{top}(\mathcal{L})\cup) \circ pr_2^* : R\pi_*\mathbb{Q} \to Rp_*\mathbb{Q}$$

vanishes on $R^0\pi_*\mathbb{Q} \oplus R^2\pi_*\mathbb{Q}^{\perp L}[-2]$, by the projection formula and because for degree reasons it factors through the composite morphism

$$R^2\pi_*\mathbb{Q} \overset{\cup c_1^{top}(\mathcal{L})}{\to} R^4\pi_*\mathbb{Q} \overset{\pi_*}{\to} \mathbb{Q}$$

which by definition vanishes on $R^2\pi_*\mathbb{Q}^{\perp L}$.

Using (2.35), it only remains to prove that the restriction to $R^2\pi_*\mathbb{Q}^{\perp L}[-2]$ of the morphism induced by the class $pr_2^* c_1^{top}(\mathcal{L})^2$

$$pr_{1*} \circ (pr_2^* c_1^{top}(\mathcal{L})^2\cup) \circ pr_2^* : R\pi_*\mathbb{Q} \to Rp_*\mathbb{Q}$$

vanishes over a Zariski open set of B. Using (2.35) and the above arguments, we conclude that on the direct summand $R^2\pi_*\mathbb{Q}^{\perp L}[-2]$ and over a nonempty Zariski open set of B we have

$$mf^* = g^* + k(pr_2^* c_1^{top}(\mathcal{L})^2)^* : R^2\pi_*\mathbb{Q}^{\perp L}[-2] \to Rp_*\mathbb{Q}. \tag{2.36}$$

Applying f_* to both sides, we conclude that

$$m\deg f Id = f_*g^* + kf_*(pr_2^* c_1^{top}(\mathcal{L})^2)^* : R^2\pi_*\mathbb{Q}^{\perp L}[-2] \to R\pi_*\mathbb{Q}. \tag{2.37}$$

But f_*g^* acts as Γ_* on the direct summand $R^2\pi_*\mathbb{Q}^{\perp L}[-2]$ by Lemma 2.15, and by definition of the direct summand $R^2\pi_*\mathbb{Q}[-2]$, Γ_* acts as $m\deg f Id$ on it. Hence we have

$$f_*g^* = m\deg f Id : R^2\pi_*\mathbb{Q}^{\perp L}[-2] \to R\pi_*\mathbb{Q},$$

and comparing with (2.37), we get that

$$f_* \circ pr_2^* c_1^{top}(\mathcal{L})^2)^* = 0 : R^2\pi_*\mathbb{Q}^{\perp L}[-2] \to R\pi_*\mathbb{Q}. \tag{2.38}$$

It is now easy to see that the last equation implies

$$(pr_2^* c_1^{top}(\mathcal{L})^2)^* = 0 : R^2\pi_*\mathbb{Q}^{\perp L}[-2] \to Rp_*\mathbb{Q}.$$

Indeed, the morphism $(pr_2^* c_1^{top}(\mathcal{L})^2)^* : R\pi_*\mathbb{Q} \to Rp_*\mathbb{Q}$ factors as $p^* \circ \psi : \mathbb{Q} \to Rp_*\mathbb{Q}$, where $\psi : R\pi_*\mathbb{Q} \to \mathbb{Q}$ is the composite morphism

$$R\pi_*\mathbb{Q} \overset{c_1^{top}(\mathcal{L})^2\cup}{\to} R\pi_*\mathbb{Q}[4] \overset{\pi_*}{\to} \mathbb{Q},$$

and we have $f_* \circ p^* = \deg f \circ \pi^* : \mathbb{Q} \to R\pi_*\mathbb{Q}$. ■

We now conclude the proof of Proposition 2.10. Using Lemma 2.16, we deduce now that, in the decomposition (2.31), the cup-product map

$$\mu : R^2\pi_*\mathbb{Q}^{\perp L}[-2] \otimes R^2\pi_*\mathbb{Q}^{\perp L}[-2] \to R\pi_*\mathbb{Q}$$

takes value in the direct summand $R^4\pi_*\mathbb{Q}[-4]$. Indeed, we have $g^* = mf^*$ on $R^2\pi_*\mathbb{Q}^{\perp L}[-2]$ and thus

$$g^* \circ \mu = m^2 f^* \circ \mu : R^2\pi_*\mathbb{Q}^{\perp L}[-2] \otimes R^2\pi_*\mathbb{Q}^{\perp L}[-2] \to Rp_*\mathbb{Q}.$$

Applying f_* on one hand, and taking the cup-product with $g^*c_1^{top}(\mathcal{L})$ on the other hand, we conclude that, on $R^2\pi_*\mathbb{Q}^{\perp L}[-2] \otimes R^2\pi_*\mathbb{Q}^{\perp L}[-2]$ we have:

$$f_*g^* \circ \mu = \deg f\, m^2\mu : R^2\pi_*\mathbb{Q}^{\perp L}[-2] \otimes R^2\pi_*\mathbb{Q}^{\perp L}[-2] \to R\pi_*\mathbb{Q}, \tag{2.39}$$

$$\begin{aligned} g^* \circ \mu \circ (c_1^{top}(\mathcal{L})\cup) = m^2(g^*c_1^{top}(\mathcal{L})\cup) \circ f^* \circ \mu : \\ R^2\pi_*\mathbb{Q}^{\perp L}[-2] \otimes R^2\pi_*\mathbb{Q}^{\perp L}[-2] \to Rp_*\mathbb{Q}[2], \end{aligned} \tag{2.40}$$

Hence, by applying f_* to the second equation (2.40), we get:

$$\begin{aligned} f_*g^* \circ \mu \circ (c_1^{top}(\mathcal{L})\cup) = m^2\lambda_m\mu \circ (c_1^{top}(\mathcal{L})\cup) : \\ R^2\pi_*\mathbb{Q}^{\perp L}[-2] \otimes R^2\pi_*\mathbb{Q}^{\perp L}[-2] \to R\pi_*\mathbb{Q}[2]. \end{aligned} \tag{2.41}$$

Using Corollary 2.14, and Lemma 2.16, 2), we get that f_*g^* preserves the decomposition (2.31), acting with eigenvalues $\deg f$ on the first summand, $m\deg f$ and λ_m on the summand $R^2\pi_*[-2]$, and $m^2\deg f$ on the summand $R^4\pi_*[-4]$. As $m^2\lambda_m \notin \{\deg f,\, m\deg f,\, \lambda_m,\, m^2\deg f\}$ by Lemma 2.7, 2), we first conclude from (2.41) that

$$\mu \circ (c_1^{top}(\mathcal{L})\cup) = c_1^{top}(\mathcal{L}) \cup \circ\mu$$

vanishes on $R^2\pi_*\mathbb{Q}^{\perp L}[-2] \otimes R^2\pi_*\mathbb{Q}^{\perp L}[-2]$.

Next, we conclude from (2.39) that $\mu : R^2\pi_*\mathbb{Q}^{\perp L}[-2] \otimes R^2\pi_*\mathbb{Q}^{\perp L}[-2] \to R\pi_*\mathbb{Q}$ takes value in the direct summand with is the sum $\mathbb{Q}L[-2] \oplus R^4\pi_*\mathbb{Q}[-4]$ (the second summand being possible if $\lambda_m = m^2\deg f$). However, as its composition with the cup-product map $c_1^{top}(\mathcal{L})\cup$ vanishes, we easily conclude that it actually takes value in the summand $R^4\pi_*\mathbb{Q}[-4]$, because the cup-product map $c_1^{top}(\mathcal{L})\cup$ induces an isomorphism $\mathbb{Q}L[-2] \cong R^4\pi_*\mathbb{Q}[-2]$, as follows from Lemma 2.15, 2) and Corollary 2.13.

It remains to see what happens on the other summands : First of all, Lemma 2.15, 2) says that the summand $\mathbb{Q}L[-2]$ of $R^2\pi_*\mathbb{Q}[-2]$ is, over a nonempty Zariski open subset B, the image of the morphism $c_1^{top}(\mathcal{L})\cup : \mathbb{Q}[-2] \to R\pi_*\mathbb{Q}$. On the other hand, Corollary 2.13 says that the direct summand $R^4\pi_*\mathbb{Q}[-4]$ is over a nonempty Zariski open set B^0 of B the image of the morphism

$$c_1^{top}(\mathcal{L})^2 : \mathbb{Q}[-4] \to R\pi_*\mathbb{Q}.$$

It follows immediately that for the summand $\mathbb{Q}L[-2] = \operatorname{Im} c_1^{top}(\mathcal{L})\cup$

$$\mu : \mathbb{Q}L[-2] \otimes \mathbb{Q}L[-2] \to R\pi_*\mathbb{Q}$$

takes value on B^0 in the direct summand $R^4\pi_*\mathbb{Q}[-4]$.

