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# Absence of exponentially localized solitons for the Novikov–Veselov equation at negative energy

A.V. Kazeykina<sup>1</sup>, R.G. Novikov<sup>2</sup>

**Abstract.** We show that Novikov–Veselov equation (an analog of KdV in dimension  $2+1$ ) does not have exponentially localized solitons at negative energy.

## 1 Introduction

In the present paper we are concerned with the following  $(2+1)$ -dimensional analog of the Korteweg–de Vries equation:

$$\begin{aligned} \partial_t v &= 4\operatorname{Re}(4\partial_z^3 v + \partial_z(vw) - E\partial_z w), \\ \partial_z w &= -3\partial_z w, \quad v = \bar{v}, \quad E \in \mathbb{R}, \\ v &= v(x, t), \quad w = w(x, t), \quad x = (x_1, x_2) \in \mathbb{R}^2, \quad t \in \mathbb{R}, \end{aligned} \tag{1.1}$$

where

$$\partial_t = \frac{\partial}{\partial t}, \quad \partial_z = \frac{1}{2} \left( \frac{\partial}{\partial x_1} - i \frac{\partial}{\partial x_2} \right), \quad \partial_{\bar{z}} = \frac{1}{2} \left( \frac{\partial}{\partial x_1} + i \frac{\partial}{\partial x_2} \right). \tag{1.2}$$

Equation (1.1) is contained implicitly in the paper of S.V. Manakov [M] as an equation possessing the following representation

$$\frac{\partial(L - E)}{\partial t} = [L - E, A] + B(L - E) \tag{1.3}$$

(Manakov L–A–B triple), where  $L = -\Delta + v(x, t)$ ,  $\Delta = 4\partial_z\partial_{\bar{z}}$ ,  $A$  and  $B$  are suitable differential operators of the third and zero order respectively,  $[\cdot, \cdot]$  denotes the commutator. Equation (1.1) was written in an explicit form by S.P. Novikov and A.P. Veselov in [NV1], [NV2], where higher analogs of (1.1) were also constructed. Note that both Kadomtsev–Petviashvili equations can be obtained from (1.1) by considering an appropriate limit  $E \rightarrow \pm\infty$  (see [ZS], [G]).

In the case when  $v(x_1, x_2, t)$ ,  $w(x_1, x_2, t)$  are independent of  $x_2$ , (1.1) can be reduced to the classic KdV equation:

$$\partial_t u - 6u\partial_x u + \partial_x^3 u = 0, \quad x \in \mathbb{R}, \quad t \in \mathbb{R}. \tag{1.4}$$

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<sup>1</sup>Centre des Mathématiques Appliquées, Ecole Polytechnique, Palaiseau, 91128, France; email: kazeykina@cmap.polytechnique.fr

<sup>2</sup>CNRS (UMR 7641), Centre des Mathématiques Appliquées, Ecole Polytechnique, Palaiseau, 91128, France; email: novikov@cmap.polytechnique.fr

It is well-known that (1.4) has the soliton solutions

$$u(x, t) = u_{\kappa, \varphi}(x - 4\kappa^2 t) = -\frac{2\kappa^2}{ch^2(\kappa(x - 4\kappa^2 t - \varphi))}, \quad x \in \mathbb{R}, t \in \mathbb{R}, \kappa \in (0, +\infty), \varphi \in \mathbb{R}. \quad (1.5)$$

Evidently,

$$\begin{aligned} u_{\kappa, \varphi} &\in C^\infty(\mathbb{R}), \\ \partial_x^j u_{\kappa, \varphi}(x) &= O(e^{-2\kappa|x|}) \text{ as } x \rightarrow \infty, \quad j = 0, 1, 2, \dots \end{aligned} \quad (1.6)$$

Properties (1.6) imply that the solitons (1.5) are exponentially localized in  $x$ .

