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On Linear Equivalence for Time-Delay Systems

C. Califano, L.A. Marquez-Martinez, C.H. Moog

Abstract—The aim of the present paper is to introduce new mathematical tools for the analysis and control of nonlinear time-delay systems (NLTDS). An Extended Lie bracket operation equivalent to the Lie bracket operation for system without delays is introduced. It will be shown that this operation, which generalizes that introduced in [19], helps to characterize certain properties of a given submodule, such as nilpotency. This basic property is then used to define the conditions under which a given unimodular matrix represents a bicausal change of coordinates. The effectiveness of the proposed approach will be shown by solving an important basic problem: to characterize if a NLTDS is equivalent or not, to a Linear Time-Delay System by bicausal change of coordinates.

I. INTRODUCTION

Geometric tools for addressing control problems have been extensively used both in the linear and nonlinear context. We recall the pioneering works [22] for the linear context and [9] with reference to the nonlinear context where the decoupling problem was addressed. Another topic of paramount importance which was first solved by using geometric tools concerns the conditions under which a given nonlinear system is diffeomorphic to a linear one. In the single input case, as well known, the solution to this problem is linked to the nilpotency of a specific distribution defined by the vector fields which characterize the dynamics of the given system $((g, ad_f g, \dots, ad_f^n g)$ for continuous time systems and $(G^0, Ad_{F_0} G^0, \dots, Ad_{F_0}^n G^0)$ for discrete time systems). This property which implies that in turn each subdistribution is also nilpotent, implies that when seeking for the weaker property of feedback equivalence, the solution is linked to the involutivity of a specific subdistribution $(g, ad_f g, \dots, ad_f^{n-2} g)$ for continuous time systems and $(G^0, Ad_{F_0} G^0, \dots, Ad_{F_0}^{n-2} G^0)$ for discrete time systems (see for example [2], [3], [7], [10], [11], [12], [16], [17]).

Time-delay systems are recently gaining more and more attention due to their importance in several applications such as those concerning the delay in the signal transmission over communication networks (see for example [1], [5], [13], [15], [18], [19], [20], [21]). A first attempt to extend some

geometric tools to this context has been pursued in [19] with reference to the input-output linearization problem.

In the present paper we introduce an Extended Lie bracket operation equivalent to the Lie bracket operation for system without delays. It will be shown that this operation, which generalizes that introduced in [19], helps to characterize certain properties of a given submodule, such as nilpotency. This basic property is then used to define the conditions under which a given unimodular matrix represents a bicausal change of coordinates. We will finally introduce a set of submodules which are linked to the accessibility property of the system. Some of their properties are discussed. The effectiveness of the proposed approach will be shown by solving an important basic problem: to characterize if a NLTDS is equivalent or not, to a Linear Time-Delay System (LTDS) by bicausal change of coordinates.

With respect to ([5], [19]) we will consider a more general class of systems where there is no assumption on the delay of the input and we will study the effect of bicausal change of coordinates on the given system. For notational simplicity and without loss of generality, we will consider the same maximal delay on the state and input variables.

The paper is organized as follows. Section II concerns recalls and notations about time-delay systems. In Section III some geometric tools for dealing with time-delay systems are introduced and discussed. In Section IV the proposed approach is used to address the problem of the equivalence under bicausal coordinates change to linear accessible time-delay systems. For space reasons most of the proofs are omitted.

II. PRELIMINARIES

The following notation and definitions, taken from [14], [23], will be used:

- \mathcal{K} denotes the field of meromorphic functions of a finite number of symbols in $\{x(t-i), u(t-i), \dot{u}(t-i), \dots, u^{(k)}(t-i), i, k \in \mathbb{N}\}$.
- \mathcal{E} is the vector space spanned by the symbols $\{dx(t-i), du(t-i), d\dot{u}(t-i), \dots, du^{(k)}(t-i), i, k \in \mathbb{N}\}$ over \mathcal{K} . The elements of this space are called 1-forms.
- d is the standard differential operator that maps elements from \mathcal{K} to \mathcal{E} .

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δ represents the backward time-shift operator, that is, given $a(t), f(t) \in \mathcal{K}$:

$$\begin{aligned}\delta a(t) df(t) &= a(t-1)\delta df(t) \\ &= a(t-1)df(t-1),\end{aligned}$$

$\deg(\cdot)$ is the polynomial degree in δ of its argument.

