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# Optimal consumption and investment for markets with random coefficients. \*

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## Abstract

We consider an optimal investment and consumption problem for a Black-Scholes financial market with stochastic coefficients driven by a diffusion process. We assume that an agent makes consumption and investment decisions based on CRRA utility functions. The dynamical programming approach leads to an investigation of the Hamilton Jacobi Bellman (HJB) equation which is a highly non linear partial differential equation (PDE) of the second order. By using the Feynman - Kac representation we prove uniqueness and smoothness of the solution. Moreover, we study the optimal convergence rate of the iterative numerical schemes for both the value function and the optimal portfolio. We show, that in this case, the optimal convergence rate is super geometrical, i.e. is more rapid than any geometrical one. We apply our results to a stochastic volatility financial market.

*Key words :* Black-Scholes market, Stochastic volatility, Optimal consumption and Investment, Hamilton-Jacobi-Bellman equation, Feynman - Kac formula, Fixed point solution.

*Mathematical Subject Classification (2000)* 91B28, 93E20

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# 1 Introduction

One of the principal questions in mathematical finance is the optimal consumption-investment problem for continuous time market models. This paper deals with an investment problem aiming at optimal consumption during a fixed investment interval  $[0, T]$  in addition to an optimal terminal wealth at maturity  $T$ . Such problems are of prime interest for the institutional investor, selling asset funds to their customers, who are entitled to certain payment during the duration of an investment contract and expect a high return at maturity. The classical approach to this problem goes back to Merton [15] and involves utility functions, more precisely, the expected utility serves as the functional which has to be optimized. By applying results from the stochastic control, explicit solutions have been obtained for financial markets with nonrandom coefficients (see, e.g. [9], [11] and references therein). Since then, there has been a growing interest in consumption and investment problems and the Merton problem has been extended in many directions. One of the generalisations considers financial models with random coefficients such as stochastic volatility markets (see, e.g., [5]). In this paper for the CRRA (Constant Relative Risk Aversion) utility functions we consider the optimal consumption-investment problem for a Black-Scholes type model with coefficients depending on a diffusion process which is referred as the external stochastic factor. The pure investment problem for such models is considered in [17] and [16]. In these papers the authors use the dynamic programming approach and they show that the nonlinear HJB equation can be transformed in a quasilinear PDE. The similar approach has been used in [12] for optimal consumption-investment problems with the default risk for financial markets with non random coefficients. Furthermore, in [3], by making use of the Girsanov measure transformation the authors study a pure optimal consumption problem for stochastic volatility financial markets. In [1] and [6] the authors use dual methods. Usually, the classical existence and uniqueness theorem for the HJB equation is shown by the linear PDE methods (see, for example, chapter VI.6 and appendix E in [4]). In this paper we use the approach proposed in [2] for the optimal consumption-investment problem for financial markets with random coefficients depending on pure jumps processes. Unfortunately, we can not apply directly this method for our case since in [2] the HJB equation is the integro-differential equation of the first order. In our case

it is a highly non linear PDE of the second order. Similarly to [2] we study the HJB equation through the Feynman - Kac representation. We introduce a special metric space in which the Feynman - Kac mapping is contracted. Taking this into account we show the fixed-point theorem for this mapping and we show that the fixed-point solution is the classical unique solution for HJB equation in our case. Moreover, using the verification theorem we provide the explicit expressions for the optimal investment and consumption which depend on the HJB solution. Therefore, to calculate the optimal strategies one needs to study numerical schemes for HJB equation. To this end we find an explicit upper bound for the approximation accuracy. Then, we minimize this bound and we get the optimal convergence rate for both the value function and the optimal financial strategies. It turns out that in this case this rate is super geometrical.

The rest of the paper is organized as follows. In section 2 we introduce the financial market, we state the main conditions on the market parameters and we write the HJB equation. In section 3 we state the main results of the paper. Section 4 presents a stochastic volatility model as an example of applications of our results. In Section 5 we study the properties of the Feynman - Kac mapping. In Section 6 the corresponding verification theorem is stated. The proofs of the main results are given in Section 7. In Section 8 we consider a numerical example. In Appendix some auxiliary results are given.

## 2 Market model

Let  $(\Omega, \mathcal{F}_T, (\mathcal{F}_t)_{0 \leq t \leq T}, \mathbf{P})$  be a standard filtered probability space with two standard independent  $(\mathcal{F}_t)_{0 \leq t \leq T}$  adapted Wiener processes  $(W_t)_{0 \leq t \leq T}$  and  $(V_t)_{0 \leq t \leq T}$  taking their values in  $\mathbb{R}^d$  and  $\mathbb{R}^m$  respectively, i.e.

$$W_t = (W_t^1, \dots, W_t^d)' \quad \text{and} \quad V_t = (V_t^1, \dots, V_t^m)'.$$

The prime  $'$  denotes the transposition. Our financial market consists of one *riskless bond*  $(S_0(t))_{0 \leq t \leq T}$  and  $d$  *risky stocks*  $(S_i(t))_{0 \leq t \leq T}$  governed by the following equations:

$$\begin{cases} dS_0(t) &= r(t, Y_t) S_0(t) dt, \\ dS_i(t) &= S_i(t) \mu_i(t, Y_t) dt + S_i(t) \sum_{j=1}^d \sigma_{ij}(t, Y_t) dW_t^j, \end{cases} \quad (2.1)$$

with  $S_0(0) = 1$  and  $S_i(0) = s_i$  for  $1 \leq i \leq d$ . In this model  $r(t, y) \in \mathbb{R}_+$  is the *riskless interest rate*,  $\mu(t, y) = (\mu_1(t, y), \dots, \mu_d(t, y))'$  is the vector of *stock-appreciation rates* and  $\sigma(t, y) = (\sigma_{ij}(t, y))_{1 \leq i, j \leq d}$  is the matrix of *stock-volatilities*. For all  $y \in \mathbb{R}^m$  the coefficients  $r(\cdot, y)$ ,  $\mu(\cdot, y) \in \mathbb{R}^d$  and  $\sigma(\cdot, y) \in \mathbb{M}_d$  are nonrandom càdlàg functions. Here  $\mathbb{M}_d$  denotes the set of quadratic matrix of order  $d$ . Moreover, in just the same way as in [17] we assume, that the stochastic factor  $Y$  valued in  $\mathbb{R}^m$  has a dynamics governed by the following stochastic differential equation:

$$dY_t = F(t, Y_t) dt + \beta dU_t, \quad (2.2)$$

where  $F$  is a  $[0, T] \times \mathbb{R}^m \rightarrow \mathbb{R}^m$  nonrandom function and  $\beta$  is fixed positive parameter. The process  $U$  is the standard Brownian motion defined as

$$U_t = \rho V_t + \sqrt{1 - \rho^2} \sigma_* W_t, \quad (2.3)$$

where  $0 \leq \rho \leq 1$  and  $\sigma_*$  is a fixed  $m \times d$  matrix for which  $\sigma_* \sigma_*' = I_m$ . Here  $I_m$  is the identity matrix of order  $m$ .

Moreover, we set  $\mathcal{K} = [0, T] \times \mathbb{R}^m$  and we note, that for the model (2.1) the risk premium is the  $\mathcal{K} \rightarrow \mathbb{R}^d$  function defined as

$$\theta(t, y) = \sigma^{-1}(t, y)(\mu(t, y) - r(t, y) \mathbf{1}_d), \quad (2.4)$$

where  $\mathbf{1}_d = (1, \dots, 1)' \in \mathbb{R}^d$ . Similarly to [10] we consider the fractional portfolio process

$$\varphi(t) = (\varphi_1(t), \dots, \varphi_d(t))' \in \mathbb{R}^d,$$

i.e.  $\varphi_i(t)$  represent the fraction of the wealth process  $X_t$  invested in the  $i$ -th stock at the time  $t$ . The fractions for consumptions we denote by  $c = (c_t)_{0 \leq t \leq T}$ . In this case the wealth process satisfies the following stochastic equation

$$dX_t = X_t(r(t, Y_t) + \pi_t' \theta(t, Y_t) - c_t) dt + X_t \pi_t' dW_t, \quad (2.5)$$

where  $\pi_t = \sigma(t, Y_t) \varphi_t$  and the initial endowment  $X_0 = x$ . Now we describe the set of all admissible strategies. A portfolio control (financial strategy)  $\vartheta = (\vartheta_t)_{t \geq 0} = ((\pi_t, c_t))_{t \geq 0}$  is said to be *admissible* if it is  $(\mathcal{F}_t)_{0 \leq t \leq T}$  - progressively measurable with values in  $\mathbb{R}^d \times [0, \infty)$ , such that

$$\|\pi\|_T := \int_0^T |\pi_t|^2 dt < \infty \quad \text{and} \quad \int_0^T c_t dt < \infty \quad \text{a.s.} \quad (2.6)$$

and the equation (2.5) has a unique strong a.s. positive continuous solution  $(X_t^\vartheta)_{0 \leq t \leq T}$  on  $[0, T]$ . We denote the set of *admissible portfolios controls* by  $\mathcal{V}$ .

In this paper we consider an agent using the CRRA utility function  $x^\gamma$  for  $0 < \gamma < 1$ . The goal is to maximize the expected utilities from the consumption on the time interval  $[0, T]$  and from the terminal wealth. Then for any  $x \in \mathbb{R}^d$ ,  $y \in \mathbb{R}^m$  and  $\vartheta \in \mathcal{V}$  the value function of agent is

$$J(x, y, \vartheta) := \mathbf{E}_{x,y} \left( \int_0^T c_t^\gamma (X_t^\vartheta)^\gamma dt + (X_T^\vartheta)^\gamma \right),$$

where  $\mathbf{E}_{x,y}$  is the conditional expectation for  $X_0^\vartheta = x$  and  $Y_0 = y$ . Our goal is to maximize this function, i.e.

$$\sup_{\vartheta \in \mathcal{V}} J(x, y, \vartheta). \quad (2.7)$$

**Remark 2.1.** *Note, that (2.7) is the classical Merton optimization problem for the market (2.1), (see, e.g. [15], [2]). Pure investment cases of this problem are studied in [17] and [16].*

## 2.1 Conditions

To list the conditions for the model (2.1) we set

$$\alpha(t, y) = F(t, y) + \beta_* \sigma_* \theta(t, y), \quad (2.8)$$

where  $\beta_* = \gamma \sqrt{1 - \rho^2} \beta / (1 - \gamma)$ . Moreover, we denote by  $\mathbf{C}^{i,k}(\mathcal{K})$  the space of the functions  $f(t, y_1, \dots, y_m)$  which are  $i$  times differentiable with respect to  $t \in [0, T]$  and  $j$  times differentiable with respect to  $(y_j)_{1 \leq j \leq m}$ .

We assume that the market parameters satisfy the following conditions:

**A<sub>1</sub>)** *The functions  $r : \mathcal{K} \rightarrow \mathbb{R}_+$ ,  $\mu : \mathcal{K} \rightarrow \mathbb{R}_+^d$  and  $\sigma : \mathcal{K} \rightarrow \mathbb{M}_d$  belong to  $\mathbf{C}^{1,1}(\mathcal{K})$  and have bounded derivatives. Moreover, for all  $(t, y) \in \mathcal{K}$  the matrix  $\sigma(t, y)$  is non-degenerated and*

$$\sup_{(t,y) \in \mathcal{K}} (|r(t, y)| + |\mu(t, y)| + |\sigma^{-1}(t, y)|) < \infty.$$

**A<sub>2</sub>)** The  $\mathcal{K} \rightarrow \mathbb{R}^m$  function  $F(\cdot, \cdot)$  belongs to  $\mathbf{C}^{1,1}(\mathcal{K})$  and its partial derivatives are bounded.

