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# ALGEBRAS DETERMINED BY THEIR SUPPORTS

IBRAHIM ASSEM, DIANE CASTONGUAY, MARCELO LANZILOTTA,  
AND ROSANA R. S. VARGAS

ABSTRACT. In this paper, we introduce and study a class of algebras which we call *ada* algebras. An artin algebra is *ada* if every indecomposable projective and every indecomposable injective module lies in the union of the left and the right parts of the module category. We describe the Auslander-Reiten components of an *ada* algebra, showing in particular that its representation theory is entirely contained in that of its left and right supports, which are both tilted algebras. Also, we prove that an *ada* algebra over an algebraically closed field is simply connected if and only if its first Hochschild cohomology group vanishes.

## Introduction

Let  $A$  be an artin algebra. We are interested in studying the representation theory of  $A$ , thus the category  $\text{mod}A$  of finitely generated right  $A$ -modules. One of the classes of algebras whose representation theory is best understood is that of the quasi-tilted algebras introduced by Happel, Reiten and Smalø in the seminal paper [21]. In particular, the ideas and techniques introduced in this paper were used to define and study successfully several generalisations of quasi-tilted algebras, such as shod, weakly shod, *laura*, left or right supported algebras. For an overview, we refer to the survey [6] or to the more recent [1].

The objective of present paper is to introduce and study a new class, which we call *ada* algebras. This also generalises quasi-tilted algebras. Indeed, an artin algebra is quasi-tilted if and only if every indecomposable projective module lies in the so-called left part of the module category, or equivalently if and only if every indecomposable injective module lies in the right part. We say that an algebra is *ada* if any indecomposable projective and any indecomposable injective lies in the union of these two parts. *Ada* algebras have the nice property that their representation theory is entirely contained in that of two tilted algebras. Namely, we recall from [5, 26] that the left support  $A_\lambda$  of an artin algebra is the endomorphism ring of the direct sum of all the indecomposable projective modules lying in the left part of  $\text{mod}A$ , and the right support  $A_\rho$  is defined dually. We prove that the left and right support of an *ada* algebra are tilted and describe the structure of the module category as in the following theorem.

**Theorem A** *Let  $A$  be an *ada* algebra which is not quasi-tilted. There exists*

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a finite family  $(\Gamma_i)_{i=1}^t$  of Auslander-Reiten components of  $\text{mod}A$  which are directed, generalised standard, convex and containing right sections such that:

- (a)  $\text{ind}A = \text{ind}A_\lambda \cup \text{ind}A_\rho$  and each of  $A_\lambda$  and  $A_\rho$  is a direct product of tilted algebras.
- (b) If  $\Gamma$  is an Auslander-Reiten component of  $\text{mod}A$  distinct from the  $\Gamma_i$ , then  $\Gamma$  is an Auslander-Reiten component of either  $\text{mod}A_\lambda$  or  $\text{mod}A_\rho$ . Moreover
  - (i) If  $\text{Hom}_A(\Gamma, \cup_i \Gamma_i) \neq 0$ , then  $\Gamma$  is an Auslander-Reiten component of  $\text{mod}A_\lambda$ , and,
  - (ii) If  $\text{Hom}_A(\cup_i \Gamma_i, \Gamma) \neq 0$ , then  $\Gamma$  is an Auslander-Reiten component of  $\text{mod}A_\rho$ .

Furthermore, the portion of the module category of an ada algebra which lies neither in the left nor in the right part is fairly well-understood (see (4.3) below), the structure of the left and right parts being known due to [1].

Considering next the case where  $A$  is a finite dimensional algebra over an algebraically closed field, we study its simple connectedness. We recall that a triangular algebra  $A$  is called simply connected if the fundamental group of any bound quiver presentation of  $A$  is trivial, see, for instance [9]. A well-known problem of Skowroński [25] links the simple connectedness of  $A$  to the vanishing of the first Hochschild cohomology group  $HH^1(A)$  of  $A$  with coefficients in the bimodule  ${}_A A_A$ . The equivalence of these conditions holds true for several classes of algebras, and among others for tilted algebras, see [22]. This brings us to our second theorem.

**Theorem B** *Let  $A$  be an ada algebra over an algebraically closed field. Then  $A$  is simply connected if and only if  $HH^1(A) = 0$ . Moreover, if this is the case, then the Hochschild cohomology ring  $HH^\bullet(A)$  reduces to the base field.*

The paper is organised as follows. After a short preliminary section, we define and study the first properties of ada algebras in section 2. The sections 3 and 4 are occupied with the proof of Theorem A, and section 5 with the proof of Theorem B.

## 1. PRELIMINARIES

**1.1. Notation.** Throughout this paper, all our algebras are basic and connected artin algebras. For an algebra  $A$ , we denote by  $\text{mod}A$  its category of finitely generated right modules and by  $\text{ind}A$  a full subcategory of  $\text{mod}A$  consisting of one representative from each isomorphism class of indecomposable modules. Whenever we speak about a module (or an indecomposable module), we always mean implicitly that it belongs to  $\text{mod}A$  (or to  $\text{ind}A$ , respectively).

Also, all subcategories of  $\text{mod}A$  are full and so are identified with their object classes. We sometimes consider an algebra  $A$  as a category, in which the object class  $A_0$  is a complete set  $\{e_1, \dots, e_n\}$  of primitive orthogonal idempotents and the set of morphisms from  $e_i$  to  $e_j$  is  $e_i A e_j$ . An algebra  $B$  is a *full subcategory* of  $A$  if there is an idempotent  $e \in A$ , sum of some of the distinguished idempotents  $e_i$ , such that  $B = e A e$ . It is *convex* in  $A$  if, for any sequence  $e_i = e_{i_0}, e_{i_1}, \dots, e_{i_t} = e_j$  of objects in  $A$  such that  $e_{i_k} A e_{i_{k+1}} \neq 0$  for all  $k$ , with  $0 \leq k < t$ , and  $e_i, e_j \in B_0$ , all  $e_{i_k}$  lie in  $B$ . We say that  $A$  is *triangular* if there is no sequence  $e_i = e_{i_0}, e_{i_1}, \dots, e_{i_t} = e_i$  of objects in  $A$  such that  $e_{i_k} A e_{i_{k+1}} \neq 0$  for all  $k$ , with  $0 \leq k < t$ . We denote by  $P_x$  (or  $I_x$ , or  $S_x$ ) the indecomposable projective (or injective, or simple, respectively)  $A$ -module corresponding to the idempotent  $e_x$ .

Let  $\mathcal{C}$  be a subcategory of  $\text{ind}A$ . We sometimes write  $M \in \mathcal{C}$  to express that  $M$  is an object in  $\mathcal{C}$ . We denote by  $\text{add}\mathcal{C}$  the subcategory of  $\text{mod}A$  with objects the direct sums of summands of modules in  $\mathcal{C}$ . If  $\mathcal{C}, \mathcal{C}'$  are two full subcategories of  $\text{ind}A$ , we write  $\text{Hom}_A(\mathcal{C}, \mathcal{C}') \neq 0$  whenever there exist  $M \in \mathcal{C}, M' \in \mathcal{C}'$  such that  $\text{Hom}_A(M, M') \neq 0$ .

Given a module  $M$ , we let  $\text{pd}M$  (or  $\text{id}M$ ) stand for its projective (or injective, respectively) dimension. The global dimension of  $A$  is denoted by  $\text{gl.dim}A$ .

For an algebra  $A$ , we denote by  $\Gamma(\text{mod}A)$  its Auslander-Reiten quiver and  $\tau_A = DTr$ ,  $\tau_A^{-1} = TrD$  its Auslander-Reiten translations. For further definitions and facts on  $\text{mod}A$  or  $\Gamma(\text{mod}A)$  we refer to [10, 12].

**1.2. Paths.** Let  $A$  be an algebra. Given  $M, N$  in  $\text{ind}A$ , a *path* from  $M$  to  $N$  in  $\text{ind}A$  (denoted by  $M \rightsquigarrow N$ ) is a sequence of non-zero morphisms

$$(*) \quad M = X_0 \xrightarrow{f_1} X_1 \rightarrow \dots \xrightarrow{f_t} X_t = N,$$

( $t \geq 1$ ) where  $X_i \in \text{ind}A$  for all  $i$ . We then say that  $M$  is a *predecessor* of  $N$  and  $N$  is a *successor* of  $M$  (denoted by  $M \leq N$ ).

A path from  $M$  to  $M$  involving at least one non-isomorphism is a *cycle*. A module  $M \in \text{ind}A$  which lies on no cycle is *directed*. If each  $f_i$  in  $(*)$  is irreducible, we say that  $(*)$  is a *path of irreducible morphisms* or *path in  $\Gamma(\text{mod}A)$* . A path of irreducible morphisms is *sectional* if  $\tau_A X_{i+1} \neq X_{i-1}$  for all  $i$  with  $0 < i < t$ .

The left and the right parts of  $\text{mod}A$  are defined by means of paths. Indeed, the *left part* is the full subcategory of  $\text{ind}A$  with object class

$$\mathcal{L}_A = \{M \in \text{ind}A \mid \text{for any } L \text{ with } L \rightsquigarrow M, \text{ we have } \text{pd}L \leq 1\}.$$

Note that  $\mathcal{L}_A$  is closed under predecessors: if  $M \in \mathcal{L}_A$  and  $L \rightsquigarrow M$  then  $L \in \mathcal{L}_A$ . The *right part*  $\mathcal{R}_A$  is defined dually and is closed under successors.