$\mathcal{K}[\delta]$ is the (left) ring of polynomials in δ with coefficients in \mathcal{K} . Every element of $\mathcal{K}[\delta]$ may be written as $\alpha[\delta] = \alpha_0(t) + \alpha_1(t)\delta + \dots + \alpha_{r_\alpha}(t)\delta^{r_\alpha}$, $\alpha_i \in \mathcal{K}$, where $r_\alpha = \deg(\alpha[\delta])$. Addition and multiplication on this ring are defined by $\alpha[\delta] + \beta[\delta] = \sum_{i=0}^{\max\{r_\alpha, r_\beta\}} (\alpha_i(t) + \beta_i(t))\delta^i$ and $\alpha[\delta]\beta[\delta] = \sum_{i=0}^{r_\alpha} \sum_{j=0}^{r_\beta} \alpha_i(t)\beta_j(t-i)\delta^{i+j}$. Although this ring is non-commutative, it is an Euclidean ring, [23]. This property has been exploited in [6][14] to obtain an inverse for matrices with entries in $\mathcal{K}[\delta]$.

$\mathcal{F}[\delta] = \text{span}_{\mathcal{K}[\delta]}\{r_1, \dots, r_s\}$, is the (right) module spanned over $\mathcal{K}[\delta]$ by the column elements $r_1, \dots, r_s \in \mathcal{K}^{n \times 1}[\delta]$.

A polynomial matrix $A(\mathbf{x}, \delta)$ is called unimodular if its inverse is polynomial too.

Example 1: Let $f(t) = x(t-2)x(t) \in \mathcal{K}$. Then

- $\delta f(t) = x(t-1)x(t-3)\delta \in \mathcal{K}[\delta]$,
- $df(t) = x(t)dx(t-2) + x(t-2)dx(t) = x(t)\delta^2 dx + x(t-2)dx$, is an exact form.

Let us consider a nonlinear dynamics with delays Σ , represented as

$$\Sigma: \dot{x}(t) = F(\mathbf{x}_{[s]}) + \sum_{j=0}^s G_j(\mathbf{x}_{[s]})u(t-j) \quad (1)$$

where $\mathbf{x}_{[s]} = (x(t), \dots, x(t-s))$ with $x \in \mathbb{R}^n$, $u \in \mathbb{R}$. It is assumed that $(0,0)$ is an equilibrium pair, and $\mathcal{X}_0 \times \mathcal{U}_0$ a neighborhood of this point. Note that there is no loss of generality in using the same upper bound s for the maximum time delay occurring in the state and that of the control input, which is done for notational simplicity.

We will denote by $\mathbf{x}_{[s]}(-p) = (x(t-p), \dots, x(t-s-p))$. $\mathbf{u}_{[s]}, \mathbf{u}_{[s]}(-p), \mathbf{z}_{[s]}$, and $\mathbf{z}_{[s]}(-p)$ are defined in a similar vein. When no confusion is possible the subindex will be omitted so that \mathbf{x} will stand for $\mathbf{x}_{[s]}$ and $\mathbf{x}(-p)$ will stand for $\mathbf{x}_{[s]}(-p)$.

With such notation, Σ_L , the differential form representation of Σ , is given by

$$\Sigma_L: d\dot{x} = f(\mathbf{x}_{[s]}, \mathbf{u}_{[s]}, \delta)dx + g(\mathbf{x}_{[s]}, \delta)du \quad (2)$$

with

$$\begin{aligned}f(\mathbf{x}_{[s]}, \mathbf{u}_{[s]}, \delta) &= \sum_{i=0}^s \frac{\partial F(\mathbf{x}_{[s]}, \delta)}{\partial x(t-i)} \delta^i \\ &\quad + \sum_{j=0}^s u(t-j) \sum_{i=0}^s \frac{\partial G_j(\mathbf{x}_{[s]})}{\partial x(t-i)} \delta^i \\ g(\mathbf{x}_{[s]}, \delta) &= \sum_{j=0}^s G_j(\mathbf{x}_{[s]})\delta^j\end{aligned}$$

Example 2: Consider

$$\begin{aligned}\dot{x}_1(t) &= x_2(t) - x_2(t-1) + \\ &\quad + 2x_2(t-1)(u(t-1) + u(t-2)) \\ \dot{x}_2(t) &= u(t) + u(t-1)\end{aligned}$$

The associated differential form representation is then characterized by

$$\begin{aligned}f(\mathbf{x}_{[s]}, \mathbf{u}_{[s]}, \delta) &= \begin{pmatrix} 0 & (2(u(t-1) + u(t-2)) - 1)\delta + 1 \\ 0 & 0 \end{pmatrix}, \\ g(\mathbf{x}_{[s]}, \delta) &= \begin{pmatrix} 2x_2(t-1)(\delta + 1)\delta \\ \delta + 1 \end{pmatrix}\end{aligned}$$

Let us end this section by recalling the definition of a bicausal change of coordinates given in [14].

Definition 1 (Bicausal change of coordinates):

Consider the dynamics Σ with state coordinates x . $z = \phi(\mathbf{x}_{[\alpha]})$, $\phi \in \mathcal{K}^n$ is a bicausal change of coordinates for Σ if there exist an integer $\ell \in \mathbb{N}$ and a function $\phi^{-1}(\mathbf{z}_{[\ell]}) \in \mathcal{K}^n$ such that $x(t) = \phi^{-1}(\mathbf{z}_{[\ell]})$.