For the 2-dimensional case we will say that a solution  $(v, w)$  of (1.1) is an exponentially localized soliton if the following properties hold:

$$\begin{aligned} v(x, t) &= V(x - ct), \quad x \in \mathbb{R}^2, \quad c = (c_1, c_2) \in \mathbb{R}^2, \\ V &\in C^3(\mathbb{R}^2), \quad \partial_x^j V(x) = O(e^{-\alpha|x|}) \text{ for } |x| \rightarrow \infty, \quad |j| \leq 3 \text{ and some } \alpha > 0 \\ &\text{(where } j = (j_1, j_2) \in (0 \cup \mathbb{N})^2, |j| = |j_1| + |j_2|, \quad \partial_x^j = \partial^{j_1+j_2} / \partial x_1^{j_1} \partial x_2^{j_2}), \\ w(\cdot, t) &\in C(\mathbb{R}^2), \quad w(x, t) \rightarrow 0 \text{ as } |x| \rightarrow \infty, \quad t \in \mathbb{R}. \end{aligned} \quad (1.7)$$

In [N1] it was shown that, in contrast with the  $(1+1)$ -dimensional case, the  $(2+1)$ -dimensional KdV equation (1.1), at least for  $E = E_{fixed} > 0$ , does not have exponentially localized solitons. More precisely, in [N1] it was shown that the following theorem is valid for  $E = E_{fixed} > 0$ :

**Theorem 1.1.** *Let  $(v, w)$  be an exponentially localized soliton solution of (1.1) in the sense (1.7). Then  $v \equiv 0$ ,  $w \equiv 0$ .*

The main result of this paper consists in the proof of Theorem 1.1 for the case  $E = E_{fixed} < 0$ . This proof is given in Section 3 and is based on Propositions 3.1 and 3.2. In addition: Proposition 3.1 is an analog of the result of [N1] about the transparency of sufficiently localized solitons for equation (1.1) for  $E > 0$ ; Proposition 3.2 is an analog of the result of [N2], [GN1] that there are no nonzero bounded real-valued exponentially localized transparent potentials (that is potentials with zero scattering amplitude) for the Schrödinger equation (2.1) for  $E = E_{fixed} > 0$ .

Note that nonzero bounded algebraically localized solitons for equation (1.1) for  $E < 0$  are also unknown (see [G]), but their absence is not proved.

As regards integrable systems in  $2+1$  dimensions admitting exponentially decaying solitons in all directions on the plane, see [BLMP], [FS].

As regards integrable systems in  $2+1$  dimensions admitting nonzero bounded algebraically decaying solitons in all directions on the plane, see [FA], [BLMP], [G], [KN] and references therein.

## 2 Inverse scattering for the 2–dimensional Schrödinger equation at a fixed negative energy

Consider the scattering problem for the two–dimensional Schrödinger equation at a fixed negative energy:

$$\begin{aligned} -\Delta\psi + v(z)\psi &= E\psi, \quad E = E_{fixed} < 0, \\ \Delta &= 4\partial_z\partial_{\bar{z}}, \quad z = x_1 + ix_2, \quad x \in \mathbb{R}^2, \end{aligned} \quad (2.1)$$

where  $\partial_z, \partial_{\bar{z}}$  are the same as in (1.2). We will assume that the potential  $v(z)$  satisfies the following conditions

$$\begin{aligned} v(z) &= \overline{v(\bar{z})}, \quad v(z) \in L^\infty(\mathbb{C}), \\ |v(z)| &< q(1 + |z|)^{-2-\varepsilon} \text{ for some } q > 0, \quad \varepsilon > 0. \end{aligned} \quad (2.2)$$

In this paper we will be concerned with the exponentially decreasing potentials, i.e. with the potentials  $v(z)$  satisfying (2.2) and the following additional condition

$$v(z) = O(e^{-\alpha|z|}) \text{ as } |z| \rightarrow \infty \text{ for some } \alpha > 0. \quad (2.3)$$

Direct and inverse scattering for the two–dimensional Schrödinger equation (2.1) at fixed negative energy under assumptions (2.2) was considered for the first time in [GN2]. For some of the results discussed in this section see also [N2], [G].

First of all, we note that by scaling transform we can reduce the scattering problem with an arbitrary fixed negative energy to the case when  $E = -1$ . Therefore, in our further reasoning we will assume that  $E = -1$ .