Consider now the cup-product

$$R^2\pi_*\mathbb{Q}^{\perp L} \otimes \mathbb{Q}L[-2] \to R\pi_*\mathbb{Q}.$$

We claim that it vanishes over a nonempty Zariski open set of B.

Indeed, Lemma 2.16 tells that over a nonempty Zariski open set of B,

$$g^* = mf^* : R^2\pi_*\mathbb{Q}^{\perp L}[-2] \to Rp_*\mathbb{Q}.$$

It follows that

$$g^* \circ \mu = \mu \circ (g^* \otimes g^*) = \mu \circ (mf^* \otimes g^*) : R^2\pi_*\mathbb{Q}^{\perp L}[-2] \otimes \mathbb{Q}L[-2] \to Rp_*\mathbb{Q}.$$

Applying the projection formula, we get that

$$f_*g^* \circ \mu = \mu \circ (mId \otimes f_*g^*) : R^2\pi_*\mathbb{Q}^{\perp L}[-2] \otimes \mathbb{Q}L[-2] \to R\pi_*\mathbb{Q}.$$

On the other hand, we know by Lemma 2.15, 2) that f_*g^* sends, locally over B, the summand $\mathbb{Q}L[-2]$ to itself, acting on it by multiplication by λ_m. It follows that

$$f_*g^* \circ \mu = m\lambda_m\mu : R^2\pi_*\mathbb{Q}^{\perp L}[-2] \otimes \mathbb{Q}L[-2] \to R\pi_*\mathbb{Q},$$

and finally we conclude that $f_*g^* \circ \mu = 0$ on $R^2\pi_*\mathbb{Q}^{\perp L}[-2] \otimes \mathbb{Q}L[-2]$ because $m\lambda_m$ is not an eigenvalue of f_*g^* acting on the cohomology of $R\pi_*\mathbb{Q}$ by Corollary 2.9 and Lemma 2.7, 2).

To conclude the proof of the multiplicativity, we just have to check that the cup-product map vanishes over a nonempty Zariski open subset of B on the summands $R^2\pi_*\mathbb{Q}[-2] \otimes R^4\pi_*\mathbb{Q}[-4]$ and $R^4\pi_*\mathbb{Q}[-4] \otimes R^4\pi_*\mathbb{Q}[-4]$. The proof works exactly as before, by an eigenvalue computation for the summand $R^2\pi_*\mathbb{Q}^{\perp L}[-2] \otimes R^4\pi_*[-4]$. For the other terms, this is clear because we have seen that over an adequate Zariski open subset of B, the factors are generated by classes $c_1^{top}(\mathcal{L})$, $c_1^{top}(\mathcal{L})^2$, whose products are classes of algebraic cycles on $\mathcal{S}$ of codimension at least 3, hence vanishing over a nonempty Zariski open subset of B.

■

2.3 Alternative proof

In this section we give a different proof of Theorem 0.7, which also provides a proof of the second statement (ii). It heavily uses the following result proved in [2, Proposition 3.2], whose proof is rather intricate.

Theorem 2.17. *(Beauville-Voisin 2004) Let S be a smooth projective $K3$ surface, L an ample line bundle on S and $o := \frac{1}{\deg_S c_1(L)^2} L^2 \in CH^2(S)_{\mathbb{Q}}$. We have*

$$\Delta = \Delta_{12} \cdot o_3 + (perm.) - (o_1 \times o_2 \times S + (perm.)) \text{ in } CH^4(S \times S \times S)_{\mathbb{Q}}. \tag{2.42}$$

(We recall that "+(*perm.*)" means that we symmetrize the considered expression in the indices. The lower index i means "pull-back of the considered cycle under the i-th projection $S^3 \to S$", and the lower index ij means "pull-back of the considered cycle under the projection $S^3 \to S^2$ onto the product of the ith and j-th factor".

Second proof of Theorem 0.7. Let us choose a relatively ample line bundle L on $\mathcal{X}$, and let

$$o_{\mathcal{X}} := \frac{1}{\deg_{\mathcal{X}_t} c_1(L)^2} L^2 \in CH^2(\mathcal{X})_{\mathbb{Q}}.$$

By Theorem 1.1 and the general principle 1.2, this cycle, which is of relative degree 1, does not depend on the choice of L up to shrinking the base B. The cohomology classes

$$pr_1^*[o_{\mathcal{X}}] = [Z_0],\ pr_2^*[o_{\mathcal{X}}] = [Z_4] \in H^4(\mathcal{X} \times_B \mathcal{X}, \mathbb{Q})$$

of the two codimension 2 cycles $Z_0 := pr_1^* o_{\mathcal{X}}$ and $Z_4 := pr_2^* o_{\mathcal{X}}$, where $pr_i : \mathcal{X} \times_B \mathcal{X} \to B$ are the two projections, provide morphisms in the derived category:

$$P_0 : R\pi_*\mathbb{Q} \to R\pi_*\mathbb{Q},\ P_4 : R\pi_*\mathbb{Q} \to R\pi_*\mathbb{Q}$$

$$P_0 := pr_{2*} \circ ([Z_0]\cup) \circ pr_1^*,\ P_4 := pr_{2*} \circ (pr_2^*[Z_4]\cup) \circ pr_1^*. \tag{2.43}$$

Lemma 2.18. *(i) The morphisms P_0, P_4 are projectors of $R\pi_*\mathbb{Q}$.*
(ii) $P_0 \circ P_4 = P_4 \circ P_0 = 0$ over a Zariski dense open set of B.

Proof. (i) We compute $P_0 \circ P_0$. From (2.43) and the projection formula [7, Prop. 8.3], we get that $P_0 \circ P_0$ is the morphism $R\pi_* \to R\pi_*$ induced by the following cycle class

$$p_{13*}(p_{12}^*[Z_0] \cup p_{23}^*[Z_0]) \in H^4(\mathcal{X} \times_B \mathcal{X}, \mathbb{Q}), \tag{2.44}$$

where the p_{ij} are the various projections from $\mathcal{X} \times_B \mathcal{X} \times_B \mathcal{X}$ to $\mathcal{X} \times_B \mathcal{X}$. We use now the fact that $p_{12}^*[Z_0] = p_1^*[o_{\mathcal{X}}]$, $p_{23}^*[Z_0] = p_2^*[o_{\mathcal{X}}]$, where the p_i's are the various projections from $\mathcal{X} \times_B \mathcal{X} \times_B \mathcal{X}$ to $\mathcal{X}$, so that (2.44) is equal to

$$p_{13*}(p_1^*[o_{\mathcal{X}}] \cup p_2^*[o_{\mathcal{X}}]). \tag{2.45}$$

Using the projection formula, this class is equal to

$$pr_1^*[o_{\mathcal{X}}] \cup pr_2^*(\pi_*[o_{\mathcal{X}}]) = pr_1^*[o_{\mathcal{X}}] \cup pr_2^*(1_B) = pr_1^*[o_{\mathcal{X}}] = [Z_1].$$

This completes the proof for P_0 and exactly the same proof works for P_4.

(ii) We compute $P_0 \circ P_4$: From (2.43) and the projection formula [7, Prop. 8.3], we get that $P_0 \circ P_4$ is the morphism $R\pi_* \to R\pi_*$ induced by the following cycle class

$$p_{13*}(p_{12}^*[Z_4] \cup p_{23}^*[Z_0]) \in H^4(\mathcal{X} \times_B \mathcal{X}, \mathbb{Q}), \tag{2.46}$$

where the p_{ij} are the various projections from $\mathcal{X} \times_B \mathcal{X} \times_B \mathcal{X}$ to $\mathcal{X} \times_B \mathcal{X}$. We use now the fact that $p_{12}^*[Z_4] = p_2^*[o_{\mathcal{X}}]$, $p_{23}^*[Z_0] = p_2^*[o_{\mathcal{X}}]$, where the p_i's are the various projections from $\mathcal{X} \times_B \mathcal{X} \times_B \mathcal{X}$ to $\mathcal{X}$, so that (2.46) is equal to

$$p_{13*}(p_2^*[o_{\mathcal{X}}] \cup p_2^*[o_{\mathcal{X}}]). \tag{2.47}$$

But the class $p_2^*[o_{\mathcal{X}}] \cup p_2^*[o_{\mathcal{X}}] = p_2^*([o_{\mathcal{X}} \cdot o_{\mathcal{X}}])$ vanishes over a Zariski dense open set of B since the cycle $o_{\mathcal{X}} \cdot o_{\mathcal{X}}$ has codimension 4 in $\mathcal{X}$. This shows that $P_0 \circ P_4 = 0$ over a Zariski dense open set of B and the proof for $P_4 \circ P_0$ works in the same way.

