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Q-convex Sets For Source Points^{*}

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Abstract. In this paper, we introduce both discrete and continuous Q-convex sets for ponctual X-ray sources. We study differences between these sets and the Q-convexes for parallel X-rays where sources are supposed infinitely far from the considered set. We also introduce a novel operator that allows computing the Q-convex hull of any set. Moreover, we provide an algorithm that generates random discrete Q-convex sets under some conditions for a given couple of ray source points.

Keywords: convexity, point X-rays, discrete sets, filling operations, Q-convex sets, random generator of sets.

1 Introduction

Tomography deals with the inverse problem of reconstructing an object from its projections (X-rays). When a few projections are available, we need more information about the set to be reconstructed. For this, we need to impose some properties on it. Many researchs were realized on reconstructing convex sets. For example, R. Gardner and P. Grizmann showed in [2] that any set of seven mutually non parallel lattice directions determine convex subsets of \mathbb{Z}^2 . In [3], the authors show that it is possible to find four projections that will uniquely determine all planar convex bodies. Furthermore, in paper [4] a polynomial-time algorithm for reconstructing a discrete set with special connectivity and convexity properties in directions $(1, 0)$, $(0, 1)$, and $(1, 1)$ is provided. A similar idea is employed by A. Daurat in his Ph.D thesis [7] where he defines a new class of subsets of \mathbb{Z}^2 called ‘Q-convex’ and studies the reconstruction problem for these subsets. Several works study this class of subsets [10–14]. S. Brunetti and A. Daurat present in [11] and [12] a study of the ‘Q-convex’ sets. They also provide two random generator for these subsets in [13]. In [5], they provide a polynomial algorithm that enables the reconstruction of this kind of sets for two directions and that can be used for reconstructing any convex subsets of \mathbb{Z}^2 for some suitable four directions or for any seven mutually nonparallel directions. However, all these results have the same context: X-rays are the number of points along parallel straight lines in a given set of directions. A more realistic context would be when these lines are no more parallel but starting from a source point.

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This context can be thought of as the generalization of the first one. Indeed, parallel X-rays are an approximation of point X-rays where the light source point is placed at an infinite distance from the object.

In this paper, we study Q-convex sets for point sources. In section 2, we introduce both continuous and discrete Q-convex sets for point sources. We also study relationships between Q-convexity and convexity and differences between Q-convex sets for point and parallel X-rays. This leads us, in Definition 9, to introduce a new operator for computing the Q-convex hull of any set. In Section 3, we present an algorithm for generating random Q-convex sets for a couple of point sources. This algorithm works under conditions that are established in the previous section.

2 Definition and notations

2.1 Classical definitions and notations

Definition 1. A set $E \subset \mathbb{R}^2$ is convex (or \mathbb{R} -convex) if for every $A, B \in E$ we have $[A, B] \subseteq E$ where $[A, B] = \{\lambda A + (1 - \lambda)B \mid 0 \leq \lambda \leq 1\}$ is called the line segment between A and B .

In all the following, \mathbb{S} denotes \mathbb{R} or \mathbb{Z} .

Let E be a subset of \mathbb{R}^2 and A and B two distinct points of \mathbb{R}^2 . In all this paper, we use the following notations:

- $\mathcal{C}(\mathbb{R}^2)$ is the set of all convex subsets of \mathbb{R}^2 .
- $\mathcal{C}_E(\mathbb{R}^2, E) = \{E' \in \mathcal{C}(\mathbb{R}^2) \mid E \subseteq E'\}$ is the set of all convex subsets of \mathbb{R}^2 containing the set E .
- $CH(E) = \bigcap_{E' \in \mathcal{C}_E(\mathbb{R}^2, E)} E'$ is the convex hull of E .
- (AB) is the straight line joining A and B .
- A ray or a half-line $R_{S,\theta}$ from a point $S = (x_0, y_0)$ in the direction $\mathbf{u}_\theta = (u_1, u_2)$ where $\sqrt{u_1^2 + u_2^2} = 1$, and $\cos \theta = u_1$ and $\sin \theta = u_2$ can be defined in different ways:

$$\begin{aligned} R_{S,\theta} &= \{(x, y) \in \mathbb{R}^2 \mid u_2(x - x_0) - u_1(y - y_0) = 0 \text{ and } x \geq x_0\}; \\ &= \{(x_0, y_0) + \lambda \mathbf{u}_\theta \mid \lambda \geq 0\}; \\ &= \{M \in \mathbb{R}^2 \mid \widehat{PSM} = \theta\}; \end{aligned}$$

where \widehat{PSM} denotes the angle between (SP) and (SM) with $P = S + (1, 0)$ (see Fig.1.). In all the following, the angle \widehat{PSM} is denoted θ_{SM} .

We also introduce:

- For a point $S \in \mathbb{R}^2$, we define the set of all the angles of all the rays issuing from S and passing through all the points of \mathbb{S}^2 relatively to the horizontal line passing through $P = S + (1, 0)$:

$$\mathcal{A}(S, \mathbb{S}^2) = \{\theta_{SM} \mid M \in \mathbb{S}^2\}.$$

- If E is a finite subset, then $|E|$ is the cardinality of E indicating the number of elements of E .
- Let A be a set. We denote by $\mathfrak{P}(A)$ the powerset of A ($\mathfrak{P}(A) = \{B | B \subseteq A\}$).

Remark 1. Let $S \in \mathbb{R}^2$:

- $\mathcal{A}(S, \mathbb{R}^2) = [0, 2\pi[$.
- $\mathcal{A}(S, \mathbb{Z}^2)$ is an infinite countable set.
- $\overline{\mathcal{A}(S, \mathbb{Z}^2)} = [0, 2\pi]$; where \overline{F} is the closure of the set F in \mathbb{R} relatively to usual topology.

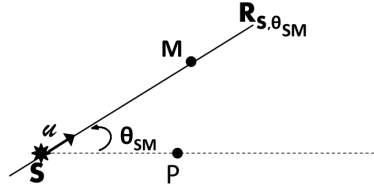


Fig. 1. The ray R is defined by the initial point S and the angle θ_{SM} .

For the discrete sets, convexity can be defined as follows :

Definition 2. Let $D \subset \mathbb{Z}^2$. D is \mathbb{Z} -convex if $D = CH(D) \cap \mathbb{Z}^2$.