It is known that for  $\lambda \in \mathbb{C} \setminus (0 \cup \mathcal{E})$ , where

$$\begin{aligned} \mathcal{E} &\text{ is the set of zeros of the modified Fredholm determinant } \Delta \\ &\text{ for the integral equation (2.10),} \end{aligned} \quad (2.4)$$

there exists a unique continuous solution  $\psi(z, \lambda)$  of (2.1) with the following asymptotics

$$\psi(z, \lambda) = e^{-\frac{1}{2}(\lambda\bar{z} + z/\lambda)}\mu(z, \lambda), \quad \mu(z, \lambda) = 1 + o(1), \quad |z| \rightarrow \infty. \quad (2.5)$$

In addition, the function  $\mu(z, \lambda)$  satisfies the following integral equation

$$\mu(z, \lambda) = 1 + \iint_{\zeta \in \mathbb{C}} g(z - \zeta, \lambda)v(\zeta)\mu(\zeta, \lambda)d\zeta_R d\zeta_I, \quad (2.6)$$

$$g(z, \lambda) = -\left(\frac{1}{2\pi}\right)^2 \iint_{\zeta \in \mathbb{C}} \frac{\exp(i/2(\zeta\bar{z} + \bar{\zeta}z))}{\zeta\bar{\zeta} + i(\lambda\bar{\zeta} + \zeta/\lambda)} d\zeta_R d\zeta_I, \quad (2.7)$$

where  $z \in \mathbb{C}$ ,  $\lambda \in \mathbb{C} \setminus 0$ ,  $\zeta_R = \operatorname{Re}\zeta$ ,  $\zeta_I = \operatorname{Im}\zeta$ .

In terms of  $\psi$  of (2.5) equation (2.6) takes the form

$$\psi(z, \lambda) = e^{-1/2(\lambda\bar{z}+z/\lambda)} + \iint_{\zeta \in \mathbb{C}} G(z - \zeta, \lambda) v(\zeta) \psi(\zeta, \lambda) d\zeta_R d\zeta_I, \quad (2.8)$$

$$G(z, \lambda) = e^{-1/2(\lambda\bar{z}+z/\lambda)} g(z, \lambda), \quad (2.9)$$

where  $z \in \mathbb{C}$ ,  $\lambda \in \mathbb{C} \setminus 0$ .

In terms of  $m(z, \lambda) = (1 + |z|)^{-(2+\varepsilon)/2} \mu(z, \lambda)$  equation (2.6) takes the form

$$m(z, \lambda) = (1 + |z|)^{-(2+\varepsilon)/2} + \iint_{\zeta \in \mathbb{C}} (1 + |\zeta|)^{-(2+\varepsilon)/2} g(z - \zeta, \lambda) \frac{v(\zeta)}{(1 + |\zeta|)^{-(2+\varepsilon)/2}} m(\zeta, \lambda) d\zeta_R d\zeta_I, \quad (2.10)$$

where  $z \in \mathbb{C}$ ,  $\lambda \in \mathbb{C} \setminus 0$ . In addition,  $A(\cdot, \cdot, \lambda) \in L^2(\mathbb{C} \times \mathbb{C})$ ,  $|\operatorname{Tr} A^2(\lambda)| < \infty$ , where  $A(z, \zeta, \lambda)$  is the Schwartz kernel of the integral operator  $A(\lambda)$  of the integral equation (2.10). Thus, the modified Fredholm determinant for (2.10) can be defined by means of the formula:

$$\ln \Delta(\lambda) = \operatorname{Tr}(\ln(I - A(\lambda)) + A(\lambda)) \quad (2.11)$$

(see [GK] for more precise sense of such definition).