III. THE GEOMETRY OF TIME-DELAY SYSTEMS

In the following section we will first examine some properties of a bicausal change of coordinates and then enlighten some geometric properties of time-delay systems.

A. Some properties of a bicausal change of coordinates

The following preliminary result is needed to show the connection between the degree of a unimodular matrix and the degree of its inverse.

Proposition 1: Let $A \in \mathcal{K}^{n \times n}[\delta]$ be a unimodular matrix with $\deg(A) = s$. Then $\deg(A^{-1}) \leq (n-1)s$.

Sketch of Proof. First, note that the standard Gauss-Jordan method can be used to compute the inverse matrix [6]. The noncommutativity does not affect the number of iterations nor the maximal degree of the inverse with respect to the commutative case.

In the commutative case, the inverse can be expressed as the adjugate matrix, divided by the determinant, which is scalar for unimodular matrices. Thus, the maximal polynomial degree of the inverse, cannot be greater than any element of the adjugate matrix. Since its elements are determinants

of $(n-1) \times (n-1)$ polynomial matrices, their polynomial degree cannot be greater than $(n-1)s$. \triangleleft

Let $z(t) = \phi(\mathbf{x}_{[\alpha]})$ be a bicausal change of coordinates and $dz = T[\mathbf{x}_{[\alpha]}, \delta]dx$ its associated differential form representation then

P1) $T[\mathbf{x}_{[\alpha]}, \delta] = \sum_{i=0}^{\alpha} \frac{\partial \phi(\mathbf{x}_{[\alpha]})}{\partial x(t-i)} \delta^i = \sum_{i=0}^{\alpha} T^i(\mathbf{x}_{[\alpha]}) \delta^i$ is unimodular

P2) The inverse $T^{-1}[\mathbf{z}, \delta]$ of $T[\mathbf{x}, \delta]$ is unimodular, with polynomial degree $\ell \leq (n-1)\alpha$ and given by

$$T^{-1}[\mathbf{z}, \delta] = \sum_{i=0}^{\ell} \frac{\partial \phi^{-1}(\mathbf{z}_{[\ell]})}{\partial z(t-i)} \delta^i = \sum_{i=0}^{\ell} \bar{T}^i(\mathbf{z}) \delta^i.$$

The following relations, which link a bicausal change of coordinates to its inverse, hold true $\forall x \in \mathcal{X}_0$:

$$T^0(\mathbf{x})|_{\phi^{-1}(\mathbf{z})} \bar{T}^0(\mathbf{z}) = \bar{T}^0(\mathbf{z})|_{\phi(\mathbf{x})} T^0(\mathbf{x}) = I$$

$$\sum_{i=1}^k T^i(\mathbf{x})|_{\phi^{-1}(\mathbf{z})} \bar{T}^{k-i}(\mathbf{z}(-i)) = 0, \quad \forall k \geq 1 \quad (3)$$

$$\sum_{i=1}^k \bar{T}^i(\mathbf{z})|_{\phi(\mathbf{x})} T^{k-i}(\mathbf{x}(-i)) = 0, \quad \forall k \geq 1$$

Let us end this section by noting that under a bicausal change of coordinates $z(t) = \phi(\mathbf{x}_{[\alpha]})$ the differential form (2) is transformed into

$$d\dot{z}(t) = \tilde{f}(\mathbf{z}, \mathbf{u}, \delta)dz + \tilde{g}(\mathbf{z}, \delta)du \quad (4)$$

with

$$\tilde{f}(\mathbf{z}, \mathbf{u}, \delta) = \left[T(\mathbf{x}, \delta) f(\mathbf{x}, \mathbf{u}, \delta) + \dot{T}(\mathbf{x}, \delta) \right] T^{-1}(\mathbf{x}, \delta) \Big|_{\phi^{-1}(\mathbf{z})}$$

$$\tilde{g}(\mathbf{z}, \delta) = (T(\mathbf{x}, \delta) g(\mathbf{x}, \delta)) \Big|_{\phi^{-1}(\mathbf{z})}.$$

B. Geometric tools for time-delay systems

Hereafter the main tools for dealing with time-delay systems are introduced. The obtained results are discussed with respect to nonlinear systems with delay.

The following definition of Delayed Lie bracket, taken from [19], will be instrumental for the definition of the Extended Lie bracket.