This condition provides the existence of a unique strong solution for the equation (2.2). On the interval  $[t, T]$  we denote this solution by

$$Y^{t,y} = (Y_v^{t,y})_{t \leq v \leq T} \text{ with } Y_t^{t,y} = y.$$

**A<sub>3</sub>)** There exist  $[0, T] \times \mathbb{R} \rightarrow \mathbb{R}$  continuously differentiable functions  $(\mathbf{a}_{i,j}(\cdot, \cdot))_{1 \leq i,j \leq m}$  such that for any  $0 \leq t \leq T$  and for any  $x = (x_1, \dots, x_m)'$  from  $\mathbb{R}^m$  the  $i$ th component  $[\alpha(\cdot, \cdot)]_i$  of the vector  $\alpha(t, x)$  has the form

$$[\alpha(t, x)]_i = \sum_{l=1}^m \mathbf{a}_{i,l}(t, x_l) \quad (2.9)$$

and, moreover,

$$\alpha_* = \max_{i,j} \max_{t,z} \max \left( |\mathbf{a}_{i,j}(t, z)|, \left| \frac{\partial \mathbf{a}_{i,j}(t, z)}{\partial t} \right|, \left| \frac{\partial \mathbf{a}_{i,j}(t, z)}{\partial z} \right| \right) < \infty.$$

In the sequel, for any vector  $x$  from  $\mathbb{R}^n$  we denote the  $i$ th component by  $[x]_i$ .

**Remark 2.2.** In Section 4 we check the condition **A<sub>3</sub>)** for a two asset stochastic volatility financial market.

## 2.2 The HJB equation

Now we introduce the HJB equation for the problem (2.7). To this end, for any differentiable  $\mathcal{K} \rightarrow \mathbb{R}$  function  $f$  we denote by  $\mathbf{D}_y f(t, y)$  its gradient with respect to  $y \in \mathbb{R}^m$ , i.e.

$$\mathbf{D}_y f(t, y) = \left( \frac{\partial}{\partial y_1} f(t, y), \dots, \frac{\partial}{\partial y_m} f(t, y) \right)'.$$

Let now  $z(t, \varsigma)$  be any  $[0, T] \times \mathbb{R} \times \mathbb{R}^m \rightarrow \mathbb{R}$  two times differentiable function. Here  $\varsigma = (x, y)'$ ,  $x \in \mathbb{R}$  and  $y \in \mathbb{R}^m$ . We set

$$\mathbf{D}_\varsigma z(t, \varsigma) = (z_x(t, \varsigma), \mathbf{D}_y z(t, \varsigma))'$$

and

$$\mathbf{D}_{\varsigma,\varsigma}z(t,\varsigma) = \begin{pmatrix} \frac{\partial^2 z(t,\varsigma)}{\partial^2 x}; & (\mathbf{D}_{x,y}z(t,\varsigma))' \\ \mathbf{D}_{x,y}z(t,\varsigma); & \mathbf{D}_{y,y}z(t,\varsigma) \end{pmatrix},$$

where

$$\mathbf{D}_{x,y}z(t,\varsigma) = \left( \frac{\partial^2 z(t,\varsigma)}{\partial x \partial y_1}, \dots, \frac{\partial^2 z(t,\varsigma)}{\partial x \partial y_m} \right)'$$

and

$$\mathbf{D}_{y,y}z(t,\varsigma) = \left( \frac{\partial^2 z(t,\varsigma)}{\partial y_j \partial y_i} \right)_{1 \leq i,j \leq m}.$$

Let now  $\mathbf{q} \in \mathbb{R}^{m+1}$  and  $\mathbf{M} \in \mathbb{M}_{m+1}$  be fixed parameters of the following form

$$\mathbf{q} = (\mathbf{q}_1, \tilde{\mathbf{q}})' \quad \text{and} \quad \mathbf{M} = \begin{pmatrix} \mu; & \tilde{\mu}' \\ \tilde{\mu}; & \mathbf{M}_0 \end{pmatrix}, \quad (2.10)$$

where  $\mathbf{q}_1, \mu \in \mathbb{R}$ ,  $\tilde{\mathbf{q}}, \tilde{\mu} \in \mathbb{R}^m$  and  $\mathbf{M}_0 \in \mathbb{M}_m$ . For these parameters with  $\mathbf{q}_1 > 0$  we define the Hamilton function as

$$\begin{aligned} H(t,\varsigma,\mathbf{q},\mathbf{M}) &= x r(t,y)\mathbf{q}_1 + (F(t,y))' \tilde{\mathbf{q}} + \frac{1}{\gamma_1} \left( \frac{\gamma}{\mathbf{q}_1} \right)^{\gamma_1-1} \\ &\quad + \frac{|\theta(t,y)\mathbf{q}_1 + \beta\sqrt{1-\rho^2}\sigma'_* \tilde{\mu}|^2}{2|\mu|} + \frac{\beta^2}{2} \text{tr} \mathbf{M}_0, \end{aligned} \quad (2.11)$$

where  $\gamma_1 = (1-\gamma)^{-1}$ . In this case the HJB equation is

$$\begin{cases} z_t(t,\varsigma) + H(t,\varsigma, \mathbf{D}_\varsigma z(t,\varsigma), \mathbf{D}_{\varsigma,\varsigma} z(t,\varsigma)) = 0, \\ z(T,\varsigma) = x^\gamma. \end{cases} \quad (2.12)$$

To study this equation we make use of the distortion power transformation introduced in [17], i.e. for a positive function  $h$  we represent  $z(t,y)$  as

$$z(t,\varsigma) = x^\gamma h^\varepsilon(t,y), \quad \varepsilon = \frac{1-\gamma}{1-\rho^2\gamma}. \quad (2.13)$$

It is easy to deduce that the function  $h$  satisfies the following quasi-linear PDE:

$$\begin{cases} h_t(t,y) + Q(t,y)h(t,y) + (\alpha(t,y))' \mathbf{D}_y h(t,y) \\ \quad + \frac{\beta^2}{2} \text{tr} \mathbf{D}_{y,y} h(t,y) + \frac{1}{q_*} \left( \frac{1}{h(t,y)} \right)^{q_*-1} = 0; \\ h(T,y) = 1, \end{cases} \quad (2.14)$$

where

$$Q(t, y) = \frac{\gamma(1 - \rho^2\gamma)}{1 - \gamma} \left( r(t, y) + \frac{|\theta(t, y)|^2}{2(1 - \gamma)} \right). \quad (2.15)$$

Note that, by the condition  $\mathbf{A}_1$ ) this function and its  $y$ -gradient are bounded. Therefore, we can set

$$Q_* = \sup_{(t, y) \in \mathcal{K}} Q(t, y) \quad \text{and} \quad D_* = \sup_{(t, y) \in \mathcal{K}} |\mathbf{D}_y Q(t, y)|. \quad (2.16)$$

Our goal is to study the equation (2.14). By making use of the probabilistic representation for the linear PDE (the Feynman-Kac formula) we show in Proposition 5.4, that the solution of this equation is the fixed-point solution for a special mapping of the integral type which will be introduced in the next section.

### 3 Main results

First, to study the equation (2.14) we introduce a special functional space. Let  $\mathcal{X}$  be the set of  $\mathcal{K} \rightarrow [1, \infty)$  functions from  $\mathbf{C}^{0,1}(\mathcal{K})$  such that

$$\|f\|_{0,1} = \sup_{(t, y) \in \mathcal{K}} (|f(t, y)| + |\mathbf{D}_x f(t, y)|) \leq \mathbf{r}^*, \quad (3.1)$$

where

$$\mathbf{r}^* = \mathbf{r}_0 + m \left( \frac{H^*}{q_*} (2\sqrt{T} + T) + D_* T e^{(\alpha_* + Q_*)T} \right), \quad (3.2)$$

$\mathbf{r}_0 = e^{Q_* T} + (e^{Q_* T} - 1)/Q_* q_*$ . The parameter  $H^*$  will be defined below in Section 3.1. To define a metrics in  $\mathcal{X}$  we set

$$\varkappa = Q_* + \zeta + 1 + m\mathbf{L}_* \quad \text{and} \quad \mathbf{L}_* = (1 + \mathbf{r}^* q_* + TD_*)e^{\alpha_* T}, \quad (3.3)$$

where the positive parameter  $\zeta$  will be specified later,  $q_* = (1 - \rho^2\gamma)^{-1}$ . Now for any  $f$  and  $g$  from  $\mathcal{X}$  we define the metrics as

$$\varrho_*(f, g) = \|f - g\|_*, \quad (3.4)$$

where

$$\|f\|_* = \sup_{(t, y) \in \mathcal{K}} e^{-\varkappa(T-t)} (|f(t, y)| + |\mathbf{D}_x f(t, y)|).$$

Moreover, to write the Feynman-Kac representation for the equation (2.14) we need to use the stochastic process  $(\eta_v)_{0 \leq v \leq T}$  governed by the following stochastic differential equation

$$d\eta_t = \alpha(t, \eta_t)dt + \beta d\mathbf{U}_t. \quad (3.5)$$

It should be noted that the condition  $\mathbf{A}_3$ ) provides the existence and the uniqueness of a strong solution on any time interval  $[t, T]$  and initial condition  $y$  from  $\mathbb{R}^m$ . We denote this solution by  $(\eta_v^{t,y})_{t \leq v \leq T}$ . Using this process we define the  $\mathcal{X} \rightarrow \mathcal{X}$  Feynman-Kac mapping  $\mathcal{L}$ :

$$\mathcal{L}_f(t, y) = \mathbf{E} \mathcal{G}(t, T, y) + \frac{1}{q_*} \int_t^T \mathcal{H}_f(t, s, y) ds, \quad (3.6)$$

where  $\mathcal{G}(t, s, y) = \exp\left(\int_t^s Q(u, \eta_u^{t,y}) du\right)$  and

$$\mathcal{H}_f(t, s, y) = \mathbf{E} \left( f(s, \eta_s^{t,y}) \right)^{1-q_*} \mathcal{G}(t, s, y). \quad (3.7)$$

To solve the equation (2.14) we need to find the fixed-point solution for the mapping  $\mathcal{L}$  in  $\mathcal{X}$ , i.e.

$$\mathcal{L}h = h. \quad (3.8)$$

To this end we construct the following iterated scheme. We set  $h_0 \equiv 1$  and

$$h_n(t, y) = \mathcal{L}_{h_{n-1}}(t, y) \quad \text{for } n \geq 1. \quad (3.9)$$

First, we study the behavior of the deviation

$$\Delta_n = h(t, y) - h_n(t, y).$$

In the following theorem we show that the contraction coefficient for the operator (3.6) is given by

$$\lambda = \frac{1 + m\mathbf{L}_*}{\zeta + 1 + m\mathbf{L}_*} \quad (3.10)$$

and an appropriate choice of  $\zeta$  gives the super-geometrical convergence rate for the sequence  $(h_n)_{n \geq 1}$ .