Taking the subsequent members in the asymptotic expansion (2.5) for  $\psi(z, \lambda)$ , we obtain (see [N2]):

$$\begin{aligned} \psi(z, \lambda) = & \exp\left(-\frac{1}{2}\left(\lambda\bar{z} + \frac{z}{\lambda}\right)\right) \left\{ 1 - 2\pi \operatorname{sgn}(1 - \lambda\bar{\lambda}) \times \right. \\ & \times \left. \left( \frac{i\lambda a(\lambda)}{z - \lambda^2\bar{z}} + \exp\left(-\frac{1}{2}\left(\left(\frac{1}{\bar{\lambda}} - \lambda\right)\bar{z} + \left(\frac{1}{\lambda} - \bar{\lambda}\right)z\right)\right) \frac{\bar{\lambda}b(\lambda)}{i(\bar{\lambda}^2z - \bar{z})} \right) + o\left(\frac{1}{|z|}\right) \right\}, \end{aligned} \quad (2.12)$$

$|z| \rightarrow \infty$ ,  $\lambda \in \mathbb{C} \setminus (\mathcal{E} \cup 0)$ .

The functions  $a(\lambda)$ ,  $b(\lambda)$  from (2.12) are called the "scattering" data for the problem (2.1), (2.2) with  $E = -1$ . It is known that for  $a(\lambda)$ ,  $b(\lambda)$  the following formulas hold (see [N2]):

$$a(\lambda) = \left(\frac{1}{2\pi}\right)^2 \iint_{z \in \mathbb{C}} \mu(z, \lambda) v(z) dz_R dz_I, \quad (2.13)$$

$$b(\lambda) = \left(\frac{1}{2\pi}\right)^2 \iint_{z \in \mathbb{C}} \exp\left(-\frac{1}{2}\left(\left(\lambda - \frac{1}{\bar{\lambda}}\right)\bar{z} - \left(\bar{\lambda} - \frac{1}{\lambda}\right)z\right)\right) \mu(z, \lambda) v(z) dz_R dz_I, \quad (2.14)$$

where  $\lambda \in \mathbb{C} \setminus (0 \cup \mathcal{E})$ ,  $z_R = \operatorname{Re} z$ ,  $z_I = \operatorname{Im} z$ . In addition, formally, formulas (2.13), (2.14) can be written as

$$a(\lambda) = h(\lambda, \lambda), \quad b(\lambda) = h\left(\lambda, \frac{1}{\lambda}\right), \quad (2.15)$$

where

$$h(\lambda, \lambda') = \left(\frac{1}{2\pi}\right)^2 \iint_{z \in \mathbb{C}} \exp\left(\frac{1}{2}(\lambda' \bar{z} + z/\lambda')\right) \psi(z, \lambda) v(z) dz_R z_I, \quad (2.16)$$

and  $\lambda \in \mathbb{C} \setminus (0 \cup \mathcal{E})$ ,  $\lambda' \in \mathbb{C} \setminus 0$ . (Note that, under assumptions (2.2), the integral in (2.16) is well-defined if  $\lambda' = \lambda$  or if  $\lambda' = 1/\bar{\lambda}$  but is not well-defined in general.)

Let

$$T = \{\lambda \in \mathbb{C} : |\lambda| = 1\}. \quad (2.17)$$

From (2.15), in particular, the following statement follows:

**Statement 2.1.** *Let (2.2) hold and  $\Delta \neq 0$  on  $T$ . Then*

$$a(\lambda) = b(\lambda), \quad \lambda \in T. \quad (2.18)$$

The following properties of functions  $\Delta(\lambda)$ ,  $a(\lambda)$ ,  $b(\lambda)$  will play a substantial role in the proof of Theorem 1.1.

**Statement 2.2.** *Let (2.2) hold. Then:*

1.  $\Delta(\lambda) \in C(\mathbb{C})$ ;
2.  $\Delta(\lambda) \rightarrow 1$  as  $\lambda \rightarrow 0$  or  $\lambda \rightarrow \infty$ ;
3.  $\Delta(\lambda) \equiv \text{const}$  for  $\lambda \in T$ ;
4.  $\Delta$  is real-valued:  $\Delta = \bar{\Delta}$ .
5.  $\Delta(\lambda) = \Delta(1/\bar{\lambda})$ ,  $\lambda \in \mathbb{C} \setminus 0$ .