Definition 3. Let $k \in \mathbb{N}$ and $k \geq 2$. A set $E \subset \mathbb{S}^2$ is said to be k - \mathbb{S} -convex if for any subset $E' \subseteq E$ such that $|E'| \leq k$, we have $CH(E') \cap \mathbb{S}^2 \subseteq E$.

Remark 2. By Definition 1, a set E is \mathbb{R} -convex if and only if E is 2- \mathbb{R} -convex.

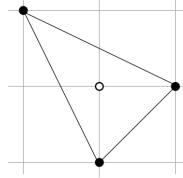


Fig. 2. The set D of points \bullet is 2- \mathbb{Z} -convex but not \mathbb{Z} -convex because the point \circ is not a point of D .

The following proposition is proven in [9].

Proposition 1. A finite set of \mathbb{Z}^2 is \mathbb{Z} -convex if and only if it is 3-convex.

Remark 3. Let $E \subseteq \mathbb{S}^2$. Then,

- If E is k - \mathbb{R} -convex with $k \geq 2$, then E is necessarily $(k+1)$ - \mathbb{R} -convex.
- If E is k - \mathbb{Z} -convex with $k \geq 3$, then E is $(k+1)$ - \mathbb{Z} -convex.
- Unlike convexity in \mathbb{R}^2 , a 2- \mathbb{Z} -convex set is not necessarily \mathbb{Z} -convex (see the example in Fig.2).

Let $E \subseteq \mathbb{R}^2$ and a point S in \mathbb{R}^2 . We define the projection of E from the source point S denoted $X_S(E, \cdot): \mathbb{R} \mapsto \mathbb{R}$ by:

$$X_S(E, \theta) = \int_0^{+\infty} \chi_{E \cap R_{S, \theta}}(S + t\mathbf{u}_{\theta_{SM}}) dt.$$

Where $\mathbf{u}_{\theta_{SM}} = (\cos \theta, \sin \theta)$ and

$$\chi_E(x) = \begin{cases} 1 & \text{if } x \in E \\ 0 & \text{otherwise.} \end{cases}$$

Then, $X_S(E, \theta) = \mu(E \cap R_{S, \theta})$ where μ is the usual measure on \mathbb{R} .

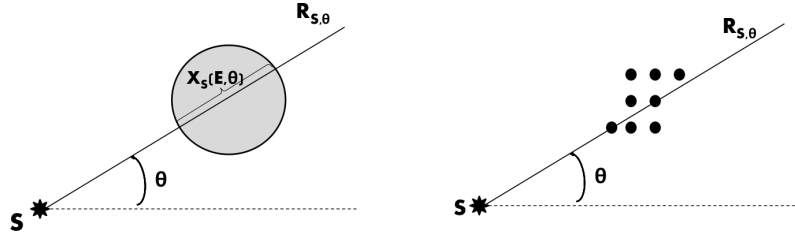


Fig. 3. Continuous (left) and discrete(right) point X-rays.

Now, let us consider a source point S and a finite subset $D \subset \mathbb{Z}^2$. We have a finite number of rays issuing from S and passing through all points of D and each of these rays passes through a finite number of points of D . The projection of D from the source point S is the function $X_S(D, \cdot): \mathbb{R} \mapsto \mathbb{N}$ such that:

$$X_S(D, \theta) = |R_{S, \theta} \cap D|.$$

Definition 4. Let $E \subset \mathbb{S}^2$. The support of E for the source point S is the set:

$$\text{Supp}_S(E, S) = \{\theta \in \mathcal{A}(S, \mathbb{S}^2) \mid X_S(E, \theta) \neq 0\}.$$

Remark 4. For the reconstructing problem in both continuous and discrete case, we aim to reconstruct the subset E of \mathbb{R}^2 or the finite subset D of \mathbb{Z}^2 having the position of the source point and a set of angles and their projections (which is infinite for the continuous case and finite for the discrete one). Only the rays $R_{S, \theta}$ such that $\theta \in \text{Supp}_S(E, S)$ are considered.

2.2 Q-convexity

In this section, we introduce both continuous and discrete Q-convex sets for two source points. This notion was first introduced by A. Daurat [7] for the discrete parallel rays.

Considering a ray $R_{S,\theta}$ from a point S , we define:

- $\mathcal{L}(R_{S,\theta}) = \{M \in \mathbb{S}^2 \mid 0 \leq \theta_{SM} - \theta \leq \pi\}$ the set of points that are on the left of the straight line containing $R_{S,\theta}$;
- $\mathcal{R}(R_{S,\theta}) = \{M \in \mathbb{S}^2 \mid -2\pi \leq \theta_{SM} - \theta \leq \pi\}$ the set of points that are on the right of the straight line containing $R_{S,\theta}$.

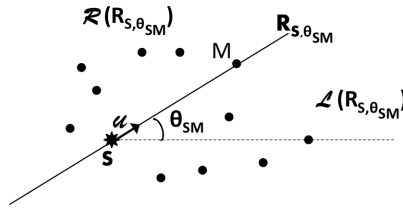


Fig. 4. Linear separation of the plane by the straight line containing the ray $R_{S,\theta}$.

We consider two distinct source points S and $S' \in \mathbb{R}^2$, a set $E \subset \mathbb{S}^2$, two rays $R_{S,\theta_{SM}}$ and $R_{S',\theta'_{S'M}}$ such that $R_{S,\theta_{SM}} \cap R_{S',\theta'_{S'M}} = \{M\}$. This intersection defines the following four zones (called quadrants):

$$\begin{aligned} Z_{\{S,S'\}}^0(M) &= \mathcal{R}(R_{S,\theta_{SM}}) \cap \mathcal{L}(R_{S',\theta'_{S'M}}), \\ Z_{\{S,S'\}}^1(M) &= \mathcal{R}(R_{S,\theta_{SM}}) \cap \mathcal{R}(R_{S',\theta'_{S'M}}), \\ Z_{\{S,S'\}}^2(M) &= \mathcal{L}(R_{S,\theta_{SM}}) \cap \mathcal{R}(R_{S',\theta'_{S'M}}), \\ Z_{\{S,S'\}}^3(M) &= \mathcal{L}(R_{S,\theta_{SM}}) \cap \mathcal{L}(R_{S',\theta'_{S'M}}). \end{aligned}$$

Definition 5. A set $E \subset \mathbb{S}^2$ is \mathbb{S} -Q-convex (quadrant-convex) for two source points S and S' if for all $M \in \mathbb{S}^2$ we have :

$$\forall t \in \{0, 1, 2, 3\}, Z_{\{S,S'\}}^t(M) \cap E \neq \emptyset \implies M \in E$$

Definition 6. Let \mathcal{P} be a set of at least two source points. We say that a set $E \subset \mathbb{S}^2$ is \mathbb{S} -Q-convex for \mathcal{P} if it is \mathbb{S} -Q-convex for any two points of \mathcal{P} .