We need to recall the definitions of Ext-projective and Ext-injective objects. Let  $\mathcal{C}$  be a full additive subcategory of  $\text{mod}A$  closed under extensions (such as  $\text{add}\mathcal{L}_A$ , or  $\text{add}\mathcal{R}_A$ , for instance), then an indecomposable  $M \in \mathcal{C}$  is called *Ext-projective* (or *Ext-injective*) in  $\mathcal{C}$  if  $\text{Ext}_A^1(M, -)|_{\mathcal{C}} = 0$  (or  $\text{Ext}_A^1(-, M)|_{\mathcal{C}} = 0$ , respectively). It is shown in [13](3.4) that  $M$  is Ext-injective in  $\text{add}\mathcal{L}_A$  if and only if  $\tau_A^{-1}M \notin \mathcal{L}_A$  and similarly,  $M$  is Ext-projective in  $\text{add}\mathcal{R}_A$  if and only if  $\tau_A M \notin \mathcal{R}_A$ . For further characterisations of these objects, we refer to [5].

**1.3. Left and right section.** A full subquiver  $\Sigma$  of a translation quiver  $(\Gamma, \tau)$  is called a *right section* if:

- (1)  $\Sigma$  is acyclic,
- (2) for any  $x \in \Gamma_0$  such that there exist  $y \in \Sigma_0$  and a path  $y \rightsquigarrow x$  in  $\Gamma$ , there is a unique  $n \geq 0$  such that  $\tau^n x \in \Sigma_0$ ,
- (3)  $\Sigma$  is convex in  $\Gamma$ .

*Left sections* are defined dually, see [1]. It is shown in [1] that, if  $A$  is an artin algebra, and  $\Sigma$  is a right section in a generalised standard component of  $\Gamma(\text{mod}A)$ , then  $A/\text{Ann}\Sigma$  is a tilted algebra having  $\Sigma$  as complete slice [1](3.6). This notion applies well to the study of the left and right parts. Namely, if  $\mathcal{E}$  is the subcategory consisting of all the Ext-projectives in  $\text{add}\mathcal{R}_A$ , and  $\Gamma$  is a component of  $\Gamma(\text{mod}A)$ , then:

- (a) If  $\Gamma \cap \mathcal{E} = \emptyset$ , then either  $\Gamma \subseteq \mathcal{R}_A$  or  $\Gamma \cap \mathcal{R}_A = \emptyset$ .

- (b) If  $\Sigma = \Gamma \cap \mathcal{E} \neq \emptyset$ , then  $\Sigma$  is a right section of  $\Gamma$ , convex in  $\text{ind}A$ , and moreover  $A/\text{Ann}\Sigma$  is a tilted algebra having  $\Sigma$  as complete slice, see [1], Theorem (B).

By component of  $\Gamma(\text{mod}A)$ , we always mean connected component.

## 2. ADA ALGEBRAS: DEFINITION AND FIRST PROPERTIES

**Definition 2.1.** An artin algebra  $A$  is called an *ada algebra* if  $A \oplus DA \in \text{add}(\mathcal{L}_A \cup \mathcal{R}_A)$ .

Clearly, this is equivalent to requiring that, for every  $x \in A_0$ , we have both  $P_x$  and  $I_x$  lying in  $\mathcal{L}_A \cup \mathcal{R}_A$ .

Also, an algebra  $A$  is ada if and only if  $A^{op}$  is ada. This follows easily from the fact that  $D\mathcal{L}_A = \mathcal{R}_{A^{op}}$  and  $D\mathcal{R}_A = \mathcal{L}_{A^{op}}$ .

Quasi-tilted algebras are clearly ada. We call *strict* an ada algebra which is not quasi-tilted.

*Examples 2.2.* (a) Let  $A$  be a shod algebra [16]. Then  $\text{ind}A = \mathcal{L}_A \cup \mathcal{R}_A$ . Therefore  $A$  is ada.

- (b) Let  $A$  be given by the quiver

$$\begin{array}{ccccccccc} 1 & & 2 & & 3 & & 4 & & 5 \\ \bullet & \longleftarrow & \bullet & \longleftarrow & \bullet & \longleftarrow & \bullet & \longleftarrow & \bullet \end{array}$$

bound by  $\text{rad}^2 A = 0$ . Then  $P_1, P_2 = I_1, P_3 = I_2$  lie in  $\mathcal{L}_A$ , while  $P_4 = I_3, P_5 = I_4$  and  $I_5$  lie in  $\mathcal{R}_A$ . Then  $A$  is a (representation-finite) ada algebra. On the other hand, the one-point extension  $A[I_5]$  is not ada.

- (c) Let  $A$  be given by the quiver

$$\begin{array}{ccccccccc} 1 & & 2 & & 3 & & 4 \\ \bullet & \xleftarrow{\quad} & \bullet & \xleftarrow{\quad} & \bullet & \xleftarrow{\quad} & \bullet \end{array}$$

bound by  $\text{rad}^2 A = 0$ . Then  $A$  is a (representation-infinite) ada algebra. This example shows that, in contrast to lura algebras [3], an ada algebra may have infinitely many indecomposables which are not in  $\mathcal{L}_A \cup \mathcal{R}_A$ .

Let  $P$  denote the direct sum of a complete set of representatives of the isomorphism classes of indecomposable projective  $A$ -modules lying in  $\mathcal{L}_A$ . Then the algebra  $A_\lambda = \text{End}P_A$  is called the *left support* of  $A$ , see [5, 26]. We recall from [5](2.2) that  $A_\lambda$  is a full convex subcategory of  $A$ , closed under successors and that  $\mathcal{L}_A \subseteq \text{ind}A_\lambda$ . Moreover, because of [5] (2.3),  $A_\lambda$  (which is not connected in general) is a direct product of quasi-tilted algebras. The *right support*  $A_\rho$  is defined dually and has dual properties.

**Lemma 2.3.** *Let  $A$  be an ada algebra, then  $A = A_\lambda \cup A_\rho$ .*

**Proof.** Let  $x \in A_0$ . If  $P_x \in \mathcal{L}_A$ , then  $x \in (A_\lambda)_0$ . If not, then  $P_x \in \mathcal{R}_A$  and the non-zero morphism  $P_x \rightarrow I_x$  with image  $S_x$  yields  $I_x \in \mathcal{R}_A$  so that  $x \in (A_\rho)_0$ .  $\square$

**Lemma 2.4.** *Let  $A$  be an ada algebra, then  $A$  is triangular.*

**Proof.** Because of [5](2.2)(a), we can write  $A$  in triangular matrix form  $A = \begin{bmatrix} A_\lambda & 0 \\ M & B \end{bmatrix}$ . Since  $A_\lambda$  is a direct product of quasi-tilted algebras, then it is triangular. On the other hand, let  $x \in B_0$ , then the indecomposable

projective  $A$ -module  $P_x$  does not lie in  $\mathcal{L}_A$ , hence it lies in  $\mathcal{R}_A$ . Now, projectives in  $\mathcal{R}_A$  are directed because of [1](6.4). In particular,  $B$  is triangular hence so is  $A$ .  $\square$

We have an easy characterisation of ada algebras.

**Theorem 2.5.** *An artin algebra  $A$  is ada if and only if we have  $\text{ind}A = \mathcal{L}_A \cup \text{ind}A_\rho = \text{ind}A_\lambda \cup \mathcal{R}_A$ . In particular, if  $A$  is ada, then  $\text{ind}A = \text{ind}A_\lambda \cup \text{ind}A_\rho$ .*

**Proof.** Assume first that  $A$  is ada, and let  $M$  be an indecomposable  $A$ -module. Suppose that  $M \notin \text{ind}A_\rho$ . Then there exists  $x \in A_0$  such that  $M(x) \neq 0$  and  $x \notin (A_\rho)_0$ . Thus  $I_x \notin \mathcal{R}_A$  and there exists a non-zero morphism  $M \rightarrow I_x$ . Since  $A$  is ada, then  $I_x \in \mathcal{L}_A$  and so  $M \in \mathcal{L}_A$ . This shows that  $\text{ind}A = \mathcal{L}_A \cup \text{ind}A_\rho$ . Similarly, we have  $\text{ind}A = \text{ind}A_\lambda \cup \mathcal{R}_A$ .

Conversely, assume that these two equalities hold, and let  $x \in A_0$ , then  $P_x \in \mathcal{R}_A$  or  $P_x \in \text{ind}A_\lambda$ . By definition of  $A_\lambda$ , this implies  $P_x \in \mathcal{L}_A$ . Therefore  $P_x \in \mathcal{L}_A \cup \mathcal{R}_A$ . Similarly,  $I_x \in \mathcal{L}_A \cup \mathcal{R}_A$ .  $\square$

Notice that both conditions  $\text{ind}A = \mathcal{L}_A \cup \text{ind}A_\rho$  and  $\text{ind}A = \text{ind}A_\lambda \cup \mathcal{R}_A$  are necessary for  $A$  to be ada.

We deduce homological properties of ada algebras.

**Corollary 2.6.** *Let  $A$  be an ada algebra, then*

- (a) *For any indecomposable module  $M$ , we have  $\text{pd}M \leq 2$  or  $\text{id}M \leq 1$ .*
- (b)  *$\text{gl.dim}A \leq 4$*

**Proof.** (a) This follows from the equality  $\text{ind}A = \text{ind}A_\lambda \cup \mathcal{R}_A$  and the fact that  $\text{gl.dim}A_\lambda \leq 2$  (using that projective  $A_\lambda$ -modules are also projective  $A$ -modules).

(b) Let  $M$  be an indecomposable  $A$ -module and suppose that  $\text{pd}M \geq 2$ . Then there exists a minimal projective resolution

$$0 \rightarrow \Omega^2(M) \rightarrow P_1 \rightarrow P_0 \rightarrow M \rightarrow 0$$

and for every indecomposable summand  $X$  of  $\Omega^2(M)$ , we have  $\text{Ext}_A^2(M, X) \neq 0$ . In particular,  $\text{id}X \geq 2$ . Because of (a), we get  $\text{pd}X \leq 2$ . This implies that  $\text{pd}M \leq 4$ .  $\square$

*Remark 2.7.* a) The bound obtained in (b) above is sharp: indeed, the algebra  $A$  of example 2.2(b) has global dimension 4.

b) Dually, for every  $M \in \text{ind}A$ , we have  $\text{pd}M \leq 1$  or  $\text{id}M \leq 2$ .