**Statement 2.3.** *Let conditions (2.2)–(2.3) be fulfilled. Then:*

- $\Delta(\lambda)$  is a real-analytic function on  $D_+$ ,  $D_-$ , where

$$D_+ = \{\lambda \in \mathbb{C} : 0 < |\lambda| \leq 1\}, \quad D_- = \{\lambda \in \mathbb{C} : |\lambda| \geq 1\}. \quad (2.19)$$

- $a(\lambda) = \frac{\mathcal{A}(\lambda)}{\Delta(\lambda)}$ ,  $b(\lambda) = \frac{\mathcal{B}(\lambda)}{\Delta(\lambda)}$ , where  $\mathcal{A}(\lambda)$ ,  $\mathcal{B}(\lambda)$  are real-analytic functions on  $D_+$ ,  $D_-$ .

Items 1–4 of Statement 2.2 are either known or follow from results mentioned in [HN], [N2] (see page 129 of [HN] and pages 420, 423, 429 of [N2]). In particular, item 1 of Statement 2.2 is a consequence of continuous dependency of  $g(z, \lambda)$  on  $\lambda \in \mathbb{C} \setminus 0$ ; item 3 of Statement 2.2 is a consequence of (2.11) and of the formula (see pages 420, 423 of [N2])  $G(z, \lambda) = (-i/4)H_0^1(i|z|)$ ,  $z \in \mathbb{C}$ ,  $\lambda \in T$ , where  $G$  is defined by (2.9),  $H_0^1$  is the Hankel function of the first type. In addition, item 5 of Statement 2.2 follows from item 4 of this statement and from symmetry  $\overline{G(z, \lambda)} = G(z, 1/\bar{\lambda})$ ,  $z \in \mathbb{C}$ ,  $\lambda \in \mathbb{C} \setminus 0$ .

Statement 2.3 is similar to Proposition 4.2 of [N2] and follow from: (i) formulas (2.13), (2.14), (ii) Cramer type formulas for solving the integral equation (2.10), (iii) the analog of Proposition 3.2 of [N2] for  $g$  of (2.7).

Under assumptions (2.2), the function  $\mu(z, \lambda)$ , defined by (2.6), satisfies the following properties:

$$\mu(z, \lambda) \text{ is a continuous function of } \lambda \text{ on } \mathbb{C} \setminus (0 \cup \mathcal{E}); \quad (2.20)$$

$$\frac{\partial \mu(z, \lambda)}{\partial \bar{\lambda}} = r(z, \lambda) \overline{\mu(z, \lambda)}, \quad (2.21a)$$

$$r(z, \lambda) = r(\lambda) \exp \left( \frac{1}{2} \left( \left( \lambda - \frac{1}{\lambda} \right) \bar{z} - \left( \bar{\lambda} - \frac{1}{\bar{\lambda}} \right) z \right) \right), \quad (2.21b)$$

$$r(\lambda) = \frac{\pi \operatorname{sgn}(1 - \lambda \bar{\lambda})}{\lambda} b(\lambda) \quad (2.21c)$$

for  $\lambda \in \mathbb{C} \setminus (0 \cup \mathcal{E})$ ;

$$\mu \rightarrow 1, \text{ as } \lambda \rightarrow \infty, \lambda \rightarrow 0. \quad (2.22)$$

The function  $b$  possesses the following properties (see [GN2], [N2]):

$$b \in C(\mathbb{C} \setminus \mathcal{E}), \quad (2.23)$$

$$b \left( -\frac{1}{\bar{\lambda}} \right) = b(\lambda), \quad b \left( \frac{1}{\bar{\lambda}} \right) = \overline{b(\lambda)}, \quad \lambda \in \mathbb{C} \setminus 0, \quad (2.24)$$

$$\lambda^{-1} b(\lambda) \in L_p(D_+) \text{ (as a function of } \lambda) \text{ if } \mathcal{E} = \emptyset, \quad 2 < p < 4. \quad (2.25)$$

In addition, the following theorem is valid:

**Theorem 2.1** ([GN2], [N2]). *Let  $v$  satisfy (2.2) and  $\mathcal{E} = \emptyset$  for this potential. Then  $v$  is uniquely determined by its scattering data  $b$  (by means of (2.20), (2.21) and equation (1.1) for  $E = -1$  and  $\psi$  of (2.5)).*