We denote the class of sets which are \mathbb{S} -Q-convex for a set of source points \mathcal{P} by $\mathcal{QC}_{\mathbb{S}}(\mathcal{P})$.

Lemma 1. Let $E \subset \mathbb{S}^2$. $E \in \mathcal{C}(\mathbb{S}^2) \implies \mathcal{QC}_{\mathbb{S}}(\mathbb{R}^2)$.

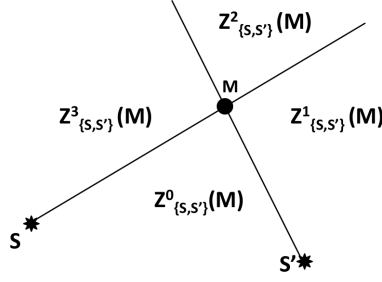


Fig. 5. Four zones resulting of the intersection between two rays in a point M

Proof. Let E be a convex set of \mathbb{S}^2 and two rays intersecting in a point $M \in \mathbb{S}^2$ such that $Z_{\{S,S'\}}^t(M) \cap E \neq \emptyset$ for all $t \in \{0, 1, 2, 3\}$. So, for all t , let $A_t \in Z_{\{S,S'\}}^t(M) \cap E$. As E is convex and thus 4-convex, we have $M \in CH(\{A_1, A_2, A_3, A_4\}) \subseteq CH(E)$ and thus $M \in E$.

Proposition 2. *Let E be a set of \mathbb{S}^2 . The following properties are equivalent:*

- E is \mathbb{S} -convex
- E is \mathbb{S} -Q-convex for \mathbb{R}^2 .

Proof. By Lemma 1 we have that if E is a \mathbb{S} -convex set then it is necessarily \mathbb{S} -Q-convex for any set of two points. So, E is \mathbb{S} -Q-convex for \mathbb{R}^2 .

Let E be a subset of \mathbb{S}^2 that is \mathbb{S} -Q-convex for any two points. According to Proposition 1, it is sufficient to prove that E is a 3- \mathbb{Z} -convex. Let us consider three distinct points $A, B, C \in E$ and a point M inside of the triangle ABC . We can separate two cases. If M is on one of the sides of the triangle, since E is Q-convex for any two points, then $M \in E$. Else, if M is not on any side of ABC , let S be a point such that S, A and M are collinear and S' be a point such that S', B and M are collinear. We know that E is Q- \mathbb{S} -convex for $\{S, S'\}$ and we have $Z_{\{S,S'\}}^t(M) \cap E \neq \emptyset$ for all $t \in \{0, 1, 2, 3\}$, hence $M \in E$.

Proposition 3. *Let D be a finite subset of \mathbb{Z}^2 . The following properties are equivalent:*

- D is \mathbb{Z} -Q-convex for \mathbb{R}^2 .
- D is \mathbb{Z} -Q-convex for \mathbb{Z}^2 .

Proof. Since $\mathbb{Z}^2 \subset \mathbb{R}^2$, any \mathbb{Z} -Q-convex set for \mathbb{R}^2 is \mathbb{Z} -Q-convex for \mathbb{Z}^2 . The second inclusion can be proved in a similar way than for Proposition 2. Actually, to prove that a set $D \subset \mathbb{Z}^2$ is \mathbb{Z} -Q-convex for \mathbb{Z}^2 it is sufficient to prove that D is convex. Let us consider the triangle ABC where $A, B, C \in D$, and a point M inside the triangle. Similarly than the proof of Proposition 2, we separate two cases. If the point M is on one of the triangle's sides we know that $M \in D$. Else, we consider the line passing through A and M and the line passing through M and B . Since each line passes through two points of \mathbb{Z}^2 there exists a point $G \in \mathbb{Z}^2$ such that M, A and G are collinear and a point $H \in \mathbb{Z}^2$ such that M, B and H are collinear and $G, H \notin D$. knowing that D is \mathbb{Z} -Q-convex for G and H with $Z_{\{S,S'\}}^t(M) \cap D \neq \emptyset$ for all $t \in \{0, 1, 2, 3\}$ we conclude that $M \in D$ hence D is convex. Referring to Proposition 2, we have D is \mathbb{Z} -Q-convex for \mathbb{R}^2 .

Remark 5. – Let $E \subseteq \mathbb{S}^2$. From Proposition 3 and Proposition 2 we have:

$$E \text{ is } \mathbb{S}\text{-convex} \Leftrightarrow E \text{ is } \mathbb{S}\text{-Q-convex for } \mathbb{S}^2.$$

So,

$$\mathcal{C}(\mathbb{S}^2) = \mathcal{QC}_{\mathbb{S}}(\mathbb{S}^2).$$

– $\mathcal{QC}_{\mathbb{S}}(\mathcal{P}) = \bigcap_{\{S_1, S_2\} \in \mathcal{P}} \mathcal{QC}_{\mathbb{S}}(\{S_1, S_2\})$ is the set of all \mathbb{S} -Q-convex sets for a set $\mathcal{P} \subseteq \mathbb{R}^2$.

Definition 7. Let $E \subseteq \mathbb{S}^2$. $\mathcal{QC}_{\mathbb{S}}(\mathcal{P}, E) = \{E' \in \mathcal{QC}_{\mathbb{S}}(\mathcal{P}) \mid E \subseteq E'\}$ denotes the set of all subsets of \mathbb{S}^2 containing E that are \mathbb{S} -Q-convex for \mathcal{P} .

Definition 8. Let $E \subset \mathbb{S}^2$. $QH_{\mathbb{S}}(\mathcal{P}, E) = \bigcap_{E' \in \mathcal{QC}_{\mathbb{S}}(\mathcal{P}, E)} E'$.