Definition 2: Let $r_1(\mathbf{x}, \delta) = \sum_{j=0}^s r_1^j(\mathbf{x}) \delta^j$ and $r_2(\mathbf{x}, \delta) = \sum_{j=0}^s r_2^j(\mathbf{x}) \delta^j$. The Delayed Lie bracket $[r_1^k(\cdot), r_2^l(\cdot)]_D$ of $r_1^k(\mathbf{x})$ and $r_2^l(\mathbf{x})$ is defined as

$$[r_1^k(\cdot), r_2^l(\cdot)]_D = -[r_2^l(\cdot), r_1^k(\cdot)]_D = \sum_{i=0}^k \frac{\partial r_2^l(\mathbf{x})}{\partial x(t-i)} r_1^{k-i}(\mathbf{x}(-i)) - \sum_{i=0}^l \frac{\partial r_1^k(\mathbf{x})}{\partial x(t-i)} r_2^{l-i}(\mathbf{x}(-i)).$$

Definition 3: Let $r_1(\mathbf{x}, \delta) = \sum_{j=0}^s r_1^j(\mathbf{x}) \delta^j$ and $r_2(\mathbf{x}, \delta) = \sum_{j=0}^s r_2^j(\mathbf{x}) \delta^j$. The Extended Lie bracket $[r_1^k(\cdot), r_2^l(\cdot)]_E$, with $k \leq l$ is defined as

$$[r_1^k(\cdot), r_2^l(\cdot)]_E = \sum_{j=0}^k [r_1^{k-j}(\cdot), r_2^{l-j}(\cdot)]_{D(\mathbf{x}(-j))} \frac{\partial}{\partial x(t-j)}$$

$$= -[r_2^l(\cdot), r_1^k(\cdot)]_E.$$

Definition 4: Consider the bicausal change of coordinates $z = \phi(\mathbf{x}_{[\alpha]})$, with $dz = T(\mathbf{x}, \delta)dx$. In the new coordinates the submodule element $r(\mathbf{x}, \mathbf{u}, \delta)$ is transformed as

$$\tilde{r}(\mathbf{z}, \mathbf{u}, \delta) = [T(\mathbf{x}, \delta) r(\mathbf{x}, \mathbf{u}, \delta)]_{\phi^{-1}(\mathbf{z})}. \quad (5)$$

Setting $T^j = 0$ for $j > \alpha = \text{deg}(T(\mathbf{x}, \delta))$ and $r^j = 0$ for $j > \text{deg}(r(\mathbf{x}, \delta))$ one has

$$\tilde{r}^l(\mathbf{z}) = \sum_{p=0}^l (T^p(\mathbf{x}) r^{l-p}(\mathbf{x}(-p))) \Big|_{\phi^{-1}(\mathbf{z})}. \quad (6)$$

Remark. Let us note that in the new coordinates $\tilde{r}(\mathbf{z}, \mathbf{u}, \delta)$ is characterized in general by a different delay than $r(\mathbf{x}, \mathbf{u}, \delta)$. This is because the change of coordinates may itself depend on the delayed variables. \triangleleft

We can now study the action of a change of coordinates either on the delayed Lie bracket and the Extended Lie bracket. The following result whose proof is omitted for space reasons, holds true.

Lemma 1: Let $r_1(\mathbf{x}, \delta) = \sum_{j=0}^s r_1^j(\mathbf{x}) \delta^j$ and $r_2(\mathbf{x}, \delta) = \sum_{j=0}^s r_2^j(\mathbf{x}) \delta^j$. Under the bicausal change of coordinates $z(t) = \phi(\mathbf{x}_{[\alpha]})$, characterized by $dz = T(\mathbf{x}, \delta)dx$ with $T(\mathbf{x}, \delta) = \sum_{j=0}^{\alpha} T^j(\mathbf{x}) \delta^j$ one has, for $k \leq l$,

$$[\tilde{r}_1^k(\mathbf{z}), \tilde{r}_2^l(\mathbf{z})]_D = \quad (7)$$

$$\sum_{p=0}^{l-k} \left(T^p(\mathbf{x}) [r_1^{k-p}(\mathbf{x}), r_2^{l-p}(\mathbf{x})]_{D(\mathbf{x}(-p))} \right) \Big|_{\phi^{-1}(\mathbf{z})}$$

and

$$[\tilde{r}_1^k(\mathbf{z}), \tilde{r}_2^l(\mathbf{z})]_E = (\Gamma^{l-k}(\mathbf{x}) [r_1^k(\mathbf{x}), r_2^l(\mathbf{x})]_E) \Big|_{\phi^{-1}(\mathbf{z})} \quad (8)$$

with

$$\Gamma^{l-k}(\mathbf{x}) = \begin{pmatrix} T^0(\mathbf{x}) & T^1(\mathbf{x}) & \dots & T^{l-k}(\mathbf{x}) \\ 0 & T^0(\mathbf{x}(-1)) & \dots & T^{l-k-1}(\mathbf{x}(-1)) \\ & & \ddots & \\ & & & T^0(\mathbf{x}(-l+k)) \end{pmatrix}.$$

Next theorem enlightens the conditions under which a set of n one-forms are exact and define a bicausal change of coordinates. The conditions are given on the corresponding submodule elements. It is shown that the nilpotency condition of a specific distribution which is the key point in the case of nonlinear systems without delays is transformed into a nilpotency condition on a certain submodule which takes into account not only the state variable $x(t)$ but also the delayed variables. The bound on the delay is defined by the state dimension and the maximal delay.