We now prove that a full subcategory of an ada algebra is ada.

**Proposition 2.8.** *Let  $A$  be an ada algebra, and  $e \in A$  be an idempotent, then  $B = eAe$  is ada.*

**Proof.** Let  $x \in B_0$  and  $P_x = e_x B$  denote the corresponding indecomposable projective  $B$ -module. Then  $P_x \otimes_B A \cong e_x A \in \mathcal{L}_A \cup \mathcal{R}_A$ . Now, because of [4](2.1), we have  $\text{Hom}_A(eA, P_x \otimes_B A) \in \mathcal{L}_B \cup \mathcal{R}_B$ .

But  $\text{Hom}_A(eA, P_x \otimes_B A) \cong (P_x \otimes_B A)e \cong e_x A e \cong e_x e A e = e_x B = P_x$ .

Then  $P_x \in \mathcal{L}_B \cup \mathcal{R}_B$ . Similarly, using that  $A^{op}$  is ada, we get  $I_x \in \mathcal{L}_B \cup \mathcal{R}_B$ .  $\square$

For the notion and main results about split-by-nilpotent extensions, we refer the reader to [11].

**Proposition 2.9.** *Let  $R$  be a split-extension of  $A$  by a nilpotent bimodule. If  $R$  is ada, then so is  $A$ .*

**Proof.** Let  $x \in A_0$ , then we clearly have  $e_x R_R \cong e_x A \otimes_A R_R$  and  $D(Re_x) \cong e_x(DR) \cong \text{Hom}_{A^{op}}(Ae_x, DR) \cong \text{Hom}_A(R, D(Ae_x))$ . The statement then follows immediately from [11](2.4).  $\square$

Ada algebras also behave well with respect to the skew group algebra construction, see [12, 8].

**Proposition 2.10.** *Let  $A$  be an artin algebra, and  $G$  be a group acting on  $A$  with  $|G|$  invertible in  $A$ . Then the basic algebra  $R = A[G]^b$  associated to the skew group algebra is ada if and only if  $A$  is ada.*

**Proof.** Assume first that  $A$  is ada, and let  $\overline{P}$  be an indecomposable projective  $R$ -module. Because of [8](4.3), there exists an indecomposable projective summand  $P_A$  of  $\text{Hom}_R(R, \overline{P})$  such that  $\overline{P}_R$  is a direct summand of  $P \otimes_A R$ .

Suppose  $P \in \mathcal{L}_A$ . Because of [8](5.2)(a), we have  $P \otimes_A R \in \text{add} \mathcal{L}_R$ . Therefore  $\overline{P} \in \mathcal{L}_R$ . Suppose next that  $P \in \mathcal{R}_A$ . Let  $X$  be an indecomposable  $R$ -module such that  $\text{Hom}_R(\overline{P}, X) \neq 0$ . We claim that  $\text{id} X \leq 1$ . Because of [8](4.6), there exist  $\sigma \in G$  and an indecomposable summand  $M_A$  of  $\text{Hom}_R(R, X)$  such that  $X$  is a summand of  ${}^\sigma M \otimes_A R$  and  $\text{Hom}_A(P, {}^\sigma M) \neq 0$ . Because  $P \in \mathcal{R}_A$ , we get  $\text{id} {}^\sigma M \leq 1$ . Since the functor  $- \otimes_A R : \text{mod} A \rightarrow \text{mod} R$  is exact and carries injectives to injectives, we get  $\text{id}({}^\sigma M \otimes_A R) \leq 1$ . Therefore  $\text{id} X \leq 1$ , as asserted. Applying [8](1.1) yields  $\overline{P} \in \mathcal{R}_R$ . The proof is entirely similar if we start with an indecomposable injective  $R$ -module.

Conversely, let  $R$  be ada, and  $P_A$  an indecomposable projective  $A$ -module. Then there exists an indecomposable projective summand  $\overline{P}$  of  $P \otimes_A R$  such that  $P_A$  is a direct summand of  $\text{Hom}_R(R, \overline{P})$ .

Suppose  $\overline{P} \in \mathcal{L}_R$ . Because of [8](5.2)(b),  $\text{Hom}_A(R, \overline{P}) \in \text{add} \mathcal{L}_A$ . Therefore  $P \in \mathcal{L}_A$ . Suppose now that  $\overline{P} \in \mathcal{R}_R$ , and let  $M$  be an indecomposable  $A$ -module such that  $\text{Hom}_A(P, M) \neq 0$ . We claim that  $\text{id} M \leq 1$ . Because of [24], or [8](4.4)(a), we have  $\text{Hom}_R(\overline{P}, M \otimes_A R) \neq 0$ . Because of [24](1.1 and 1.8), there exists an indecomposable decomposition  $M \otimes_A R = \bigoplus_{i=1}^m X_i$  such that  $\text{Hom}_R(R, X_i) = \bigoplus_{\sigma \in H_i} {}^\sigma M$  for some  $H_i \subseteq G$ . Hence there exists  $i$  such that  $1 \leq i \leq m$  and  $\text{Hom}_R(\overline{P}, X_i) \neq 0$ . Because  $\overline{P} \in \mathcal{R}_R$ , we get  $\text{id} X_i \leq 1$ . This implies that, for every  $\sigma \in H_i$ , we have  $\text{id} {}^\sigma M \leq 1$ . Therefore  $\text{id} M \leq 1$ , as required. Another application of [8](1.1) yields  $P \in \mathcal{R}_A$ . Again the proof is similar if we start with an indecomposable injective  $A$ -module.  $\square$

### 3. THE MODULE CATEGORY OF AN ADA ALGEBRA

3.1. Assume  $A$  is a strict ada algebra. Then there exists  $x \in A_0$  such that  $P_x \notin \mathcal{L}_A$ . By definition,  $P_x \in \mathcal{R}_A$  and is clearly Ext-projective in  $\text{add} \mathcal{R}_A$ . Therefore the set  $\Sigma$  of indecomposable Ext-projectives in  $\text{add} \mathcal{R}_A$  is non-void. Let  $\Sigma = \Sigma_1 \amalg \Sigma_2 \amalg \cdots \amalg \Sigma_t$  where we assume that each  $\Sigma_i$  is the set of Ext-projectives in  $\text{add} \mathcal{R}_A$  lying in the same component  $\Gamma_i$  of  $\Gamma(\text{mod} A)$ . Note that  $\Sigma_i$  is not necessarily connected.

Because of [1](6.7), each  $\Sigma_i$  is a right section in  $\Gamma_i$ , convex in  $\text{ind}A$ . Moreover,  $A/\text{Ann}\Sigma_i$  is tilted and has  $\Sigma_i$  as a complete slice. The objective of this section is to prove the following theorem.

**Theorem 3.1.** *Let  $A$  be a strict ada algebra. Then there exists a finite family  $(\Gamma_i)_{i=1}^t$  of components of  $\Gamma(\text{mod}A)$  which are directed, generalised standard, convex, containing right sections such that, if  $\Gamma$  is an Auslander-Reiten component distinct from the  $\Gamma_i$ , then  $\Gamma$  is a component of either  $\Gamma(\text{mod}A_\lambda)$  or  $\Gamma(\text{mod}A_\rho)$  (and, in this latter case, it is contained in  $\mathcal{R}_A$ ). Moreover,*

- (i) if  $\text{Hom}_A(\Gamma, \cup_i \Gamma_i) \neq 0$ , then  $\Gamma$  is a component of  $\Gamma(\text{mod}A_\lambda)$ , and
- (ii) if  $\text{Hom}_A(\cup_i \Gamma_i, \Gamma) \neq 0$ , then  $\Gamma$  is a component of  $\Gamma(\text{mod}A_\rho)$ .

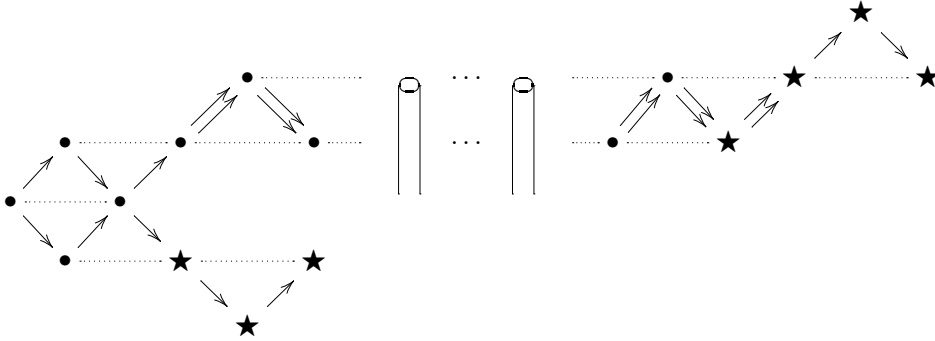
Clearly, the dual statement holds as well: there exists a finite family  $(\Gamma'_j)_{j=1}^s$  of directed, generalised standard, convex components of  $\Gamma(\text{mod}A)$ , each containing a left section  $\Sigma'_j$  consisting of indecomposable Ext-injectives in  $\text{add}\mathcal{L}_A$ , and equipped with the obvious properties. We leave the primal-dual translation to the reader.