**Theorem 3.1.** *Under the conditions  $\mathbf{A}_1)$ – $\mathbf{A}_3)$  the equation (3.8) has a unique solution  $h$  in  $\mathcal{X}$  such that for any  $n \geq 1$  and  $\zeta > 0$*

$$\sup_{(t,y) \in \mathcal{K}} (|\Delta_n(t, y)| + |\mathbf{D}_y \Delta_n(t, y)|) \leq \mathbf{B}^* \lambda^n, \quad (3.11)$$

where  $\mathbf{B}^* = e^{\varkappa T} (1 + \mathbf{r}^*) / (1 - \lambda)$  and  $\varkappa$  is given in (3.3).

The proof of this theorem is given in Section 7.

Now we can minimize the upper bound (3.11) over  $\zeta > 0$ . Indeed, setting  $\tilde{\zeta} = \zeta/(1 + m\mathbf{L}_*)$  and  $\tilde{T} = (1 + m\mathbf{L}_*)T$ , we obtain

$$\mathbf{B}^* \lambda^n = \mathbf{C}^* \exp\{\mathbf{g}_n(\tilde{\zeta})\},$$

where  $\mathbf{C}^* = (1 + \mathbf{r}^*)e^{(Q_*+1+m\mathbf{L}_*)T}$  and

$$\mathbf{g}_n(x) = x\tilde{T} - \ln x - (n - 1) \ln(1 + x).$$

Now we minimize this function over  $x > 0$ , i.e.

$$\min_{x>0} \mathbf{g}_n(x) = x_n^* \tilde{T} - \ln x_n^* - (n - 1) \ln(1 + x_n^*),$$

where

$$x_n^* = \frac{\sqrt{(\tilde{T} - n)^2 + 4\tilde{T} + n} - \tilde{T}}{2\tilde{T}}.$$

Therefore, for

$$\zeta = \zeta_n^* = (1 + m\mathbf{L}_*)x_n^*$$

we obtain the optimal upper bound (3.11).

**Corollary 3.2.** *Under the conditions  $\mathbf{A}_1)$ – $\mathbf{A}_3)$  the equation (3.8) has a unique solution  $h$  in  $\mathcal{X}$  such that for any  $n \geq 1$*

$$\sup_{(t,y) \in \mathcal{K}} (|\Delta_n(t, y)| + |\mathbf{D}_y \Delta_n(t, y)|) \leq \mathbf{U}_n^*, \quad (3.12)$$

where  $\mathbf{U}_n^* = \mathbf{C}^* \exp\{\mathbf{g}_n^*\}$ .

**Remark 3.1.** *One can check directly that for some  $\delta > 0$*

$$\mathbf{U}_n^* = O(n^{-\delta n}) \quad \text{as } n \rightarrow \infty.$$

*This means that the convergence rate is more rapid than any geometrical one, i.e. it is super geometrical.*

**Theorem 3.3.** *Assume, that the conditions  $\mathbf{A}_1)$ – $\mathbf{A}_3)$  hold. Then the optimal value of  $J(x, y, \vartheta)$  for optimization problem (2.7) is given by*

$$\max_{\vartheta \in \mathcal{V}} J(x, y, \vartheta) = J(x, y, \vartheta^*) = x^\gamma (h(0, y))^\varepsilon,$$

where  $h(t, y)$  is the unique solution of the equation (2.14). Moreover, for all  $0 \leq t \leq T$  an optimal financial strategy  $\vartheta^* = (\pi^*, c^*)$  is of the form

$$\begin{cases} \pi_t^* = \pi^*(t, Y_t) = \frac{\theta(t, Y_t)}{1 - \gamma} + \frac{\varepsilon \sqrt{1 - \rho^2} \beta \sigma_* D_y h(t, Y_t)}{(1 - \gamma) h(t, Y_t)}; \\ c_t^* = c^*(t, Y_t) = (h(t, Y_t))^{-q_*}. \end{cases} \quad (3.13)$$

The optimal wealth process  $(X_t^*)_{0 \leq t \leq T}$  satisfies the following stochastic equation

$$dX_t^* = a^*(t, Y_t) X_t^* dt + X_t^* (b^*(t, Y_t))' dW_t, \quad X_0^* = x, \quad (3.14)$$

where

$$\begin{aligned} a^*(t, y) &= \frac{|\theta(t, y)|^2}{1 - \gamma} + \frac{\varepsilon \sqrt{1 - \rho^2} \beta}{(1 - \gamma) h(t, y)} (\sigma_* D_y h(t, y))' \theta(t, y) \\ &\quad + r(t, y) - (h(t, y))^{-q_*}; \\ b^*(t, y) &= \frac{\theta(t, y)}{1 - \gamma} + \frac{\varepsilon \sqrt{1 - \rho^2} \beta}{(1 - \gamma) h(t, y)} \sigma_* D_y h(t, y). \end{aligned}$$

The proof of this theorem is given in Section 7.

**Remark 3.2.** Note that the optimal strategy (3.13) coincides with the well-known Merton strategy in the case  $\rho = 1$ , i.e. when the process  $Y$  is independent of the brownian motion  $W$  generating the financial market (2.1). Note also that, in the case  $0 \leq \rho < 1$ , the optimal investment strategy in (3.13) depends on the external factor  $Y$  through the derivative of the function  $h$ . The first term in this expression is the well-known Merton strategy and the second term is the impact of the external factor.

To calculate the optimal strategy in (3.13) we use the sequence  $(h_n)_{n \geq 1}$ , i.e. we set

$$\pi_n^*(t, y) = \frac{\theta(t, y)}{1 - \gamma} + \frac{\varepsilon \sqrt{1 - \rho^2} \beta}{(1 - \gamma) h_n(t, y)} \sigma_* D_y h_n(t, y)$$

and

$$c_n^*(t, y) = (h_n(t, y))^{-q_*}.$$

Theorem 3.1–Theorem 3.3 imply the following result

**Theorem 3.4.** *Assume, that the conditions  $\mathbf{A}_1)$ – $\mathbf{A}_3)$  hold. Then for any  $n \geq 1$*

$$\sup_{(t,y) \in \mathcal{K}} (|\pi^*(t,y) - \pi_n^*(t,y)| + |c^*(t,y) - c_n^*(t,y)|) \leq \mathbf{B}_1^* \mathbf{U}_n^*,$$

where  $\mathbf{B}_1^* = \gamma_1 \beta \varepsilon \sqrt{1 - \rho^2} |\sigma_*| \mathbf{r}^* + q_*$ .

**Remark 3.3.** *Note that in this paper we use only the power utility function  $x^\gamma$  with  $0 < \gamma < 1$ . It seems that the question about the upper bound (3.11) is open for general utility functions of the CRRA type. The method to obtain the bound (3.11) is based on the explicit form of the utility function. The heart of this method is to show that the operator (3.6) is contracted in a suitable functional space. To this end one needs to make use of the power utility.*

### 3.1 Formula for $H^*$

To write the upper bound for the partial derivatives of the function  $\mathcal{H}_f(t, s, y)$  (see Lemma A.2) we define for any  $q > 0$

$$\phi_q = \sqrt{2} e^{T\tilde{\phi}_q} \quad \text{and} \quad \iota_q = q\alpha_* (\beta^{-2} + T), \quad (3.15)$$

where  $\tilde{\phi}_q = q(Q_* + \beta^2\alpha_*/2) + (2m^2 + 1)q^2\beta^{-2}\alpha_*^2$ . Using these parameters we set

$$H^* = 2 \max \left( 2\phi_1(\iota_1 + \alpha_*) + \Psi_*(\phi_2)^{1/2}, \frac{1}{\beta\sqrt{2\pi}} \right) e^{\tilde{\iota}}, \quad (3.16)$$

where  $\Psi_* = \sqrt{T\alpha_* + (TD_* + T\alpha_*(\alpha_* + 1 + \beta^2/2) + \beta^{-2}\alpha_*^2)^2}$  and  $\tilde{\iota} = \beta^2 T (\max(\iota_1, \iota_2) + \alpha_*)^2$ .

## 4 Two-asset stochastic volatility model

In this section we consider an important example of the model (2.1) used in financial markets with the stochastic volatilities (see, e.g., [16]). This model is defined as follows:

$$dS_0(t) = r S_0(t) dt$$

and

$$dS_1(t) = \mu_1 S_1(t)dt + S_1(t)\sigma_1(t, Y_t^1) (dW_t^1 + \mathbf{b}_1 dW_t^2) ;$$

$$dS_2(t) = \mu_2 S_2(t)dt + S_2(t)\sigma_2(t, Y_t^2) (dW_t^1 + \mathbf{b}_2 dW_t^2) ,$$

where  $\mathbf{b}_1 \neq \mathbf{b}_2$ , the process  $Y_t = (Y_t^1, Y_t^2)'$  is governed by the following stochastic differential equations

$$dY_t^1 = F_1(t, Y_t^1) dt + \beta d\mathbf{U}_t^1 ,$$

$$dY_t^2 = F_2(t, Y_t^2) dt + \beta d\mathbf{U}_t^2 .$$

Here  $\mathbf{U}_t^i = \rho V_t^i + \sqrt{1 - \rho^2} W_t^i$  for some  $0 \leq \rho \leq 1$ , i.e. the parameter  $\sigma_* = 1$  in (2.3). Assume, that the  $[0, T] \times \mathbb{R} \rightarrow \mathbb{R}$  functions  $F_i$  are bounded and have bounded derivatives, i.e.

$$F_{max} = \max_{1 \leq i \leq 2} \sup_{(t, y)} \max \left( |F_i(t, y)|, \left| \frac{\partial F_i(t, y)}{\partial t} \right|, \left| \frac{\partial F_i(t, y)}{\partial y} \right| \right) < \infty .$$

We assume also, that the  $[0, T] \times \mathbb{R} \rightarrow \mathbb{R}$  functions  $\sigma_i$  are such, that

$$\sigma_{max} = \max_{1 \leq i \leq 2} \sup_{(t, y)} \max \left( \left| \frac{\partial \sigma_i(t, y)}{\partial t} \right|, \left| \frac{\partial \sigma_i(t, y)}{\partial y} \right| \right) < \infty$$

and

$$\sigma_{min} = \inf_{(t, y)} \min (|\sigma_1(t, y)|, |\sigma_2(t, y)|) > 0 .$$

Obviously, that in this case the matrix  $\sigma(t, y_1, y_2)$  is non-degenerated and

$$\sigma^{-1}(t, y_1, y_2) = \begin{pmatrix} \frac{\mathbf{b}_1^*}{\sigma_1(t, y_1)} & ; & -\frac{\mathbf{b}_1^*}{\sigma_2(t, y_2)} \\ -\frac{\mathbf{b}_2^*}{\sigma_1(t, y_1)} & ; & \frac{\mathbf{b}_2^*}{\sigma_2(t, y_2)} \end{pmatrix} ,$$

where  $\mathbf{b}_1^* = \mathbf{b}_2(\mathbf{b}_2 - \mathbf{b}_1)^{-1}$  and  $\mathbf{b}_2^* = (\mathbf{b}_2 - \mathbf{b}_1)^{-1}$ . Therefore, the components of the function (2.4) are

$$[\theta(t, y_1, y_2)]_i = \frac{\tilde{\mathbf{b}}_{1,i}}{\sigma_1(t, y_1)} + \frac{\tilde{\mathbf{b}}_{2,i}}{\sigma_2(t, y_2)} , \quad 1 \leq i \leq 2 ,$$

where  $\tilde{\mathbf{b}}_{1,1} = \mathbf{b}_1^*(\mu_1 - r)$ ,  $\tilde{\mathbf{b}}_{2,1} = -\mathbf{b}_1^*(\mu_2 - r)$ ,  $\tilde{\mathbf{b}}_{1,2} = -\mathbf{b}_2^*(\mu_1 - r)$  and  $\tilde{\mathbf{b}}_{2,2} = \mathbf{b}_2^*(\mu_2 - r)$ . From this we obtain, that in this case function

