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Electrified thin films: Global existence of non-negative solutions

C. Imbert* and A. Mellet†

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Abstract

We consider an equation modeling the evolution of a viscous liquid thin film wetting a horizontal solid substrate destabilized by an electric field normal to the substrate. The effects of the electric field are modeled by a lower order non-local term. We introduce the good functional analysis framework to study this equation on a bounded domain and prove the existence of weak solutions defined globally in time for general initial data (with finite energy).

Keywords: Higher order equation, Non-local equation, Thin film equation, Non-negative solutions

MSC: 35G25, 35K25, 35A01, 35B09

1 Introduction

In this paper, we construct solutions for a thin film type equation with a destabilizing singular integral term. This term models the effects of an electric field (see [19]). From the analytical point of view, this paper belongs to the large body of literature devoted to the thin film equation with destabilizing terms such as long-wave unstable thin film problems [7, 8, 20] or the Kuramato-Sivashinsky equation in combustion and solidification [12, 11].

More precisely, we are considering the following equation, which is introduced by Tseluiko and Papageorgiou in [19] (see also [16]):

$$u_t + \left(u^3(cu_{xx} - \alpha u - \lambda I(u))_x\right)_x = 0 \qquad x \in [0, L], \quad t > 0 \tag{1}$$

(in [19], (1) is supplemented with periodic boundary conditions). This equation models the evolution of a liquid thin film (of height u) wetting a horizontal solid

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substrate which is subject to a gravity field and an electric field normal to the substrate. The term $\lambda I(u)$ models the effects of the electric field on the thin film. The operator $I(u)$ is a nonlocal elliptic operator of order 1 which will be defined precisely later on (for now, we can think of it as being the half-Laplace operator: $I(u) = -(-\Delta)^{1/2}u$). When $\lambda > 0$, it has a destabilizing effect (it has the "wrong" sign). The term αu accounts for the effects of gravity, and it is also destabilizing when $\alpha < 0$ ("hanging film"). In [19], it is proved that despite these destabilizing terms, positive smooth solutions of (1) do not blow up and remain bounded in H^1 for all time.

As mentioned above, there are many papers devoted to the study of thin film equations with destabilizing terms. In its simplest form, the thin film equation reads:

$$u_t + (f(u)u_{xxx})_x = 0, \tag{2}$$

The existence of non-negative weak solutions for (2) was first established by F. Bernis and A. Friedman [4] for $f(u) = u^n$, $n > 1$. Further results (existence for $n > 0$ and further regularity results) were later obtained, by similar technics, in particular by E. Beretta, M. Bertsch and R. Dal Passo [2] and A. Bertozzi and M. Pugh [5, 6]. Results in higher dimension were obtained in particular by Grün in [14, 13, 10]. The thin film equation with lower order destabilizing terms has also received a lot of interest. In particular in [7, 8], the following equation

$$u_t + (f(u)u_{xxx} - g(u)u_x)_x = 0 \tag{3}$$

is considered. Such a destabilizing term (which, unlike that of (1), is a local term of order 2) models, for instance, the effects of gravity for a hanging thin film, or van der Waals type interactions with the solid substrate. In [7], the nonlinearities $f(u) = u^n$ and $g(u) = u^m$ are considered and it is proved (among other things) that there is no blow-up for $m < n + 2$. In [8], for $f(u) = u$ and $g(u) = u^m$, it is proved that there is blow-up for $m \geq 3$ and initial data in $H^1(\mathbb{R})$ with negative "energy". The reader is referred to [8] for a precise statement.

In our equation (1), the nonlinearities in front of the stabilizing and destabilizing terms are the same ($f(u) = g(u) = u^3$), but the destabilizing term is elliptic of order 3 and is nonlocal in space. It is known (see [19]) that positive smooth solutions of (1) do not blow up. The goal of the present paper is to prove the existence of global in time weak solutions for (1).

Note that besides the existence of solutions, many important properties of the thin film equation (2) have been investigated (finite speed of propagation of the support, waiting time phenomenon, existence of source-type solutions *etc.*). The key tools in many of these studies are various delicate integral inequalities (in particular the so-called α-entropy inequalities and local entropy and energy inequalities. See [4, 3, 6, 7]). It is not clear that similar functional inequalities holds for (1). One reason is that the algebra involving the operator $I(u)$ is considerably more difficult than that of the Laplace operator. Another reason, is the obvious difficulty in deriving local estimates (due to the nonlocal nature of the operator $I(u)$). For that reason, we only address the existence issue in this paper.

As in [7, 8], the main difficulty in proving the existence of solutions for (1) comes from the fact that the energy (see (8) below) can take negative values. In order to obtain H^1 a priori estimate, one thus has to use the conservation of mass which, for non-negative solutions, gives a global in time L^1 bound for the solution (see Lemma 1).

With such an estimate in hand, the existence of global in time solutions should follow from the construction of approximated solutions satisfying the right functional inequalities. Typically, one needs to regularize the mobility coefficient u^3. One way to proceed is to replace the coefficient u^3 with $u^3+\varepsilon$ so that the equation becomes strictly parabolic. However, for such a regularized equation one cannot show the existence of non-negative solutions (the maximum principle does not hold for fourth order parabolic equations) and Lemma 1 is of no use. An alternative regularization is to replace the mobility coefficient u^3 with a function $f_\varepsilon(u)$ which satisfies in particular $f_\varepsilon(u) \sim u^4/\varepsilon$. For such (more degenerate) mobility coefficient, solutions are expected to be strictly positive and therefore smooth. This second regularization procedure was first suggested by Bernis and Friedman [4] and is used in particular by Bertozzi and Pugh [5, 7]. However, the local in time existence for such a degenerate equation is not clear to us, since the corresponding proofs in [4] rely on Schauder estimates which are not classical (and perhaps tedious) with our non-local singular term $I(u)$. For this reason, we choose the first regularization approach; this requires us to pay attention to the lack of positivity of the approximated solutions. In particular, the L^1 norm is not controlled, and the H^1 norm will be controlled by combining the energy inequality with the entropy inequality. The idea of combining the energy together with the entropy in order to get a Lyapunov functional appeared previously in [17] where a thin film equation with a nonlinear drift term is thouroughly studied in all dimensions.

The main contribution of this paper is thus to introduce the precise functional analysis framework to be used to treat the term $I(u)$ and to provide a method for constructing weak solutions satisfying the proper a priori estimates.

Rather than working in the periodic setting, we will consider equation (1) on a bounded domain with Neumann boundary conditions (these Neumann conditions can be interpreted as the usual contact angle conditions and seem physically more relevant - the periodic framework could be treated as well with minor modifications). Further details about the derivation of (1) will be given in Section 2. Since the gravity term is of lower order than the electric field term, it is of limited interest in the mathematical theory developed in this paper. We will thus take

$$\alpha = 0 \quad \text{and} \quad c = \lambda = 1.$$

We thus consider the following problem:

$$\begin{cases} u_t + \big(f(u)(u_{xx} - I(u))_x\big)_x = 0 & \text{for } x \in \Omega, \quad t > 0 \\ u_x = 0,\ f(u)(u_{xx} - I(u))_x = 0 & \text{for } x \in \partial\Omega, \quad t > 0 \\ u(x,0) = u_0(x) & \text{for } x \in \Omega. \end{cases} \tag{4}$$

The domain Ω is a bounded interval in $\mathbb{R}$; in the sequel, we will always take $\Omega = (0,1)$. The mobility coefficient $f(u)$ is a $\mathcal{C}^1$ function $f : [0,+\infty) \to (0,+\infty)$ satisfying

$$f(u) \sim u^n \quad \text{as } u \to 0 \tag{5}$$

for some $n > 1$. The operator I is a non-local elliptic operator of order 1 which will be defined precisely in Section 3 as the square root of the Laplace operator with Neumann boundary conditions (we have to be very careful with the definition of I in a bounded domain).