We illustrate the theorem with the following example:

*Examples 3.2.* Let  $A$  be given by the quiver

$$\bullet \longrightarrow \bullet \longrightarrow \bullet \longleftarrow \bullet \rightleftarrows \bullet \longleftarrow \bullet$$

bound by  $\text{rad}^2 A = 0$ . The Auslander-Reiten quiver  $\Gamma(\text{mod}A)$  of  $A$  looks as follows.



where we have illustrated the objects of the subcategory  $\mathcal{R}_A$  by  $\star$ . Let  $\Gamma_1$  denote the postprojective component and  $\Gamma_2$  the preinjective component. Then  $\Sigma = \Sigma_1 \cup \Sigma_2$  with  $\Sigma_1 \subseteq \Gamma_1$  and  $\Sigma_2 \subseteq \Gamma_2$ . Notice that  $\text{Hom}_A(\Gamma_1, \Gamma_2) \neq 0$  (and so the components  $\Gamma_i$  are not orthogonal). Also, if  $\Gamma$  is a regular tube, then  $\text{Hom}_A(\Gamma_1, \Gamma) \neq 0$  but  $\Gamma$  is not contained in  $\mathcal{R}_A$ .

The proof of Theorem 3.1 will be split into a series of lemmata.

**Lemma 3.3.** *Let  $P_x \in \Sigma_i$  be projective. Then every projective successor of  $P_x$  lies in the same connected component of  $\Sigma_i$ .*

**Proof.** Assume we have a path  $P_x \rightsquigarrow P_y$  with  $P_y$  projective. Since  $P_x \in \mathcal{R}_A$ , we have also  $P_y \in \mathcal{R}_A$ . Therefore,  $P_y$  is Ext-projective in  $\text{add}\mathcal{R}_A$  and so there exists  $j$  so that  $P_y \in \Sigma_j$ . By [1](6.3), the path  $P_x \rightsquigarrow P_y$  can be refined to a path of irreducible morphisms and every module on each such refinement is Ext-projective in  $\text{add}\mathcal{R}_A$ . But then,  $P_x$  and  $P_y$  belong to the same connected component of  $\Sigma$ . In particular,  $i = j$ .  $\square$



We denote by  $(\Gamma_i)_{\geq \Sigma_i}$  the full subquiver of  $\Gamma_i$  consisting of the successors of  $\Sigma_i$  (and by  $(\Gamma_i)_{\not\geq \Sigma_i}$  the full subquiver of  $\Gamma_i$  consisting of the non-successors). By definition of  $\Sigma$ , the successors of  $\Sigma_i$  on  $\Gamma_i$  are  $A_\rho$ -modules. In fact we have the following result.

**Lemma 3.4.**  $(\Gamma_i)_{\geq \Sigma_i} = \Gamma_i \cap \mathcal{R}_A$

**Proof.** Assume  $X \in (\Gamma_i)_{\geq \Sigma_i}$ . Then there exist  $Y \in \Sigma_i$  and a path  $Y \rightsquigarrow X$ . Since  $Y \in \mathcal{R}_A$ , we have  $X \in \mathcal{R}_A$  and so  $X \in \Gamma_i \cap \mathcal{R}_A$ . Conversely, let  $X \in \Gamma_i \cap \mathcal{R}_A$ . Because of [1](6.6), there exists  $m \geq 0$  such that  $\tau_A^m X \in \Sigma_i$ . Clearly,  $X \in (\Gamma_i)_{\geq \Sigma_i}$ .  $\square$

We have a similar statement for non-successors.

**Corollary 3.5.** *Let  $X \in (\Gamma_i)_{\not\geq \Sigma_i}$ , then  $X \notin \mathcal{R}_A$  and  $X \in \text{ind}A_\lambda$ .*

**Proof.** The first statement follows from 3.4, and the second from 2.5.  $\square$

Since modules in  $\Sigma$  are directed (because of [1](6.4)) we deduce the following statement.

**Corollary 3.6.** *Let  $X \in \Gamma_i$  be a proper predecessor of  $\Sigma$ , then  $X \notin \mathcal{R}_A$  and  $X \in \text{ind}A_\lambda$ .*

**Lemma 3.7.** *The modules in  $\tau_A \Sigma_i$  are directed in  $\text{ind}A$ .*

**Proof.** Since  $\Sigma_i$  is acyclic, and  $\tau_A \Sigma_i$  contains no injectives, then  $\tau_A \Sigma_i$  is acyclic. Let  $X \in \Sigma_i$  and assume that we have a cycle in  $\text{ind}A$

$$\tau_A X = M_0 \xrightarrow{f_1} M_1 \rightarrow \dots \xrightarrow{f_t} M_t = \tau_A X.$$

Assume first that none of the  $f_j$  factors through an injective module. Then the above cycle induces another one in  $\text{ind}A$

$$X = \tau_A^{-1} M_0 \rightarrow \tau_A^{-1} M_1 \rightarrow \dots \rightarrow \tau_A^{-1} M_t = X.$$

Because of the convexity, this cycle lie inside  $\Sigma_i$ , thus contradicting the acyclicity of  $\Sigma_i$ . Therefore, we can assume that there exists  $j$  such that  $M_j$  is injective. Since,  $\tau_A X \notin \mathcal{R}_A$ , we have  $M_j \notin \mathcal{R}_A$  and thus  $M_j \in \mathcal{L}_A$ . Because of [1](6.4),  $M_j$  is directed, a contradiction.  $\square$

**Lemma 3.8.** *For any  $i$ ,  $\tau \Sigma_i$  lies in a union of directed components of  $\Gamma(\text{mod}A_\lambda)$ .*

**Proof.** Because of 3.7,  $\tau_A \Sigma_i$  is directed in  $\text{ind}A$ , hence it is also directed in  $\text{ind}A_\lambda$ .

Assume that  $X \in \Sigma_i$  is such that  $\tau_A X$  does not lie in a directed component of  $\Gamma(\text{mod}A_\lambda)$ . Because of the structure of the module category of the quasi-tilted algebra  $A_\lambda$  (see [15], [23]), we have one of two cases:

- (1)  $\tau_A X$  belongs to an inserted tube or component of type  $\mathbb{Z}A_\infty$  in  $\Gamma(\text{mod}A_\lambda)$ . Since  $\tau_A X$  is directed, there exists a non-directed indecomposable projective  $A_\lambda$ -module  $P$  and a path of irreducible morphisms  $\tau_A X \rightsquigarrow P$ .

Note that  $P$  is also projective as an  $A$ -module and is also not directed in  $\text{ind}A$ . In particular,  $P \notin \mathcal{R}_A$  (by [1](6.4)). Thus  $P \in \mathcal{L}_A$  and hence  $\tau_A X \in \mathcal{L}_A$ .

On the other hand, the path  $\tau_A X \rightsquigarrow P$  of irreducible morphisms contains no injective  $A_\lambda$ -module, because of the semiregularity of the component. Since any injective  $A$ -module lying in  $\text{ind}A_\lambda$  is also injective as an  $A_\lambda$ -module, then this path contains no injective  $A$ -module either. Therefore, we have a path  $X \rightsquigarrow \tau_A^{-1}P$  of irreducible morphisms. Since  $X \in \mathcal{R}_A$ , then  $\tau_A^{-1}P \in \mathcal{R}_A$ . Hence  $\tau_A^{-1}P \in \Sigma_i$  and  $P \in \tau_A \Sigma_i$  is directed in  $\text{ind}A$ , hence in  $\text{ind}A_\lambda$ , a contradiction.

- (2)  $\tau_A X$  belongs to a co-inserted tube or component of type  $\mathbb{Z}\mathbb{A}_\infty$  in  $\Gamma(\text{mod}A_\lambda)$ . We denote this component by  $\Gamma'$ .

Recall that  $\mathcal{L}_{A_\lambda}$  intersects no co-inserted tube or component of type  $\mathbb{Z}\mathbb{A}_\infty$ . Therefore, no module in  $\Gamma'$  belongs to  $\mathcal{L}_{A_\lambda}$ . Because of 2.5 and  $\mathcal{L}_A \subseteq \mathcal{L}_{A_\lambda}$ , this means that  $\Gamma'$  consists entirely of  $A_\rho$ -modules.

We claim that any irreducible morphism  $f : Y \rightarrow Z$  between two predecessors of  $\tau_A X$  in  $\Gamma'$  remains irreducible in  $\text{mod}A$ . Indeed, assume that this

is not the case, and let  $g = \begin{pmatrix} g_1 \\ \vdots \\ g_t \end{pmatrix} : Y \rightarrow \bigoplus_{i=1}^t E_i$  be left minimal almost

split in  $\text{mod}A$ , where the  $E_i$  are assumed indecomposable. Then  $f$  factors through  $g$ , that is, there exists  $h = (h_1, \dots, h_t) : \bigoplus_{i=1}^t E_i \rightarrow Z$  such that  $f = \sum_{i=1}^t h_i g_i$ . Let  $i$  be such that  $h_i g_i \neq 0$ .

Since  $Z$  precedes  $\tau_A X$ , then so does  $E_i$ . Hence  $E_i$  is in  $\text{mod}A_\lambda$  by 3.5. Since so are  $Y$  and  $Z$ , then the left minimal almost split morphism  $g$  in  $\text{mod}A$  remains left minimal almost split in  $\text{mod}A_\lambda$ . Consequently,  $h$  is a retraction and we are done.

Since  $Y, Z$  are predecessors of  $\tau_A X$  in  $\Gamma'$ , then they are also indecomposable  $A_\rho$ -modules, and hence  $f : Y \rightarrow Z$  remains irreducible in  $\text{mod}A_\rho$ .

This implies that the full subquiver  $\Gamma'_{\leq \tau_A X}$  of all predecessors of  $\tau_A X$  in  $\Gamma'$  is contained in exactly one component  $\Gamma$  of  $\Gamma(\text{mod}A_\rho)$ .

Now, there exist a non-directed injective  $A_\lambda$ -module  $I \in \Gamma'$  and a path  $I \rightsquigarrow \tau_A X$  of irreducible morphisms in  $\Gamma'$ . Because of the previous argument, this path induces a path  $I \rightsquigarrow \tau_A X$  of irreducible morphisms in  $\Gamma$ . Thus,  $\Gamma$  is a component of  $\Gamma(\text{mod}A_\rho)$  containing at the same time directed modules (such as  $\tau_A X$ ) and non-directed ones (such as  $I$ ) and also a path from a non-directed to a directed module. Using [15], [23], this shows that  $\Gamma$  is also a co-inserted tube or component of type  $\mathbb{Z}\mathbb{A}_\infty$  in  $\Gamma(\text{mod}A_\rho)$ .