(2.8) has the form (2.9), i.e.

$$[\alpha(t, y_1, y_2)]_i = F_i(t, y_i) + \beta_* \left( \frac{\tilde{\mathbf{b}}_{1,i}}{\sigma_1(t, y_1)} + \frac{\tilde{\mathbf{b}}_{2,i}}{\sigma_2(t, y_2)} \right), \quad 1 \leq i \leq 2.$$

It is easy to see, that for this model the conditions  $\mathbf{A}_1)$ – $\mathbf{A}_3)$  hold with

$$\alpha_* \leq F_{max} + \tilde{\mathbf{b}}_* \frac{1 + \sigma_{max}}{\sigma_{min}} \quad \text{and} \quad \tilde{\mathbf{b}}_* = \beta_* \max_{1 \leq i, j \leq 2} (|\tilde{\mathbf{b}}_{1,i}|).$$

## 5 Properties of the mapping $\mathcal{L}$

**Proposition 5.1.** *Assume, that the conditions  $\mathbf{A}_1)$ – $\mathbf{A}_3)$  hold. Then  $\mathcal{L}_f \in \mathcal{X}$  for any  $f \in \mathcal{X}$ .*

**Proof.** Obviously, that for any  $f \in \mathcal{X}$  the mapping  $\mathcal{L}_f \geq 1$ . Moreover, setting

$$\tilde{f}_s = f(s, \eta_s^{t,y}), \quad (5.1)$$

we represent  $\mathcal{L}_f(t, y)$  as

$$\mathcal{L}_f(t, y) = \mathbf{E} \mathcal{G}(t, T, y) + \frac{1}{q_*} \int_t^T \mathbf{E} \left( \tilde{f}_s \right)^{1-q_*} \mathcal{G}(t, s, y) ds. \quad (5.2)$$

Therefore, taking into account that  $\tilde{f}_s \geq 1$  and  $q_* \geq 1$  we get

$$\mathcal{L}_f(t, y) \leq e^{Q_*(T-t)} + \int_t^T \frac{1}{q_*} e^{Q_*(s-t)} ds \leq \mathbf{r}_0, \quad (5.3)$$

where the upper bound  $\mathbf{r}_0$  is defined in (3.2). Moreover, Lemmas A.4–A.5 yield

$$\frac{\partial}{\partial y_i} \mathcal{L}_f(t, y) = \mathbf{E} \frac{\partial}{\partial y_i} \mathcal{G}(t, T, y) + \frac{1}{q_*} \int_t^T \frac{\partial}{\partial y_i} \mathcal{H}_f(t, s, y) ds.$$

Therefore, applying here (5.3), (A.16) and Lemmas A.2 we get

$$\mathcal{L}_f(t, y) + |\mathbf{D}_y \mathcal{L}_f(t, y)| \leq \mathbf{r}^*.$$

Now we have to show that the function  $\mathcal{L}_f(t, y)$  is continuously differentiable with respect to  $t$  for any  $f \in \mathcal{X}$ . Indeed, to this end we consider for any  $f$  from  $\mathcal{X}$  the equation (2.14), i.e.

$$\begin{cases} u_t(t, y) + Q(t, y)u(t, y) + (\alpha(t, y))' \mathbf{D}_y u(t, y) \\ \quad + \frac{\beta^2}{2} \text{tr} \mathbf{D}_{y, y} u(t, y) + \frac{1}{q_*} \left( \frac{1}{f(t, y)} \right)^{q_* - 1} = 0; \\ u(T, y) = 1. \end{cases} \quad (5.4)$$

Setting here  $\tilde{u}(t, y) = u(T - t, y)$  we obtain a uniformly parabolic equation for  $\tilde{u}$  with initial condition  $\tilde{u}(0, y) = 1$ . Moreover, due to the conditions  $\mathbf{A}_1$ ) and  $\mathbf{A}_3$ ) the functions  $Q$  and  $\alpha$  have bounded partial derivatives with respect to  $y$ . Therefore, for any  $f$  from  $\mathcal{X}$  Theorem 5.1 from [13] (p. 320) with  $0 < l < 1$  provides the existence of the unique solution of (5.4) belonging to  $\mathbf{C}^{1,2}(\mathcal{K})$ . Applying the Itô formula to the process

$$\left( u(s, \eta_s^{t, y}) e^{\int_t^s Q(v, \eta_v^{t, y}) dv} \right)_{t \leq s \leq T}$$

and taking into account the equation (5.4) we get

$$u(t, y) = \mathcal{L}_f(t, y). \quad (5.5)$$

Therefore, the function  $\mathcal{L}_f(t, y) \in \mathbf{C}^{1,2}(\mathcal{K})$ , i.e.  $\mathcal{L}_f \in \mathcal{X}$  for any  $f \in \mathcal{X}$ . Hence Proposition 5.1.  $\square$

**Proposition 5.2.** *Under the conditions  $\mathbf{A}_1$ )– $\mathbf{A}_3$ ) the mapping  $\mathcal{L}$  is a contraction in the metric space  $(\mathcal{X}, \varrho_*)$ , i.e. for any  $f, g$  from  $\mathcal{X}$*

$$\varrho_*(\mathcal{L}_f, \mathcal{L}_g) \leq \lambda \varrho_*(f, g), \quad (5.6)$$

where the parameter  $0 < \lambda < 1$  is given in (3.10).

**Proof.** First note that, for any  $f$  and  $g$  from  $\mathcal{X}$  and for any  $y \in \mathbb{R}^m$

$$\begin{aligned} |\mathcal{L}_f(t, y) - \mathcal{L}_g(t, y)| &\leq \frac{1}{q_*} \mathbf{E} \int_t^T \mathcal{G}(t, s, y) \left| \left( \tilde{f}_s \right)^{1-q_*} - \left( \tilde{g}_s \right)^{1-q_*} \right| ds \\ &\leq \int_t^T \mathbf{E} \mathcal{G}(t, s, y) \left| \tilde{f}_s - \tilde{g}_s \right| ds. \end{aligned}$$

We recall that  $\tilde{f}_s = f(s, \eta_s^{t,y})$  and  $\tilde{g}_s = g(s, \eta_s^{t,y})$ . Taking into account here that  $\mathcal{G}(t, s, y) \leq e^{Q_*(s-t)}$  we obtain

$$|\mathcal{L}_f(t, y) - \mathcal{L}_g(t, y)| \leq \int_t^T e^{Q_*(s-t)} \mathbf{E} |\tilde{f}_s - \tilde{g}_s| ds.$$

Taking into account in the last inequality, that

$$|\tilde{f}_s - \tilde{g}_s| \leq e^{\varkappa(T-s)} \varrho_*(f, g) \quad \text{a.s.}, \quad (5.7)$$

we get

$$\left| e^{-\varkappa(T-t)} (\mathcal{L}_f(t, y) - \mathcal{L}_g(t, y)) \right| \leq \frac{1}{\varkappa - Q_*} \varrho_*(f, g). \quad (5.8)$$

Moreover, by virtue of Lemma A.4 we obtain, that for any  $1 \leq i \leq m$

$$\begin{aligned} \frac{\partial}{\partial y_i} \mathcal{L}_f(t, y) &= \mathbf{E} \frac{\partial}{\partial y_i} \mathcal{G}(t, T, y) \\ &+ \frac{1 - q_*}{q_*} \int_t^T \mathbf{E} (\tilde{f}_s)^{-q_*} (\tilde{f}_0(s))' v_i(s) \mathcal{G}(t, s, y) ds \\ &+ \frac{1}{q_*} \int_t^T \mathbf{E} (\tilde{f}_s)^{1-q_*} \frac{\partial}{\partial y_i} \mathcal{G}(t, s, y) ds, \end{aligned} \quad (5.9)$$

where  $\tilde{f}_0(s) = f_0(s, \eta_s^{t,y})$ ,  $f_0(s, z) = \mathbf{D}_z f(s, z)$  and  $v_i(s) = \partial \eta_s^{t,y} / \partial y_i$ . Therefore, for any  $f$  and  $g$  from  $\mathcal{X}$

$$\begin{aligned} \frac{\partial \mathcal{L}_f(t, y)}{\partial y_i} - \frac{\partial \mathcal{L}_g(t, y)}{\partial y_i} &= \frac{1 - q_*}{q_*} \int_t^T \mathbf{E} (\varpi_1(s))' v_i(s) \mathcal{G}(t, s, y) ds \\ &+ \frac{1}{q_*} \int_t^T \mathbf{E} \varpi_2(s) \frac{\partial}{\partial y_i} \mathcal{G}(t, s, y) ds, \end{aligned}$$

where

$$\varpi_1(s) = \frac{\tilde{f}_0(s)}{(\tilde{f}_s)^{q_*}} - \frac{\tilde{g}_0(s)}{(\tilde{g}_s)^{q_*}} \quad \text{and} \quad \varpi_2(s) = (\tilde{f}_s)^{1-q_*} - (\tilde{g}_s)^{1-q_*}.$$

Note, that similarly to (5.7)

$$|\tilde{f}_0(s) - \tilde{g}_0(s)| \leq e^{\varkappa(T-s)} \varrho_*(f, g) \quad \text{a.s.}$$

Using this, one can check directly that

$$|\varpi_1(s)| \leq (1 + \mathbf{r}^* q_*) e^{\varkappa(T-s)} \varrho_*(f, g)$$

and

$$|\varpi_2(s)| \leq (q_* - 1)e^{\varkappa(T-s)} \varrho_*(f, g).$$

Therefore, these upper bounds and Lemma A.3 imply

$$\left| \frac{\partial \mathcal{L}_f(t, y)}{\partial y_i} - \frac{\partial \mathcal{L}_g(t, y)}{\partial y_i} \right| \leq e^{\varkappa(T-t)} \frac{\mathbf{L}^*}{\varkappa - Q_*} \varrho_*(f, g),$$

where  $\mathbf{L}^*$  is given in (3.3). Therefore,

$$\sup_{(t, y) \in \mathcal{K}} e^{-\varkappa(T-t)} |\mathbf{D}_y \mathcal{L}_f(t, y) - \mathbf{D}_y \mathcal{L}_g(t, y)| \leq \frac{m\mathbf{L}^*}{\varkappa - Q_*} \varrho_*(f, g).$$

Taking into account the definition of  $\varkappa$  in (3.3), we obtain the inequality (5.6). Hence Proposition 5.2.  $\square$

**Proposition 5.3.** *Under the conditions  $\mathbf{A}_1)$ – $\mathbf{A}_3)$  the equation  $\mathcal{L}_h = h$  has a unique solution in  $\mathcal{X}$ .*