A priori estimates. As for the thin film equation (2), we prove the existence of solutions for (4) using a regularization/stability argument. The main tools are integral inequalities which provide the necessary compactness. Besides the conservation of mass, we will see that the solution u of (4) satisfies two important integral inequalities: We define the energy $\mathcal{E}(u)$ and the entropy $e(u)$ by

$$\mathcal{E}(u)(t) = \frac{1}{2}\int_\Omega (u_x^2(t) + u(t)Iu(t))dx \quad \text{and} \quad e(u)(t) = \frac{1}{2}\int_\Omega G(u(t))dx$$

where G is a non-negative convex function such that $fG'' = 1$. Classical solutions of (4) then satisfy:

$$\mathcal{E}(u)(t) + \int_0^t \int_\Omega f(u)\big[(u_{xx} - I(u))_x\big]^2 \, dx\, ds \le \mathcal{E}(u_0), \tag{6}$$

$$e(u)(t) + \int_0^t \int (u_{xx})^2 \, dx\, ds + \int_0^t \int u_x I(u)_x \, dx\, ds \le e(u_0). \tag{7}$$

Similar inequalities hold for the thin film equation (2). However, we see here the destabilizing effect of the nonlocal term $I(u)$: First, we note that as in [7, 8] the energy $\mathcal{E}(u)$ can be written as the difference of two non-negative quantities:

$$\mathcal{E}(u)(t) = \|u(t)\|^2_{\dot{H}^1(\Omega)} - \|u(t)\|^2_{\dot{H}^{\frac{1}{2}}(\Omega)}, \tag{8}$$

and may thus take negative values. Similarly, the entropy dissipation can be written as

$$\int_0^t \int (u_{xx})^2 \, dx\, ds + \int_0^t \int u_x I(u)_x \, dx\, ds = \|u(t)\|^2_{\dot{H}^2(\Omega)} - \|u(t)\|^2_{\dot{H}^{\frac{3}{2}}_N(\Omega)},$$

so the entropy may not be decreasing.

Nevertheless, it is reasonable to expect (4) to have solutions that exist for all times. Indeed, as shown in [19], the conservation of mass, the inequality (6) and the following functional inequality (see Lemma 1)

$$\|u\|^2_{\dot{H}^1(\Omega)} \le \alpha\mathcal{E}(u) + \beta\|u\|^2_{L^1(\Omega)}, \quad \forall u \in H^1(\Omega),$$

implies that non-negative solution remains bounded in $L^\infty(0,T;H^1(\Omega))$ for all time T.

Furthermore, the interpolation inequality

$$\begin{aligned}\|u(t)\|^2_{\dot H^{\frac{3}{2}}_N} &\leq C\|u(t)\|_{\dot H^1}\|u(t)\|_{\dot H^2}\\ &\leq \frac{1}{2}\|u(t)\|^2_{\dot H^2}+\frac{C}{2}\|u(t)\|^2_{\dot H^1}\end{aligned}$$

yields

$$e(u)(t)+\frac{1}{2}\int_0^t \|u(r)\|^2_{\dot H^2}\,dr \leq e(u_0)+\frac{1}{2}\int_0^t \|u(r)\|^2_{\dot H^1}\,dr,$$

and so the entropy remains bounded for all time as well.

Main results. We now state the two main results proved in this paper. They should be compared with Theorems 3.1 and 4.2 in [4, pp.185&194]. The first one deals with non-negative initial data whose entropies are finite.

We recall that G is a non-negative convex function such that

$$G''(u)=\frac{1}{f(u)} \qquad \text{for all } u>0.$$

Theorem 1. *Let $n>1$ and $u_0\in H^1(\Omega)$ be such that $u_0\geq 0$ and*

$$\int_\Omega G(u_0)\,dx<\infty. \tag{9}$$

For all $T>0$ there exists a function $u(t,x)\geq 0$ with

$$u\in\mathcal{C}(0,T;L^2(\Omega))\cap L^\infty(0,T;H^1(\Omega)), \quad u_x\in L^2(0,T;H^1_0(\Omega))$$

such that, for all $\phi\in\mathcal{D}([0,T)\times\bar\Omega)$ satisfying $\phi_x=0$ on $(0,T)\times\partial\Omega$,

$$\iint_Q u\phi_t-f(u)[u_{xx}-I(u)]\phi_{xx}-f'(u)u_x(u_{xx}-I(u))\phi_x\,dt\,dx + \int_\Omega u_0(x)\phi(0,x)\,dx=0. \tag{10}$$

Moreover, the function u satisfies for every $t\in[0,T]$,

$$\int_\Omega u(t,x)\,dx = \int_\Omega u_0(x)\,dx,$$

$$\mathcal{E}(u(t))+\int_0^t\int_\Omega f(u)\big[(u_{xx}-I(u))_x\big]^2\,ds\,dx \leq \mathcal{E}(u_0), \tag{11}$$

$$\int_\Omega G(u(t))\,dx+\int_0^t\int_\Omega (u_{xx})^2+u_xI(u)_x\,dsdx \leq \int_\Omega G(u_0)\,dx. \tag{12}$$

We point out that the weak formulation (10) involve two integrations by parts. Our second main result is concerned with non-negative initial data whose

entropies are possibly infinite (this is the case if u_0 vanishes on an open subset of Ω and $n \geq 2$). In that case, only one integration by parts is possible, and the solutions that we construct are weaker than those constructed in Theorem 1. In particular, the equation is only satisfied on the positivity set of the solution and the boundary conditions are satisfied in a weaker sense.

Theorem 2. *Assume $n > 1$ and let $u_0 \in H^1(\Omega)$ be such that $u_0 \geq 0$. For all $T > 0$ there exists a function $u(t,x) \geq 0$ such that*

$$u \in \mathcal{C}(0,T;L^2(\Omega)) \cap L^\infty(0,T;H^1(\Omega)) \cap \mathcal{C}^{\frac{1}{2},\frac{1}{8}}(\Omega \times (0,T))$$

such that

$$f(u)[u_{xx} - I(u)]_x \in L^2(P)$$

and such that, for all $\phi \in \mathcal{D}([0,T) \times \bar{\Omega})$,

$$\iint_Q u\phi_t \, dt \, dx + \iint_P f(u)[u_{xx} - I(u)]_x \phi_x \, dt \, dx + \int_\Omega u_0(x)\phi(0,x) \, dx = 0 \quad (13)$$

where $P = \{(x,t) \in \bar{Q} : u(x,t) > 0, t > 0\}$. Moreover, the function u satisfies the conservation of mass and the energy inequality (11).

Finally, u_x vanishes at all points (x,t) of $\partial\Omega \times (0,T)$ such that $u(x,t) \neq 0$.

Comments. These results are comparable to those of [4] when $\lambda = 0$. The reader might be surprised that they are presented in a different order than in [4]. The reason has to do with the proofs; indeed, in contrast with [4], weak solutions given by Theorem 2 are constructed as limits of the solutions given by Theorem 1. This is because the entropy is needed in order to construct the non-negative solutions. See the discussion at the beginning of Section 5 for further details.

As pointed out earlier, the non-linearities in front of the stabilizing (u_{xxx}) and destabilizing ($(I(u))_x$) are the same. This is in contrast with the work of Bertozzi and Pugh [7, 8]. By analogy with (3), one could consider the equation

$$u_t + (u^n u_{xxx} - u^m (I(u))_x)_x = 0$$

in which case a scaling analysis similar to that of [7] suggests that blow up can only occur if $m \geq n + 1$. However, to our knowledge, there is no physical motivation for such a generalization (in our case).

Organization of the article. In Section 2, we give more details about the physical model leading to (4). We gather, in Section 3, material that will be used throughout the paper. In particular, we detail the functional analysis framework and the definition of the non-local operator I (which is similar to that used in [15]). Section 4, 5 and 6 are devoted to the proofs of the main results. Finally, we give in Appendix a technical result which is more or less classical.

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2 Physical model

In this section, we briefly recall the derivation of (4) (see [19] for further details). We consider a viscous liquid film which completely wets a solid horizontal substrate and is constrained between two solid walls (at $x = 0$ and $x = 1$), see Figure 1. The fluid is Newtonian and is assumed to be a perfect conductor. The

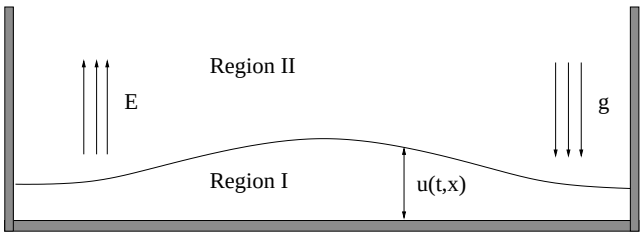

Figure 1: A viscous thin film submitted to an electric field $\mathbf{E}$ and gravity g

substrate is a grounded electrode held at zero voltage. Thanks to the presence of another electrode (at infinity), an electric field $\mathbf{E}$ is created which is constant at infinity (in the direction perpendicular to the substrate):

$$\mathbf{E}(x, y) \longrightarrow (0, E_0) \qquad \text{as } y \to +\infty.$$

The height of the fluid is denoted by $u(t, x)$. Under the assumptions of the lubrication approximation, it is classical that the evolution of u is described by Poiseuille's law:

$$u_t - \partial_x \left(\frac{u^3}{3\mu} \partial_x p \right) = 0 \tag{14}$$

where p is the pressure at the free surface of the fluid $y = u(t, x)$. This pressure is the sum of three terms:

1. The capillary pressure due to surface tension, which can be approximated by
$$p_1 \sim -\sigma u_{xx}$$
(replacing the mean curvature operator by the Laplacian).