Theorem 1: Consider the matrix

$$T(\mathbf{x}, \delta) = [r_1(\mathbf{x}, \delta), \dots, r_n(\mathbf{x}, \delta)] \in \mathcal{K}^{n \times n}(\delta) \quad (9)$$

with $r_i = \sum_{j=0}^s r_i^j(\mathbf{x}_{[\beta]})\delta^j$. Then locally around the origin there exist a bicausal change of coordinates $z = \phi(\mathbf{x})$ such that $dz = T^{-1}(\mathbf{x}, \delta)dx$ if and only if

- a) $T(\mathbf{x}, \delta)$ is unimodular
- b) $\forall x \in \mathcal{X}_0, \forall l, j \in [1, n]$ and $\forall i, k \in [0, 2s]$

$$[r_j^i(\mathbf{x}), r_l^k(\mathbf{x})]_E = 0 \quad (10)$$

While the detailed proof is omitted for space reasons, note that conditions (10) correspond to consider the vector fields $R_j^k(\mathbf{x}) = \sum_{i=0}^k r_j^{k-i}(\mathbf{x}(-i))\frac{\partial}{\partial x^i}$ defined on the infinite dimensional space, that is

$$\begin{pmatrix} r^0 & r^1 & \dots & r^s & 0 & \dots & 0 & \dots \\ 0 & r^0(-1) & \dots & r^{s-1}(-1) & r^s(-1) & 0 & \vdots & \dots \\ 0 & 0 & \dots & r^{s-2}(-2) & r^{s-1}(-2) & r^s(-1) & \ddots & \dots \\ \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & 0 & \dots \\ 0 & \dots & 0 & r^0(-s) & r^1(-s) & \dots & r^s(-s) & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \dots \end{pmatrix}$$

with $r^i = r_1^i \dots r_j^i$. Despite the infinite dimensionality of the vector fields, all the brackets are characterized by a finite number of equations. In fact it is immediately clear that the Lie bracket $[R_j^k, R_l^i] = 0$ whenever $|k - l| > 2s$, while the others yield the same equations, only time-shifted.

Let us now consider the submodules

$$\mathcal{R}_i = \text{span}_{\mathcal{K}(\delta)}(g_1(\mathbf{x}_{[s]}, \delta), \dots, g_i(\mathbf{x}_{[s]}, \mathbf{u}_{[s]}, \delta)), \quad i \in [1, n+1]$$

with $g_1(\mathbf{x}_{[s]}, \delta) := g(\mathbf{x}_{[s]}, \delta)$ and for $k > 1$, g_k recursively defined as

$$g_k(\mathbf{x}_{[s]}, \mathbf{u}_{[s]}, \delta) = f(\mathbf{x}_{[s]}, \mathbf{u}_{[s]}, \delta)g_{k-1}(\mathbf{x}_{[s]}, \mathbf{u}_{[s]}, \delta) + \dot{g}_{k-1}(\mathbf{x}_{[s]}, \mathbf{u}_{[s]}, \delta).$$

Remark. Note that for nonlinear time-delay systems, the left-kernel of R_i is the left-submodule \mathcal{H}_{i+1} , as shown in [23]. For nonlinear systems without delays

$$\begin{aligned} &g_k(x(t), u(t), \dots, u^{(k-2)}(t)) = \\ &(-1)^{k-1} \left(ad_f^{k-1}g + [g, ad_f^{k-2}g]u + \dots \right) \end{aligned}$$

while for the linear time-varying and time-invariant cases, \mathcal{R}_n reduces to the corresponding accessibility matrices $[B(t) \ A(t)B(t) - \dot{B}(t) \ \dots]$ and $[B \ AB \ \dots \ A^{n-1}B]$. We will thus call $R(x) = (g_1(\mathbf{x}_{[s]}, \delta), \dots, g_n(\mathbf{x}_{[s]}, \mathbf{u}_{[s]}, \delta))$ the accessibility matrix and a system characterized by a unimodular $R(x)$ accessible. \triangleleft

The following property holds true.

Proposition 2: If $g_{i+1}(\mathbf{x}, \mathbf{u}) \in \mathcal{R}_i$ then $\forall j \geq 1$, $g_{i+j}(\mathbf{x}, \mathbf{u}) \in \mathcal{R}_i$.