**Proof.** Indeed, using the contraction of the operator  $\mathcal{L}$  in  $\mathcal{X}$  and the definition of the sequence  $(h_n)_{n \geq 1}$  in (3.9) we get, that for any  $n \geq 1$

$$\varrho_*(h_n, h_{n-1}) \leq \lambda^{n-1} \varrho_*(h_1, h_0), \quad (5.10)$$

i.e. the sequence (3.9) is fundamental in  $(\mathcal{X}, \varrho_*)$ . Therefore, due to Proposition A.1 this sequence has a limit in  $\mathcal{X}$ , i.e. there exists a function  $h$  from  $\mathcal{X}$  for which

$$\lim_{n \rightarrow \infty} \varrho_*(h, h_n) = 0.$$

Moreover, taking into account that  $h_n = \mathcal{L}_{h_{n-1}}$  we obtain, that for any  $n \geq 1$

$$\varrho_*(h, \mathcal{L}_h) \leq \varrho_*(h, h_n) + \varrho_*(\mathcal{L}_{h_{n-1}}, \mathcal{L}_h) \leq \varrho_*(h, h_n) + \lambda \varrho_*(h, h_{n-1}).$$

The last expression tends to zero as  $n \rightarrow \infty$ . Therefore  $\varrho_*(h, \mathcal{L}_h) = 0$ , i.e.  $h = \mathcal{L}_h$ . Proposition 5.2 implies immediately that this solution is unique.  $\square$

**Proposition 5.4.** *Under the conditions  $\mathbf{A}_1)$ – $\mathbf{A}_3)$  the equation (2.14) has a unique solution which is the solution  $h$  of the fixed-point equation  $\mathcal{L}_h = h$ .*

**Proof.** Choosing in (5.4) the function  $f = h$  and taking into account the representation (5.5) and the equation (3.8) we obtain, that the solution of the equation (5.4)

$$u = \mathcal{L}_h = h.$$

Therefore, the function  $h$  satisfies the equation (2.14). Moreover, this solution is unique since  $h$  is the unique solution of the equation (3.8).

□

**Remark 5.1.** *The representation (5.5) for the solution of the equation (5.4) is called the probabilistic representation or the Feynman - Kac formula (see, e. g., [7], p. 194).*

## 6 Verification theorem

In this section we state the verification theorem from [10]. Consider on the interval  $[0, T]$  the stochastic control process given by the  $N$  - dimensional Itô process

$$\begin{cases} d\varsigma_t^\vartheta = \mathbf{a}(t, \varsigma_t^\vartheta, \vartheta_t) dt + \mathbf{b}(t, \varsigma_t^\vartheta, \vartheta_t) dW_t, & t \geq 0, \\ \varsigma_0^\vartheta = x \in \mathbb{R}^N, \end{cases} \quad (6.1)$$

where  $(W)_{0 \leq t \leq T}$  is a standard  $k$  - dimensional Brownian motion. We assume that the control process  $\vartheta$  takes values in some set  $\Theta$ . Moreover, we assume that the coefficients  $\mathbf{a}$  and  $\mathbf{b}$  satisfy the following conditions

- for all  $t \in [0, T]$  the functions  $\mathbf{a}(t, \cdot, \cdot)$  and  $\mathbf{b}(t, \cdot, \cdot)$  are continuous on  $\mathbb{R}^N \times \Theta$ ;
- for every deterministic vector  $v \in \Theta$  the stochastic differential equation

$$d\varsigma_t^\vartheta = \mathbf{a}(t, \varsigma_t^\vartheta, \vartheta) dt + \mathbf{b}(t, \varsigma_t^\vartheta, \vartheta) dW_t$$

has a unique strong solution.

Now we introduce admissible control processes for the equation (6.1). We set  $\mathcal{F}_t = \sigma\{W_u, 0 \leq u \leq t\}$  for any  $0 < t \leq T$ .

**Definition 6.1.** *A stochastic control process  $\vartheta = (\vartheta_t)_{0 \leq t \leq T}$  is called admissible on  $[0, T]$  with respect to equation (6.1) if it is  $(\mathcal{F}_t)_{0 \leq t \leq T}$  -*

progressively measurable with values in  $\Theta$ , and equation (6.1) has a unique strong a.s. continuous solution  $(\varsigma_t^\vartheta)_{0 \leq t \leq T}$  such that

$$\int_0^T \left( |\mathbf{a}(t, \varsigma_t^\vartheta, \vartheta_t)| + |\mathbf{b}(t, \varsigma_t^\vartheta, \vartheta_t)|^2 \right) dt < \infty \quad a.s.. \quad (6.2)$$

We denote by  $\mathcal{V}$  the set of all admissible control processes with respect to the equation (6.1).

Moreover, let  $\mathbf{f} : [0, T] \times \mathbb{R}^m \times \Theta \rightarrow [0, \infty)$  and  $\mathbf{h} : \mathbb{R}^m \rightarrow [0, \infty)$  be continuous utility functions. We define the cost function by

$$J(t, x, \vartheta) = \mathbf{E}_{t,x} \left( \int_t^T \mathbf{f}(s, \varsigma_s^\vartheta, \vartheta_s) ds + \mathbf{h}(\varsigma_T^\vartheta) \right), \quad 0 \leq t \leq T,$$

where  $\mathbf{E}_{t,x}$  is the expectation operator conditional on  $\varsigma_t^\vartheta = x$ . Our goal is to solve the optimization problem

$$J^*(t, x) := \sup_{\vartheta \in \mathcal{V}} J(t, x, \vartheta). \quad (6.3)$$

To this end we introduce the Hamilton function, i.e. for any  $0 \leq t \leq T$ ,  $\varsigma, \mathbf{q} \in \mathbb{R}^N$  and symmetric  $N \times N$  matrix  $\mathbf{M}$  we set

$$H(t, \varsigma, \mathbf{q}, \mathbf{M}) := \sup_{\vartheta \in \Theta} H_0(t, \varsigma, \mathbf{q}, \mathbf{M}, \vartheta), \quad (6.4)$$

where

$$H_0(t, \varsigma, \mathbf{q}, \mathbf{M}, \vartheta) := \mathbf{a}'(t, \varsigma, \vartheta) \mathbf{q} + \frac{1}{2} \text{tr}(\mathbf{b} \mathbf{b}'(t, \varsigma, \vartheta) \mathbf{M}) + \mathbf{f}(t, \varsigma, \vartheta).$$

In order to find the solution to (6.3) we investigate the *Hamilton-Jacobi-Bellman* equation

$$\begin{cases} z_t(t, \varsigma) + H(t, \varsigma, \mathbf{D}_\varsigma z(t, \varsigma), \mathbf{D}_{\varsigma, \varsigma} z(t, \varsigma)) = 0, & t \in [0, T], \\ z(T, \varsigma) = \mathbf{h}(\varsigma), & \varsigma \in \mathbb{R}^N. \end{cases} \quad (6.5)$$

Here  $z_t$  denotes the partial derivative of  $z$  with respect to  $t$ ,  $D_\varsigma z(t, \varsigma)$  the gradient vector with respect to  $\varsigma$  in  $\mathbb{R}^N$  and  $D_{\varsigma, \varsigma} z(t, \varsigma)$  denotes the symmetric hessian matrix, that is the matrix of the second order partial derivatives with respect to  $\varsigma$ .

We assume that the following conditions hold:

**H<sub>1</sub>)** *The functions  $\mathbf{f}$  and  $\mathbf{h}$  are non negative.*

**H<sub>2</sub>)** There exists  $[0, T] \times \mathbb{R}^N \rightarrow (0, \infty)$  function  $z(t, \varsigma)$  from  $\mathbf{C}^{1,2}([0, T] \times \mathbb{R}^N)$  which satisfies the HJB equation (6.5).

**H<sub>3</sub>)** There exists a measurable function  $\vartheta^* : [0, T] \times \mathbb{R}^N \rightarrow \Theta$  such that for all  $0 \leq t \leq T$  and  $\varsigma \in \mathbb{R}^N$

$$H(t, \varsigma, \mathbf{D}_\varsigma z(t, \varsigma), \mathbf{D}_{\varsigma, \varsigma} z(t, \varsigma)) = H_0(t, \varsigma, \mathbf{D}_\varsigma z(t, \varsigma), \mathbf{D}_{\varsigma, \varsigma} z(t, \varsigma), \vartheta^*(t, \varsigma)).$$

**H<sub>4</sub>)** There exists a unique strong solution to the Itô equation

$$d\varsigma_t^* = \mathbf{a}^*(t, \varsigma_t^*) dt + \mathbf{b}^*(t, \varsigma_t^*) dW_t, \quad t \geq 0, \quad \varsigma_0^* = x, \quad (6.6)$$

where  $\mathbf{a}^*(t, \cdot) = \mathbf{a}(t, \cdot, \vartheta^*(t, \cdot))$  and  $\mathbf{b}^*(t, \cdot) = \mathbf{b}(t, \cdot, \vartheta^*(t, \cdot))$ . Moreover, the optimal control process  $\vartheta_t^* = \vartheta^*(t, \varsigma_t^*)$  for  $0 \leq t \leq T$  belongs to  $\mathcal{V}$ .

**H<sub>5</sub>)** There exists some  $\delta > 1$  such that for all  $0 \leq t \leq T$  and  $\varsigma \in \mathbb{R}^N$

$$\sup_{\tau \in \mathcal{T}_t} \mathbf{E}_{t, \varsigma} (z(\tau, \varsigma_\tau^*))^\delta < \infty$$

where  $\mathcal{T}_t$  is the set of all stopping times in  $[t, T]$ .

**Theorem 6.2.** Assume that  $\mathcal{V} \neq \emptyset$  and **H<sub>1</sub>)** – **H<sub>5</sub>)** hold. Then for all  $t \in [0, T]$  and for all  $x \in \mathbb{R}^N$  the solution of the Hamilton-Jacobi-Bellman equation (6.5) coincides with the optimal value of the cost function, i.e.

$$z(t, x) = J^*(t, x) = J^*(t, x, \vartheta^*),$$

where the optimal strategy  $\vartheta^*$  is defined in **H<sub>3</sub>)** and **H<sub>4</sub>)**.