2. The effect of gravity, given by
$$p_2 = gu.$$

3. The additional pressure due to the action of the electric field E.

To compute the third term appearing in the pressure, we introduce the potential V such that $\mathbf{E} = -\nabla V$, which satisfies

$$\Delta V = 0 \text{ for } y \geq u(x)$$

and

$$V(x, y) = 0 \quad \text{ on } y = u(x).$$

The condition at $y \to \infty$ means that we can write

$$V \sim E_0(Y_0 - y)$$

with (using standard linear approximation)

$$\begin{cases} \Delta Y_0 = 0 & \text{for } y > 0,\ x \in \Omega \\ \nabla Y_0 \to 0 & \text{as } y \to \infty,\ x \in \Omega \\ Y_0(x, 0) = u(x) & \text{for } x \in \Omega. \end{cases} \tag{15}$$

At the boundary of the cylinder, we assume that the electric field has no horizontal component:

$$\partial_x V = 0, \quad \text{ for } x \in \partial\Omega,\ y > 0.$$

The pressure exerted by the electric field is then proportional to

$$p_3 = \gamma \mathbf{E}_y = -\gamma \partial_y V(x, 0) = -\gamma E_0(\partial_y Y_0 - 1).$$

The application $u \mapsto \partial_y Y_0(x, 0)$ is a Dirichlet-to-Neumann map for the harmonic extension problem (15). We denote this operator by $I(u)$. We will see in Section 3 that $I(u)$ is in fact the square root of the Laplace operator on the interval Ω with homogeneous Neumann boundary conditions.

We thus have

$$p = p_1 + p_2 + p_3 = -\sigma u_{xx} + gu - \gamma E_0 I(u) + c_0$$

for some constant c_0, and we obtain (1) with $c = \frac{\sigma}{3\mu}$, $\alpha = \frac{g}{3\mu}$ and $\lambda = -\frac{\gamma E_0}{3\mu}$. Note that Poiseuille's law (14) is obtained under the no-slip condition for the fluid along the solid support. Other conditions, such as the Navier slip condition leads to

$$f(u) = u^3 + \Lambda u^s$$

with $s = 1$ or $s = 2$. This explains the interest of the community for general diffusion coefficient $f(u)$.

Boundary conditions. Along the boundary $\partial\Omega$, the fluid is in contact with a solid wall. It is thus natural to consider a contact angle condition at $x = 0$ and $x = 1$: Assuming that the contact angle is equal to $\pi/2$, we then get the boundary condition

$$u_x = 0 \qquad \text{on } \partial\Omega.$$

In [19], the authors derive their analytic results in a periodic setting which is obtained by considering the even extension of u to the interval $(-1,1)$ (recall that $\Omega = (0,1)$) and then taking the periodic extension (with period 2) to $\mathbb{R}$.

Finally, since the equation is of order 4, we need an additional boundary condition. We thus assume that u satisfies the following null-flux condition

$$u^3(u_{xx} - I(u))_x = 0 \qquad \text{on } \partial\Omega$$

which will guarantee the conservation of mass.

3 Preliminaries

In this section, we recall how the operator I is defined (see [15]) and give the functional analysis results that we will need to prove the main theorem. A very similar operator, with Dirichlet boundary conditions rather than Neumann boundary conditions, was studied by Cabré and Tan [9].

3.1 Functional spaces

The space $H^s_N(\Omega)$. We denote by $\{\lambda_k, \varphi_k\}_{k=0,1,2\ldots}$ the eigenvalues and corresponding eigenfunctions of the Laplace operator in Ω with Neumann boundary conditions on $\partial\Omega$:

$$\begin{cases} -\Delta\varphi_k = \lambda_k\varphi_k & \text{in } \Omega \\ \partial_\nu\varphi_k = 0 & \text{on } \partial\Omega, \end{cases} \tag{16}$$

normalized so that $\int_\Omega \varphi_k^2\,dx = 1$. When $\Omega = (0,1)$, we have

$$\lambda_0 = 0\,, \qquad \varphi_0(x) = 1$$

and

$$\lambda_k = (k\pi)^2\,, \qquad \varphi_k(x) = \sqrt{2}\cos(k\pi x) \qquad k = 1,\,2,\,3,\,\ldots$$

The φ_k's clearly form an orthonormal basis of $L^2(\Omega)$. Furthermore, the φ_k's also form an orthogonal basis of the space $H^s_N(\Omega)$ defined by

$$H^s_N(\Omega) = \left\{ u = \sum_{k=0}^{\infty} c_k\varphi_k\,;\ \sum_{k=0}^{\infty} c_k^2(1+\lambda_k^s) < +\infty \right\}$$

equipped with the norm

$$||u||^2_{H^s_N(\Omega)} = \sum_{k=0}^{\infty} c_k^2(1+\lambda_k^s)$$

or equivalently (noting that $c_0 = \int_\Omega u(x)\,dx$ and $\lambda_k \geq 1$ for $k \geq 1$):

$$||u||^2_{H^s_N(\Omega)} = \|u\|^2_{L^1(\Omega)} + \|u\|^2_{\dot{H}^s_N(\Omega)}$$

where the homogeneous norm is given by:

$$\|u\|^2_{\dot{H}^s_N(\Omega)} = \sum_{k=1}^{\infty} c_k^2\lambda_k^s.$$

A characterisation of $H^s_N(\Omega)$. The precise description of the space $H^s_N(\Omega)$ is a classical problem.

Intuitively, for $s < 3/2$, the boundary condition $u_\nu = 0$ does not make sense, and one can show that (see Agranovich and Amosov [1] and references therein):

$$H^s_N(\Omega) = H^s(\Omega) \quad \text{for all } 0 \le s < \frac{3}{2}.$$

In particular, we have $H^{\frac{1}{2}}_N(\Omega) = H^{\frac{1}{2}}(\Omega)$ and we will see later that

$$\|u\|^2_{\dot{H}^{\frac{1}{2}}(\Omega)} = \int_\Omega \int_\Omega (u(y) - u(x))^2 \nu(x,y) dx dy$$

where $\nu(x,y)$ is a given positive function; see (20) below.

For $s > 3/2$, the Neumann condition has to be taken into account, and we have in particular

$$H^2_N(\Omega) = \{u \in H^2(\Omega)\,;\, u_\nu = 0 \text{ on } \partial\Omega\}$$

which will play a particular role in the sequel. More generally, a similar characterization holds for $3/2 < s < 7/2$. For $s > 7/2$, additional boundary conditions would have to be taken into account, but we will not use such spaces in this paper. In Section 4, we will also work with the space $H^3_N(\Omega)$ which is exactly the set of functions in $H^3(\Omega)$ satisfying $u_\nu = 0$ on $\partial\Omega$.

The case $s = 3/2$ is critical (note that $u_\nu|_{\partial\Omega}$ is not well defined in that space) and one can show that

$$H^{\frac{3}{2}}_N(\Omega) \quad = \quad \left\{ u \in H^{\frac{3}{2}}(\Omega)\,;\, \int_\Omega \frac{u_x^2}{d(x)}\, dx < \infty \right\}$$

where $d(x)$ denotes the distance to $\partial\Omega$. A similar result appears in [9]; more precisely, such a characterization of $H^{\frac{3}{2}}_N(\Omega)$ can be obtained by considering functions u such that $u_x \in \mathcal{V}_0(\Omega)$ where $\mathcal{V}_0(\Omega)$ is defined in [9] as the equivalent of our space $H^{1/2}_N(\Omega)$ with Dirichlet rather than Neumann boundary conditions. We do not dwell on this issue since we will not need this result in this paper.

3.2 The operator I

As it is explained in the Introduction, the operator I is related to the computation of the pressure as a function of the height of the fluid.

Spectral definition. With λ_k and φ_k defined by (16), we define the operator

$$I : \sum_{k=0}^{\infty} c_k \varphi_k \;\longmapsto\; -\sum_{k=0}^{\infty} c_k \lambda_k^{\frac{1}{2}} \varphi_k \tag{17}$$

which clearly maps $H^1(\Omega)$ onto $L^2(\Omega)$ and $H^2_N(\Omega)$ onto $H^1(\Omega)$.