Since injective  $A_\rho$ -modules are also injective  $A$ -modules, there is a non-directed injective  $A$ -module  $J \in \Gamma$  and a path  $J \rightsquigarrow \tau_A X$  in  $\text{ind}A_\rho$  and therefore in  $\text{ind}A$ . Since  $\tau_A X \notin \mathcal{R}_A$ , then  $J \notin \mathcal{R}_A$ . On the other hand,  $J$  is not directed, so  $J \notin \mathcal{L}_A$ , because of [1](6.4), and this contradicts the hypothesis that  $A$  is *ada*. □

We may now start the proof of Theorem 3.1.

**Lemma 3.9.** *Each of the components  $\Gamma_i$  is directed and generalised standard and convex in  $\text{ind}A$ .*

**Proof.** Suppose first that we have a cycle in  $\text{ind}A$  lying in the component  $\Gamma_i$ . Since  $\Sigma_i$  is a right section,  $(\Gamma_i)_{\geq \Sigma_i}$  is directed, because of [1](2.2). On the other hand,

$(\Gamma_i)_{\not\geq \Sigma_i}$  consists of  $A_\lambda$ -modules, because of 3.5. We now claim that each connected component  $\Gamma$  of  $(\Gamma_i)_{\not\geq \Sigma_i}$  contains at least a module of the form  $\tau_A X$ , with  $X \in \Sigma_i$ .

Assume  $\Gamma \cap \tau \Sigma_i = \emptyset$ . Let  $Y \in \Gamma$  (thus,  $Y \in \Gamma_i$ ). Since, by definition  $\Gamma_i \cap \Sigma_i \neq \emptyset$  and  $\Gamma_i$  is connected, then there exists a walk in  $\Gamma_i$ ,

$$Y = Y_0 - Y_1 - \dots - Y_t = X$$

for some  $X \in \Sigma_i$ . We know that  $Y$  is not a successor of  $\Sigma_i$ , hence  $Y \notin \mathcal{R}_A$  while  $X \in \mathcal{R}_A$ . Hence there exists a least  $i$  such that  $1 \leq i \leq t$  and  $Y_0, Y_1, \dots, Y_{i-1} \notin \mathcal{R}_A$  while  $Y_i \in \mathcal{R}_A$ . Then we have an arrow  $Y_{i-1} \rightarrow Y_i$ . Assume first that  $Y_i$  is not projective, then there is an arrow  $\tau_A Y_i \rightarrow Y_{i-1}$ , so  $\tau_A Y_i \notin \mathcal{R}_A$ . Therefore,  $Y_i \in \Sigma_i$ . Next, if  $Y_i$  is projective, then  $Y_{i-1}$  is not injective and so there is an arrow  $Y_i \rightarrow \tau_A^{-1} Y_{i-1}$ . Since  $\tau_A^{-1} Y_{i-1} \in \mathcal{R}_A$  we get  $\tau_A^{-1} Y_{i-1} \in \Sigma_i$ . This establishes our claim. Applying 3.8, we get that  $(\Gamma_i)_{\not\geq \Sigma_i}$  is directed.

This shows that, if we have a cycle in  $\Gamma_i$ , then it must be of the form

$$M = M_0 \rightarrow M_1 \rightarrow \dots \rightarrow M_j \rightarrow \dots \rightarrow M_t = M$$

where there exists  $j$  such that  $M \in (\Gamma_i)_{\geq \Sigma_i}$  and  $M_j \in (\Gamma_i)_{\not\geq \Sigma_i}$ . But now,  $M \in (\Gamma_i)_{\geq \Sigma_i}$  yields  $M \in \mathcal{R}_A$ , and so  $M_j \in \mathcal{R}_A$ , a contradiction to 3.4. This shows that  $\Gamma_i$  is directed.

Now, we assume that  $\Gamma_i$  is not generalised standard and let  $L, M \in \Gamma_i$  be such that  $\text{rad}_A^\infty(L, M) \neq 0$ . Since  $(\Gamma_i)_{\geq \Sigma_i}$  is generalised standard, because of [1](3.2), and  $(\Gamma_i)_{\not\geq \Sigma_i}$  also, because it is part of a directed, hence generalised standard component of the Auslander-Reiten quiver of the quasi-tilted algebra  $A_\lambda$ , then we must have  $L \in (\Gamma_i)_{\not\geq \Sigma_i}$  and  $M \in (\Gamma_i)_{\geq \Sigma_i}$ . Let  $f \in \text{rad}_A^\infty(L, M)$  be non-zero. For any  $t \geq 0$ , the morphism  $f$  induces a path in  $\text{ind} A$

$$L \xrightarrow{g_t} M_t \xrightarrow{f_t} \dots \rightarrow M_1 \xrightarrow{f_1} M_0 = M$$

with  $f_1, \dots, f_t$  irreducible,  $g_t \in \text{rad}_A^\infty(L, M_t)$  and  $f_1 \dots f_t g_t \neq 0$ . Therefore, there exists  $t$  such that  $M_t \in (\Gamma_i)_{\not\geq \Sigma_i}$  and  $\text{rad}_A^\infty(L, M_t) \neq 0$ , a contradiction to the fact that  $(\Gamma_i)_{\not\geq \Sigma_i}$  is generalised standard.

It remains to prove the convexity of  $\Gamma_i$ . Assume that we have a path in  $\text{ind} A$ :

$$M = M_0 \xrightarrow{f_1} M_1 \rightarrow \dots \xrightarrow{f_t} M_t = N$$

with  $M, N \in \Gamma_i$  and  $M_1, \dots, M_{t-1} \notin \Gamma_i$  (thus  $t \geq 2$ ). Then,  $f_t \in \text{rad}_A^\infty(M_{t-1}, N)$ . Suppose first that  $N \in (\Gamma_i)_{\geq \Sigma_i}$  then, for any  $s \geq 0$ , we have a path in  $\text{ind} A$

$$M_{t-1} \xrightarrow{h_s} N_s \xrightarrow{g_s} \dots \rightarrow N_1 \xrightarrow{g_1} N_0 = N$$

with  $g_1, \dots, g_s$  irreducible and  $h_s \in \text{rad}_A^\infty(M_{t-1}, N_s)$  such that  $h_s g_s \dots g_1 \neq 0$ . Then there exists  $s$  such that  $N_s \in (\Gamma_i)_{\not\geq \Sigma_i}$ .

We may thus suppose from the start that  $N \in (\Gamma_i)_{\not\geq \Sigma_i}$ . In particular,  $N \notin \mathcal{R}_A$  and thus  $M \notin \mathcal{R}_A$  and they are  $A_\lambda$ -modules because of 3.5. We claim that all  $M_j$  are  $A_\lambda$ -modules. Indeed, if this is not the case, by 2.5 there exists  $M_j \in \mathcal{R}_A$ , a contradiction. Then the given path consists entirely of  $A_\lambda$ -modules, with  $M, N \in (\Gamma_i)_{\not\geq \Sigma_i}$ . The conclusion then follows from the fact that  $(\Gamma_i)_{\not\geq \Sigma_i}$  is part of a directed component, hence convex component of  $\Gamma(\text{mod} A_\lambda)$ .  $\square$

Recall that an artin algebra  $A$  is *laura* if the class  $\text{ind}A \setminus (\mathcal{L}_A \cup \mathcal{R}_A)$  contains only finitely many objects [3]. A laura algebra which is not quasi-tilted always has a unique Auslander-Reiten component which is non-semiregular and faithful. The algebra  $A$  is called *weakly shod* [17] if this component is directed.

**Corollary 3.10.** *Let  $A$  be a strict ada algebra. If  $A$  is laura, then it is weakly shod.*

**Proof.** Let  $\Gamma$  be the faithful non-semiregular component of  $\Gamma(\text{mod}A)$ . Since  $A$  is strict, there exists a projective  $A$ -module  $P_x$  such that  $P_x \in \mathcal{R}_A \setminus \mathcal{L}_A$ . Because  $\Gamma$  is faithful, there exists  $M \in \Gamma$  such that  $\text{Hom}_A(P_x, M) \neq 0$  and so  $M \in \mathcal{R}_A \setminus \mathcal{L}_A$ . This shows that  $\Gamma \cap \mathcal{R}_A \neq \emptyset$  and that  $\Gamma \not\subseteq \mathcal{L}_A$ . Dually  $\Gamma \not\subseteq \mathcal{R}_A$ .

Because of [1], Theorem B, the intersection of  $\Gamma$  with the class  $\Sigma$  of indecomposable Ext-projectives in  $\text{add}\mathcal{R}_A$  is a right section of  $\Gamma$ . Since  $\Gamma = \Gamma_i$  is directed because of 3.9, we get that  $A$  is weakly shod.  $\square$

The proof of Theorem 3.1 will be completed once we prove the following lemma

**Lemma 3.11.** *Let  $A$  be a strict ada algebra. If  $\Gamma$  is a component of  $\Gamma(\text{mod}A)$  distinct from the  $\Gamma_i$ , then  $\Gamma$  is a component of either  $\Gamma(\text{mod}A_\lambda)$  or  $\Gamma(\text{mod}A_\rho)$  (and in this latter case, it is contained in  $\mathcal{R}_A$ ). Moreover, we have either*

- i) *If  $\text{Hom}_A(\Gamma, \cup_i \Gamma_i) \neq 0$  then  $\Gamma$  is a component of  $\Gamma(\text{mod}A_\lambda)$ , or*
- ii) *If  $\text{Hom}_A(\cup_i \Gamma_i, \Gamma) \neq 0$  then  $\Gamma$  is a component of  $\Gamma(\text{mod}A_\rho)$*

**Proof.** Because  $\Gamma \neq \Gamma_i$  for all  $i$ , we have  $\Gamma \cap \Sigma = \emptyset$ . Because of [1](Theorem B), we get that either  $\Gamma \subseteq \mathcal{R}_A$  or  $\Gamma \cap \mathcal{R}_A = \emptyset$ . In the first case, clearly,  $\Gamma$  is a component of  $\Gamma(\text{mod}A_\rho)$  contained in  $\mathcal{R}_A$ . We claim that, if  $\Gamma \cap \mathcal{R}_A = \emptyset$ , then  $\Gamma$  is a component of  $\Gamma(\text{mod}A_\lambda)$ . It suffices to prove that each  $X \in \Gamma$  is an  $A_\lambda$ -module. Now, if this is not the case, then there exists an indecomposable projective  $P \notin \mathcal{L}_A$  such that  $\text{Hom}_A(P, X) \neq 0$ . But then  $P \in \mathcal{R}_A$  and so  $X \in \mathcal{R}_A$ , a contradiction which establishes our claim.