■

Using Lemma 2.18, we get (up to passing to a Zariski dense open set of B) a third projector

$$P_2 := Id - P_0 - P_4$$

acting on $R\pi_*\mathbb{Q}$ and commuting with the two other ones.

It is well-known (cf. [11]) that the action of these three projectors on cohomology are given by

$$P_0 = 0 \text{ on } R^2\pi_*\mathbb{Q},\ R^4\pi_*\mathbb{Q},\quad P_{0*} = Id \text{ on } R^0\pi_*\mathbb{Q}.$$

$$P_4 = 0 \text{ on } R^2\pi_*\mathbb{Q},\ R^0\pi_*\mathbb{Q},\quad P_{4*} = Id \text{ on } R^4\pi_*\mathbb{Q}.$$

$$P_2 = 0 \text{ on } R^0\pi_*\mathbb{Q},\ R^4\pi_*\mathbb{Q},\quad P_2 = Id \text{ on } R^2\pi_*\mathbb{Q}.$$

As a consequence, we get (for example using Lemma 2.1) a decomposition

$$R\pi_*\mathbb{Q} \cong \oplus R^i\pi_*\mathbb{Q}[-i], \tag{2.48}$$

where the corresponding projectors of $R\pi_*\mathbb{Q}$ identify respectively to P_0, P_2, P_4.

We now prove the following result,

Proposition 2.19. *Assume the cohomology class of the relative small diagonal $\Delta \subset \mathcal{X} \times_B \mathcal{X} \times_B \mathcal{X}$ satisfies the equality*

$$[\Delta] = p_1^*[o_{\mathcal{X}}] \cup p_{23}^*[\Delta_{\mathcal{X}}] + (perm.) - (p_1^*[o_{\mathcal{X}}] \cup p_2^*[o_{\mathcal{X}}] + (perm.)) \tag{2.49}$$

where the p_{ij}, p_i's are as above and $\Delta_{\mathcal{X}}$ is the relative diagonal $\mathcal{X} \subset \mathcal{X} \times_B \mathcal{X}$, then, over some Zariski dense open set $B^0 \subset B$, we have:

(i) The decomposition (2.48) is multiplicative.

(ii) The class of the diagonal $[\Delta_{\mathcal{X}}] \in H^4(\mathcal{X} \times_B \mathcal{X}, \mathbb{Q})$ belongs to the direct summand

$$H^0(B, R^4(\pi,\pi)_*\mathbb{Q}) \subset H^4(\mathcal{X} \times_B \mathcal{X}, \mathbb{Q})$$

induced by the decomposition (2.48).

Admitting Proposition 2.19, the end of the proof of Theorem 0.7 is as follows: By Theorem 2.17, we know that the relation

$$\Delta_t = p_1^* o_{\mathcal{X}_t} \cdot p_{23}^* \Delta_{\mathcal{X}_t} + (perm.) - (p_1^* o_{\mathcal{X}_t} \cdot p_2^* o_{\mathcal{X}_t} + (perm.))$$

holds in $CH_2(\mathcal{X}_t \times \mathcal{X}_t \times \mathcal{X}_t, \mathbb{Q})$ for any $t \in B$. By the general principle 1.2, we conclude that there exists a Zariski dense open set B^0 of B such that (2.49) holds in $H^8(\mathcal{X} \times_B \mathcal{X} \times_B \mathcal{X}, \mathbb{Q})$. The statements (i) and (ii) of Theorem 0.7 thus follow respectively from the statements (i) and (ii) of Proposition 2.19. As proved in Lemma 1.4, the statement (iii) of Theorem 0.7 is implied by (i). ■

Proof of Proposition 2.19. (i) We want to show that

$$P_k \circ \cup \circ (P_i \otimes P_j) : R\pi_*\mathbb{Q} \otimes R\pi_*\mathbb{Q} \to R\pi_*\mathbb{Q}$$

vanishes for $k \neq i + j$.

We note that

$$\cup : R\pi_*\mathbb{Q} \otimes R\pi_*\mathbb{Q} \to R\pi_*\mathbb{Q}$$

is induced, via the relative Künneth decomposition

$$R\pi_*\mathbb{Q} \otimes R\pi_*\mathbb{Q} \cong R(\pi,\pi)_*\mathbb{Q}$$

by the class $[\Delta]$ of the small relative diagonal in $\mathcal{X} \times_B \mathcal{X} \times_B \mathcal{X}$, seen as a relative correspondence between $\mathcal{X} \times_B \mathcal{X}$ and $\mathcal{X}$, while P_0, P_4, P_2 are induced by the cycle classes $[Z_0]$, $[Z_4]$, $[Z_2] \in H^4(\mathcal{X} \times_B \mathcal{X}, \mathbb{Q})$, where $Z_2 := \Delta_{\mathcal{X}} - Z_0 - Z_4 \subset \mathcal{X} \times_B \mathcal{X}$. It thus suffices to show that the cycle classes

$$[Z_4 \circ \Delta \circ (Z_0 \times_B Z_0)],\ [Z_2 \circ \Delta \circ (Z_0 \times_B Z_0)],$$

$$[Z_0 \circ \Delta \circ (Z_2 \times_B Z_2)],\ [Z_2 \circ \Delta \circ (Z_2 \times_B Z_2)],$$

$$[Z_0 \circ \Delta \circ (Z_4 \times_B Z_4)],\ [Z_2 \circ \Delta \circ (Z_4 \times_B Z_4)],\ [Z_4 \circ \Delta \circ (Z_4 \times_B Z_4)],$$

$$[Z_0 \circ \Delta \circ (Z_0 \times_B Z_4)],\ [Z_2 \circ \Delta \circ (Z_0 \times_B Z_4)],$$
$$[Z_0 \circ \Delta \circ (Z_2 \times_B Z_4)],\ [Z_2 \circ \Delta \circ (Z_2 \times_B Z_4)],\ [Z_4 \circ \Delta \circ (Z_2 \times_B Z_4)],$$
$$[Z_0 \circ \Delta \circ (Z_0 \times_B Z_2)],\ [Z_4 \circ \Delta \circ (Z_0 \times_B Z_2)],$$

vanish in $H^8(\mathcal{X} \times_B \mathcal{X} \times_B \mathcal{X}, \mathbb{Q})$ over a dense Zariski open set of B. Here, all the compositions of correspondences are over B. Equivalently, it suffices to prove the following equality of cycle classes in $H^8(\mathcal{X}^0 \times_B \mathcal{X}^0 \times_B \mathcal{X}^0, \mathbb{Q})$, $\mathcal{X}^0 = \pi^{-1}(B^0)$, for a Zariski dense open set of B^0 of B:

$$\begin{aligned}[\Delta] = [Z_0 \circ \Delta \circ (Z_0 \times_B Z_0)] + [Z_4 \circ \Delta \circ (Z_2 \times_B Z_2)] + [Z_2 \circ \Delta \circ (Z_0 \times_B Z_2)] \\ +[Z_2 \circ \Delta \circ (Z_2 \times_B Z_0)] + [Z_4 \circ \Delta \circ (Z_0 \times_B Z_4)] + [Z_4 \circ \Delta \circ (Z_4 \times_B Z_0)].\end{aligned} \tag{2.50}$$