Proof: Since $g_{i+1}(\mathbf{x}, \mathbf{u}) \in \mathcal{R}_i$ then $g_{i+1}(\mathbf{x}, \mathbf{u}) = \sum_{j=1}^i g_j(\mathbf{x}, \mathbf{u})\alpha_j(\mathbf{x}, \mathbf{u})$. By definition

$$\begin{aligned} g_{i+2}(\mathbf{x}, \mathbf{u}) &= f(\mathbf{x}, \mathbf{u})g_{i+1}(\mathbf{x}, \mathbf{u}) - \dot{g}_{i+1}(\mathbf{x}, \mathbf{u}) \\ &= f(\mathbf{x}, \mathbf{u}) \left(\sum_{j=1}^i g_j(\mathbf{x}, \mathbf{u}) \right) \alpha_j(\mathbf{x}, \mathbf{u}) + \\ &\quad - \sum_{j=1}^i \dot{g}_j(\mathbf{x}, \mathbf{u})\alpha_j(\mathbf{x}, \mathbf{u}) - \sum_{j=1}^i g_j(\mathbf{x}, \mathbf{u})\dot{\alpha}_j(\mathbf{x}, \mathbf{u}) \\ &= \sum_{j=1}^i g_{j+1}(\mathbf{x}, \mathbf{u})\alpha_j(\mathbf{x}, \mathbf{u}) - \sum_{j=1}^i g_j(\mathbf{x}, \mathbf{u})\dot{\alpha}_j(\mathbf{x}, \mathbf{u}) \in \mathcal{R}_i \end{aligned}$$

which ends the proof. \blacksquare

Proposition 3: Under the change of coordinates $z = \phi(\mathbf{x}_{[\alpha]})$, with $dz = T(\mathbf{x}_{[\alpha]}, \delta)dx$, $\tilde{g}_j(\cdot)$, $j \geq 1$ is transformed as

$$\tilde{g}_j(\mathbf{z}, \mathbf{u}, \delta) = [T(\mathbf{x}, \delta)g_j(\mathbf{x}, \mathbf{u}, \delta)]_{\phi^{-1}(\mathbf{z})}. \quad (11)$$

Proof: According to (5), (11) is verified for $j = 1$. Recursively, assume that it is verified for $k - 1$, then by definition

$$\begin{aligned} \tilde{g}_k(\mathbf{z}, \mathbf{u}, \delta) &= \tilde{f}(\mathbf{z}, \mathbf{u}, \delta)\tilde{g}_{k-1}(\mathbf{z}, \mathbf{u}, \delta) - \dot{\tilde{g}}_{k-1}(\mathbf{z}, \mathbf{u}, \delta) \\ &= \left[\left(T(\mathbf{x}, \delta)f(\mathbf{x}, \mathbf{u}, \delta) + \dot{T}(\mathbf{x}, \delta) \right) T^{-1}(\mathbf{x}, \delta) \right]_{\phi^{-1}(\mathbf{z})} \times \\ &\quad [T(\mathbf{x}, \delta)g_{k-1}(\mathbf{x}, \mathbf{u}, \delta)]_{\phi^{-1}(\mathbf{z})} + \\ &\quad - \left[\dot{T}(\mathbf{x}, \delta)g_{k-1}(\mathbf{x}, \mathbf{u}, \delta) + T(\mathbf{x}, \delta)\dot{g}_{k-1}(\mathbf{x}, \mathbf{u}, \delta) \right]_{\phi^{-1}(\mathbf{z})} \end{aligned}$$

that is

$$\begin{aligned} \tilde{g}_k(\mathbf{z}, \mathbf{u}, \delta) &= (T(\mathbf{x}, \delta)f(\mathbf{x}, \mathbf{u}, \delta)g_{k-1}(\mathbf{x}, \mathbf{u}, \delta) + \\ &\quad - T(\mathbf{x}, \delta)\dot{g}_{k-1}(\mathbf{x}, \mathbf{u}, \delta))_{\phi^{-1}(\mathbf{z})} \\ &= (T(\mathbf{x}, \delta)g_k(\mathbf{x}, \mathbf{u}, \delta))_{\phi^{-1}(\mathbf{z})} \end{aligned}$$

An immediate consequence is the following.

Corollary 1: Under a bicausal change of coordinates $z = \phi(\mathbf{x}_{[\alpha]})$

$$\begin{aligned} \mathcal{R}_i &= \text{span}_{\mathcal{K}(\delta)}\{g_1(\mathbf{x}) \dots g_i(\mathbf{x}, \mathbf{u})\} \\ &\equiv \tilde{\mathcal{R}}_i = \text{span}_{\mathcal{K}(\delta)}\{\tilde{g}_1(\mathbf{z}) \dots \tilde{g}_i(\mathbf{z}, \mathbf{u})\}. \end{aligned}$$

IV. LINEAR EQUIVALENCE OF TIME-DELAY SYSTEMS

We will now show how the results proposed in the previous section can be effectively used to address the problem of the equivalence under change of coordinates to a linear system with time delays. The following result holds true.