Now, assume that  $\text{Hom}_A(\Gamma, \cup_i \Gamma_i) \neq 0$  and  $\Gamma$  is not a component of  $\Gamma(\text{mod}A_\lambda)$ . Let  $X \in \Gamma$  be not an  $A_\lambda$ -module. Then there exists an indecomposable projective  $A$ -module  $P \notin \mathcal{L}_A$  such that  $\text{Hom}_A(P, X) \neq 0$ . As above,  $X \in \mathcal{R}_A$  and so  $\Gamma \cap \mathcal{R}_A \neq \emptyset$ . Because of [1](Theorem B), we have  $\Gamma \subseteq \mathcal{R}_A$ .

Since  $\text{Hom}_A(\Gamma, \cup_i \Gamma_i) \neq 0$ , there exist  $M \in \Gamma$  and  $N \in \Gamma_i$  for some  $i$  such that  $\text{Hom}_A(M, N) \neq 0$ . Since  $M \in \mathcal{R}_A$ , thus  $N \in \mathcal{R}_A$ . Because of 3.4, we have  $N \in (\Gamma_i)_{\geq \Sigma_i}$ . Since  $\Gamma \neq \Gamma_i$ , we have  $\text{Hom}_A(M, N) = \text{rad}_A^\infty(M, N) \neq 0$ . Thus, for any  $s \geq 0$ , there exists a path in  $\text{ind}A$

$$M \xrightarrow{h_s} N_s \xrightarrow{g_s} \dots \rightarrow N_1 \xrightarrow{g_1} N_0 = N$$

with  $g_1, \dots, g_s$  irreducible and  $h_s \in \text{rad}_A^\infty(M, N_s)$  such that  $g_1 \dots g_s h_s \neq 0$ . Therefore, there exists  $s$  such that  $N_s \in (\Gamma_i)_{\not\geq \Sigma_i}$ . But then  $N_s \in \mathcal{R}_A$ , a contradiction to 3.4. This completes the proof of i).

Finally, assume similarly that  $\text{Hom}_A(\cup_i \Gamma_i, \Gamma) \neq 0$  and  $\Gamma$  is not a component of  $\Gamma(\text{mod}A_\rho)$ . In particular,  $\Gamma$  is not contained in  $\mathcal{R}_A$  and since moreover  $\Gamma \cap \Sigma = \emptyset$ , we deduce from [1], Theorem B, that  $\Gamma \cap \mathcal{R}_A = \emptyset$ .

By hypothesis, there exist  $i$ ,  $M \in \Gamma_i$  and  $X \in \Gamma$  such that  $\text{Hom}_A(M, X) \neq 0$ . If  $M \in (\Gamma_i)_{\geq \Sigma_i}$ , then  $M \in \mathcal{R}_A$  by 3.4, so that  $X \in \mathcal{R}_A$ , a contradiction. Therefore,  $M$  is not a successor of  $\Sigma_i$ . We then consider two cases.

Suppose first that  $(\Gamma_i)_{\not\geq \Sigma_i}$  contains no injective. In this case,  $\Sigma_i$  is a section in the directed component  $\Gamma_i$ , because of [1](2.3) and moreover  $\Gamma_i$  is the connecting component of the tilted algebra  $A/\text{Ann}\Sigma_i$ , and  $\Sigma_i$  is a complete slice, because of [1](3.6). Now, observe that  $\Sigma_i \subseteq \mathcal{R}_A$ , so  $(\Gamma_i)_{\geq \Sigma_i} \subseteq \mathcal{R}_A$ , thus  $(\Gamma_i)_{\geq \Sigma_i}$  consists of  $A_\rho$ -modules. Since  $\Sigma_i$  cogenerates  $(\Gamma_i)_{\not\geq \Gamma_i}$ , then  $(\Gamma_i)_{\not\geq \Sigma_i}$  also consists of  $A_\rho$ -modules. In particular,  $A/\text{Ann}\Sigma_i$  is a connected component of  $A_\rho$ . Because  $\Sigma_i$  is a complete slice,  $M \in \Sigma_i$  is not a successor of  $\Sigma_i$  if and only if  $M$  is a predecessor of  $\Sigma_i$ . Therefore  $\text{rad}_A^\infty(M, X) \neq 0$  gives, for any  $t \geq 0$ , a path in  $\text{ind}A$

$$M = M_0 \xrightarrow{f_1} M_1 \rightarrow \dots \xrightarrow{f_t} M_t \xrightarrow{g_t} X$$

where the  $f_i$  are irreducible and  $g_t \in \text{rad}_A^\infty(M_t, X)$  is such that  $g_t f_t \cdots f_1 \neq 0$ . Let  $t \geq 0$  be such that  $M_t$  is a successor of  $\Sigma_i$ , then  $M_t \in \mathcal{R}_A$ , hence  $X \in \mathcal{R}_A$  and we get a contradiction in this case.

Suppose next that  $(\Gamma_i)_{\not\geq \Sigma}$  contains an injective  $A$ -module  $I$ . Because of 3.5, we have  $I \notin \mathcal{R}_A$ . Hence  $I \in \mathcal{L}_A$  and so is Ext-injective in  $\text{add}\mathcal{L}_A$ . Using the notation in 3.1, this shows that the Ext-injectives in  $\text{add}\mathcal{L}_A$  form a left section  $\Sigma'_j$  in some component  $\Gamma'_j$ . Note that  $\Gamma'_j = \Gamma_i$ . Since  $\text{rad}_A^\infty(M, X) \neq 0$ , there exists, for each  $t \geq 0$ , a path in  $\text{ind}A$

$$M = M_0 \xrightarrow{f_1} M_1 \rightarrow \dots \xrightarrow{f_t} M_t \xrightarrow{g_t} X$$

where the  $f_i$  are irreducible and  $g_t \in \text{rad}_A^\infty(M_t, X)$  is such that  $g_t f_t \cdots f_1 \neq 0$ . Let  $t \geq 0$  be such that  $M_t$  is a proper successor of  $\Sigma'_j$ . Because of 3.4, this gives  $M_t \notin \mathcal{L}_A$ . Therefore,  $X \notin \mathcal{L}_A$ . This shows that  $\Gamma$  contains at least an indecomposable  $X$  which is not in  $\mathcal{L}_A$ . Now, we claim that  $\Gamma \cap \mathcal{L}_A = \emptyset$ . By induction, it suffices to show that no neighbour  $Y$  of  $X$  belongs to  $\mathcal{L}_A$ . If there is an arrow  $X \rightarrow Y$ , then  $X \notin \mathcal{L}_A$  implies  $Y \notin \mathcal{L}_A$ . Assume that we have an arrow  $Y \rightarrow X$  and that  $Y \in \mathcal{L}_A$ . We claim that in this case  $Y$  is Ext-injective in  $\text{add}\mathcal{L}_A$ . This is obvious if  $Y$  is injective, and, if it is not, then there is an arrow  $X \rightarrow \tau_A^{-1}Y$  so that  $\tau_A^{-1}Y \notin \mathcal{L}_A$  and again  $Y$  is Ext-injective in  $\text{add}\mathcal{L}_A$ . In particular,  $\Gamma = \Gamma'_l$  for some  $l$  and  $Y \in \Sigma'_l$ . Now there exists a non-zero morphism  $g_s \in \text{rad}_A^\infty(M_s, X)$ . This morphism factors through  $\Sigma'_l$  (because  $X$  is a successor of  $\Sigma'_l$ ). Then  $\Sigma'_l \subseteq \mathcal{L}_A$  yields  $M_s \in \mathcal{L}_A$  and this is a contradiction. Therefore  $Y \notin \mathcal{L}_A$ . This shows that  $\Gamma \cap \mathcal{L}_A = \emptyset$ . Because of 2.5,  $\Gamma$  consists of  $A_\rho$ -modules and hence is a component of  $\Gamma(\text{mod}A_\rho)$ .  $\square$

#### 4. THE SUPPORTS OF AN ADA ALGEBRA

Throughout this section, we let  $A$  be a strict ada algebra.

**Proposition 4.1.** *Each of  $A_\lambda$  and  $A_\rho$  is a direct product of tilted algebras.*

**Proof.** Indeed, assume that  $B$  is a connected component of  $A_\lambda$  and is not tilted. Since  $A$  is strict, we have  $B \neq A$  and so there exist an indecomposable  $B$ -module  $X$  and an irreducible morphism  $X \rightarrow P_x$  with  $P_x$  an indecomposable projective  $A$ -module which is not a  $B$ -module. Since  $X$  is isomorphic to an indecomposable summand of  $\text{rad}_A(P_x)$ , then  $P_x \notin \mathcal{L}_A$  hence  $P_x \in \mathcal{R}_A$  and therefore is Ext-projective in  $\text{add}\mathcal{R}_A$ .