Replacing Z_2 by $\Delta_{\mathcal{X}} - Z_0 - Z_4$, we get

$$\begin{aligned}Z_2 \times_B Z_2 = \Delta_{\mathcal{X}} \times_B \Delta_{\mathcal{X}} - \Delta_{\mathcal{X}} \times_B Z_0 - \Delta_{\mathcal{X}} \times_B Z_4 - Z_0 \times_B \Delta_{\mathcal{X}} \\ -Z_4 \times_B \Delta_{\mathcal{X}} + Z_0 \times_B Z_0 + Z_4 \times_B Z_4 + Z_0 \times_B Z_4 + Z_4 \times_B Z_0\end{aligned}$$

and thus (2.50) becomes

$$\begin{aligned}[\Delta] = [Z_0 \circ \Delta \circ (Z_0 \times_B Z_0)] + [Z_4 \circ \Delta \circ (\Delta_{\mathcal{X}} \times_B \Delta_X)] \\ -[Z_4 \circ \Delta \circ (\Delta_{\mathcal{X}} \times_B Z_0)] - [Z_4 \circ \Delta \circ (\Delta_{\mathcal{X}} \times_B Z_4)] - [Z_4 \circ \Delta \circ (Z_0 \times_B \Delta_{\mathcal{X}})] \\ -[Z_4 \circ \Delta \circ (Z_4 \times_B \Delta_{\mathcal{X}})] + [Z_4 \circ \Delta \circ (Z_0 \times_B Z_0)] + [Z_4 \circ \Delta \circ (Z_4 \times_B Z_4)] \\ +[Z_4 \circ \Delta \circ (Z_0 \times_B Z_4)] + [Z_4 \circ \Delta \circ (Z_4 \times_B Z_0)] + [Z_2 \circ \Delta \circ (Z_0 \times_B \Delta_{\mathcal{X}})] \\ -[Z_2 \circ \Delta \circ (Z_0 \times_B Z_0)] - [Z_2 \circ \Delta \circ (Z_0 \times_B Z_4)] + [Z_2 \circ \Delta \circ (\Delta_{\mathcal{X}} \times_B Z_0)] \\ -[Z_2 \circ \Delta \circ (Z_0 \times_B Z_0)] - [Z_2 \circ \Delta \circ (Z_4 \times_B Z_0)] + [Z_4 \circ \Delta \circ (Z_0 \times_B Z_4)] \\ +[Z_4 \circ \Delta \circ (Z_4 \times_B Z_0)].\end{aligned} \tag{2.51}$$

We now have the following lemma:

Lemma 2.20. *We have the following equalities of cycles in $CH^4(\mathcal{X} \times_B \mathcal{X} \times_B \mathcal{X})_{\mathbb{Q}}$ (or relative correspondences between $\mathcal{X} \times_B \mathcal{X}$ and $\mathcal{X}$)*

$$\Delta \circ (Z_0 \times_B Z_0) = p_1^* o_{\mathcal{X}} \cdot p_2^* o_{\mathcal{X}}, \tag{2.52}$$

$$\Delta \circ (\Delta_{\mathcal{X}} \times_B \Delta_{\mathcal{X}}) = \Delta, \tag{2.53}$$

$$\Delta \circ (\Delta_{\mathcal{X}} \times_B Z_0) = p_{13}^* \Delta_{\mathcal{X}} \cdot p_2^* o_{\mathcal{X}}, \tag{2.54}$$

$$\Delta \circ (\Delta_{\mathcal{X}} \times_B Z_4) = p_1^* o_{\mathcal{X}} \cdot p_3^* o_{\mathcal{X}}, \tag{2.55}$$

$$\Delta \circ (Z_0 \times_B \Delta_{\mathcal{X}}) = p_1^* o_{\mathcal{X}} \cdot p_{23}^* \Delta_{\mathcal{X}}, \tag{2.56}$$

$$\Delta \circ (Z_4 \times_B \Delta_{\mathcal{X}}) = p_2^* o_{\mathcal{X}} \cdot p_3^* o_{\mathcal{X}}, \tag{2.57}$$

$$\Delta \circ (Z_4 \times_B Z_4) = p_3^* (o_{\mathcal{X}} \cdot o_{\mathcal{X}}), \tag{2.58}$$

$$\Delta \circ (Z_0 \times_B Z_4) = p_1^* o_{\mathcal{X}} \cdot p_3^* o_{\mathcal{X}}, \tag{2.59}$$

$$\Delta \circ (Z_4 \times_B Z_0) = p_2^* o_{\mathcal{X}} \cdot p_3^* o_{\mathcal{X}}, \tag{2.60}$$

where the p_i's, for $i = 1, 2, 3$ are the projections from $\mathcal{X} \times_B \mathcal{X} \times_B \mathcal{X}$ to $\mathcal{X}$ and the p_{ij} are the projections from $\mathcal{X} \times_B \mathcal{X} \times_B \mathcal{X}$ to $\mathcal{X} \times_B \mathcal{X}$.

Proof. Equation (2.53) is obvious. Equations (2.52), (2.58), (2.59), (2.60) are all similar. Let us just prove (2.59). The cycle Z_4 is $\mathcal{X} \times_B o_{\mathcal{X}} \subset \mathcal{X} \times_B \mathcal{X}$, and similarly $Z_0 = o_{\mathcal{X}} \times_B \mathcal{X} \subset \mathcal{X} \times_B \mathcal{X}$, hence $Z_0 \times_B Z_4$ is the cycle

$$\{(o_{\mathcal{X}_b}, x, y, o_{\mathcal{X}_b}),\ x \in \mathcal{X}_b,\ y \in \mathcal{X}_b,\ b \in B\} \subset \mathcal{X} \times_B \mathcal{X} \times_B \mathcal{X} \times_B \mathcal{X}. \tag{2.61}$$

(It turns out that in this case, we do not have to take care about the ordering we take for the last inclusion.) Composing over B with $\Delta \subset \mathcal{X} \times_B \mathcal{X} \times_B \mathcal{X}$ is done by taking the pull-back of (2.61) under $p_{1234} : \mathcal{X}^{5/B} \to \mathcal{X}^{4/B}$, intersecting with $p_{345}^*\Delta$, and projecting the resulting cycle to $\mathcal{X}^{3/B}$ via p_{125}. The resulting cycle is obviously

$$\{(o_{\mathcal{X}_b}, x, o_{\mathcal{X}_b}),\ x \in \mathcal{X}_b,\ b \in B\} \subset \mathcal{X} \times_B \mathcal{X} \times_B \mathcal{X},$$

which proves (2.59).

For the last formulas which are all of the same kind, let us just prove (2.54). Recall that $Z_0 = o_{\mathcal{X}} \times_B \mathcal{X} \subset \mathcal{X} \times_B \mathcal{X}$. Thus $\Delta_{\mathcal{X}} \times_B Z_0$ is the cycle

$$\{(x, x, o_{\mathcal{X}_b}, y),\ x \in \mathcal{X}_b,\ y \in \mathcal{X}_b,\ b \in B\} \subset \mathcal{X} \times_B \mathcal{X} \times_B \mathcal{X} \times_B \mathcal{X}.$$

But we have to see this cycle as a relative self-correspondence of $\mathcal{X} \times_B \mathcal{X}$, for which the right ordering is

$$\{(x, o_{\mathcal{X}_b}, x, y),\ x \in \mathcal{X}_b,\ y \in \mathcal{X}_b,\ b \in B\} \subset \mathcal{X} \times_B \mathcal{X} \times_B \mathcal{X} \times_B \mathcal{X}. \tag{2.62}$$

Composing over B with $\Delta \subset \mathcal{X} \times_B \mathcal{X} \times_B \mathcal{X}$ is done again by taking the pull-back of (2.62) by $p_{1234} : \mathcal{X}^{5/B} \to \mathcal{X}^{4/B}$, intersecting with $p_{345}^*\Delta$, and projecting the resulting cycle to $\mathcal{X}^{3/B}$ via p_{125}. Since $\Delta = \{(z, z, z),\ z \in \mathcal{X}$, the considered intersection is $\{(x, o_{\mathcal{X}_b}, x, x, x),\ x \in \mathcal{X}_b,\ b \in B\}$, and thus the projection via p_{125} is $\{(x, o_{\mathcal{X}_b}, x),\ x \in \mathcal{X}_b,\ b \in B\}$, thus proving (2.54).