$QH_{\mathbb{S}}(\mathcal{P}, E)$ is called the \mathbb{S} -Q-convex hull of E .

Proposition 4. Let $E \subseteq \mathbb{Z}^2$. Then, $QH_{\mathbb{S}}(\mathcal{P}, E) \subseteq CH(E)$.

Proof. Let $M \in QH(\mathcal{P}, E)$. Then for all t we have $Z_{\{S, S'\}}^t(M) \cap E \neq \emptyset$. Let us consider for each t a point $A_t \in Z_{\{S, S'\}}^t(M) \cap E$. $CH(E)$ is convex then it is 4-convex. So, $M \in CH(\{A_0, A_1, A_2, A_3\}) \subseteq CH(E)$. Then, $M \in CH(E)$.

Definition 9. We consider the operator $\mathcal{F}_{\mathcal{P}}: \mathfrak{P}(\mathbb{S}^2) \mapsto \mathfrak{P}(\mathbb{S}^2)$ such that for $E \subset \mathbb{S}^2$ we have $\mathcal{F}_{\mathcal{P}}(E) = \{M \in \mathbb{S}^2 \mid \forall S, S' \in \mathcal{P} \text{ and } S \neq S', Z_{\{S, S'\}}^t(M) \cap E \neq \emptyset \text{ for all } t\}$.

Given $E \subseteq \mathbb{S}^2$, we put $E_0 = E$ and $E_{i+1} = \mathcal{F}_{\mathcal{P}}(E_i)$. So, $E_i = \mathcal{F}_{\mathcal{P}}^i(E)$ for $i \geq 0$. We denote $\mathfrak{F}_{\mathcal{P}}(E) = \bigcup_{i \geq 0} \mathcal{F}_{\mathcal{P}}^i(E)$.

Remark 6. Unlike parallel X-rays [7], for point X-rays we don't always have $QH_{\mathbb{S}}(\mathcal{P}, E) = \mathcal{F}_{\mathcal{P}}(E)$. Indeed, in Fig.6 we can see that $N \in QH_{\mathbb{S}}(\mathcal{P}, E)$ but $N \notin \mathcal{F}_{\mathcal{P}}(E)$.

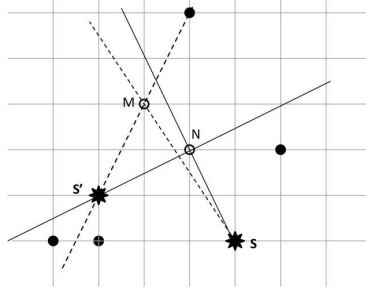


Fig. 6. $QH_{\mathbb{S}}(\mathcal{P}, E) \neq \mathcal{F}_{\mathcal{P}}(E)$.

Remark 7. Let $E \subseteq \mathbb{S}^2$ and $\mathcal{P} \subseteq \mathbb{R}^2$. We have $E \subseteq \mathcal{F}_{\mathcal{P}}(E) \subseteq \mathfrak{F}_{\mathcal{P}}(E)$.

The following proposition gives some properties of the completion operators $\mathcal{F}_{\mathcal{P}}$ and $\mathfrak{F}_{\mathcal{P}}$.

Proposition 5. *Let $E \subseteq \mathbb{S}^2$ and $\mathcal{P} \subseteq \mathbb{R}^2$, we have:*

1. $\mathfrak{F}_{\mathcal{P}}$ is an increasing operator: If $E', E'' \subseteq \mathbb{S}^2$, then,

$$E' \subseteq E'' \implies \mathfrak{F}_{\mathcal{P}}(E') \subseteq \mathfrak{F}_{\mathcal{P}}(E'').$$

2. $E \in QC_{\mathbb{S}}(\mathcal{P})$ if and only if $\mathcal{F}_{\mathcal{P}}(E) = E$ if and only if $\mathfrak{F}_{\mathcal{P}}(E) = E$;
3. $\mathfrak{F}_{\mathcal{P}}(\mathfrak{F}_{\mathcal{P}}(E)) = \mathfrak{F}_{\mathcal{P}}(E)$
4. $QH_{\mathbb{S}}(\mathcal{P}, E) = \mathfrak{F}_{\mathcal{P}}(E)$
5. If E is finite set and $\mathbb{S} = \mathbb{Z}$, then there exists $k \in \mathbb{N}$ such that $\mathfrak{F}_{\mathcal{P}}(E) = \mathcal{F}_{\mathcal{P}}^k(E)$

Proof. 1. Since $\mathcal{F}_{\mathcal{P}}$ is an increasing operator relatively to set inclusion, if $E \subseteq E'$, then $\mathcal{F}_{\mathcal{P}}^i(E_0) = E_i \subseteq \mathcal{F}_{\mathcal{P}}^i(E'_0) = E'_i$. Hence, $\mathfrak{F}_{\mathcal{P}}(E) = \bigcup_{i \geq 0} E_i \subseteq \mathfrak{F}_{\mathcal{P}}(E') = \bigcup_{i \geq 0} E'_i$.

2. – Suppose that $E \in QC_{\mathbb{S}}(\mathcal{P})$ and let $M \in \mathcal{F}_{\mathcal{P}}(E)$. Then, we have for all t , $Z_{\{S, S'\}}^t(M) \cap E \neq \emptyset$. We necessarily have $M \in E$ because $E \in QC_{\mathbb{S}}(\mathcal{P})$. Then $\mathcal{F}_{\mathcal{P}}(E) \subseteq E$. Also, by definition, we have $E \subseteq \mathcal{F}_{\mathcal{P}}(E)$. Hence, $\mathcal{F}_{\mathcal{P}}(E) = E$. Now suppose that $\mathcal{F}_{\mathcal{P}}(E) = E$. Then for any point M such that for any t , $Z_{\{S, S'\}}^t(M) \cap E \neq \emptyset$ we have $M \in \mathcal{F}_{\mathcal{P}}(E)$ which means that $M \in E$. Hence $E \in QC_{\mathbb{S}}(\mathcal{P})$.

– $\mathcal{F}_{\mathcal{P}}(E) = E \Leftrightarrow \mathcal{F}_{\mathcal{P}}^i(E) = E$ for any $i \Leftrightarrow \mathfrak{F}_{\mathcal{P}}(E) = \bigcup_{i \geq 0} \mathcal{F}_{\mathcal{P}}^i(E) = E$.