Dirichlet-to-Neuman map. We now check that this definition of the operator I is the same as the one given in Section 2, namely I is the Dirichlet-to-Neumann operator associated with the Laplace operator supplemented with Neumann boundary conditions:

We consider the following extension problem:

$$\begin{cases} -\Delta v = 0 & \text{in } \Omega \times (0, +\infty), \\ v(x, 0) = u(x) & \text{on } \Omega, \\ v_\nu = 0 & \text{on } \partial\Omega \times (0, \infty). \end{cases} \tag{18}$$

Then, we can show (see [15]):

Proposition 1 ([15])**.** *For all $u \in H_N^{\frac{1}{2}}(\Omega)$, there exists a unique extension $v \in H^1(\Omega \times (0, +\infty))$ solution of (18).*
Furthermore, if $u(x) = \sum_{k=1}^{\infty} c_k \varphi_k(x)$, then

$$v(x, y) = \sum_{k=1}^{\infty} c_k \varphi_k(x) \exp(-\lambda_k^{\frac{1}{2}} y). \tag{19}$$

and we have:

Proposition 2 ([15])**.** *For all $u \in H_N^2(\Omega)$, we have*

$$I(u)(x) = -\frac{\partial v}{\partial \nu}(x, 0) = \partial_y v(x, 0) \quad \text{for all } x \in \Omega,$$

where v is the unique harmonic extension solution of (18).
Furthermore $I \circ I(u) = -\Delta u$.

Integral representation. Finally, the operator I can also be represented as a singular integral operator:

Proposition 3 ([15])**.** *Consider a smooth function $u : \Omega \to \mathbb{R}$. Then for all $x \in \Omega$,*

$$I(u)(x) = \int_\Omega (u(y) - u(x))\nu(x, y) dy$$

where $\nu(x, y)$ is defined as follows: for all $x, y \in \Omega$,

$$\nu(x, y) = \frac{\pi}{2} \left(\frac{1}{1 - \cos(\pi(x - y))} + \frac{1}{1 - \cos(\pi(x + y))} \right). \tag{20}$$

3.3 Functional equalities and inequalities

Equalities. The semi-norms $||\cdot||_{\dot{H}^{\frac{1}{2}}(\Omega)}$, $||\cdot||_{\dot{H}^1(\Omega)}$, $||\cdot||_{\dot{H}_N^{\frac{3}{2}}(\Omega)}$ and $||\cdot||_{\dot{H}_N^2(\Omega)}$ are related to the operator I by equalities which will be used repeatedly.

Proposition 4 (The operator I and several semi-norms – [15])**.**

For all $u \in H^{\frac{1}{2}}(\Omega)$, we have

$$-\int uI(u)\,dx = \frac{1}{2}\int_\Omega\int_\Omega (u(x)-u(y))^2\nu(x,y)dxdy = ||u||^2_{\dot{H}^{\frac{1}{2}}(\Omega)}$$

where ν is defined in (20).

For all $u \in H^2_N(\Omega)$, we have

$$-\int_\Omega u_x I(u)_x\,dx = ||u||^2_{\dot{H}^{\frac{3}{2}}_N(\Omega)}.$$

For all $k \in \mathbb{N}$ and $u \in H^{k+1}_N(\Omega)$, we have

$$\int_\Omega (\partial_x^k I(u))^2\,dx = \|u\|^2_{\dot{H}^{k+1}_N(\Omega)}. \tag{21}$$

Inequalities. First, we recall the following Nash inequality:

$$\|u\|_{L^2(\Omega)} \le C\|u\|^{\frac{1}{3}}_{H^1(\Omega)}\|u\|^{\frac{2}{3}}_{L^1(\Omega)}.$$

It implies in particular that,

$$\|u\|^2_{L^2(\Omega)} \le \frac{1}{2}\|u\|^2_{H^1(\Omega)} + C\|u\|^2_{L^1(\Omega)}. \tag{22}$$

This inequality will allow us to control the H^1 norm by the energy $\mathcal{E}(u)$ and the L^1 norm. Indeed, we recall that the energy is defined by

$$\mathcal{E}(u) = \frac{1}{2}\int_\Omega |u_x|^2 + u\,I(u)\,dx = \frac{1}{2}\|u\|^2_{\dot{H}^1(\Omega)} - \frac{1}{2}\|u\|^2_{\dot{H}^{\frac{1}{2}}(\Omega)}. \tag{23}$$

We then have:

Lemma 1. *There exist positive constants α, β such that for all $u \in H^1(\Omega)$,*

$$\|u\|^2_{\dot{H}^1(\Omega)} \le \alpha\mathcal{E}(u) + \beta\|u\|^2_{L^1(\Omega)}.$$

Remark 1. See also Lemma 4.1 in [19].

Proof. We have

$$\|u\|^2_{\dot{H}^1(\Omega)} = 2\mathcal{E}(u) + \|u\|^2_{\dot{H}^{\frac{1}{2}}(\Omega)},$$

and using (21) with $k = 0$ and (22), we get:

$$\begin{aligned}
\|u\|^2_{\dot{H}^{\frac{1}{2}}(\Omega)} &= -\int u\,I(u)\,dx \\
&\le \|u\|_{L^2(\Omega)}\|I(u)\|_{L^2(\Omega)} \\
&\le \|u\|_{L^2(\Omega)}\|u\|_{\dot{H}^1(\Omega)} \\
&\le \frac{1}{2}\|u\|^2_{L^2(\Omega)} + \frac{1}{2}\|u\|^2_{\dot{H}^1(\Omega)} \\
&\le \frac{3}{4}\|u\|^2_{\dot{H}^1(\Omega)} + \frac{C}{2}\|u\|^2_{L^1(\Omega)},
\end{aligned}$$

hence the result. □

4 A regularized problem

We can now turn to the proof of Theorem 1 and 2. As usual, we introduce the following regularized equation:

$$\begin{cases} u_t + (f_\varepsilon(u)(u_{xx} - I(u))_x)_x = 0 & \text{for } x \in \Omega, \quad t > 0 \\ u_x = 0,\ f_\varepsilon(u)(u_{xx} - I(u))_x = 0 & \text{for } x \in \partial\Omega, \quad t > 0 \\ u(x,0) = u_0(x) & \text{for } x \in \Omega \end{cases} \tag{24}$$

where the mobility coefficient $f(u)$ is approximated by

$$f_\varepsilon(u) = \min(\max(\varepsilon, f(|u|)), M)$$

which satisfies

$$\varepsilon \le f_\varepsilon(u) \le M \qquad \text{for all } u \in \mathbb{R}.$$

Ultimately, we will show that the solution u satisfies

$$0 \le u(t,x) \le M_0$$

for some constant M_0 independent of M, so that we do not have to worry about M (provided we take it large enough). The ε is of course the most important parameter in the regularization since it makes (24) non-degenerate.

The proof of Theorem 1 consists of two parts: First, we have to show that the regularized equation (24) has a solution (which may take negative values). Then we must pass to the limit $\varepsilon \to 0$ and show that we obtain a non-negative solution of (4).

In this section, we prove the first part. Namely, we prove (to be compared with Theorem 1.1 in [4]):

Theorem 3. *Let $u_0 \in H^1(\Omega)$. For all $T > 0$ there exists a function $u^\varepsilon(t,x)$ such that*

$$u^\varepsilon \in \mathcal{C}(0,T; L^2(\Omega)) \cap L^\infty(0,T; H^1(\Omega)) \cap L^2(0,T; H^3_N(\Omega))$$

such that, for all $\phi \in \mathcal{D}([0,T) \times \bar{\Omega})$,

$$\iint_Q u^\varepsilon \phi_t + f_\varepsilon(u^\varepsilon)[u^\varepsilon_{xx} - I(u^\varepsilon)]_x \phi_x \, dt\, dx + \int_\Omega u_0(x)\phi(0,x)\, dx = 0. \tag{25}$$

Moreover, the function u^ε satisfies for every $t \in [0,T]$,

$$\begin{aligned} \int_\Omega u^\varepsilon(t,x)\, dx &= \int_\Omega u_0(x)\, dx, \\ \mathcal{E}(u^\varepsilon(t)) + \int_0^t \int_\Omega f_\varepsilon(u^\varepsilon)\big[(u^\varepsilon_{xx} - I(u^\varepsilon))_x\big]^2 \, ds\, dx &\le \mathcal{E}(u_0), \end{aligned} \tag{26}$$

and

$$\int_\Omega (u_x^\varepsilon)^2\,dx + \int_0^t \int_\Omega f_\varepsilon(u^\varepsilon)(u_{xxx}^\varepsilon)^2\,ds\,dx \\ \leq \int_\Omega ((u_0)_x)^2\,dx + \int_0^t \int_\Omega f_\varepsilon(u^\varepsilon) u_{xxx}^\varepsilon (I(u^\varepsilon))_x\,ds\,dx \quad (27)$$

and

$$\int_\Omega G_\varepsilon(u^\varepsilon(t))\,dx + \int_0^t \int_\Omega (u_{xx}^\varepsilon)^2 + u_x^\varepsilon I(u^\varepsilon)_x\,dsdx \leq \int_\Omega G_\varepsilon(u_0)\,dx. \quad (28)$$

where G_ε is a non-negative function such that $f_\varepsilon G_\varepsilon'' = 1$.