■

Using Lemma 2.20 and the fact that the cycle $p_3^*(o_{\mathcal{X}} \cdot o_{\mathcal{X}})$ vanishes by dimension reasons over a dense Zariski open set of B, (2.51) becomes, after passing to a Zariski open set of B if necessary:

$$\begin{aligned}
[\Delta] = {} & [Z_0 \circ (p_1^* o_{\mathcal{X}} \cdot p_2^* o_{\mathcal{X}})] + [Z_4 \circ \Delta] \\
& -[Z_4 \circ (p_{13}^* \Delta_{\mathcal{X}} \cdot p_2^* o_{\mathcal{X}})] - [Z_4 \circ (p_1^* o_{\mathcal{X}} \cdot p_3^* o_{\mathcal{X}})] - [Z_4 \circ (p_1^* o_{\mathcal{X}} \cdot p_{23}^* \Delta_{\mathcal{X}})] \\
& -[Z_4 \circ (p_2^* o_{\mathcal{X}} \cdot p_3^* o_{\mathcal{X}})] + [Z_4 \circ (p_1^* o_{\mathcal{X}} \cdot p_2^* o_{\mathcal{X}})] + [Z_4 \circ (p_1^* o_{\mathcal{X}} \cdot p_3^* o_{\mathcal{X}})] \\
& +[Z_4 \circ (p_2^* o_{\mathcal{X}} \cdot p_3^* o_{\mathcal{X}})] + [Z_2 \circ (p_1^* o_{\mathcal{X}} \cdot p_{23}^* \Delta_{\mathcal{X}})] - [Z_2 \circ (p_1^* o_{\mathcal{X}} \cdot p_2^* o_{\mathcal{X}})] \\
& -[Z_2 \circ (p_1^* o_{\mathcal{X}} \cdot p_3^* o_{\mathcal{X}})] + [Z_2 \circ (p_{13}^* \Delta_{\mathcal{X}} \cdot p_2^* o_{\mathcal{X}})] - [Z_2 \circ (p_1^* o_{\mathcal{X}} \cdot p_2^* o_{\mathcal{X}})] \\
& -[Z_2 \circ (p_2^* o_{\mathcal{X}} \cdot p_3^* o_{\mathcal{X}})] + [Z_4 \circ (p_1^* o_{\mathcal{X}} \cdot p_3^* o_{\mathcal{X}})] + [Z_4 \circ (p_2^* o_{\mathcal{X}} \cdot p_3^* o_{\mathcal{X}})],
\end{aligned} \tag{2.63}$$

which rewrites as

$$\begin{aligned}
[\Delta] = {} & [Z_0 \circ (p_1^* o_{\mathcal{X}} \cdot p_2^* o_{\mathcal{X}})] + [Z_4 \circ \Delta] \\
& -[Z_4 \circ (p_{13}^* \Delta_{\mathcal{X}} \cdot p_2^* o_{\mathcal{X}})] - [Z_4 \circ (p_1^* o_{\mathcal{X}} \cdot p_{23}^* \Delta_{\mathcal{X}})] + [Z_4 \circ (p_1^* o_{\mathcal{X}} \cdot p_3^* o_{\mathcal{X}})] \\
& +[Z_4 \circ (p_2^* o_{\mathcal{X}} \cdot p_3^* o_{\mathcal{X}})] + [Z_4 \circ (p_1^* o_{\mathcal{X}} \cdot p_2^* o_{\mathcal{X}})] + [Z_2 \circ (p_1^* o_{\mathcal{X}} \cdot p_{23}^* \Delta_{\mathcal{X}})] \\
& -[Z_2 \circ (p_1^* o_{\mathcal{X}} \cdot p_3^* o_{\mathcal{X}})] + [Z_2 \circ (p_{13}^* \Delta_{\mathcal{X}} \cdot p_2^* o_{\mathcal{X}})] - 2[Z_2 \circ (p_1^* o_{\mathcal{X}} \cdot p_2^* o_{\mathcal{X}})].
\end{aligned} \tag{2.64}$$

To conclude, we use the following lemma:

Lemma 2.21. *Up to passing to a dense Zariski open set of B, we have the following equalities in $CH^4(\mathcal{X} \times_B \mathcal{X} \times_B \mathcal{X})_{\mathbb{Q}}$:*

$$Z_0 \circ (p_1^* o_{\mathcal{X}} \cdot p_2^* o_{\mathcal{X}}) = p_1^* o_{\mathcal{X}} \cdot p_2^* o_{\mathcal{X}}, \tag{2.65}$$

$$Z_4 \circ \Delta = p_{12}^* \Delta_{\mathcal{X}} \cdot p_3^* o_{\mathcal{X}}, \tag{2.66}$$

$$Z_4 \circ (p_{13}^* \Delta_{\mathcal{X}} \cdot p_2^* o_{\mathcal{X}}) = p_2^* o_{\mathcal{X}} \cdot p_3^* o_{\mathcal{X}}, \tag{2.67}$$

$$Z_4 \circ (p_2^* o_{\mathcal{X}} \cdot p_3^* o_{\mathcal{X}}) = p_2^* o_{\mathcal{X}} \cdot p_3^* o_{\mathcal{X}}, \tag{2.68}$$

$$Z_4 \circ (p_1^* o_{\mathcal{X}} \cdot p_{23}^* \Delta_{\mathcal{X}}) = p_1^* o_{\mathcal{X}} \cdot p_3^* o_{\mathcal{X}}, \tag{2.69}$$

$$Z_4 \circ (p_1^* o_{\mathcal{X}} \cdot p_2^* o_{\mathcal{X}}) = 0, \tag{2.70}$$

$$Z_4 \circ (p_1^* o_{\mathcal{X}} \cdot p_3^* o_{\mathcal{X}}) = p_1^* o_{\mathcal{X}} \cdot p_3^* o_{\mathcal{X}}, \tag{2.71}$$

$$Z_2 \circ (p_1^* o_{\mathcal{X}} \cdot p_3^* o_{\mathcal{X}}) = 0, \tag{2.72}$$

$$Z_2 \circ (p_1^* o_{\mathcal{X}} \cdot p_{23}^* \Delta_{\mathcal{X}}) = p_1^* o_{\mathcal{X}} \cdot p_{23}^* \Delta_{\mathcal{X}} - p_1^* o_{\mathcal{X}} \cdot p_2^* o_{\mathcal{X}} - p_1^* o_{\mathcal{X}} \cdot p_3^* o_{\mathcal{X}}, \tag{2.73}$$

$$Z_2 \circ (p_{13}^* \Delta_{\mathcal{X}} \cdot p_2^* o_{\mathcal{X}}) = p_{13}^* \Delta_{\mathcal{X}} \cdot p_2^* o_{\mathcal{X}} - p_1^* o_{\mathcal{X}} \cdot p_2^* o_{\mathcal{X}} - p_2^* o_{\mathcal{X}} \cdot p_3^* o_{\mathcal{X}}, \tag{2.74}$$

$$Z_2 \circ (p_1^* o_{\mathcal{X}} \cdot p_2^* o_{\mathcal{X}}) = 0. \tag{2.75}$$

Proof. The proof of (2.67) is explicit, recalling that $Z_4 = \{(x, o_{\mathcal{X}_b}),\, x \in \mathcal{X}_b,\, b \in B\}$, and that $p_{13}^* \Delta_{\mathcal{X}} \cdot p_2^* o_{\mathcal{X}} = \{(y, o_{\mathcal{X}_b}, y),\, y \in \mathcal{X}_b,\, b \in B\}$. We then find that $Z_4 \circ (p_{13}^* \Delta_{\mathcal{X}} \cdot p_2^* o_{\mathcal{X}})$ is the cycle

$$\begin{aligned} p_{124}(p_{13}^* \Delta_{\mathcal{X}} \cdot p_2^* o_{\mathcal{X}} \cdot p_{34}^*(Z_4)) &= p_{124}(\{(y, o_{\mathcal{X}_b}, y, o_{\mathcal{X}_b}),\, y \in \mathcal{X}_b,\, b \in B\}) \\ &= \{(y, o_{\mathcal{X}_b}, o_{\mathcal{X}_b}),\, y \in \mathcal{X}_b,\, b \in B\}, \end{aligned}$$

which proves (2.67). (2.68) is the same formula as (2.65) with the indices 1 and 3 exchanged. The proofs of (2.65) to (2.71) work similarly.

For the other proofs, we recall that

$$Z_2 = \Delta_{\mathcal{X}} - Z_0 - Z_4 \subset \mathcal{X} \times_B \mathcal{X}.$$

Thus we get, as $\Delta_{\mathcal{X}}$ acts as the identity:

$$Z_2 \circ (p_1^* o_{\mathcal{X}} \cdot p_{23}^* \Delta_{\mathcal{X}}) = p_1^* o_{\mathcal{X}} \cdot p_{23}^* \Delta_{\mathcal{X}} - Z_0 \circ (p_1^* o_{\mathcal{X}} \cdot p_{23}^* \Delta_{\mathcal{X}}) - Z_4 \circ (p_1^* o_{\mathcal{X}} \cdot p_{23}^* \Delta_{\mathcal{X}}).$$

We then compute the terms $Z_0 \circ (p_1^* o_{\mathcal{X}} \cdot p_{23}^* \Delta_{\mathcal{X}})$, $Z_4 \circ (p_1^* o_{\mathcal{X}} \cdot p_{23}^* \Delta_{\mathcal{X}})$ explicitly as before, which gives (2.73).