3. We consider a point $M \in \mathcal{F}_{\mathcal{P}}(\mathfrak{F}_{\mathcal{P}}(E))$. Then, we have for all t , $Z_{\{S, S'\}}^t(M) \cap \mathfrak{F}_{\mathcal{P}}(E) \neq \emptyset$. Then there exists an i such that for all t , $Z_{\{S, S'\}}^t(M) \cap \mathcal{F}_{\mathcal{P}}^i(E) \neq \emptyset$, which means that $M \in \mathcal{F}_{\mathcal{P}}^{i+1}(E)$ and $M \in \mathfrak{F}_{\mathcal{P}}(E)$. Also, we know that $\mathfrak{F}_{\mathcal{P}}(E) \subseteq \mathcal{F}_{\mathcal{P}}(\mathfrak{F}_{\mathcal{P}}(E))$, then we have $\mathcal{F}_{\mathcal{P}}(\mathfrak{F}_{\mathcal{P}}(E)) = \mathfrak{F}_{\mathcal{P}}(E)$. So, item 2 implies that $\mathfrak{F}_{\mathcal{P}}(\mathfrak{F}_{\mathcal{P}}(E)) = \mathfrak{F}_{\mathcal{P}}(E)$.

4. Items 2 and 3 $\Rightarrow \mathfrak{F}_{\mathcal{P}}(E) \in QC_{\mathbb{S}}(\mathcal{P}, E)$. So, $QH_{\mathbb{S}}(\mathcal{P}, E) \subseteq \mathfrak{F}_{\mathcal{P}}(E)$. Also, we have $E \subseteq QH_{\mathbb{S}}(\mathcal{P}, E)$. Then $\mathcal{F}_{\mathcal{P}}(E) \subseteq \mathcal{F}_{\mathcal{P}}(QH_{\mathbb{S}}(\mathcal{P}, E)) = QH_{\mathbb{S}}(\mathcal{P}, E)$. Hence, $\mathfrak{F}_{\mathcal{P}}(E) \subseteq QH_{\mathbb{S}}(\mathcal{P}, E)$.

5. Let $n = |CH(E) \cap \mathbb{Z}^2| - |E|$. We have $QH_{\mathbb{S}}(\mathcal{P}, E) \subseteq CH(E)$. Also, for all i we have $\mathcal{F}_{\mathcal{P}}^i(E) \subset \mathcal{F}_{\mathcal{P}}^{i+1}(E)$ or $\mathcal{F}_{\mathcal{P}}^i(E) = \mathcal{F}_{\mathcal{P}}^{i+1}(E)$. When $\mathcal{F}_{\mathcal{P}}^i(E) = \mathcal{F}_{\mathcal{P}}^{i+1}(E)$ for all $i \geq 0$, we have $\mathcal{F}_{\mathcal{P}}^i(E) = \mathfrak{F}_{\mathcal{P}}(E) = QH_{\mathbb{S}}(\mathcal{P}, E)$.

Definition 10. *Let $E \subseteq \mathbb{S}^2$. $\mathcal{H}_{\mathbb{S}}(E) = \{E' \subseteq \mathbb{S}^2 \mid QH_{\mathbb{S}}(\mathcal{P}, E') = QH_{\mathbb{S}}(\mathcal{P}, E)\}$ is the set of all $E' \subseteq \mathbb{S}^2$ having the same \mathbb{S} - Q -convex hull as E .*

Proposition 6. *Let $E \subseteq \mathbb{S}^2$ and $H \subseteq \mathcal{H}_{\mathbb{S}}(E)$. Then,*

$$\bigcup_{E' \in H} E' \in \mathcal{H}_{\mathbb{S}}(E).$$

Proof. We know that for any $E' \in H$ we have $E' \subseteq \mathfrak{F}_{\mathcal{P}}(E)$. So, $\bigcup_{E' \in H} E' \subseteq \mathfrak{F}_{\mathcal{P}}(E)$. Thus, $\mathfrak{F}_{\mathcal{P}}(\bigcup_{E' \in H} E') \subseteq \mathfrak{F}_{\mathcal{P}}(E)$. In the other hand, if $E'' \in H$, then as $E'' \subseteq \bigcup_{E' \in H} E'$, we have $\mathfrak{F}_{\mathcal{P}}(E) = \mathfrak{F}_{\mathcal{P}}(E'') \subseteq \mathfrak{F}_{\mathcal{P}}(\bigcup_{E' \in H} E')$. Thus, $\mathfrak{F}_{\mathcal{P}}(\bigcup_{E' \in H} E') = \mathfrak{F}_{\mathcal{P}}(E)$.

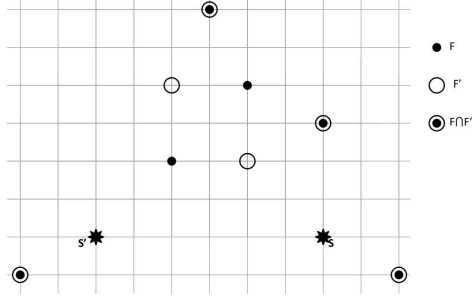


Fig. 7. $F, F' \in \mathcal{H}_S(E)$; $F \cap F' \notin \mathcal{H}_S(E)$

Remark 8. We proved in Proposition 6 that the set $\mathcal{H}_S(E)$ is closed under union. However, $\mathcal{H}_S(E)$ is not generally closed under intersection. Indeed, Fig.7 shows two sets F and F' such that $\mathfrak{F}_{\mathcal{P}}(F) = \mathfrak{F}_{\mathcal{P}}(F')$ and $\mathfrak{F}_{\mathcal{P}}(F \cap F') \neq \mathfrak{F}_{\mathcal{P}}(F)$.

Since technical devices allow us to choose source positions, we consider, in all the following, a set \mathcal{P} of source points such that $|\mathcal{P}| = 2$ and a class of sets $E \subset \mathbb{S}^2$ such that for any source point $S \in \mathcal{P}$ we have:

$$0 \leq \max(\text{Supp}_{\mathbb{Z}}(E, S)) - \min(\text{Supp}_{\mathbb{Z}}(E, S)) \leq \pi. \quad (\star)$$

Condition (\star) is a key condition for what follows. It can be verified for example when all the points of E are in the same half plane delimited by the straight line (SS') .