Finally, u^ε is $\frac{1}{2}$-Hölder continuous with respect to x and $\frac{1}{8}$-Hölder continuous with respect to t; more precisely, there exists a constant C_0 only depending on Ω and $\|u^\varepsilon\|_{L^\infty(0,T;H^1(\Omega))}$ and $\|f_\varepsilon(u^\varepsilon)[u_{xx}^\varepsilon - I(u^\varepsilon)]_x\|_{L^2(Q)}$ such that

$$\|u^\varepsilon\|_{\mathcal{C}_{t,x}^{\frac{1}{2},\frac{1}{8}}(Q)} \leq C_0. \quad (29)$$

Remark 2. This theorem is very similar to Theorem 1.1 in [4], and some steps in our proof follow along the lines of [4]. For instance, getting the Hölder estimates from the $L^\infty H^1$ estimate is done in the same way. However, the main difficulty in proving this theorem is precisely to get the $L^\infty H^1$ estimate; this step is not straightforward at all and this is a significant difference with [4].

Proof of Theorem 3. Theorem 3 follows from a fixed point argument: For some T_*, we denote

$$V = L^2(0, T_*; H_N^2(\Omega))$$

and we define the application $\mathcal{F} : V \to V$ such that for $v \in V$, $\mathcal{F}(v)$ is the solution u of

$$\begin{cases} u_t + (f_\varepsilon(v)(u_{xxx} - I(v)_x))_x = 0 & \text{for } x \in \Omega, \quad t > 0 \\ u_x = 0,\ f_\varepsilon(v)(u_{xx} - I(v))_x = 0 & \text{for } x \in \partial\Omega, \quad t > 0 \\ u(x,0) = u_0(x) & \text{for } x \in \Omega. \end{cases} \quad (30)$$

The fact that $\mathcal{F}$ is well defined follows from the observation that for $v \in V$, we have

$$a(t,x) = f_\varepsilon(v(t,x)) \in [\varepsilon, M] \quad \text{and} \quad g(t,x) = I(v)_x \in L^2(Q)$$

and the following proposition:

Proposition 5. *Consider $u_0 \in H^1(\Omega)$ and $a(t,x) \in L^\infty(Q)$ such that $\varepsilon \leq a(t,x) \leq M$ a.e. in Q. If $g \in L^2(Q)$, then there exists a function*

$$u \in \mathcal{C}(0,T; L^2(\Omega)) \cap L^\infty(0,T; H^1(\Omega)) \cap L^2(0,T; H_N^3(\Omega))$$

and

$$\iint_Q \left[u\phi_t + a(u_{xxx} - g)\phi_x\right] dt\, dx + \int_\Omega u_0(x)\phi(0,x)\, dx = 0$$

for all $\phi \in \mathcal{D}([0,T) \times \bar{\Omega})$*. Moreover, for every* $t \in [0,T]$*,* u *satisfies*

$$\int_\Omega u(t,x)\, dx = \int_\Omega u_0(x)\, dx$$

and

$$\int_\Omega (u_x)^2(t)\, dx + \frac{1}{2}\int_0^t \int_\Omega a(u_{xxx})^2\, ds\, dx \leq \frac{M}{2}\int_0^t \int_\Omega g^2\, ds\, dx + \int_\Omega (u_0)_x^2\, dx. \quad (31)$$

Furthermore, u *is* $\frac{1}{2}$*-Hölder continuous with respect to* x *and* $\frac{1}{8}$*-Hölder continuous with respect to* t*; more precisely, there exists a constant* C_0 *only depending on* Ω *and* $\|u\|_{L^\infty(0,T;H^1(\Omega))}$ *and* $\|u_{xxx} - g\|_{L^2(Q)}$ *such that*

$$\|u\|_{\mathcal{C}^{\frac{1}{2},\frac{1}{8}}_{t,x}(Q)} \leq C_0. \quad (32)$$

This proposition is a very natural existence result for the fourth order linear parabolic equation

$$u_t + (au_{xxx})_x = (ag)_x.$$

Its proof is fairly classical, we give some details in Appendix A for the interested reader.

Next, we show the following result:

Lemma 2. *There exists a (small) time* $T_* > 0$*, depending only on* ε*,* M *and* Ω*, such that* $\mathcal{F}$ *has a fixed point* u *in* $V = L^2(0,T_*; H^2_N(\Omega))$ *for any initial data* $u_0 \in H^1(\Omega)$*. Furthermore,* u *satisfies*

$$\|u\|_V \leq R||u_0||_{\dot{H}^1(\Omega)}$$

and

$$\|u\|_{L^\infty(0,T_*;\dot{H}^1(\Omega))} \leq \sqrt{2}||u_0||_{\dot{H}^1(\Omega)}. \quad (33)$$

Before proving this lemma, let us complete the proof of Theorem 3.

Construction of a solution for large times. Lemma 2 gives the existence of a solution u_1^ε of (24) defined for $t \in [0,T_*]$. Since T_* does not depend on the initial condition, we can apply Lemma 2 to construct a solution u_2^ε in $[T_*, 2T_*]$ with initial condition $u_1^\varepsilon(T_*,x)$ which is $H^1(\Omega)$ by (33). This way, we obtain a solution u^ε of (24) on the time interval $[0,2T_*]$. Note that we also have

$$\|u^\varepsilon\|_{L^\infty(0,2T_*;\dot{H}^1(\Omega))} \leq \sqrt{2}^2||u_0||_{\dot{H}^1(\Omega)}.$$

Iterating this argument, we construct a solution u^ε on any interval $[0,T]$ satisfying in particular, for all $k \in \mathbb{N}$ such that $kT_* \leq T$,

$$||u^\varepsilon||_{L^\infty(0,kT_*,\dot{H}^1(\Omega))} \leq \sqrt{2}^k R||u_0||_{\dot{H}^1(\Omega)}.$$

Energy and entropy estimates. The conservation of mass follows from Proposition 5, but we need to explain how to derive (26), (27) and (28) from (25). Formally, one has to choose successively $\phi = -u^\varepsilon_{xx} + I(u^\varepsilon)$, $\phi = -u^\varepsilon_{xx}$ and $\phi = G'_\varepsilon(u^\varepsilon)$. Making such a formal computation rigourous is quite standard; details are given in Appendix B for the reader's convenience. Finally, (29) follows from (32). □

Proof of Lemma 2. We need to check that the conditions of Leray-Schauder's fixed point theorem are satisfied:

$\mathcal{F}$ is compact. Let $(v_n)_n$ be a bounded sequence in V and let u_n denote $\mathcal{F}(v_n)$. The sequence $(I(v_n))_x$ is bounded in $L^2(Q)$, and so

$$g_n = f_\varepsilon(v_n)\partial_x I(v_n) \text{ is bounded in } L^2(Q).$$

Estimate (31) implies that u_n is bounded in $L^2(0, T_*; H^3_N(\Omega))$. In particular $\partial_{xxx} u_n$ is bounded in $L^2(0, T_*; L^2(\Omega))$ and Equation (30) implies that $\partial_t(u_n)$ is bounded in $L^2(0, T_*, H^{-1}(\Omega))$. Using Aubin's lemma, we deduce that $(u_n)_n$ is pre-compact in $V = L^2(0, T_*; H^2_N(\Omega))$.

$\mathcal{F}$ is continuous. Consider now a sequence $(v_n)_n$ in V such that $v_n \to v$ in V and let $u_n = \mathcal{F}(v_n)$. We have in particular $v_n \to v$ in $L^2(Q)$ and, up to a subsequence, we can assume that $v_n \to v$ almost everywhere in Q. Hence, $f_\varepsilon(v_n) \to f_\varepsilon(v)$ almost everywhere in Q. We also have that $(I(v_n))_x$ converges to $(I(v))_x$ in $L^2(Q)$, and since $|f_\varepsilon(v_n)| \le M$ a.e., we can show that

$$g_n = f_\varepsilon(v_n)\partial_x I(v_n) \to f_\varepsilon(v)\partial_x I(v) = g \quad \text{ in } L^2(Q).$$

Next, the compacity of $\mathcal{F}$ implies that $(u_n)_n$ is pre-compact in the space $L^2(0, T; H^2_N(\Omega))$, and so u_n converges (up to a subsequence) to U in V. In particular, $u_n \to U$ in $L^2(Q)$ and (up to a another subsequence), $u_n \to U$ almost everywhere in Q. We thus have $f_\varepsilon(u_n) \to f_\varepsilon(U)$ in $L^2(Q)$, and passing to the limit in the equation, we conclude that $U = u = \mathcal{F}(v)$ (by the uniqueness result in Proposition 5). Since this holds for any subsequence of u_n, we deduce that the whole sequence u_n converges to u hence

$$\mathcal{F}(v_n) \to \mathcal{F}(v) \text{ in } V \text{ as } n \to \infty$$

and $\mathcal{F}$ is continuous.