The other proofs are similar.

■

Using the cohomological version of Lemma 2.21, (2.64) becomes:

$$\begin{aligned}[\Delta] = [p_1^* o_{\mathcal{X}} \cdot p_2^* o_{\mathcal{X}}] + [p_{12}^* \Delta_{\mathcal{X}} \cdot p_3^* o_{\mathcal{X}}] \\ -[p_2^* o_{\mathcal{X}} \cdot p_3^* o_{\mathcal{X}})] - [p_1^* o_{\mathcal{X}} \cdot p_3^* o_{\mathcal{X}})] + [p_1^* o_{\mathcal{X}} \cdot p_3^* o_{\mathcal{X}}] \\ +[p_2^* o_{\mathcal{X}} \cdot p_3^* o_{\mathcal{X}}] + [p_1^* o_{\mathcal{X}} \cdot p_{23}^* \Delta_{\mathcal{X}} - p_1^* o_{\mathcal{X}} \cdot p_2^* o_{\mathcal{X}} - p_1^* o_{\mathcal{X}} \cdot p_3^* o_{\mathcal{X}}] \\ +[p_{13}^* \Delta_{\mathcal{X}} \cdot p_2^* o_{\mathcal{X}} - p_1^* o_{\mathcal{X}} \cdot p_2^* o_{\mathcal{X}} - p_2^* o_{\mathcal{X}} \cdot p_3^* o_{\mathcal{X}}].\end{aligned} \tag{2.76}$$

This last equality is now satisfied by assumption (compare with (2.49)) and this concludes the proof of formula (2.50). Thus (i) is proved.

(ii) We just have to prove that

$$\begin{aligned}P_0 \otimes P_0([\Delta_{\mathcal{X}}]) = P_4 \otimes P_4([\Delta_{\mathcal{X}}]) = 0, \\ P_0 \otimes P_2([\Delta_{\mathcal{X}}]) = P_4 \otimes P_2([\Delta_{\mathcal{X}}]) = 0 \text{ in } H^4(\mathcal{X} \times_B \mathcal{X}, \mathbb{Q}).\end{aligned} \tag{2.77}$$

Indeed, the relative Künneth decomposition gives

$$R(\pi, \pi)_* \mathbb{Q} = R\pi_* \mathbb{Q} \otimes R\pi_* \mathbb{Q}$$

and the decomposition (2.48) induces a decomposition of the above tensor product on the right:

$$R\pi_* \mathbb{Q} \otimes R\pi_* \mathbb{Q} = \oplus_{k,l} R^k \pi_* \mathbb{Q} \otimes R^l \pi_* \mathbb{Q}[-k-l], \tag{2.78}$$

where the decomposition is induced by the various tensor products of P_0, P_2, P_4. Taking cohomology in (2.78) gives

$$H^4(\mathcal{X} \times_B \mathcal{X}, \mathbb{Q}) = \oplus_{s+k+l=4} H^s(B, R^k \pi_* \mathbb{Q} \otimes R^l \pi_* \mathbb{Q}).$$

The term $H^0(R^4(\pi, \pi)_* \mathbb{Q})$ is then exactly the term in the above decomposition of $H^4(\mathcal{X} \times_B \mathcal{X}, \mathbb{Q})$ which is annihilated by the four projectors $P_0 \otimes P_0$, $P_0 \otimes P_2$, $P_4 \otimes P_2$, $P_4 \otimes P_4$ and those obtained by changing the order of factors.

The proof of (2.77) is elementary. Indeed, consider for example the term $P_0 \otimes P_0$, which is given by the cohomology class of the cycle

$$Z := pr_1^* o_{\mathcal{X}} \cdot pr_2^* o_{\mathcal{X}} \subset \mathcal{X} \times_B \mathcal{X} \times_B \mathcal{X} \times_B \mathcal{X},$$

which we see as a relative self-correspondence of $\mathcal{X} \times_B \mathcal{X}$ We have

$$Z_*(\Delta_{\mathcal{X}}) = p_{34*}(p_{12}^* \Delta_{\mathcal{X}} \cdot Z).$$

But the cycle on the right is trivially rationally equivalent to 0 on fibers $\mathcal{X}_t \times \mathcal{X}_t$. It thus follows from the general principle 1.2 that for some dense Zariski open set B^0 of B,

$$[Z]_*([\Delta_{\mathcal{X}}]) = 0 \text{ in } H^4(\mathcal{X}^0 \times_{B^0} \mathcal{X}^0, \mathbb{Q}).$$

The other vanishing statements are proved similarly. ■

3 Calabi-Yau hypersurfaces

In the case of smooth Calabi-Yau hypersurfaces X in projective space $\mathbb{P}^n$, that is hypersurfaces of degree $n+1$ in $\mathbb{P}^n$, we have the following result which partially generalizes Theorem 2.17 and provides some information on the Chow ring of X. Denote by $o \in CH_0(X)_{\mathbb{Q}}$ the class of the 0-cycle $\frac{h^{n-1}}{n+1}$, where $h := c_1(\mathcal{O}_X(1)) \in CH^1(X)$. We denote again by Δ the small diagonal of X in X^3.

Theorem 3.1. *The following relation holds in $CH^{2n-2}(X \times X \times X)_{\mathbb{Q}}$:*

$$\Delta = \Delta_{12} \cdot o_3 + (perm.) + Z + \Gamma', \tag{3.79}$$

where Z is the restriction to $X \times X \times X$ of a cycle on $\mathbb{P}^n \times \mathbb{P}^n \times \mathbb{P}^n$, and Γ' is a multiple of the following effective cycle of dimension $n-1$:

$$\Gamma := \cup_{t \in F(X)} \mathbb{P}^1_t \times \mathbb{P}^1_t \times \mathbb{P}^1_t. \tag{3.80}$$

Here $F(X)$ is the variety of lines contained in X. It is of dimension $n-4$ for general X. For $t \in F(X)$ we denote $\mathbb{P}^1_t \subset X \subset \mathbb{P}^n$ the corresponding line.

Proof of Theorem 3.1. Observe first of all that it suffices to prove the following equality of $n-1$-cycles on $X^3_0 := X^3 \setminus \Delta$:

$$\Gamma_{|X^3_0} = 2(n+1)(\Delta_{12|X^3_0} \cdot o_3 + (perm.)) + Z \text{ in } CH^{2n-2}(X^3_0)_{\mathbb{Q}} \tag{3.81}$$

where Z is the restriction to X^3_0 of a cycle on $(\mathbb{P}^n)^3$. Indeed, by the localization exact sequence (cf. [14, Lemma 9.12]), (3.81) implies an equality, for an adequate multiple Γ' of Γ:

$$N\Delta = \Delta_{12} \cdot o_3 + (perm.) + Z + \Gamma' \text{ in } CH^{2n-2}(X \times X \times X)_{\mathbb{Q}}, \tag{3.82}$$

for some rational number N. Projecting to X^2 and taking cohomology classes, we easily conclude then that $N = 1$. (We use here the fact that X has some transcendental cohomology, so that the cohomology class of the diagonal of X does not vanish on products $U \times U$, where $U \subset X$ is Zariski open.)

In order to prove (3.81), we do the following: First of all we compute the class in $CH^{n-1}(X^3_0)$ of the $2n-2$-dimensional subvariety

$$X^3_{0,col,sch} \subset X^3_0$$

parameterizing 3-uples of collinear points satisfying the following property:

Let $\mathbb{P}^1_{x_1x_2x_3} =< x_1, x_2, x_3 >$ be the line generated by the x_i's. Then the subscheme $x_1+x_2+x_3$ of $\mathbb{P}^1_{x_1x_2x_3} \subset \mathbb{P}^n$ is contained in X.

We will denote $X^3_{0,col} \subset X^3_0$ the $2n-2$-dimensional subvariety parameterizing 3-uples of collinear points. Obviously $X^3_{0,col,sch} \subset X^3_{0,col}$. We will see that the first one is in fact an irreducible component of the second one.