Remark 9. Let $E \subseteq \mathbb{S}^2$ be a set that verifies condition (\star) . Then,

- If $E' \subseteq E$, then E' verifies condition (\star) .
- $CH(E)$ verifies condition (\star) .

Proposition 7. *We consider two source points S and $S' \in \mathbb{R}^2$ and a set $E \subset \mathbb{S}^2$ that verifies condition (\star) .*

Let $M, M' \in \mathbb{S}^2$. We have for all $t \in \{0, 1, 2, 3\}$:

$$M' \in Z_{S, S'}^t(M) \implies Z_{S, S'}^t(M') \cap E \subseteq Z_{S, S'}^t(M) \cap E.$$

Proof. We will prove the proposition for the case $t = 3$. The same arguments can be used to prove the proposition for the cases $t = 0, 1$ and 2 .

Let $M' \in Z_{S, S'}^3(M)$ and $N \in Z_{S, S'}^3(M') \cap E$ such that $N \notin Z_{S, S'}^3(M) \cap E$. So, N is

necessarily in the shaded area on Fig.8 and thus:
$$\begin{cases} 0 \leq \theta_{SN} - \theta_{SM'} \leq \pi \\ \theta_{SN} - \theta_{SM} \geq \pi \end{cases}$$

which is absurd since the set E verifies the condition (\star) .

Proposition 8. *Let $E \subseteq \mathbb{S}^2$ be a set that verifies condition (\star) . Then,*

$$QH_S(\mathcal{P}, E) = \mathcal{F}_{\mathcal{P}}(E).$$

(see Fig.9). If $\{N\} \subset Z_{\{S,S'\}}^3(N) \cap E$ then, as E verifies (\star) , there exists a point $N' \in Z_{\{S,S'\}}^3(N) \cap E$ such that $\theta_{SN} < \theta_{SN'}$ which is impossible as $\theta_{SN} = \max \text{Supp}_{\mathbb{Z}}(Z_{\{S,S'\}}^3(M) \cap E, S)$. Then $Z_{\{S,S'\}}^3(N) \cap E = \{N\}$. Then, $N \in \xi(E)$ and so $N \in QH(\mathcal{P}, \xi(E))$. Similarly, we find for each $t \in \{0, 1, 2\}$ a point $N \in Z_{\{S,S'\}}^t(M) \cap E$ such that $N_t \in QH(\mathcal{P}, \xi(E))$. Thus, we have for all t , $Z_{\{S,S'\}}^t(M) \cap QH(\mathcal{P}, \xi(E)) \neq \emptyset$. $QH(\mathcal{P}, \xi(E))$ is \mathbb{S} -Q-convex, then $M \in QH(\mathcal{P}, \xi(E))$.

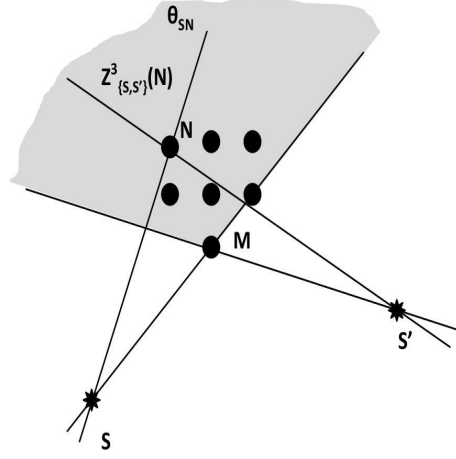


Fig. 9. $N \in Z_{\{S,S'\}}^3(M) \cap E$ and $\theta_{SN} = \max \text{Supp}_{\mathbb{Z}}(Z_{\{S,S'\}}^3(M) \cap E, S)$.

2. – Let $M \in \xi(E)$ and a t such that $Z_{\{S,S'\}}^t(M) \cap E = \{M\}$. Let us suppose that $\{M\} \subset Z_{\{S,S'\}}^t(M) \cap QH(\mathcal{P}, E)$. As E verifies condition (\star) , $CH(E)$ verifies condition (\star) as well. We have $QH(\mathcal{P}, E) \subseteq CH(E)$, then $QH(\mathcal{P}, E)$ verifies condition (\star) also. Let us consider a point $N \in Z_{\{S,S'\}}^t(M) \cap QH(\mathcal{P}, E)$, similarly than in the previous proof, such that $N \in \xi(QH(\mathcal{P}, E))$. If $N \notin E$, then $Z_{\{S,S'\}}^t(N) \cap E \subseteq Z_{\{S,S'\}}^t(N) \cap (QH(\mathcal{P}, E) \setminus \{N\}) = \emptyset$ which is absurd because $N \in QH(\mathcal{P}, E)$. So we necessarily have $N \in E$ and so, $Z_{\{S,S'\}}^t(M) \cap E \supset \{M\}$ which is absurd because we have $Z_{\{S,S'\}}^t(M) \cap E = \{M\}$. Then, $Z_{\{S,S'\}}^t(M) \cap QH(\mathcal{P}, E) = \{M\}$ and thus $M \in \xi(QH(\mathcal{P}, E))$. $S, \xi(E) \subseteq \xi(QH(\mathcal{P}, E))$.
- Let $M \in \xi(QH(\mathcal{P}, E))$. So, there exists a t such that $Z_{\{S,S'\}}^t(M) \cap QH(\mathcal{P}, E) = \{M\}$. If $M \notin E$, then $Z_{\{S,S'\}}^t(M) \cap E \subseteq Z_{\{S,S'\}}^t(M) \cap (QH(\mathcal{P}, E) \setminus \{M\}) = \emptyset$. So, $Z_{\{S,S'\}}^t(M) \cap E = \emptyset$ which means that $M \notin \mathcal{F}_{\mathcal{P}}(E) = QH(\mathcal{P}, E)$ which is absurd. Thus, $M \in E$ and $Z_{\{S,S'\}}^t(M) \cap E = \{M\}$. Then, $M \in \xi(E)$ and so $\xi(QH(\mathcal{P}, E)) \subseteq \xi(E)$.