Theorem 2: System (1) is equivalent, under a bicausal change of coordinates, to a linear strongly controllable delay system if and only if

- a) for $1 \leq i \leq n$, $g_i(\cdot) := g_i(\mathbf{x}, \delta)$
- b) $R(\mathbf{x}) = (g_1(\mathbf{x}, \delta), \dots, g_n(\mathbf{x}, \delta))$ is unimodular
- c) $g_{n+1}(\cdot) \in \text{span}_{R(\delta)} \mathcal{R}_n$, that is $g_{n+1}(\cdot) := g_{n+1}(\mathbf{x}, \delta) = \sum_{i=1}^n g_i(\mathbf{x}, \delta) c_i(\delta)$
- d) denoting by $\bar{s} \leq ns$ the maximal delay in $R(x)$, $\forall x \in \mathcal{X}_0$, for $i, j \in [1, n]$ and $r \leq \beta \in [0, 2\bar{s}]$, the following relations are satisfied

$$[g_j^\beta(\mathbf{x}), g_i^r(\mathbf{x})]_E = 0$$

$$\text{with } g_l(\mathbf{x}, \delta) = g_l^0(\mathbf{x}) + g_l^1(\mathbf{x})\delta + \dots + g_l^k(\mathbf{x})\delta^k.$$

Proof: It is easily verified that conditions a)÷ d) are satisfied by a linear strongly controllable time-delay system since $g_i(\cdot, \delta) = g_i(\delta)$. Due to Lemma 3 under any bicausal change of coordinates $\tilde{g}_i(\mathbf{z}, \delta) = (T(\mathbf{z}, \delta)g_i(\mathbf{x}, \delta))_{\phi^{-1}(\mathbf{x})}$ which implies that a)÷ c) must be satisfied. Finally d) must be also satisfied, due to Lemma 1.

Sufficiency. Let us assume that the conditions are satisfied. According to Theorem 1, since $R(x, \delta)$ is unimodular and d) are satisfied, we can consider the change of coordinates $z = \phi(\mathbf{x}_{[\alpha]})$ such that $dz = T(\mathbf{x}, \delta)dx$ with $T(\mathbf{x}, \delta) = R^{-1}(\mathbf{x}, \delta)$. Under such a change of coordinates, due to a) and b)

$$\begin{aligned} &(\tilde{g}_1(\mathbf{z}, \delta), \dots, \tilde{g}_n(\mathbf{z}, \delta)) = \\ &[T(\mathbf{x}, \delta)(g_1(\mathbf{x}, \delta), \dots, g_n(\mathbf{x}, \delta))]_{\phi^{-1}(\mathbf{z}, \delta)} = Id \end{aligned}$$

and due to c)

$$\begin{aligned} \tilde{g}_{n+1}(\mathbf{z}, \delta) &= \left[T(\mathbf{x}, \delta) \sum_{i=1}^n g_i(\mathbf{x}, \delta) c_i(\delta) \right]_{\phi^{-1}(\mathbf{z})} \\ &= \sum_{i=1}^n \tilde{g}_i(\delta) c_i(\delta). \end{aligned}$$

It follows that $\tilde{g}_1(\cdot) = B$ which proves the linearity of the control dependent part of the dynamics in the new coordinates, and due to the independence of $\tilde{g}_i(\cdot)$ from z and u

$$\begin{aligned} (\tilde{g}_2(\mathbf{z}, \delta), \dots, \tilde{g}_{n+1}(\mathbf{z}, \delta)) &= \sum_{i=0}^s \frac{\partial \tilde{F}(z, \delta)}{\partial z(t-i)} \delta^i (\tilde{g}_1(\cdot), \dots, \tilde{g}_n(\cdot)) \\ &= Q_1(\delta) \end{aligned}$$

that is

$$\sum_{i=0}^s \frac{\partial \tilde{F}(z, \delta)}{\partial z(t-i)} \delta^i = Q_1(\delta) = \sum_{i=0}^s A_i \delta^i$$

which proves the linearity of the dynamics with

$$\frac{\partial \tilde{F}(z, \delta)}{\partial z(t-i)} = A_i, \text{ for } i \geq 0. \quad \blacksquare$$

Corollary 2: System (1) is equivalent, under a bicausal change of coordinates, to a linear strongly controllable system without delays if and only if conditions a) b) and d) of Theorem 2 are satisfied, and additionally

$$\text{c')} \quad g_{n+1}(\cdot) \in \text{span}_R \mathcal{R}_n \text{ that is } g_{n+1}(\cdot) := g_{n+1}(\mathbf{x}, \delta) = \sum_{i=1}^n g_i(\mathbf{x}, \delta) c_i \text{ with } c_i \in R.$$