Next we observe that there is a natural morphism $\phi : X^3_{0,col} \to G(2, n+1)$ to the Grassmannian of lines in $\mathbb{P}^n$, which to (x_1, x_2, x_3) associates the line $\mathbb{P}^1_{x_1x_2x_3}$. This morphism is well-defined on $X^3_{0,col}$ because at least two of the points x_i are distinct, so that this line is well-determined. The morphism ϕ corresponds to a tautological rank 2 vector bundle $\mathcal{E}$ on $X^3_{0,col}$, with fiber $H^0(\mathcal{O}_{\mathbb{P}^1_{x_1x_2x_3}}(1))$ over the point (x_1, x_2, x_3). We then observe that $\Gamma \subset X^3_{0,col,sch}$ is defined by

the condition that the line $\mathbb{P}^1_{x_1x_2x_3}$ be contained in X. In other words, the equation f defining X has to vanish on this line. This condition can be seen globally as the vanishing of the section σ of the vector bundle $S^{n+1}\mathcal{E}$ defined by

$$\sigma((x_1, x_2, x_3)) = f_{|\mathbb{P}^1_{x_1x_2x_3}},$$

This section σ is not transverse, (in fact the rank of $S^{n+1}\mathcal{E}$ is $n+2$, while the codimension of Γ is $n-1$), but the reason for this is very simple: indeed, at a point (x_1, x_2, x_3) of $X^3_{0,col,sch}$, the equation f vanishes by definition on the degree 3 cycle $x_1+x_2+x_3$ of $\mathbb{P}^1_{x_1x_2x_3}$. Another way to express this is to say that σ is in fact a section of the rank $n-1$ bundle

$$\mathcal{F} \subset S^{n+1}\mathcal{E} \tag{3.83}$$

where $\mathcal{F}_{(x_1,x_2,x_3)}$ consists of degree $n+1$ polynomials vanishing on the subscheme $x_1+x_2+x_3$ of $\mathbb{P}^1_{x_1x_2x_3}$.

The section σ of $\mathcal{F}$ is transverse and thus we conclude that we have the following equality

$$\Gamma_{|X^3_0} = j_*(c_{n-1}(\mathcal{F})) \text{ in } CH^{2n-2}(X^3_0)_{\mathbb{Q}}, \tag{3.84}$$

where j is the inclusion of $X^3_{0,col,sch}$ in X^3_0.

We now observe that the vector bundles $\mathcal{E}$ and $\mathcal{F}$ come from vector bundles on the variety $(\mathbb{P}^n)^3_{0,col}$ parameterizing 3-uples of collinear points in $\mathbb{P}^n$, at least two of them being distinct.

The variety $(\mathbb{P}^n)^3_{0,col}$ is smooth irreducible of dimension $2n+1$ (hence of codimension $n-1$ in $(\mathbb{P}^n)^3$), being Zariski open in a $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$-bundle over the Grassmannian $G(2, n+1)$. We have now the following:

Lemma 3.2. *The intersection $(\mathbb{P}^n)^3_{0,col} \cap X^3_0$ is reduced, of pure dimension $2n-2$. It decomposes as*

$$(\mathbb{P}^n)^3_{0,col} \cap X^3_0 = X^3_{0,col,sch} \cup \Delta_{0,12} \cup \Delta_{0,13} \cup \Delta_{0,23}, \tag{3.85}$$

where $\Delta_{0,ij} \subset X^3_0$ is defined as $\Delta_{ij} \cap X^3_0$ with Δ_{ij} the big diagonal $\{x_i = x_j\}$.

Proof. The set theoretic equality in (3.85) is obvious. The fact that each component on the right has dimension $2n-2$ and thus is a component of the right dimension of this intersection is also obvious. The only point to check is thus the fact that these intersections are transverse at the generic point of each component in the right hand side. The generic point of the irreducible variety $X^3_{0,col,sch}$ parameterizes a triple of distinct collinear points which are on a line D not tangent to X. At such a triple, the intersection $(\mathbb{P}^n)^3_{0,col} \cap X^3_0$ is smooth of dimension $2n-2$ because $(\mathbb{P}^n)^3_{0,col}$ is Zariski open in the triple self-product $P \times_{G(2,n+1)} P \times_{G(2,n+1)} P$ of the tautological $\mathbb{P}^1$-bundle P over the Grassmannian $G(2, n+1)$, and the intersection with X^3_0 is defined by the three equations

$$p \circ pr_1^* f,\ p \circ pr_2^* f,\ p \circ pr_3^* f,$$

where the pr_i's are the projections $P^{3/G(2,n+1)} \to P$ and $p : P \to \mathbb{P}^n$ is the natural map. These three equations are independent since they are independent after restriction to $D \times D \times D \subset P \times_{G(2,n+1)} P \times_{G(2,n+1)} P$ at the point (x_1, x_2, x_3) because D is not tangent to X.

Similarly, the generic point of the irreducible variety $\Delta_{0,1,2,j} \subset X^3_{0,col}$ parameterizes a triple (x, x, y) with the property that $x \neq y$ and the line $\mathbb{P}^1_{xy} :=< x, y >$ is not tangent to X. Again,

the intersection $(\mathbb{P}^n)^3_{0,col} \cap X^3_0$ is smooth of dimension $2n-2$ near (x,x,y) because the restrictions to $\mathbb{P}^1_{xy} \times \mathbb{P}^1_{xy} \times \mathbb{P}^1_{xy} \subset P \times_{G(2,n+1)} P \times_{G(2,n+1)} P$ of the equations

$$p \circ pr_1^* f,\, p \circ pr_2^* f,\, p \circ pr_3^* f,$$

defining X^3 are independent.

■

Combining (3.85), (3.84) and the fact that the vector bundle $\mathcal{F}$ already exists on $(\mathbb{P}^n)^3_{0,col}$, we find that

$$\Gamma_{|X^3_0} = J_*(c_{n-1}(\mathcal{F}_{|(\mathbb{P}^n)^3_{0,col} \cap X^3_0})) - \sum_{i \neq j} J_{0,ij*} c_{n-1}(\mathcal{F}_{|\Delta_{0,ij}}) \text{ in } CH^{2n-2}(X^3_0)_{\mathbb{Q}},$$

where $J : (\mathbb{P}^n)^3_{0,col} \cap X^3_0 \hookrightarrow X^3_0$ is the inclusion and similarly for $J_{0,ij} : \Delta_{0,ij} \hookrightarrow X^3_0$. This provides us with the formula:

$$\Gamma_{|X^3_0} = (K_* c_{n-1}(\mathcal{F}))_{|X^3_0} - \sum_{i \neq j} J_{0,ij*} c_{n-1}(\mathcal{F}_{|\Delta_{0,ij}}) \text{ in } CH^{2n-2}(X^3_0)_{\mathbb{Q}}, \tag{3.86}$$

where $K : (\mathbb{P}^n)^3_{0,col} \hookrightarrow (\mathbb{P}^n)^3_0$ is the inclusion map.

The first term comes from $CH((\mathbb{P}^n)^3_0)$, so to conclude we only have to compute the terms $J_{0,ij*} c_{n-1}(\mathcal{F}_{|\Delta_{0,ij}})$. This is however very easy, because the vector bundles $\mathcal{E}$ and $\mathcal{F}$ are very simple on $\Delta_{0,ij}$: Assume for simplicity $i = 1, j = 2$. Points of $\Delta_{0,12}$ are points (x,x,y), $x \neq y \in X$. The line $\phi((x,x,y))$ is the line $< x, y >$, $x \neq y$, and it follows that

$$\mathcal{E}_{|\Delta_{0,12}} = pr_2^* \mathcal{O}_X(1) \oplus pr_3^* \mathcal{O}_X(1). \tag{3.87}$$

The projective bundle $\mathbb{P}(\mathcal{E}_{|\Delta_{0,12}})$ has two sections on $\Delta_{0,12}$ which give two divisors

$$D_2 \in |\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1) \otimes pr_3^* \mathcal{O}_X(-1)|,\, D_3 \in |\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1) \otimes pr_2^* \mathcal{O}_X(-1)|.$$

The length 3 subscheme $2D_2 + D_3 \subset \mathbb{P}(\mathcal{E}_{|\Delta_{0,1,2}})$ with fiber $2x + y$ over the point (x,x,y) is thus the zero set of a section α of the line bundle $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(3) \otimes pr_3^* \mathcal{O}_X(-2) \otimes pr_2^* \mathcal{O}_X(-1)$. We thus conclude that the vector bundle $\mathcal{F}_{|\Delta_{0,12}}$ is isomorphic to

$$pr_3^* \mathcal{O}_X(2) \otimes pr_2^* \mathcal{O}_X(1) \otimes S^{n-2} \mathcal{E}_{|\Delta_{0,12}}.$$

Combining with (3.87), we conclude that $c_{n-1}(\mathcal{F}_{|\Delta_{0,12}})$ can be expressed as a polynomial of degree $n-1$ in $h_2 = c_1(pr_2^* \mathcal{O}_X(1))$ and $h_3 = c_1(pr_3^* \mathcal{O}_X(1)))$ on $\Delta_{0,12}$. The proof of (3.81) is completed by the following lemma:

Lemma 3.3. *Let $\Delta_X \subset X \times X$ be the diagonal. Then the codimension n cycles*

$$pr_1^* c_1(\mathcal{O}_X(1)) \cdot \Delta_X,\; pr_2^* c_1(\mathcal{O}_X(1)) \cdot \Delta_X$$

of $X \times X$ are restrictions to $X \times X$ of cycles $Z \in CH^n(\mathbb{P}^n \times \mathbb{P}^n)_{\mathbb{Q}}$.