We claim that  $X$  is a directed  $A$ -module. Indeed,  $X$  is not injective, so we have an arrow  $P_x \rightarrow \tau_A^{-1}X$  and then we have two cases. If  $X \notin \mathcal{R}_A$  then  $\tau_A^{-1}X \in \mathcal{R}_A$  yields  $\tau_A^{-1}X \in \Sigma$  and so  $X \in \tau_A \Sigma$  is a directed  $A$ -module. If  $X \in \mathcal{R}_A$ , then

$X \in \Sigma$  and so is again directed. In fact, it follows from 3.9 that  $X$  lies in a directed component of  $\Gamma(\text{mod}A)$  and 3.8 that it lies in a directed component of  $\Gamma(\text{mod}B)$ . Since  $B$  is quasi-tilted but not tilted, then this is the postprojective or the preinjective component of  $\Gamma(\text{mod}B)$ .

Let  $e = e_x + \sum_{y \in B_0} e_y$ . Then  $A' = eAe$  is ada, because of 2.8 and is a one-point extension of  $B$ . Because of 2.9, we may assume that  $A' = B[X]$ .

Assume first that  $X$  lies in the postprojective component of  $\Gamma(\text{mod}B)$ . Let  $P'_x$  be the indecomposable projective  $A'$ -module corresponding to the point  $x$ . Then, considering  $P'_x$  as an  $A$ -module under the standard embedding of  $\text{mod}A'$  into  $\text{mod}A$ , we have an epimorphism  $P_x \rightarrow P'_x$ . Since  $P_x \in \mathcal{R}_A \setminus \mathcal{L}_A$ , then  $P'_x \in \mathcal{R}_A \setminus \mathcal{L}_A$  as well. Applying [4](2.1), we get  $P'_x \in \mathcal{R}_{A'}$ . On the other hand, since  $B$  is quasi-tilted but not tilted, there exists a non-directed indecomposable projective  $B$ -module  $P_y$  lying in an inserted tube or component of type  $\mathbb{Z}\mathbb{A}_\infty$ . Note that  $y$  is a source in  $B$  and hence also is  $A'$ . Thus  $P_y = P'_y$  is a non-directed indecomposable projective  $A'$ -module. On the other hand,  $P'_x$  lies in the postprojective component of  $\Gamma(\text{mod}A')$ . We claim that there exists a path  $P'_x \rightsquigarrow P'_y$  in  $\text{mod}A'$ . Indeed, since  $B$  is connected and  $y$  is a source, there exists  $z \in B_0$  such that  $P'_z$  lies in the postprojective component of  $\Gamma(\text{mod}A')$  and a non-zero morphism  $f : P'_z \rightarrow P'_y$ . Since  $f \in \text{rad}_{A'}^\infty(P'_z, P'_y)$ , there exists, for any  $t \geq 0$ , a path in  $\text{ind}A$

$$P'_z = M_0 \xrightarrow{f_1} M_1 \rightarrow \dots \xrightarrow{f_t} M_t \xrightarrow{g_t} P'_y$$

with the  $f_i$  irreducible and  $g_t \in \text{rad}_{A'}^\infty(M_t, P'_y)$  such that  $g_t f_t \dots f_1 \neq 0$ .

Let  $t$  be such that  $M_t$  is a successor of  $P'_x$ . This yields the required path  $P'_x \rightsquigarrow P'_y$  in  $\text{mod}A'$ . But we have already seen that  $P'_x \in \mathcal{R}_{A'}$ , a contradiction because  $P'_y$  is not directed.

Therefore, we may assume  $X$  to lie in the preinjective component of  $\Gamma(\text{mod}B)$ . Now, since  $B$  is quasi-tilted but not tilted, there exists a non-directed indecomposable injective  $B$ -module  $I_y$  lying in a co-inserted tube or component of type  $\mathbb{Z}\mathbb{A}_\infty$ . Because  $A' = B[X]$  and  $X$  is preinjective, then  $I_y$  is also an injective  $A'$ -module. However, we have  $P'_x \in \mathcal{R}_{A'}$ , and there exists a non-sectional path  $I_y \rightsquigarrow X \rightarrow P'_x$ . Because of [4](1.5), this implies that  $I_y \notin \mathcal{R}_{A'}$ . The algebra  $A'$  being ada, we get  $I_y \in \mathcal{L}_{A'}$  a contradiction, because  $I_y$  is not directed. The proof is now complete.  $\square$

It follows from 3.1 and 4.1 that, if  $A$  is an ada algebra, then we have a good description of the indecomposable modules (or components) lying in  $\mathcal{L}_A \cup \mathcal{R}_A$ : these are modules (or components) over one of the tilted algebras  $A_\lambda$  and  $A_\rho$ . We now wish to describe those modules which do not belong to  $\mathcal{L}_A \cup \mathcal{R}_A$ . As in 3.1, we denote by  $\Sigma$  the class of Ext-projectives in  $\text{add}\mathcal{R}_A$  and by  $\Sigma'$  the class of Ext-injectives in  $\text{add}\mathcal{L}_A$ .

**Lemma 4.2.** *Let  $A$  be a strict ada algebra and  $X$  an indecomposable  $A$ -module not lying in  $\mathcal{L}_A \cup \mathcal{R}_A$ . Then there exist an indecomposable projective module  $P \in \Sigma$  and a non-sectional path  $X \rightsquigarrow P$ .*

**Proof.** Indeed, since  $X \notin \mathcal{R}_A$ , then there exists a path  $X \rightsquigarrow Y$  in  $\text{ind}A$  where  $Y$  is such that  $\text{id}Y > 1$ . Hence there exists an indecomposable projective  $A$ -module  $P$  such that we have a path  $X \rightsquigarrow Y \rightarrow * \rightarrow \tau_A^{-1}Y \rightarrow P$  in  $\text{ind}A$ . Since  $X \notin \mathcal{L}_A$ , we also have  $P \notin \mathcal{L}_A$ . Therefore  $P \in \mathcal{R}_A$  and so  $P \in \Sigma$ .  $\square$

Now, notice that  $C = A_\lambda \cap A_\rho$  is a full convex subcategory of  $A_\lambda$  (or  $A_\rho$ ) and therefore is tilted, because of [19](III.6.5).

**Proposition 4.3.** *Let  $A$  be a strict ada algebra, and  $X$  be an indecomposable  $A$ -module. The following conditions are equivalent.*

- (a)  $X \notin \mathcal{L}_A \cup \mathcal{R}_A$ .
- (b) There exist  $P \in \Sigma$  projective,  $I \in \Sigma'$  injective and two non-sectional paths  $I \rightsquigarrow X$  and  $X \rightsquigarrow P$ .
- (c)  $X$  is a proper predecessor of  $\Sigma$  and a proper successor of  $\Sigma'$ .

Moreover, if this is the case, then  $X$  is an indecomposable  $C$ -module, generated by  $\Sigma'$  and cogenerated by  $\Sigma$ .

**Proof.** That (a) implies (b) follows from 4.2 and its dual. That (b) implies (c) follows from [1](6.3), because the given paths are non-sectional. Finally, assume that (c) holds. Since  $X$  is a proper predecessor of  $\Sigma$ , then there exists a non-sectional path from  $X$  to some  $M \in \Sigma$ . Because of [1](6.3), this implies that  $X \notin \mathcal{R}_A$ . Similarly,  $X \notin \mathcal{L}_A$ .

Now, if this is the case, then  $X$  being a proper predecessor of  $\Sigma$  implies  $X \in \text{ind}A_\lambda$ , because of 3.6. Similarly,  $X \in \text{ind}A_\rho$ . Therefore  $X \in \text{ind}C$ . The statements about generation and cogeneration follow from the fact that there exist neither projectives nor injectives lying strictly between  $\Sigma'$  and  $\Sigma$ .  $\square$

## 5. HOCHSCHILD COHOMOLOGY AND SIMPLE CONNECTEDNESS

Throughout this last section, all our algebras are finite dimensional algebras over an algebraically closed field  $k$ .

Let  $A$  be ada. We recall from [7] that an indecomposable projective  $P_x \in \mathcal{R}_A$  is called a *maximal projective* if it has no projective successor. We then say that  $A$  is a *maximal extension* of  $B = A \setminus \{x\}$ . Denoting by  $M$  the radical of  $P_x$ , we have  $A = B[M]$ . We shall prove in 5.6 below that any strict ada algebra may be written as a maximal extension of another ada algebra.

**Lemma 5.1.** *Let  $A = B[M]$  be a maximal extension. Then for every  $i \geq 1$ , we have  $\text{Ext}_B^i(M, M) = 0$ .*

**Proof.** Same as [7](2.3).  $\square$

Let  $\text{HH}^i(A)$  denote the  $i^{\text{th}}$  Hochschild cohomology group of  $A$  with coefficients in the bimodule  ${}_A A_A$  (see [18] for details). It is shown in [18](5.3) that, if  $A = B[M]$ , then there exists a long exact sequence

$$0 \rightarrow \text{HH}^0(A) \rightarrow \text{HH}^0(B) \rightarrow \text{End}M/k \rightarrow \text{HH}^1(A) \rightarrow \text{HH}^1(B) \rightarrow \text{Ext}_B^1(M, M) \rightarrow \dots \\ \dots \rightarrow \text{HH}^i(A) \rightarrow \text{HH}^i(B) \rightarrow \text{Ext}_B^i(M, M) \rightarrow \dots$$

We refer to this sequence in the sequel as *Happel's sequence*. We also recall that the extension point  $x$  is called *separating* if the number of indecomposable summands of  $\text{rad}P_x$  equals the number of connected components of  $B = A \setminus \{x\}$ , see, for instance [9].