## 7 Proofs

### 7.1 Proof of Theorem 3.1

Proposition 5.3 implies the first part of this theorem. Moreover, from (5.10) it is easy to see, that for each  $n \geq 1$

$$\varrho_*(h, h_n) \leq \frac{\lambda^n}{1 - \lambda} \varrho_*(h_1, h_0).$$

Thanks to Proposition 5.1 all the functions  $h_n$  belong to  $\mathcal{X}$ , i.e. by the definition of the space  $\mathcal{X}$

$$\varrho_*(h_1, h_0) \leq \sup_{(t, y) \in \mathcal{X}} (|h_1(t, y) - 1| + |\mathbf{D}_y h_1(t, y)|) \leq 1 + \mathbf{r}^*.$$

Taking into account that

$$\sup_{(t,y) \in \mathcal{K}} (|\Delta_n(t,y)| + |\mathbf{D}_y \Delta_n(t,y)|) \leq e^{\varkappa T} \varrho_*(h, h_n),$$

we obtain the inequality (3.11). Hence Theorem 3.1.  $\square$

## 7.2 Proof of Theorem 3.3

We apply the Verification Theorem 6.2 to Problem 2.7 for the stochastic control differential equation (2.5). For fixed  $\vartheta = (\pi, c)$ , where  $\pi \in \mathbb{R}^d$  and  $c \in [0, \infty)$ , the coefficients in model (6.1) are defined as

$$\begin{aligned} \mathbf{a}(t, \varsigma, \vartheta) &= (x(r(t,y) + \pi' \theta(t,y) - c), F(t,y))' \\ \mathbf{b}(t, \varsigma, \vartheta) &= \begin{pmatrix} x \pi'; & \mathbf{0}'_m \\ \beta \sqrt{1 - \rho^2} \sigma_*; & \beta \rho I_m \end{pmatrix}, \end{aligned}$$

where  $\varsigma = (x, y)' \in \mathbb{R}^N$ ,  $N = m + 1$ ,  $k = d + m$ ,  $\mathbf{0}_m = (0, \dots, 0)' \in \mathbb{R}^m$ ,  $I_m$  is the identity matrix of the order  $m$ . Note that

$$\mathbf{b}\mathbf{b}'(t, \varsigma, \vartheta) = \begin{pmatrix} x^2 |\pi|^2; & x \beta \sqrt{1 - \rho^2} \pi' \sigma_*' \\ x \beta \sqrt{1 - \rho^2} \sigma_*' \pi; & \beta^2 I_m \end{pmatrix}.$$

Therefore, according to the definition of  $H_0$  in (6.4), for any  $\mathbf{q}$  and  $\mathbf{M}$  of the form (2.10)

$$\begin{aligned} H_0(t, \varsigma, \mathbf{q}, \mathbf{M}, \vartheta) &= x r(t,y) \mathbf{q}_1 + (F(t,y))' \tilde{\mathbf{q}} + (x^\gamma c^\gamma - x c \mathbf{q}_1) \\ &\quad + \frac{1}{2} x^2 \mu |\pi|^2 + x \pi' \left( \theta(t,y) \mathbf{q}_1 + \beta \sqrt{1 - \rho^2} \sigma_*' \tilde{\mu} \right) + \frac{\beta^2}{2} \text{tr} \mathbf{M}_0. \end{aligned}$$

To check the conditions  $\mathbf{H}_2) - \mathbf{H}_4)$  we need to calculate the Hamilton function (6.4) for Problem 2.7 which is defined as

$$H(t, \varsigma, \mathbf{q}, \mathbf{M}) = \sup_{\vartheta \in \mathbb{R}^d \times [0, \infty)} H_0(t, \varsigma, \mathbf{q}, \mathbf{M}, \vartheta) = H_0(t, \varsigma, \mathbf{q}, \mathbf{M}, \vartheta_0).$$

One can check directly, that for  $\mu < 0$  and  $\mathbf{q}_1 > 0$  we obtain the form (2.11), where the optimal function

$\vartheta_0 = \vartheta_0(t, \varsigma, \mathbf{q}, \mathbf{M}) = (\pi^*(t, \varsigma, \mathbf{q}, \mathbf{M}), c^*(t, \varsigma, \mathbf{q}, \mathbf{M}))$  and

$$\begin{cases} \pi^* = \pi^*(t, \varsigma, \mathbf{q}, \mathbf{M}) &= \frac{\theta(t,y) \mathbf{q}_1 + \beta \sqrt{1 - \rho^2} \sigma_*' \tilde{\mu}}{x |\mu|}; \\ c^* = c^*(t, \varsigma, \mathbf{q}, \mathbf{M}) &= \frac{1}{x} \left( \frac{\gamma}{\mathbf{q}_1} \right)^{\gamma_1}. \end{cases} \quad (7.1)$$

Proposition 5.4 implies that the HJB equation (2.12) has a solution from  $\mathbf{C}^{1,2}(\mathcal{K})$  defined in (2.13). It should be noted that for this function  $\mathbf{q}_1 = z_x(t, x, y) = \gamma x^{\gamma-1} h(t, y) > 0$  and  $\mu = z_{xx}(t, x, y) = \gamma(\gamma-1)x^{\gamma-2} h(t, y) < 0$ . Therefore, in this case the condition  $\mathbf{H}_3$ ) holds with  $\vartheta^*(t, \varsigma) = (\pi^*(t, \varsigma), c^*(t, \varsigma))$  and

$$\begin{cases} \pi^*(t, \varsigma) &= \frac{\theta(t, y)}{1-\gamma} + \varepsilon \sqrt{1-\rho^2} \beta \frac{\sigma_* \mathbf{D}_y h(t, y)}{(1-\gamma)h(t, y)}; \\ c^*(t, \varsigma) &= (h(t, y))^{-q_*}, \end{cases} \quad (7.2)$$

where the function  $h$  is solution of the equation (2.14).

Now we check the conditions  $\mathbf{H}_4$ ) and  $\mathbf{H}_5$ ). Note that the equation (6.6) is identical to the equations (2.2) and (2.5). Due to the condition  $\mathbf{A}_1$ ) the coefficients  $a^*(t, y)$  and  $b^*(t, y)$  are bounded. Therefore, the equation (2.2) has a unique strong solution which for  $0 \leq t \leq s \leq T$  can be represented as

$$X_s^* = X_t^* e^{\int_t^s a^*(v, Y_v) dv} \mathcal{E}_{t,s},$$

where

$$\mathcal{E}_{t,s} = \exp \left\{ \int_t^s (b^*(v, Y_v))' dW_v - \frac{1}{2} \int_t^s |b^*(v, Y_v)|^2 dv \right\}.$$

It is clear, that for the bounded function  $b^*(v, y)$  the process  $(\mathcal{E}_{t,s})_{t \leq s \leq T}$  is a quadratic intergrable martingale and by the Doob inequality

$$\mathbf{E} \sup_{t \leq s \leq T} \mathcal{E}_{t,s}^2 \leq 4 \mathbf{E} \mathcal{E}_{t,T}^2 < \infty.$$

This implies directly, that for any  $0 \leq t \leq T$ ,  $x > 0$  and  $y \in \mathbb{R}^m$

$$\mathbf{E} \left( \sup_{t \leq s \leq T} X_s^* | X_t^* = x, Y_t = y \right) < \infty.$$

Now we recall that

$$z(t, \varsigma) = x^\gamma (h(t, y))^\varepsilon$$

where  $h(t, y)$  is some positive function bounded by  $\mathbf{r}^*$ . Thus

$$\mathbf{E}_{t,\varsigma} z^m(\tau, X_\tau^*, Y_\tau) \leq (\mathbf{r}^*)^{m\varepsilon} \mathbf{E}_{t,\varsigma} (X_\tau^*)^m.$$

Therefore, taking  $m = 1/\gamma > 1$  one gets

$$\mathbf{E}_{t,\varsigma} z^m(\tau, X_\tau^*, Y_\tau) \leq (\mathbf{r}^*)^{\varepsilon/\gamma} \mathbf{E}_{t,\varsigma} \sup_{t \leq s \leq T} X_s^* < \infty$$

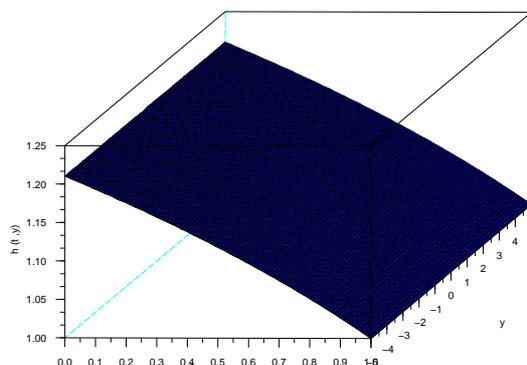
which implies the conditions  $\mathbf{H}_4$ ) and  $\mathbf{H}_5$ ). Therefore, thanks to Theorem 6.2 we get Theorem 3.3.  $\square$

## 8 Numerical example

In this section through Scilab we calculate the function  $h(t, y)$  using the sequence (3.9) with  $n = 14$  iterations. The curve is obtained in the following stochastic volatility market settings: the market consists on one riskless asset (the bond) and a risky one (that means  $d = 1$ ). Moreover, we set  $m = 1$ ,  $T = 1$ ,

$$r(t, y) = 0.01(1 + 0.5 \sin(yt)), \quad \mu(t, y) = 0.02(1 + 0.5 \sin(yt))$$

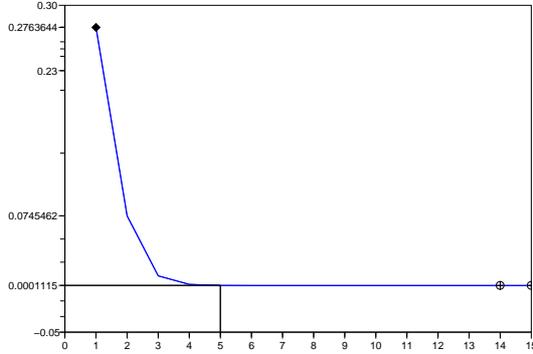
and  $\sigma(t, y) = 0.5 + \sin^2(yt)$ . The parameters of the economic factor are  $F(t, y) = 0.1 \sin(yt)$ ,  $\beta = 1$  and  $\rho = 0.5$ . The utility function parameter is  $\gamma = 0,75$ .



**Figure 1: The function  $h(t,y)$**

The second figure illustrates the super geometrical convergence rate for the functions  $(h_n)_{n \geq 0}$ . The curve represents the accuracy  $\delta_n$  calculated at each step by

$$\delta_n = \sup_{(t,y)} |h_n(t, y) - h_{n-1}(t, y)|.$$



**Figure 2: The accuracy  $\delta_n$  at each iteration**

We observe the values of the accuracy:  $\delta_5 \simeq 0.0001$ ,  $\delta_8 \simeq 10^{-08}$  and  $\delta_{14} \simeq 10^{-16}$ . Moreover  $\delta_n = 0 \forall n \geq 15$ , i.e. numerically the limit function  $h$  is reached at the 14<sup>th</sup> iteration.

## 9 Appendix

### A.1 Properties of the space $(\mathcal{X}, \varrho_*)$

**Proposition A.1.**  $(\mathcal{X}, \varrho_*)$  is a complete metrical space.

**Proof.** Indeed, let  $(f_n)_{n \geq 1}$  be a fundamental sequence from  $(\mathcal{X}, \varrho_*)$ , i.e.

$$\lim_{m, n \rightarrow \infty} \varrho_*(f_n, f_m) = 0.$$

Taking into account that the norm  $\|\cdot\|_*$  defined in (3.4) is equivalent to the norm (3.1), we obtain that this sequence is fundamental in Banach space  $\mathbf{C}^{0,1}(\mathcal{K})$ . Therefore, there exists a function  $f \in \mathbf{C}^{0,1}(\mathcal{K})$  such that

$$\lim_{n \rightarrow \infty} \varrho_*(f_n, f) = 0.$$

The definition of the metrics  $\varrho_*$  in (3.4) implies immediately that  $f \in \mathcal{X}$ . Hence Proposition A.1.  $\square$

## A.2 Properties of the function $\mathcal{H}_f$

In this subsection we study the smoothness of  $\mathcal{H}_f$  with respect to  $y$ .

**Lemma A.2.** *Assume that the condition  $\mathbf{A}_3$ ) holds. Then, for any  $0 < t \leq s \leq T$*

$$\max_{1 \leq i \leq m} \sup_{y \in \mathbb{R}^m} \sup_{f \in \mathcal{X}} \left| \frac{\partial}{\partial y_i} \mathcal{H}_f(t, s, y) \right| \leq H^* \left( \frac{1}{\sqrt{s-t}} + 1 \right), \quad (\text{A.1})$$

where  $H^*$  is given in (3.16).