Proof: As for the necessity of c'), note that for a linear system $g_i = A^i B$ for $i \geq 0$, and due to Cayley Hamilton $A^n = c_1 I + c_2 A + \dots + c_n A^{n-1}$ with real coefficients c_i , so that $g_{n+1} = \sum_{i=1}^n g_i c_i = A^n B = \sum_{i=0}^{n-1} A^i B c_{i+1}$. Under any bicausal change of coordinates $z = \phi(\mathbf{x}_{[\alpha]})$ with $dz = T(\mathbf{x}, \delta)dx$, $\tilde{g}_i(\mathbf{z}) = (T(\mathbf{x}, \delta)g_i(\mathbf{x}))_{\phi^{-1}(\mathbf{z})}$, so that

$$\begin{aligned} \tilde{g}_{n+1}(\mathbf{z}, \delta) &= \left[T(\mathbf{x}, \delta) \sum_{i=0}^{n-1} g_i(\mathbf{x}) c_i \right] \phi^{-1}(\mathbf{z}, \delta) \\ &= \sum_{i=1}^n \tilde{g}_i(\mathbf{z}, \delta) c_i \end{aligned}$$

which instead proves c').

As for the sufficiency, we must prove that in the new coordinates the obtained linear system is without delays. To this end note that by assumption in the new coordinates $(\tilde{g}_1(\mathbf{z}, \delta), \dots, \tilde{g}_n(\mathbf{z}, \delta)) = Id$ and $(\tilde{g}_2(\mathbf{z}, \delta), \dots, \tilde{g}_{n+1}(\mathbf{z}, \delta)) = A$. Since

$$\sum_{j=0}^s \frac{\partial \tilde{F}(z)}{\partial z(t-j)} \delta^j (\tilde{g}_1(\mathbf{z}, \delta), \dots, \tilde{g}_n(\mathbf{z}, \delta)) = (\tilde{g}_2(\mathbf{z}, \delta), \dots, \tilde{g}_{n+1}(\mathbf{z}, \delta))$$

we have that $\sum_{j=0}^s \frac{\partial \tilde{F}(z)}{\partial z(t-j)} \delta^j = A$, which proves that $\frac{\partial \tilde{F}(z)}{\partial z(t)} = A$ and $\frac{\partial \tilde{F}(z)}{\partial z(t-i)} = 0$ for $i \geq 1$. \blacksquare

Example 3: Consider the dynamics

$$\begin{aligned} \dot{x}_1(t) &= x_2(t) - x_2(t-1) + 2x_2(t-1)u(t-1) \\ \dot{x}_2(t) &= u(t) \end{aligned}$$

for which

$$g_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \begin{pmatrix} 2x_2(t-1) \\ 0 \end{pmatrix} \delta, \quad g_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad g_3 = 0$$

Since condition a) of Theorem 2 is satisfied, the accessibility matrix $R(\mathbf{x})$ is independent of u and given by

$$R(\mathbf{x}) = \begin{pmatrix} 2x_2(t-1)\delta & 1 \\ 1 & 0 \end{pmatrix}$$

Thus $R(\mathbf{x})$ is unimodular which shows that condition b) is verified. Condition c) is also satisfied so we must only check condition d) with $\bar{s} = 1$. We have

$$\begin{aligned} [g_1^0, g_2^0]_E &= \left[\begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right] = 0 \\ [g_1^1, g_2^0]_E &= \left[\begin{pmatrix} 2x_2(t-1) \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right] = 0 \\ [g_1^1, g_1^0]_E &= \left[\begin{pmatrix} 2x_2(t-1) \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right] = 0 \end{aligned}$$

Though we should consider also

$$\begin{aligned} g_1^2 &= 2x_2(t-2) \frac{\partial}{\partial x_1(t-1)} + \frac{\partial}{\partial x_2(t-2)} \\ g_2^2 &= \frac{\partial}{\partial x_1(t-2)}, \end{aligned}$$

it is immediately clear that all the extended Lie brackets are zero. It follows that the unimodular matrix $R(\mathbf{x}, \delta)$ defines the change of coordinates

$$\begin{aligned} dz &= \begin{pmatrix} 0 & 1 \\ 1 & -2x_2(t-1)\delta \end{pmatrix} dx \\ &= \begin{pmatrix} d(x_2(t)) \\ d(x_1(t) - x_2^2(t-1)) \end{pmatrix} \end{aligned}$$

and yields

$$\dot{z}(t) = \begin{pmatrix} 0 & 0 \\ 1 - \delta & 0 \end{pmatrix} z(t) + \begin{pmatrix} 1 \\ 0 \end{pmatrix} u(t)$$

V. CONCLUSION

In the present paper a geometric approach for the study of time-delay systems has been used. It has been shown that starting from the definition of delayed state bracket introduced in [19] an analysis of the geometric properties of a delayed system can be successively pursued. This has been shown with respect to the problem of the equivalence of a nonlinear time-delay system to a linear strongly controllable system.

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