A priori estimates. It only remains to show that there exists a constant $R > 0$ such that for all functions $u \in V$ and $\sigma \in [0, 1]$ such that $u = \sigma\mathcal{F}(u)$, we have

$$\|u\|_V \le R.$$

This is where the smallness of T_* will be needed.

Using (31), we see that

$$\int_\Omega (u_x(t))^2\,dx + \frac{\varepsilon}{2}\int_0^{T_*}\int_\Omega (u_{xxx})^2\,dx\,dt \\ \leq \frac{M}{2}\int_0^{T_*}\int_\Omega ((I(u)_x)^2\,dx\,dt + \int_\Omega (u_0)_x^2\,dx, \quad (34)$$

and using (21) and the interpolation inequality

$$\|u_x\|_{L^2(\Omega)} \leq C\|u\|_{L^2(\Omega)}^{\frac{1}{2}}\|u_{xx}\|_{L^2(\Omega)}^{\frac{1}{2}}, \quad (35)$$

we get:

$$\begin{aligned}\int_0^{T_*}\int_\Omega ((I(u)_x)^2\,dx\,dt &\leq 2C_\varepsilon\int_0^{T_*}\|I(u)(t)\|_2^2\,dt + \frac{\varepsilon}{2M}\int_0^{T_*}\|(I(u))_{xx}(t)\|_2^2\,dt \\ &\leq 2C_\varepsilon\int_0^{T_*}\|u_x(t)\|_2^2\,dt + \frac{\varepsilon}{2M}\int_0^{T_*}\|u_{xxx}(t)\|_2^2\,dt \\ &\leq 2C_\varepsilon T_*\sup_{t\in[0,T_*]}\|u_x(t)\|_2^2 + \frac{\varepsilon}{2M}\int_0^{T_*}\|u_{xxx}(t)\|_2^2\,dt \quad (36)\end{aligned}$$

where C_ε only depends on the constant C in (35) and the parameters ε and M. Combining (34) and (36), we conclude that

$$(1 - MC_\varepsilon T_*)\sup_{t\in[0,T]}\|u_x(t)\|_2^2 + \frac{\varepsilon}{4}\int_0^{T_*}\|u_{xxx}(t)\|_2^2\,dt \leq \int_\Omega (u_0)_x^2\,dx.$$

Therefore, choosing $T_* := \frac{1}{2MC_\varepsilon}$, we get the following estimates

$$\|u\|_{L^\infty(0,T_*;\dot{H}^1(\Omega))} \leq \sqrt{2}||u_0||_{\dot{H}^1(\Omega)} \quad \text{and } \|u\|_{L^2(0,T_*;\dot{H}^3_N(\Omega))} \leq \frac{2}{\sqrt{\varepsilon}}||u_0||_{\dot{H}^1(\Omega)}$$

Since we also have

$$\int_\Omega u(t)\,dx = \int_\Omega u_0\,dx,$$

we deduce that $\|u\|_V \leq R$ for some constant R depending on ε, which completes the proof. □

5 Proof of Theorem 1

As pointed out in the introduction, one of the main difficulties in the proof of Theorem 1 is that the natural energy estimate (31) does not give any information by itself, since $\mathcal{E}(u)$ may be negative. Even if Lemma 1 implies that the quantity

$$\alpha\mathcal{E}(u^\varepsilon) + \beta||u^\varepsilon||_{L^1(\Omega)}$$

is bounded below by the H^1 norm of u, the mass conservation only allows us to control the L^1 norm of u^ε if we know that u^ε is non-negative. Unfortunately, it is well known that equation (24) does not satisfy the maximum principle, and that the existence of non-negative solutions of (4) is precisely a consequence of the degeneracy of the diffusion coefficient, so that while we can hope (and we will prove) to have $\lim_{\varepsilon\to 0} u^\varepsilon \geq 0$ we do not have, in general, that $u^\varepsilon \geq 0$. Lemma 3 below will show that it is nevertheless possible to derive some a priori estimates that are enough to pass to the limit, provided the initial entropy is finite (9).

Proof of Theorem 1. Consider the solution u^ε of (24) given by Theorem 3. In order to prove Theorem 1, we need to show that $\lim_{\varepsilon\to 0} u^\varepsilon$ exists and solves (10).

Since we cannot use the energy inequality to get the necessary estimates on u^ε, we will need the following lemma:

Lemma 3. *Let H_ε denote the following functional:*

$$H_\varepsilon(v) = \int_\Omega [v_x^2 + 2MG_\varepsilon(v)]\, dx.$$

Then the solution u^ε given by Theorem 3 satisfies

$$H_\varepsilon(u^\varepsilon(t)) + \frac{M}{2}\int_0^t \int_\Omega (u^\varepsilon_{xx})^2\, dx\, ds + \frac{1}{2}\int_0^t \int_\Omega f_\varepsilon(u^\varepsilon)(u^\varepsilon_{xxx})^2\, dx\, ds \leq H_\varepsilon(u_0)e^{t/2}$$

for every $t \in [0,T]$.

Proof of Lemma 3. Using (27) and (28) we see that

$$\int_\Omega G_\varepsilon(u^\varepsilon(t))\, dx + \frac{1}{2}\int_0^t \int_\Omega (u^\varepsilon_{xx})^2\, dx\, ds \leq \int_\Omega G_\varepsilon(u_0)\, dx + \frac{1}{2}\int_0^t \int_\Omega (u^\varepsilon_x)^2\, dx\, ds$$

and

$$\int_\Omega (u^\varepsilon_x)^2\, dx + \frac{1}{2}\int_0^t \int_\Omega f_\varepsilon(u^\varepsilon)[u^\varepsilon_{xxx}]^2\, dx\, ds \leq \int_\Omega ((u_0)_x)^2\, dx + \frac{M}{2}\int_0^t \int_\Omega (u^\varepsilon_{xx})^2\, dx\, ds.$$

This implies

$$H_\varepsilon(u^\varepsilon(t)) \leq H_\varepsilon(u_0) + \int_0^t H_\varepsilon(u^\varepsilon(s))ds,$$

and Gronwall's lemma yields the desired result. □

Sobolev and Hölder bounds. We now gather all the a priori estimates: Using the conservation of mass, Lemma 3 and inequality (9), we see that there exists a constant C independent of ε such that:

$$\sup_{t\in[0,T]} \int_\Omega (u_x^\varepsilon(t))^2\,dx \quad \leq \quad C, \tag{37}$$

$$\sup_{t\in[0,T]} \int_\Omega G_\varepsilon(u^\varepsilon(t))\,dx \quad \leq \quad C, \tag{38}$$

$$\int_0^T \int_\Omega f_\varepsilon(u^\varepsilon)[u_{xxx}^\varepsilon]^2\,dx\,dt \quad \leq \quad C, \tag{39}$$

$$\int_0^T \int_\Omega (u_{xx}^\varepsilon)^2\,dx\,dt \quad \leq \quad C. \tag{40}$$

Next, we note that (37) yields

$$\mathcal{E}(u^\varepsilon) \geq -||u^\varepsilon||_{L^\infty(0,T;\dot{H}^{1/2}(\Omega))} \geq -C||u^\varepsilon||_{L^\infty(0,T;H^1(\Omega))} \geq -C$$

and so (26) gives

$$\int_0^T \int_\Omega f_\varepsilon(u^\varepsilon)\big[u_{xxx}^\varepsilon - I(u^\varepsilon)_{xx}\big]^2\,ds\,dx \leq C. \tag{41}$$

Finally, estimates (29), (37) and (41) yield that u^ε is bounded in $\mathcal{C}_{x,t}^{1/2,1/8}(Q)$.