Proof. Indeed, let $j_X : X \hookrightarrow \mathbb{P}^n$ be the inclusion of X in $\mathbb{P}^n$, and $j_{X,1}$, $j_{X,2}$ the corresponding inclusions of $X \times X$ in $\mathbb{P}^n \times X$, resp. $X \times \mathbb{P}^n$. Then as X is a degree $n+1$ hypersurface, the composition $j_{X,1}^* \circ j_{X,1*} : CH^*(X \times X) \to CH^{*+1}(X \times X)$ is equal to the morphism given by intersection with the class $(n+1) pr_1^* c_1(\mathcal{O}_X(1))$, and similarly for the second inclusion. On the other hand, $j_{X,1*}(\Delta_X) \subset \mathbb{P}^n \times X$ is obviously the (transpose of the) graph of the inclusion of X in $\mathbb{P}^n$, hence its class is the restriction to $\mathbb{P}^n \times X$ of the diagonal of $\mathbb{P}^n \times \mathbb{P}^n$. This implies that

$$(n+1) pr_1^* c_1(\mathcal{O}_X(1)) \cdot \Delta_X = j_{X,1}^*((\Delta_{\mathbb{P}^n \times \mathbb{P}^n})_{|\mathbb{P}^n \times X}),$$

which proves the result for $pr_1^* c_1(\mathcal{O}_X(1)) \cdot \Delta_X$. We argue similarly for the second cycle. ■

It follows from this lemma that a monomial of degree $n-1$ in $h_2 = c_1(pr_2^*\mathcal{O}_X(1))$ and $h_3 = c_1(pr_3^*\mathcal{O}_X(1)))$ on $\Delta_{0,12}$, seen as a cycle in X_0^3, will be the restriction to X_0^3 of a cycle with $\mathbb{Q}$-coefficients on $(\mathbb{P}^n)^3$, unless it is proportional to h_3^{n-1}. Recalling that $c_1(\mathcal{O}_X(1))^{n-1} = (n+1)o \in CH_0(X)$, we finally proved that modulo restrictions of cycles on $(\mathbb{P}^n)^3$, the term $J_{0,12*}c_{n-1}(\mathcal{F}_{|\Delta_{0,12}})$ is a multiple of $(\Delta_{12}\cdot o_3)_{|X_0^3}$ in $CH^{2n-2}(X_0^3)_{\mathbb{Q}})$. The precise coefficient is in fact given by the argument above. Indeed, we just saw that modulo restrictions of cycles coming from $\mathbb{P}^n\times\mathbb{P}^n\times\mathbb{P}^n$, the term $J_{0,12*}c_{n-1}(\mathcal{F}_{|\Delta_{0,12}})$ is equal to

$$\mu\Delta_{12}\cdot pr_3^*(c_1(\mathcal{O}_X(1))^{n-1}) = \mu(n+1)(\Delta_{12}\times o_3)_{|X_0^3}, \tag{3.88}$$

with $c_1(\mathcal{O}_X(1))^{n-1} = (n+1)o$ in $CH_0(X)$, and where the coefficient μ is the coefficient of h_3^{n-1} in the polynomial in h_2, h_3 computing $c_{n-1}(\mathcal{F}_{|\Delta_{0,12}})$.

We use now the isomorphism

$$\mathcal{F}_{|\Delta_{0,12}} \cong pr_3^*\mathcal{O}_X(2)\otimes pr_2^*\mathcal{O}_X(1)\otimes S^{n-2}\mathcal{E}_{|\Delta_{0,12}},$$

where $\mathcal{E}_{|\Delta_{0,12}} \cong pr_2^*\mathcal{O}_X(1)\oplus pr_3^*\mathcal{O}_X(1)$ according to (3.87). Hence we conclude that the coefficient μ is equal to 2, and this concludes the proof of (3.81), using (3.88) and (3.86).

■

We have the following consequence of Theorem 3.1, which is a generalization of Theorem 1.1 to Calabi-Yau hypersurfaces.

Theorem 3.4. *Let Z_i, Z_i' be cycles of codimension >0 on X such that* $\operatorname{codim} Z_i + \operatorname{codim} Z_i' = n-1$. *Then if we have a cohomological relation*

$$\sum_i n_i[Z_i]\cup[Z_i'] = 0 \text{ in } H^{2n-2}(X,\mathbb{Q})$$

this relation already holds at the level of Chow groups:

$$\sum_i n_i Z_i\cdot Z_i' = 0 \text{ in } CH_0(X)_{\mathbb{Q}}.$$

Proof. Indeed, let us view formula (3.79) as an equality of correspondences between $X\times X$ and X. The left hand side applied to $\sum_i n_iZ_i\times Z_i'$ is the desired cycle: $\Delta_*(\sum_i n_iZ_i\times Z_i') = \sum_i n_iZ_i\cdot Z_i'$ in $CH_0(X)_{\mathbb{Q}}$. The right hand side is a sum of three terms:

$$(\Delta_{12}\cdot o_3 + (perm.))_*(\sum_i n_iZ_i\times Z_i') + Z_*(\sum_i n_iZ_i\times Z_i') + \Gamma'_*(\sum_i n_iZ_i\times Z_i'). \tag{3.89}$$

For the first term, we observe that $(\Delta_{12}\cdot o_3)_*(\sum_i n_iZ_i\times Z_i') = (\deg\sum_i n_iZ_i\cdot Z_i')\,o_3$ vanishes in $CH_0(X)_{\mathbb{Q}}$, and that the two other terms $(\Delta_{13}\cdot o_2)_*(\sum_i n_iZ_i\times Z_i')$ and $(\Delta_{23}\cdot o_1)_*(\sum_i n_iZ_i\times Z_i')$ vanish by the assumption that $\operatorname{codim} Z_i > 0$ for all i.

For the second term, we observe that as Z is the restriction of a cycle $Z'\in CH^{2n-2}(\mathbb{P}^n\times\mathbb{P}^n\times\mathbb{P}^n)_{\mathbb{Q}}$, $Z_*(\sum_i n_iZ_i\times Z_i')$ is equal to

$$j^*(Z'^*((j,j)_*(\sum_i n_iZ_i\times Z_i'))) \in CH^{n-1}(X)_{\mathbb{Q}}.$$

Hence it belongs to $\operatorname{Im} j^*$, and is proportional to o.

Consider finally the term $\Gamma'_*(\sum_i n_i Z_i \times Z'_i)$, which is a multiple of $\Gamma'_*(\sum_i n_i Z_i \times Z'_i)$: Let $\Gamma_0 \subset X$ be the locus swept-out by lines. We observe that for any line $D \cong \mathbb{P}^1 \subset X$, any point on D is rationally equivalent to the zero cycle $h \cdot D$ which is in fact proportional to o, since

$$(n+1)h \cdot D = j^* \circ j_*(D) \text{ in } CH_0(X)$$

and $j_*(D) = c_1(\mathcal{O}_{\mathbb{P}^n}(1))^{n-1}$ in $CH^{n-1}(\mathbb{P}^n)$. Hence all points of Γ_0 are rationally equivalent to o in X, and thus $\Gamma'_*(\sum_i n_i Z_i \times Z'_i)$ is also proportional to o.

It follows from the above analysis that the 0-cycle (3.89) is a multiple of o in $CH_0(X)_{\mathbb{Q}}$. As it is of degree 0, it is in fact rationally equivalent to 0.

■

We leave as a conjecture the following :

Conjecture 3.5. *For any smooth $n-1$-dimensional Calabi-Yau hypersurface for which the variety of lines $F(X)$ has dimension $n-4$, the $n-1$-cycle $\Gamma \in CH_{n-1}(X \times X \times X)_{\mathbb{Q}}$ of (3.80) is the restriction to $X \times X \times X$ of a $n+2$-cycle on $\mathbb{P}^n \times \mathbb{P}^n \times \mathbb{P}^n$.*

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