Proposition 9, leads to the following:

Corollary 1. *Let $E, E' \subset \mathbb{S}^2$. Then,*

$$\xi(E) = \xi(E') \iff QH_{\mathbb{Z}}(\mathcal{P}, E) = QH_{\mathbb{Z}}(\mathcal{P}, E').$$

We recall that for a set $E \subseteq \mathbb{S}^2$, $\mathcal{H}_{\mathbb{S}}(E) = \{E' \subseteq \mathbb{S}^2 \mid QH(\mathcal{P}, E') = QH(\mathcal{P}, E)\}$. Corollary 1 shows that if E verifies condition (\star) then we have $\mathcal{H}_{\mathbb{S}}(E) = \{E' \subseteq \mathbb{S}^2 \mid \xi(E') = \xi(E)\}$. We proved in Proposition 6 that if $H \subseteq \mathcal{H}_{\mathbb{S}}(E)$, then $\bigcup_{E' \in H} E' \in \mathcal{H}_{\mathbb{S}}(E)$. So, $\mathcal{H}_{\mathbb{S}}(E)$ is closed by union. In the following, we prove that, under condition (\star) , $\mathcal{H}_{\mathbb{S}}(E)$ is closed by intersection.

Proposition 10. *Let $E \subseteq \mathbb{S}^2$ and $H \subseteq \mathcal{H}_{\mathbb{S}}(E)$. Then,*

$$\bigcap_{E' \in H} E' \in \mathcal{H}_{\mathbb{S}}(E).$$

Moreover, $\xi(E) = \bigcap_{E' \in \mathcal{H}_{\mathbb{S}}(E)} E'$, and thus $\xi(E) \in \mathcal{H}_{\mathbb{S}}(E)$.

Proof. We know that for any $E' \in H$, we have $\xi(E') = \xi(E) \subseteq E' \subseteq \mathfrak{F}_{\mathcal{P}}(E')$. Then, $\xi(E) \subseteq \bigcap_{E' \in H} E' \subseteq \mathfrak{F}_{\mathcal{P}}(E')$. So, $\mathfrak{F}_{\mathcal{P}}(\xi(E)) \subseteq \mathfrak{F}_{\mathcal{P}}(\bigcap_{E' \in H} E') \subseteq \mathfrak{F}_{\mathcal{P}}(E)$. Thus, $\mathfrak{F}_{\mathcal{P}}(\bigcap_{E' \in H} E') = \mathfrak{F}_{\mathcal{P}}(E)$.

Corollary 2. *Let $E \subseteq \mathbb{S}^2$ such that E verifies condition (\star) . Then:*

$$\mathcal{H}_{\mathbb{S}}(E) = \{E' \subseteq \mathbb{S}^2 \mid \xi(E) \subseteq E' \subseteq \mathcal{F}_{\mathcal{P}}(E) = QH(\mathcal{P}, E)\}.$$

3 Random generation of \mathbb{Z} -Q-convex for two source points

In this section, we present a method of randomly generating a \mathbb{Z} -Q-convex set for a given set of source points $\mathcal{P} = \{S, S'\}$.

The idea here is to generate uniformly a random set of points F in a rectangle $\llbracket -k_1, k_1 \rrbracket \times \llbracket -k_2, k_2 \rrbracket$ with $k_1, k_2 \in \mathbb{N}$ (see Fig.10) and then complete F to obtain the Q-convex set $QH_{\mathbb{Z}}(\mathcal{P}, F)$. Actually, the generated Q-convex will be the Q-convex hull of F . Corollary 1 says that a \mathbb{Z} -Q-convex set D is generated by any set $F \in \mathcal{H}_{\mathbb{Z}}(D)$. Hence, a \mathbb{Z} -Q-convex set D has a probability to be produced that is proportional to $|\mathcal{H}_{\mathbb{Z}}(D)|$. When F verifies the condition (\star) , this probability is more exactly proportional to $2^{|D| - |\xi(D)|}$.

Referring to the Proposition 2.2, we know that, for a set F , $QH(\mathcal{P}, F) \subseteq CH(F) \cap \mathbb{Z}^2$. Hence after computing $CH(F)$, we calculate $\mathcal{F}_{\mathcal{P}}(F)$ by verifying for each point $M \in CH(F) \cap \mathbb{Z}^2$ if $Z_{\{S, S'\}}^t(M) \cap D \neq \emptyset$ for all $t \in \{0, 1, 2, 3\}$ or not. If this condition is verified, the point M is added to $\mathcal{F}_{\mathcal{P}}(F)$. We do the same to the obtained set $\mathcal{F}_{\mathcal{P}}(F)$. We would repeat this operation until no points are added any more and we obtain $D = \mathfrak{F}_{\mathcal{P}}(F)$.

Algorithm 1:

Uniformly generate a random set of points $F \subseteq \mathbb{Z}^2$ contained into a rectangle $\llbracket -k_1, k_1 \rrbracket \times \llbracket -k_2, k_2 \rrbracket$ such that no point of F is on the line (SS')
 $F' = F$
repeat
 $F = F'$

```

     $F' = \mathcal{F}_{\mathcal{P}}(F)$ 
  until  $F = F'$ 
   $D = F$ 

```

According to Proposition 5, since F is a finite set the halting condition $F = F'$ of the algorithm will be verified in a finite number k of steps (According to Proposition 9, $k \leq |D| - |\xi(D)|$).

We recall that under condition (\star) , we have $QH(\mathcal{P}, F) = \mathcal{F}_{\mathcal{P}}(F)$.

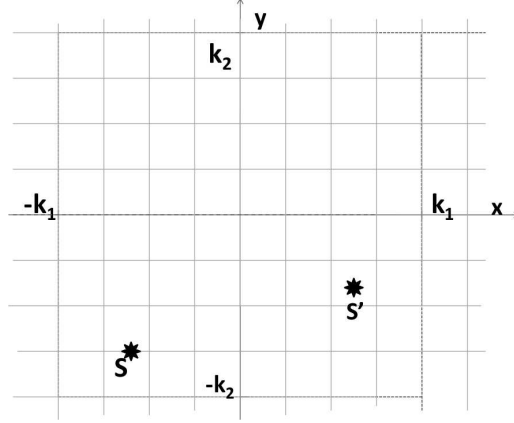


Fig. 10. Generated \mathbb{Z} -Q-convex set will be included in a rectangle $\llbracket -k_1, k_1 \rrbracket \times \llbracket -k_2, k_2 \rrbracket$.