Limit $\varepsilon \to 0$. The previous Hölder estimate implies that there exists a function $u(x,t)$ such that u^ε converges uniformly to u as ε goes to zero (up to a subsequence). Inequality (40) also implies that

$$u^\varepsilon \rightharpoonup u \text{ in } L^2(0,T;H^2(\Omega))\text{-weak}$$

and Aubin's lemma gives

$$u^\varepsilon \longrightarrow u \text{ in } L^2(0,T;H^1(\Omega))\text{-strong.}$$

After integration by parts, (25) can be written as

$$\iint_Q u^\varepsilon\phi_t - f_\varepsilon(u^\varepsilon)[u_{xx}^\varepsilon - I(u^\varepsilon)]\phi_{xx} - f_\varepsilon'(u^\varepsilon)u_x^\varepsilon(u_{xx}^\varepsilon - I(u^\varepsilon))\phi_x\,dt\,dx$$
$$= \int_\Omega u_0(x)\phi(0,x)\,dx$$

and passing to the limit $\varepsilon \to 0$ gives (10).

Non-negative solution. It only remains to show that u is non-negative. This can be done as in [4], using (38) (and the fact that f satisfies (5) with $n > 1$).

L^∞ **a priori estimate.** Finally, (37) and Sobolev's embedding implies that there exits a constant M_0 depending only on $||u_0||_{H^1(\Omega)}$ such that

$$0 \leq u(t,x) \leq M_0. \tag{42}$$

Choosing $M > M_0$, we deduce that u solves (4). □

6 Proof of Theorem 2

In order to get Theorem 2, we need to derive the following corollary from Theorem 1.

Corollary 1. *The solution u constructed in Theorem 1 satisfies for all $\phi \in \mathcal{D}((0,T) \times \bar{\Omega})$,*

$$\iint_Q u\phi_t \, dt \, dx + \iint_P f(u)[u_{xx} - I(u)]_x \phi_x \, dt \, dx = 0 \tag{43}$$

where $P = \{(x,t) \in \bar{Q} : u(x,t) > 0, t > 0\}$.

Proof. In view of the proof of Theorem 1, u is the uniform limit of a subsequence of $(u^\varepsilon)_{\varepsilon>0}$ where u^ε is given by Theorem 3. Since u^ε satisfies (25), it is thus enough to pass to the limit in this weak formulation as $\varepsilon \to 0$ in order to get the desired result. Let h^ε denote $f_\varepsilon(u^\varepsilon)[u^\varepsilon_{xx} - I(u)]_x$. Estimates (41) and (42) imply

$$\iint_Q h_\varepsilon^2 \, dx \, dt \leq C. \tag{44}$$

In other words, $(h^\varepsilon)_\varepsilon$ is bounded in $L^2(Q)$. Hence, up to a subsequence,

$$h^\varepsilon \rightharpoonup h \qquad \text{in } L^2(Q)-\text{weak}.$$

Furthermore, we recall that there exists a continuous function $u(x,t)$ such that u^ε converges uniformly to u as ε goes to zero (up to a subsequence).

Passing to the limit in (25), we deduce that the function u satisfies

$$\iint_Q u\phi_t \, dt \, dx + \iint_\Omega h\phi_x \, dt \, dx = 0.$$

We now have to show that

$$h = \begin{cases} 0 & \text{in } \{u = 0\}, \\ f(u)\big[u_{xx} - I(u)\big]_x & \text{in } P = \{u > 0\}. \end{cases}$$

First we note that for any test function ϕ and $\eta > 0$, we have

$$\left| \int_0^T \int_{\{u \leq \eta\}} f_\varepsilon(u^\varepsilon)\big[u^\varepsilon_{xxx} - I(u^\varepsilon)_x\big]\phi \, dx \, dt \right|$$
$$\leq C(\phi)\Big(f_\varepsilon(3\eta/2)\Big)^{1/2} \left(\int_0^T \int_{\{u \leq \eta\}} f_\varepsilon(u^\varepsilon)\big[u_{xxx} - I(u)_x\big]^2 \, dx \, dt \right)^{1/2}$$

for ε small enough (so that $|u^\varepsilon - u| \le \eta/2$). Inequality (41) thus implies

$$\limsup_{\varepsilon\to 0}\left|\int_0^T\int_{\{u\le 2\eta\}} f_\varepsilon(u^\varepsilon)\big[u^\varepsilon_{xxx} - I(u^\varepsilon)_x\big]\phi\,dx\,dt\right| \le C(\phi)f(\eta/2)^{1/2}.$$

We deduce (since $f(0)=0$)

$$h = 0 \qquad \text{on } \{u=0\}. \tag{45}$$

Next, (44) yields (for $\varepsilon > 0$ and η small enough)

$$\iint_{\{u>2\eta\}} |u^\varepsilon_{xxx} - I(u^\varepsilon)_x|^2\,dx\,ds \le C(\eta).$$

This implies that, if Q_η denotes $\{u > 2\eta\}$, $(u^\varepsilon_{xxx} - I(u^\varepsilon)_x)$ is bounded in the space $L^2(Q_\eta)$. Hence, we can extract from $(u^\varepsilon_{xxx} - I(u^\varepsilon)_x)_{\varepsilon>0}$ a subsequence converging weakly in $L^2(Q_\eta)$. Moreover, remark that Q_η is an open subset of Q (recall that u is Hölder continuous) and $u^\varepsilon_{xxx} - I(u^\varepsilon)_x$ converges in the sense of distributions to $u_{xxx} - I(u)_x$ (use the integral representation for $I(\cdot)$). We thus conclude that,

$$u^\varepsilon_{xxx} - I(u^\varepsilon)_x \rightharpoonup u_{xxx} - I(u)_x \quad \text{ in } L^2(Q_\eta).$$

This yields

$$h = f(u)\big[u_{xxx} - I(u)_{xx}\big] \quad \text{ in } \{u>0\}$$

which concludes the proof of Corollary 1. □

We now turn to the proof of Theorem 2.

Proof of Theorem 2. When u_0 does not satisfy (9), we lose the $L^2(0,T;H^2(\Omega))$ bound on u^ε, and the previous analysis fails. However, we can introduce

$$u_0^\delta = u_0 + \delta$$

which satisfies (9). Theorem 1 then provides the existence of a *non-negative* solution u^δ of (10). In view of Corollary 1, u^δ satisfies:

$$\iint_Q u^\delta\phi_t\,dt\,dx + \iint_P f(u^\delta)[u^\delta_{xx} - I(u^\delta)]_x\phi_x\,dt\,dx = 0. \tag{46}$$

Since u^δ is non-negative, the conservation of mass gives a bound in the space $L^\infty(0,T;L^1(\Omega))$ and allows us to make use of the energy inequality: Indeed, using (11) and Lemma 1 we see that there exists a constant C independent of δ such that

$$||u^\delta||_{L^\infty(0,T;H^1(\Omega))} \le C$$

and

$$\int_0^T\int_\Omega f(u^\delta)\big[u^\delta_{xxx} - I(u^\delta)_x\big]^2\,ds\,dx \le C. \tag{47}$$

We now define the flux

$$h^\delta = f(u^\delta)\big[u^\delta_{xxx} - I(u^\delta)_x\big].$$

Inequality (47) implies that h^δ is bounded in $L^2(Q)$, and that there exists a function $h \in L^2(Q)$ such that

$$h^\delta \rightharpoonup h \qquad \text{in } L^2(Q)-\text{weak}.$$

Proceeding as in the proof of Theorem 1, we deduce that u^δ is bounded in $\mathcal{C}^{1/2,1/8}(\Omega\times(0,T))$ and that there exists a function $u(x,t)$ such that u^δ converges uniformly to u as δ goes to zero (up to a subsequence). We can now argue (with minor changes) as in the proof of Corollary 1 and conclude. □

A Proof of Proposition 5

Our goal here is to prove the existence of a weak solution of

$$u_t + (au_{xxx})_x = (ag)_x.$$

We first prove the following proposition.