**Lemma 5.2.** *Let  $A = B[M]$  be an ada maximal extension. Then:*

(a) *There exists an exact sequence*

$$0 \rightarrow \mathrm{HH}^0(A) \rightarrow \mathrm{HH}^0(B) \rightarrow \mathrm{End}M/k \rightarrow \mathrm{HH}^1(A) \rightarrow \mathrm{HH}^1(B) \rightarrow 0$$

(b) *For any  $i \geq 2$ , we have  $\mathrm{HH}^i(A) \cong \mathrm{HH}^i(B)$ .*

(c)  *$\mathrm{HH}^1(A) \cong \mathrm{HH}^1(B)$  if and only if the extension point is separating.*

**Proof.**

The statements (a) and (b) follow from Lemma 5.1 and Happel's sequence. We proceed to prove (c). The surjective morphism  $\mathrm{HH}^1(A) \rightarrow \mathrm{HH}^1(B)$  has kernel with dimension equal to

$$\dim_k(\mathrm{End}M/k) - \dim_k \mathrm{HH}^0(B) + \dim_k \mathrm{HH}^0(A) = \dim_k \mathrm{End}M - \dim_k \mathrm{HH}^0(B)$$

because  $A$  is connected. Therefore,  $\mathrm{HH}^1(A) \cong \mathrm{HH}^1(B)$  if and only if  $\dim_k \mathrm{End}M$  equals the number of connected components of  $B$ , and this is the case if and only if the extension point  $x$  is separating and  $M$  is a direct sum of bricks. Because of Theorem 3.1, every indecomposable projective lying in  $\mathcal{R}_A$  belongs to a directed generalised standard component. Therefore, every indecomposable summand of  $M$  is a brick. The statement follows.  $\square$

*Remark 5.3.* In particular, we proved that the module  $M$  is separated, see [9] for the definition.

A triangular algebra  $A$  is called *simply connected* if, for every presentation  $A \cong kQ/I$  of  $A$  as a bound quiver algebra, the fundamental group of  $(Q, I)$  is trivial, see [25, 9]. Let  $A = B[M]$  where we denote by  $x$  the extension point. We fix a presentation of  $A$  and consider the induced presentation of  $B$ . Let  $\sim$  be the least equivalence relation on the arrows of source  $x$  such that  $\alpha_1 \sim \alpha_2$  if there exists a minimal relation of the form  $\lambda_1 \alpha_1 v_1 + \lambda_2 \alpha_2 v_2 + \sum_{j \geq 3} \lambda_j w_j$ . Let  $t$  be the number of equivalence classes of arrows of source  $x$  under this relation. For each  $i$ , with  $1 \leq i \leq t$ , let  $l(i)$  be the number of tuples of paths  $(u_1, v_1, \dots, u_n, v_n)$  such that there are minimal relations of the forms  $\lambda_{1,1} \alpha_1 u_1 + \lambda_{2,1} \alpha_n v_n + \sum_{j \geq 3} \lambda_{j,1} w_{j,1}, \lambda_{1,2} \alpha_1 v_1 + \lambda_{2,2} \alpha_2 u_2 + \sum_{j \geq 3} \lambda_{j,2} w_{j,2}, \dots$  where  $\alpha_1, \dots, \alpha_n$  are distinct arrows in the same equivalence class, see [9](2.4).

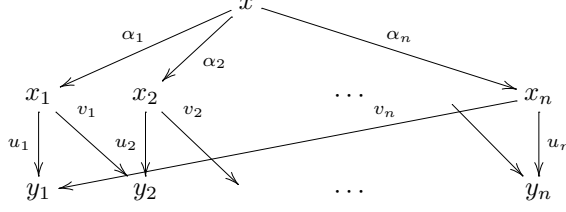
**Lemma 5.4.** *Let  $A$  be a strict ada algebra.*

- (a) *If  $B$  is a direct product of simply connected algebras, then  $A$  is simply connected if and only if the extension point is separating.*
- (b) *If  $A$  is a simply connected strict ada maximal extension, then  $B$  is a direct product of simply connected algebras.*

**Proof.** (a) This statement follows from [2](3.6).

(b) Let  $B \cong kQ_B/I'$  be an arbitrary presentation of  $B$ , then there exist a presentation  $A \cong kQ_A/I$  of  $A$  such that  $I \cap kQ_B = I'$ . Because of [9](2.4) it suffices to show that  $l(i) = 0$  for all  $i$ . However, if  $l(i) \neq 0$  for some  $i$ , then there exists a tuple of paths  $(u_1, v_1, \dots, u_n, v_n)$  and a full subcategory  $C$  of  $A$  which is a split extension of a subcategory  $D$  of the form





(indeed, there might be in  $C$  additional arrows from some  $y_i$  to some  $y_j$ ). We denote respectively by  $P_x$ ,  $P'_x$ ,  $P''_x$  the indecomposable projective module corresponding to  $x$  in  $\text{mod}A$ ,  $\text{mod}C$  and  $\text{mod}D$ . Then  $P'_x = P''_x \otimes_D C$  and we have an epimorphism from  $P_x$  to  $\overline{P}'_x$  where  $\overline{P}'_x = P'_x \otimes_C A$ . Now,  $P_x \in \mathcal{R}_A \setminus \mathcal{L}_A$  (because  $A$  is strict), hence  $\overline{P}'_x \in \mathcal{R}_A \setminus \mathcal{L}_A$ . But then, because of [4](2.1),  $P'_x \in \mathcal{R}_C$ . Hence, because of [11](2.4), we have  $P''_x \in \mathcal{R}_D$ . However,  $\text{rad}P''_x$  is a simple homogeneous module over the hereditary full subcategory of  $D$  with class of objects  $D \setminus \{x\}$ . In particular,  $\text{rad}P''_x$  is not directed in  $\text{ind}D$ , hence neither is  $P''_x$ . This however contradicts the fact that  $P''_x \in \mathcal{R}_D$  (and [1] (6.4)). Therefore  $l(i) = 0$  for all  $i$  as asserted and so  $B$  is a direct product of simply connected algebras.  $\square$

We say that an ada algebra is of *tree type* if the orbit graph (see, for instance, [14] or [7](4.1)) of each of the  $\Gamma_i$  is a tree.

**Lemma 5.5.** *Let  $A = B[M]$  be an ada maximal extension. Then  $A$  is of tree type if and only if  $B$  is of tree type and the extension point is separating.*

**Proof.** Same as [7](4.1).  $\square$

A sequence of ada algebras of the form

$$A_\lambda = A_0 \subsetneq A_1 \subsetneq \cdots \subsetneq A_m = A$$

is called a *maximal filtration* of  $A$  provided that for each  $i$ , with  $1 \leq i \leq m$ , there exists an  $A_{i-1}$ -module  $M_i$  such that  $A_i = A_{i-1}[M_i]$  is a maximal extension.

**Proposition 5.6.** *Let  $A$  be a strict ada algebra. Then  $A$  admits a maximal filtration.*

**Proof.** Since  $A$  is strict, there exists an indecomposable projective in  $\mathcal{R}_A$  which is not in  $\mathcal{L}_A$ . Since every such projective is directed, because of [1](6.4), there exists (at least) a maximal projective  $P_x$ . Let  $A = B[M]$  where  $B = A \setminus \{x\}$  and  $M = \text{rad}P_x$ . Because of 2.8,  $B$  is also an ada algebra. If  $B$  is not strict, then every indecomposable projective  $B$ -module lies in  $\mathcal{L}_A \cap \text{ind}B = \mathcal{L}_B \subseteq \mathcal{L}_A$  and so  $B = A_\lambda$ . Otherwise, we apply induction.  $\square$

**Corollary 5.7.** *Let  $A$  be a strict ada algebra, then*

- (a)  $\text{HH}^1(A) = 0$  if and only if  $\text{HH}^1(A_\lambda) = 0$  and each of the extension points of a maximal filtration is separating.
- (b)  $\text{HH}^i(A) = 0$  for all  $i \geq 2$ .

**Proof.** (a) This follows immediately from 5.5 and 5.2.  
 (b) Follows from 5.5 and 5.2, using that  $A_\lambda$  is tilted and [20], Theorem 2.2.  $\square$

We also have the immediate corollary.

**Corollary 5.8.** *Let  $A$  be a strict ada algebra. Then  $A$  is of tree type if and only if  $A_\lambda$  is of tree type and each of the extension points in a maximal filtration is separating.*  $\square$

We are now in a position to prove our main result of this section.

**Theorem 5.9.** *Let  $A$  be an ada algebra. The following are equivalent:*

- (a)  $A$  is simply connected.
- (b)  $\mathrm{HH}^1(A) = 0$
- (c)  $A$  is of tree type.

**Proof.** We may assume that  $A$  is strict ada.

Assume first that  $\mathrm{HH}^1(A) = 0$ . Because of 5.7(a), we have  $\mathrm{HH}^1(A_\lambda) = 0$  and each of the extension points in a maximal filtration is separating. Because of [22],  $\mathrm{HH}^1(A_\lambda) = 0$  if and only if  $A_\lambda$  is a direct product of simply connected algebras. Applying 5.4(a) and induction, we get that  $A$  is simply connected.

Conversely, assume that  $A$  is a simply connected ada algebra. Therefore there exists a maximal projective  $P_x \in \mathcal{R}_A$ , such that  $A = B[M]$  is a maximal extension where, as usual,  $B = A \setminus \{x\}$  and  $M = \mathrm{rad}P_x$ . Now,  $x$  is a source in  $A$ , hence, by [9](2.6),  $x$  is separating. On the other hand, because of 5.4(b),  $B$  is a direct product of simply connected algebras. Hence, inductively,  $\mathrm{HH}^1(B) = 0$ . Applying 5.2(c), we get  $\mathrm{HH}^1(A) = 0$ .

The equivalence with condition (c) is proved in the same way using 5.8, and the fact proved in [22], that  $A_\lambda$  is of tree type if and only if  $\mathrm{HH}^1(A_\lambda) = 0$ .  $\square$

**Corollary 5.10.** *Let  $A$  be an ada algebra. Then  $A$  is simply connected if and only if the Hochschild cohomology ring is equal to  $k$ .*

**Proof.** This follows from 5.9 and 5.7(b).  $\square$

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