In the following, we describe how to compute $\mathcal{F}_{\mathcal{P}}(F)$.

After generating the random set of points F , as $\mathcal{F}_{\mathcal{P}}(F) \subseteq CH(F)$, we compute $CH(F)$. The points of $CH(F) \cap \mathbb{Z}^2$ will be the candidate points for D . We have: $Supp_{\mathbb{Z}}(CH(F) \cap \mathbb{Z}^2, S) = \{\theta_1, \dots, \theta_n\}$ where angles are sorted in an ascending order. We denote by j the range of the ray R_{S, θ_j} issued from S .

$Supp_{\mathbb{Z}}(CH(F) \cap \mathbb{Z}^2, S') = \{\theta'_1, \dots, \theta'_m\}$ where angles are sorted in a descending order. We denote by i the range of the ray R_{S', θ'_i} issued from S' .

We denote by $\mathcal{I}(i, j)$, the point such that $\mathcal{I}(i, j) = R_{S, \theta_j} \cap R_{S', \theta'_i}$. We note that $\mathcal{I}(i, j)$ is not always in \mathbb{Z}^2 . Hence, we obtain a deformed grill (see Fig.11). Since we have fixed the two point sources S and S' , we will denote $Z_{S, S'}^t(M)$ simply by $Z^t(M)$. We have then:

$$\begin{aligned}
 Z^0(\mathcal{I}(i, j)) &= \{\mathcal{I}(i', j') \in \mathbb{Z}^2 : i' \leq i \text{ and } j' \leq j\}, \\
 Z^1(\mathcal{I}(i, j)) &= \{\mathcal{I}(i', j') \in \mathbb{Z}^2 : i' \geq i \text{ and } j' \leq j\}, \\
 Z^2(\mathcal{I}(i, j)) &= \{\mathcal{I}(i', j') \in \mathbb{Z}^2 : i' \geq i \text{ and } j' \geq j\}, \\
 Z^3(\mathcal{I}(i, j)) &= \{\mathcal{I}(i', j') \in \mathbb{Z}^2 : i' \leq i \text{ and } j' \geq j\}.
 \end{aligned}$$

We define for each quadrant $Z^t(\mathcal{I}(i, j))$ of a point M a boolean array $(V_t(M))_{t \in \{0, 1, 2, 3\}}$ such that $V_t(M) = \text{"True"}$ if and only if $Z^t(M) \cap F \neq \emptyset$.

This array of booleans is determined by induction based on the property of the inclusion between quadrants of neighbor points. $V_t(\mathcal{I}(i, j))$ is set to "True" if and only

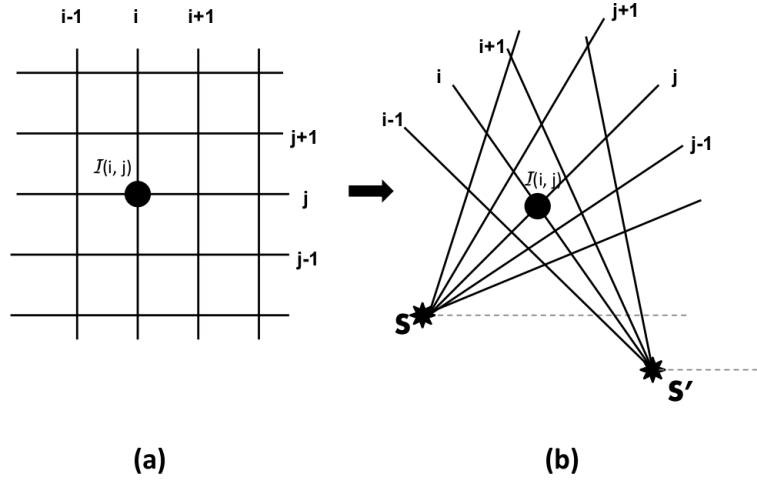


Fig. 11. (a) Grill obtained by parallel rays - (b) Deformed grill obtained by rays from two point sources

if $Z^t(\mathcal{I}(i, j)) \cap F \neq \emptyset$. So, under the condition (\star) , $V_t(\mathcal{I}(i, j))$ is “True” if and only if $V_0(\mathcal{I}(i-1, j))$ is “True” or $V_0(\mathcal{I}(i, j-1))$ is “True” or $\mathcal{I}(i, j) \in D$. And we have:

$$\begin{aligned} V_0(\mathcal{I}(i, j)) &= V_0(\mathcal{I}(i-1, j)) \vee V_0(\mathcal{I}(i, j-1)) \vee \mathcal{I}(i, j) \in F; \\ V_1(\mathcal{I}(i, j)) &= V_1(\mathcal{I}(i+1, j)) \vee V_1(\mathcal{I}(i, j-1)) \vee \mathcal{I}(i, j) \in F; \\ V_2(\mathcal{I}(i, j)) &= V_2(\mathcal{I}(i+1, j)) \vee V_2(\mathcal{I}(i, j+1)) \vee \mathcal{I}(i, j) \in F; \\ V_3(\mathcal{I}(i, j)) &= V_3(\mathcal{I}(i-1, j)) \vee V_3(\mathcal{I}(i, j+1)) \vee \mathcal{I}(i, j) \in F. \end{aligned}$$

Finally, $\mathcal{F}_P(F) = \{\mathcal{I}(i, j) \in CH(F) \cap \mathbb{Z}^2 \mid V_0(\mathcal{I}(i, j)) \wedge V_1(\mathcal{I}(i, j)) \wedge V_2(\mathcal{I}(i, j)) \wedge V_3(\mathcal{I}(i, j))\}.$

After generating the \mathbb{Z} -Q-convex set D , we compute its projections $X_S(D, \cdot)$ for each ray passing through it. This will be the data we use for testing the reconstruction algorithm.

4 Conclusion

We presented in this paper a study on Q-convex sets for point sources. This study shows that, under some conditions, we can connect these sets to Q-convex sets for parallel X-rays. The study also leads to a new operator that computes the Q-convex hull for any set. One of our perspectives is to study the new horizons that this operator opens and learn more about this new class of subsets. We also provided an algorithm for generating random discrete Q-convex sets for point sources under specified conditions. This algorithm allows us to have ground truth for testing the reconstruction algorithm.

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