**Proof.** First, for any  $y \in \mathbb{R}^m$  we introduce the auxiliary Brownian motion on the interval  $[t, s]$  as

$$\xi_v = y + \beta(U_v - U_t).$$

By making use of this process and the Girsanov theorem (see [14] p. 254) we can represent the mapping  $\mathcal{H}_f$  as

$$\mathcal{H}_f(t, s, y) = \mathbf{E} f_1(s, \xi_s) e^{\Phi(\xi)},$$

where  $f_1 = f^{1-q_*}$ ,

$$\Phi(\xi) = \int_t^s Q_1(v, \xi_v) dv + \beta^{-2} \sum_{l=1}^m \int_t^s [\alpha(v, \xi_v)]_l d[\xi_v]_l \quad (\text{A.2})$$

and

$$Q_1(v, y) = Q(v, y) - \frac{|\alpha(v, y)|^2}{2\beta^2}.$$

Now we fix some  $1 \leq i \leq m$  and we set

$$\varsigma_v = [\xi_v]_i = y_i + \beta ([\mathbf{U}_v]_i - [\mathbf{U}_t]_i). \quad (\text{A.3})$$

We recall, that  $[x]_i$  is the  $i$ th component of the vector  $x \in \mathbb{R}^m$ . Taking the conditional expectation with respect to  $\varsigma_s$  we represent  $\mathcal{H}_f$  as

$$\mathcal{H}_f(t, s, y) = \int_{\mathbb{R}} \widehat{\mathcal{H}}_f(s, y, z) \mathbf{p}(z, y_i) dz, \quad (\text{A.4})$$

where  $\widehat{\mathcal{H}}_f(s, y, z) = \mathbf{E} (f_1(s, \xi_s) e^{\Phi(\xi)} | \varsigma_s = z)$ ,

$$\mathbf{p}(z, y_i) = \frac{1}{\nu\sqrt{2\pi}} e^{-\frac{(z-y_i)^2}{2\nu^2}} \quad \text{and} \quad \nu^2 = \beta^2(t-s).$$

To calculate this conditional expectation note, that one can check directly that for any  $t < v_1 < \dots < v_k < s$  and for any bounded  $\mathbb{R}^k \rightarrow \mathbb{R}$  function  $G$

$$\mathbf{E} \left( G(\varsigma_{v_1}, \dots, \varsigma_{v_k}) | \varsigma_s = z \right) = \mathbf{E} G(B_{v_1}, \dots, B_{v_k}), \quad (\text{A.5})$$

where  $B_v$  is the well-known Brownian Bridge on the interval  $[t, s]$  (see, for example, [8], p. 359), i.e.

$$B_v = y_i + \frac{v-t}{s-t} (z - y_i) + \varsigma_v^0 - \frac{v-t}{s-t} \varsigma_s^0 \quad (\text{A.6})$$

and  $\varsigma_v^0 = \beta ([\mathbf{U}_v]_i - [\mathbf{U}_t]_i)$ . Moreover, using the representation (2.9) we can represent the stochastic integral with respect to the  $i$ th component in (A.2) as

$$\int_t^s [\alpha(v, \xi_v)]_i d[\xi_v]_i = \int_t^s \bar{\mathbf{a}}_i(v, \xi_v) d\varsigma_v + \int_t^s \mathbf{a}_{i,i}(v, \varsigma_v) d\varsigma_v, \quad (\text{A.7})$$

where

$$\bar{\mathbf{a}}_i(v, x) = \sum_{l=1, l \neq i}^m \mathbf{a}_{i,l}(v, x_l) \quad \text{for } x = (x_1, \dots, x_m)'$$

By putting

$$A_i(v, z, y_i) = \int_{y_i}^z \mathbf{a}_{i,i}(v, u) du$$

and using the Ito formula, we can rewrite the last stochastic integral in (A.7) as

$$\int_t^s \mathbf{a}_{i,i}(v, \varsigma_v) d\varsigma_v = A_i(s, \varsigma_s, y_i) - \int_t^s \tilde{A}_i(v, \varsigma_v, y_i) dv,$$

where

$$\tilde{A}_i(v, z, y_i) = \int_{y_i}^z \frac{\partial}{\partial v} \mathbf{a}_{i,i}(v, u) du + \frac{\beta^2}{2} \frac{\partial}{\partial z} \mathbf{a}_{i,i}(v, z).$$

Now for any vector  $x = (x_1, \dots, x_m)'$  we set

$$\tilde{x}^i = (x_1, \dots, x_{i-1}, 0, x_{i+1}, \dots, x_m)'. \quad (\text{A.8})$$

Using these notations one can represent the function (A.2) as

$$\Phi(\xi) = A_i(s, \varsigma_s, y_i) + \Phi_{1,i}(\xi, y_i),$$

where

$$\begin{aligned}\Phi_{1,i}(\xi, y_i) &= \beta^{-2} \int_t^s (\check{\alpha}^i(v, \xi_v))' d\xi_v + \beta^{-2} \int_t^s \bar{\mathbf{a}}_i(v, \xi_v) d\zeta_v \\ &\quad + \int_t^s Q_{2,i}(v, \xi_v, y_i) dv\end{aligned}$$

and  $Q_{2,i}(v, x, y_i) = Q_1(v, x) - \tilde{A}_i(v, x_i, y_i)$ . Therefore, by the property (A.5)

$$\widehat{\mathcal{H}}_f(s, y, z) = e^{A_i(s, z, y_i)} \mathbf{E} \left( f_1(s, \widehat{\xi}_s^i) e^{\Phi_{1,i}(\widehat{\xi}^i, y_i)} \right), \quad (\text{A.9})$$

where  $\widehat{\xi}_v^i = ([\xi_v]_1, \dots, [\xi_v]_{i-1}, B_v, [\xi_v]_{i+1}, \dots, [\xi_v]_m)'$ . Taking into account that  $\bar{\mathbf{a}}_i(v, \widehat{\xi}_v^i) = \bar{\mathbf{a}}_i(v, \xi_v)$  and

$$\int_t^s (\check{\alpha}^i(v, \widehat{\xi}_v^i))' d\widehat{\xi}_v^i = \int_t^s (\check{\alpha}^i(v, \xi_v))' d\xi_v$$

we get

$$\begin{aligned}\Phi_{1,i}(\widehat{\xi}^i, y_i) &= \beta^{-2} \int_t^s (\check{\alpha}^i(v, \widehat{\xi}_v^i))' d\xi_v + \beta^{-2} \int_t^s \bar{\mathbf{a}}_i(v, \xi_v) dB_v \\ &\quad + \int_t^s Q_{2,i}(v, \widehat{\xi}_v^i, y_i) dv.\end{aligned}$$

Through the definition of  $B_v$  in (A.6) we rewrite this equality as

$$\begin{aligned}\Phi_{1,i}(\widehat{\xi}^i, y_i) &= \frac{(z - y_i - \zeta_s^0)}{\beta^2(s-t)} \int_t^s \bar{\mathbf{a}}_i(v, \xi_v) dv + \int_t^s Q_{2,i}(v, \widehat{\xi}_v^i, y_i) dv \\ &\quad + \int_t^s (\mathbf{g}^i(v, \widehat{\xi}_v^i))' dU_v,\end{aligned} \quad (\text{A.10})$$

where the function  $\mathbf{g}^i(v, x) \in \mathbb{R}^m$  is defined as

$$[\mathbf{g}^i(v, x)]_j = \beta^{-1} \begin{cases} [\alpha(v, x)]_j & \text{for } j \neq i; \\ \bar{\mathbf{a}}_i(v, x) & \text{for } j = i. \end{cases}$$

Due to the condition  $\mathbf{A}_3$ ) this function is bounded, i.e.

$$|\mathbf{g}^i(v, x)| \leq 2\beta^{-1}\alpha_*. \quad (\text{A.11})$$

Let us show now, that for any  $q > 0$

$$\mathbf{E} e^{q\Phi_{1,i}(\widehat{\xi}^i, y_i)} \leq \phi_q e^{\iota_q |z - y_i|}, \quad (\text{A.12})$$

where  $\phi_q$  and  $\iota_q$  are given in (3.15). First note, that by the condition  $\mathbf{A}_3$ )

$$|\widetilde{A}_i(v, z, y_i)| \leq \alpha_* |z - y_i| + \alpha_* \beta^2/2$$

and, therefore,

$$Q_{2,i}(v, x) = Q_1(v, x) - \widetilde{A}_i(v, z, y_i) \leq Q_* + \alpha_* |z - y_i| + \alpha_* \beta^2/2.$$

Putting  $\zeta_v^i = \mathbf{g}^i(v, \widehat{\xi}_v^i)$  and using the coefficients defined in (3.15), we obtain, that for any  $0 < t < s < T$

$$\Phi_{1,i}(\widehat{\xi}^i, y_i) \leq \int_t^s (\zeta_v^i)' dU_v + \widetilde{\alpha}_1 |\zeta_s^0| + \widetilde{\alpha}_2 |z - y_i| + \widetilde{\alpha}_3. \quad (\text{A.13})$$

Obviously, that for each  $1 \leq j \leq m$  the process  $([\zeta_v^i]_j)_{t \leq v \leq s}$  is adapted to the filtration  $(\mathcal{B}_v^j)_{t \leq v \leq s}$ , where

$$\mathcal{B}_v^j = \sigma \{ ([U_u]_j)_{t \leq u \leq v}, (\check{U}_u^j)_{t \leq u \leq s} \}.$$

Therefore, for any  $q > 0$  the process

$$\varrho_v^j = e^{q \int_t^v [\zeta_u^i]_j d[U_v]_j - \frac{q^2}{2} \int_t^v [\zeta_u^i]_j^2 du}$$

is martingale, i.e.  $\mathbf{E} \varrho_s^j = 1$ . This and (A.11) yield

$$\mathbf{E} e^{q \int_t^s [\zeta_v^i]_j d[U_v]_j} = \mathbf{E} \varrho_s^j e^{\frac{q^2}{2} \int_t^s [\zeta_v^i]_j^2 dv} \leq e^{q^2 \beta^{-2} T \alpha_*^2}.$$

Now we use the following multidimensional version of the Hölder inequality. For any integrated variables  $(\eta_j)_{1 \leq j \leq m}$

$$\mathbf{E} \prod_{j=1}^m |\eta_j| \leq \prod_{j=1}^m (\mathbf{E} |\eta_j|^{r_j})^{1/r_j},$$

where  $(r_j)_{1 \leq j \leq m}$  are positive numbers such, that  $\sum_{j=1}^m r_j^{-1} = 1$ . This inequality implies

$$\begin{aligned} \mathbf{E} e^{q \int_t^s \mathbf{g}^i(v, \widehat{\xi}_v^i) dU_v} &= \mathbf{E} \prod_{j=1}^m e^{q \int_t^s [\zeta_v^i]_j d[U_v]_j} \\ &\leq \prod_{j=1}^m \left( \mathbf{E} e^{q^m \int_t^s [\zeta_v^i]_j d[U_v]_j} \right)^{1/m} \leq e^{q^2 \beta^{-2} m^2 T \alpha_*^2}. \end{aligned}$$