Proposition 6. *For all $h \in H^1(\Omega)$, there exists $v \in V_0 := H^1 \cap H^3_N$ such that for all $\phi \in \mathcal{D}(\bar\Omega)$,*

$$-\int_\Omega \frac{v-h}{\tau}\phi\,dx + \int av_{xxx}\phi_x\,dx = \int_\Omega ag\phi_x\,dx. \tag{48}$$

In particular,

$$\begin{aligned} \int_\Omega v\,dx &= \int_\Omega h\,dx, \\ \frac{1}{2}\int_\Omega v_x^2 + \tau\int_\Omega av_{xxx}^2 &\le \frac{1}{2}\int_\Omega h_x^2 + \tau\int_\Omega agv_{xxx}. \end{aligned} \tag{49}$$

Proof. In order to prove this proposition, we have to reformulate the equation. More precisely, instead of choosing test functions $\phi \in \mathcal{D}(\bar\Omega)$, we choose $\phi = -\psi_{xx} + \int_\Omega \psi\,dx$ where ψ is given by the following lemma

Lemma 4. *For all $\phi \in \mathcal{D}(\bar\Omega)$, there exists $\psi \in \mathcal{D}(\bar\Omega)$ such that*

$$-\psi_{xx} + \int_\Omega \psi\,dx = \phi.$$

Hence, we consider $V_0 := H^1 \cap H^3_N$ equipped with the norm $\|v\|^2_{V_0} = \|v_{xxx}\|^2_{L^2} + (\int v\,dx)^2$ and we look for $v \in V_0$ such that for all $\psi \in \mathcal{D}(\bar\Omega)$,

$$\begin{aligned} &\int_\Omega v_x\psi_x\,dx + \tau\int av_{xxx}\psi_{xxx}\,dx + \left(\int_\Omega v\,dx\right)\left(\int_\Omega \psi\,dx\right) \\ &\qquad = \int_\Omega h_x\psi_x\,dx + \left(\int_\Omega h\,dx\right)\left(\int_\Omega \psi\,dx\right) + \tau\int_\Omega ag\psi_{xxx}\,dx. \end{aligned} \tag{50}$$

We thus consider the bilinear form A in V_0 defined as follows: for all $v, w \in V_0$,

$$A(v,w) = \int_\Omega v_x w_x \, dx + \tau \int a v_{xxx} w_{xxx} \, dx + \left(\int_\Omega v \, dx \right) \left(\int_\Omega w \, dx \right).$$

We check that it is continuous and coercive:

$$\begin{aligned} |A(v,w)| &\leq \|v_x\|_2 \|w_x\|_2 + M\tau \|v_{xxx}\|_2 \|w_{xxx}\|_2 + \|v\|_1 \|w\|_1 \\ &\leq C \|v\|_{V_0} \|w\|_{V_0} \\ A(v,v) &\geq \int_\Omega [(v_x)^2 + \varepsilon\tau (v_{xxx})^2] \, dx + \left(\int_\Omega v \, dx \right)^2 \geq \|v\|_{V_0}^2. \end{aligned}$$

We now consider the following linear form L in V_0: for all $w \in V_0$,

$$L(w) = \int_\Omega h_x w_x \, dx + \left(\int_\Omega h \, dx \right) \left(\int_\Omega w \, dx \right) + \tau \int_\Omega a g w_{xxx} \, dx$$

Since $0 \leq a \leq M$ and $g \in L^2$, L is continuous as soon as $h \in H^1(\Omega)$. Lax-Milgram theorem thus implies that there exists $v \in V_0$ such that (50) holds true for all $w \in V_0$.

Eventually, remark that conservation of mass and (49) are direct consequences of (48). The proof of Proposition 6 is now complete. □

We can now prove Proposition 5.

Proof of Proposition 5. For any $\tau > 0$, we consider $N_\tau = \lceil \frac{T}{\tau} \rceil$. We then define inductively a sequence $(u^n)_{n=0,\ldots,N_\tau}$ of V_0 as follows: $u^0 = u_0$ and u^{n+1} is obtained by applying Proposition 6 to $h = u^n$. We then define $u^\tau : [0, N_\tau \tau) \times \Omega$ as follows:

$$u^\tau(t,x) = u^n(x) \quad \text{for } t \in [n\tau, (n+1)\tau).$$

We have $\int_\Omega u^\tau(t,x) \, dx = \int_\Omega u_0(x) \, dx$ for all t. We also derive from (49) that we have

$$\int_\Omega (u_x^\tau)^2(T,x) \, dx + \int_0^T \int_\Omega a (u_{xxx}^\tau)^2(t,x) \, dtdx \\ \leq \int_\Omega ((u_0)_x)^2 \, dx + \int_0^T \int_\Omega (a g u_{xxx})(t,x) \, dtdx$$

In particular, $(u^\tau)_\tau$ is bounded in $L^\infty(0,T;H^1(\Omega))$ and $(S_\tau u^\tau - u^\tau)_\tau$ is bounded in $L^2(0, T-\tau; H^{-1}(\Omega))$ where $S_\tau v(t,x) = v(t+\tau, x)$. We derive from [18, Theorem 5] that $(u^\tau)_\tau$ is relatively compact in $\mathcal{C}(0,T;L^2(\Omega))$.

We now have to pass to the limit in (48). Since $(u_{xxx}^\tau)_\tau$ is bounded in $L^2(Q)$ and we can find a sequence $\tau_n \to 0$ such that $u^{\tau_n} \to u$ in $\mathcal{C}(0,T,L^2(\Omega))$ and $u_{xxx}^{\tau_n} \to u_{xxx}$ in $L^2(Q)$. This is enough to conclude.

We next explain how to get (32). Sobolev's embedding imply that there exists a constant K (depending on $\|u\|_{L^\infty(0,T;H^1(\Omega))}$) such that

$$|u(x_1,t) - u(x_2,t)| \leq K |x_1 - x_2|^{1/2}$$

for all x_1, $x_2 \in \Omega$ and for all $t \in (0,T)$. Since u satisfies

$$u_t = h_x$$

with $h \in L^2(Q)$, it is a fairly classical result that Hölder regularity in space implies Hölder regularity in time. More precisely, we have (see [4], Lemma 2.1 for details):

Lemma 5. *There exists a constant C such that for all x_1, x_2 in Ω and all t_1, $t_2 > 0$,*

$$|u(x_1,t_1) - u(x_2,t_2)| \le C|x_1 - x_2|^{1/2} + C|t_1 - t_2|^{1/8}.$$

The proof of Proposition 5 is thus complete. □

B Proof of (26), (27) and (28)

We have to derive (26), (27) and (28) from (25). Using the fact that u^ε lies in $\mathcal{C}([0,T], H^1(\Omega))$, we first state the following lemma

Lemma 6. *For all $\phi \in \mathcal{D}(\bar{Q})$,*

$$\iint_Q u^\varepsilon \phi_t + f_\varepsilon(u^\varepsilon)[u^\varepsilon_{xx} - I(u^\varepsilon)]_x \phi_x \, dt\, dx = \int_\Omega u^\varepsilon(x,T)\phi(x,T) - \int_\Omega u_0(x)\phi(x,0)\, dx. \quad (51)$$

The proof of such a lemma is fairly classical. It relies on mollifiers that are decentered in the time variable. More precisely, one considers a smooth even function $\rho : \mathbb{R} \to [0,1]$ compactly supported in $[-1,1]$ and such that $\int \rho = 1$. Then for $\alpha > 0$ and $\delta \in \mathbb{R}$, one can define

$$\rho_{\alpha,\delta}(t) = \tau_\delta \rho_\alpha(t) = \alpha\rho\left(\frac{t-\delta}{\alpha}\right).$$

If now a function f is defined in $[0,T]$, it can be extended by 0 to $\mathbb{R}$; in other words, it can be replaced with $f\mathbf{1}_\Omega$ where $\mathbf{1}_\Omega(x) = 1$ if $x \in \Omega$ and $\mathbf{1}_\Omega(x) = 0$ if not; then the convolution product in $\mathbb{R}$: $(f\mathbf{1}_\Omega) \star \rho_{\alpha,\delta}$ is a smooth function in $\mathbb{R}$ which vanishes near $t = 0$ (resp. $t = T$) if $\delta > 0$ (resp. $\delta < 0$).

Consider a smooth function ρ such as in the proof of Lemma 6. Consider $\alpha > 0$ and define $\rho_\alpha(x) = \alpha\rho(\frac{x}{\alpha})$. Then consider $\theta(x,t) = \rho_\alpha(x)\rho_\alpha(t)$.

Lemma 7. *Recall that $u^\varepsilon \in L^\infty(Q)$ and $h^\varepsilon = f_\varepsilon(u^\varepsilon)[u^\varepsilon_{xx} - I(u^\varepsilon)]_x \in L^2(Q)$. Then for all $v \in L^1(Q)$,*

$$\iint_Q u^\varepsilon (v \star \theta)_t = \iint_Q (u^\varepsilon \star \theta)_t v \quad (52)$$

where $f \star g$ means $(f\mathbf{1}_Q) \star (g\mathbf{1}_Q)$.

We next apply Lemma 6 with $\phi = v \star \theta$ where v is chosen to be successively $u^\varepsilon_{xx} \star \theta$, $I(u^\varepsilon \star \theta)$ and $G'_\varepsilon(u^\varepsilon \star \theta)$. After direct computations, we can let $\alpha \to 0$ and get the desired estimates.

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