Now from (A.13) for any  $q > 0$  we can estimate the expectation in (A.12) in the following way

$$\begin{aligned} \mathbf{E}e^{q\Phi_{1,i}(\widehat{\xi}^i)} &\leq e^{q\tilde{\alpha}_2|z-y_i|+q\tilde{\alpha}_3} \mathbf{E}e^{q\int_t^s (\zeta_v^i)' dU_v + q\tilde{\alpha}_1|\zeta_s^0|} \\ &\leq e^{q\tilde{\alpha}_2|z-y_i|+q\tilde{\alpha}_3} \left( \mathbf{E}e^{2q\int_t^s (\zeta_v^i)' dU_v} \right)^{1/2} \left( \mathbf{E}e^{2q\tilde{\alpha}_1|\zeta_s^0|} \right)^{1/2}, \end{aligned}$$

where  $\tilde{\alpha}_1 = \alpha_*\beta^{-2}$ ,  $\tilde{\alpha}_2 = \alpha_*(\beta^{-2} + T)$  and  $\tilde{\alpha}_3 = T(Q_* + \beta^2\alpha_*/2)$ . Taking into account here, that for any  $a \in \mathbb{R}$

$$\mathbf{E}e^{a|\zeta_s^0|} \leq 2e^{a^2\beta^2(s-t)/2},$$

we come to the upper bound (A.12). Obviously, that in (A.9) the function  $|A_i(s, z)| \leq \alpha_*|z - y_i|$ . Therefore,

$$\sup_{f \in \mathcal{X}} \widehat{\mathcal{H}}_f(s, y, z) \leq \phi_1 e^{(\iota_1 + \alpha_*)|z - y_i|}. \quad (\text{A.14})$$

Moreover, from (A.9) we get

$$\begin{aligned} \frac{\partial}{\partial y_i} \widehat{\mathcal{H}}_f(s, y, z) &= -\mathbf{a}_{i,i}(s, y_i) \widehat{\mathcal{H}}_f(s, y, z) \\ &\quad + e^{A_i(s, z, y_i)} \widehat{\mathcal{H}}_{f,i}^1(s, y, z), \end{aligned} \quad (\text{A.15})$$

where

$$\widehat{\mathcal{H}}_{f,i}^1(s, y, z) = \mathbf{E} \left( f_1(s, \widehat{\xi}_s^i) e^{\Phi_{1,i}(\widehat{\xi}^i, y_i)} \frac{\partial}{\partial y_i} \Phi_{1,i}(\widehat{\xi}^i, y_i) \right).$$

Therefore,

$$\sup_{f \in \mathcal{X}} \left| \widehat{\mathcal{H}}_{f,i}^1(s, y, z) \right| \leq \sqrt{\mathbf{E}e^{2\Phi_{1,i}(\widehat{\xi}^i)}} \sqrt{\mathbf{E}\Psi_i^2(t, s)},$$

where  $\Psi_i(t, s, y_i) = \partial\Phi_{1,i}(\widehat{\xi}^i, y_i)/\partial y_i$ . Taking into account here the bound (A.12) we obtain

$$\sup_{f \in \mathcal{X}} \left| \widehat{\mathcal{H}}_{f,i}^1(s, y, z) \right| \leq (\phi_2)^{1/2} e^{\frac{\iota_2}{2}|z-y_i|} \sqrt{\mathbf{E}\Psi_i^2(t, s)}.$$

Moreover, through (A.6) and (A.10) we can calculate directly, that

$$\begin{aligned} \Psi_i(t, s, y_i) &= \sum_{j=1, j \neq i}^m \int_t^s \frac{\partial \mathbf{a}_{j,i}(v, B_v)}{\partial z} \frac{(s-v)}{(s-t)} d[U_v]_j \\ &\quad + \int_t^s \left( \frac{(s-v)}{(s-t)} \Psi_{1,i}(v, \widehat{\xi}_v^i, y_i) - \frac{\bar{\mathbf{a}}_i(v, \xi_v)}{\beta^2(s-t)} \right) dv, \end{aligned}$$

where for any  $t \leq v \leq t$  and any vector  $x = (x_1, \dots, x_m)'$

$$\begin{aligned} \Psi_{1,i}(v, x, y_i) &= \frac{\partial Q(v, x)}{\partial x_i} - \sum_{j=1}^m [\alpha(v, x)]_j \frac{\partial \mathbf{a}_{j,i}(v, x_i)}{\partial z} - \frac{\partial \mathbf{a}_{i,i}(v, x_i)}{\partial v} \\ &\quad - \frac{\beta^2}{2} \frac{\partial^2 \mathbf{a}_{i,i}(v, x_i)}{\partial z^2} + \frac{\partial \mathbf{a}_{i,i}(v, y_i)}{\partial v}. \end{aligned}$$

From the condition  $\mathbf{A}_3$ ) we can estimate this function

$$\max_{1 \leq i \leq m} \sup_{(v, x) \in \mathcal{K}} |\Psi_{1,i}(v, x)| \leq D_* + \alpha_* (\alpha_* + 1 + \beta^2/2).$$

Therefore,

$$\begin{aligned} \mathbf{E} \Psi_i^2(t, s) &= \mathbf{E} \int_t^s \sum_{j=1, j \neq i}^m \left( \frac{\partial \mathbf{a}_{j,i}(v, B_v)}{\partial z} \right)^2 \frac{(s-v)^2}{(s-t)^2} dv \\ &\quad + \mathbf{E} \left( \int_t^s \left( \frac{(s-v)}{(s-t)} \Psi_{1,i}(v, \hat{\xi}_v^i) - \frac{\bar{\mathbf{a}}_i(v, \xi_v)}{\beta^2(s-t)} \right) dv \right)^2. \end{aligned}$$

This implies directly

$$\mathbf{E} \Psi_i^2(t, s) \leq \Psi_*^2,$$

where the upper bound  $\Psi_*$  is given in (3.16). Therefore,

$$\sup_{f \in \mathcal{X}} \left| \hat{\mathcal{H}}_{f,i}^1(s, y, z) \right| \leq \Psi_* (\phi_2)^{1/2} e^{\frac{\iota_2}{2}|z-y_i|}.$$

Now from (A.14) and (A.15) it follows that

$$\sup_{f \in \mathcal{X}} \left| \frac{\partial}{\partial y_i} \hat{\mathcal{H}}_f(s, y, z) \right| \leq \hat{H}_* e^{\iota_* |z-y_i|},$$

where  $\hat{H}_* = \alpha_* \phi_1 + \Psi_* (\phi_2)^{1/2}$  and  $\iota_* = \max(\iota_1, \iota_2/2) + \alpha_*$ . Now from (A.4) we obtain

$$\begin{aligned} \frac{\partial \mathcal{H}_f(t, s, y)}{\partial y_i} &= \int_{\mathbb{R}} \frac{\partial \hat{\mathcal{H}}_f(s, y, z)}{\partial y_i} \mathbf{p}(z, y_i) dz \\ &\quad + \int_{\mathbb{R}} \hat{\mathcal{H}}_f(s, y, z) \frac{(z - y_i)}{\nu^2} \mathbf{p}(z, y_i) dz. \end{aligned}$$

Taking into account here, that for any  $a > 0$

$$\int_{\mathbb{R}} e^{a|z-y_i|} \mathbf{p}(z, y_i) dz \leq 2e^{\frac{a^2\nu^2}{2}}$$

and

$$\int_{\mathbb{R}} \frac{|z - y_i| e^{a|z-y_i|}}{\nu^2} \mathbf{p}(z, y_i) dz \leq 2a e^{\frac{a^2\nu^2}{2}} + \frac{\sqrt{2}}{\nu\sqrt{\pi}},$$

we obtain the upper bound (A.1). Hence Lemma A.2.  $\square$

### A.3 Properties of the process (3.5)

In this subsection we study the properties of the process  $\eta = (\eta_s^{t,y})_{t \leq s \leq T}$

**Lemma A.3.** *Under the conditions  $\mathbf{A}_1)$ – $\mathbf{A}_2)$  the process  $(\eta_s^{t,y})_{t \leq s \leq T}$  is almost sure continuously differentiable with respect to  $y \in \mathbb{R}^m$  for any  $t \leq s \leq T$ , i.e. for any  $1 \leq i \leq m$  there exists almost sure the derivative  $v_i(s) = \partial \eta_s^{t,y} / \partial y_i$  such that*

$$\sup_{0 \leq s \leq T} \sup_{y \in \mathbb{R}^m} \max_{1 \leq i \leq m} |v_i(s)| \leq e^{\alpha_* T} \quad a.s..$$

**Proof.** First we introduce the matrix of the first partial derivatives of the function  $\alpha(v, y)$  as

$$\alpha_0(t, z) = \left( \frac{\partial [\alpha(t, z)]_k}{\partial z_l}, \quad 1 \leq k, l \leq m \right).$$

One can check directly that the processes  $v_i(s)$  satisfies the following differential equations

$$\frac{d}{ds} v_i(s) = A_s v_i(s), \quad v_i(t) = e_i,$$

where  $A_s = \alpha_0(s, \eta_s^{t,y})$ ,  $e_i = (0, \dots, 0, 1, 0, \dots, 0)'$  (only  $i$ th component is equal to 1). Now by applying here the Gronwall-Bellman inequality we obtain the upper bounds for the derivatives  $v_i(s)$ . Hence Lemma A.3.  $\square$

## A.4 Properties of the function $\mathcal{G}$

Now we study the partial derivatives of the function  $\mathcal{G}(t, s, y)$  defined in (3.6). To this end we need the following general result.

**Lemma A.4.** *Let  $F = F(y, \omega)$  be a  $\mathbb{R} \times \Omega \rightarrow \mathbb{R}$  random bounded function such that for some nonrandom constant  $c^*$*

$$\left| \frac{d}{dy} F(y, \omega) \right| \leq c^* \quad a.s..$$

Then

$$\frac{d}{dy} \mathbf{E} F(y, \omega) = \mathbf{E} \frac{d}{dy} F(y, \omega).$$

This Lemma follows immediately from the Lebesgue dominated convergence theorem.

**Lemma A.5.** *Under the conditions  $\mathbf{A}_1)$ – $\mathbf{A}_2)$  there exist the partial derivatives  $(\partial \mathcal{G}(t, s, y) / \partial y_i)_{1 \leq i \leq m}$  such that*

$$\max_{1 \leq i \leq m} \sup_{y \in \mathbb{R}^m} \left| \frac{\partial \mathcal{G}(t, s, y)}{\partial y_i} \right| \leq D_* T e^{(\alpha_* + Q_*)T} \quad (\text{A.16})$$

and

$$\frac{\partial}{\partial y_i} \mathbf{E} \mathcal{G}(t, s, y) = \mathbf{E} \frac{\partial}{\partial y_i} \mathcal{G}(t, s, y).$$

**Proof.** Lemma A.3 implies immediately, that

$$\frac{\partial \mathcal{G}(t, s, y)}{\partial y_i} = \mathcal{G}(t, s, y) \mathbf{G}_i(t, s, y)$$

where  $\mathbf{G}_i(t, s, y) = \int_t^s (Q_0(u, \eta_u^{t,y}))' v_i(u) du$ ,  $Q_0(u, z) = \mathbf{D}_z Q(u, z)$  and  $v_i(u) = \partial \eta_u^{t,y} / \partial y_i$ . Now Lemma A.3 and Lemma A.4 imply directly this lemma.

□

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