Finally, if  $(v(z, t), w(z, t))$  is a solution of equation (1.1) with  $E = -1$ , where  $(v(z, t), w(z, t))$  satisfy the following conditions:

$v, w \in C(\mathbb{R}^2 \times \mathbb{R})$  and for each  $t \in \mathbb{R}$  the following properties are fulfilled:  
 $v(\cdot, t) \in C^3(\mathbb{R}^2)$ ,  $\partial_x^j v(x, t) = O(|x|^{-2-\varepsilon})$  for  $|x| \rightarrow \infty, |j| \leq 3$  and some  $\varepsilon > 0$ ,

$$(2.26)$$

$w(x, t) \rightarrow 0$  for  $|x| \rightarrow \infty$ ,

then the dynamics of the scattering data is described by the following equations

$$a(\lambda, t) = a(\lambda, 0), \quad (2.27)$$

$$b(\lambda, t) = \exp \left\{ \left( \lambda^3 + \frac{1}{\lambda^3} - \bar{\lambda}^3 - \frac{1}{\bar{\lambda}^3} \right) t \right\} b(\lambda, 0), \quad (2.28)$$

where  $\lambda \in \mathbb{C} \setminus 0, t \in \mathbb{R}$ .

### 3 Proof of Theorem 1.1

The proof of Theorem 1.1 is based on Proposition 3.1 and Proposition 3.2 given below.

**Lemma 3.1.** *Let  $v(z)$  satisfy (2.2) and  $a(\lambda), b(\lambda)$  be the scattering data corresponding to  $v(z)$ . Then the scattering data  $a_\zeta(\lambda), b_\zeta(\lambda)$  for the potential  $v_\zeta(z) = v(z - \zeta)$  are related to  $a(\lambda), b(\lambda)$  by the formulas*

$$a_\zeta(\lambda) = a(\lambda), \quad (3.1)$$

$$b_\zeta(\lambda) = \exp \left( -\frac{1}{2} \left( \left( \lambda - \frac{1}{\lambda} \right) \bar{\zeta} - \left( \bar{\lambda} - \frac{1}{\bar{\lambda}} \right) \zeta \right) \right) b(\lambda), \quad (3.2)$$

where  $z, \zeta \in \mathbb{C}, \lambda \in \mathbb{C} \setminus 0$ .

*Proof.* We first note that  $\psi(z - \zeta, \lambda)$  satisfies equation (2.1) with the operator  $L = -\Delta + v_\zeta(z)$ . Then the function  $\psi_\zeta(z, \lambda)$  corresponding to  $v_\zeta(z)$  and possessing the asymptotics (2.5) is  $\psi_\zeta(z, \lambda) = e^{-\frac{1}{2}(\lambda \bar{\zeta} + \zeta/\lambda)} \psi(z - \zeta, \lambda)$ . In terms of function  $\mu$  this relation is written  $\mu_\zeta(z, \lambda) = \mu(z - \zeta, \lambda)$ . Thus we have

$$a_\zeta(\lambda) = \left( \frac{1}{2\pi} \right)^2 \iint_{z \in \mathbb{C}} v(z - \zeta) \mu(z - \zeta, \lambda) dz_R dz_I = a(\lambda),$$



and, similarly,

$$\begin{aligned}
b_\zeta(\lambda) &= \left(\frac{1}{2\pi}\right)^2 \iint_{z \in \mathbb{C}} \exp\left(-\frac{1}{2} \left( \left(\lambda - \frac{1}{\lambda}\right) \bar{z} - \left(\bar{\lambda} - \frac{1}{\bar{\lambda}}\right) z \right)\right) \times \\
&\quad \times v(z - \zeta) \mu(z - \zeta, \lambda) dz_R dz_I = \\
&= \exp\left(-\frac{1}{2} \left( \left(\lambda - \frac{1}{\lambda}\right) \bar{\zeta} - \left(\bar{\lambda} - \frac{1}{\bar{\lambda}}\right) \zeta \right)\right) b(\lambda).
\end{aligned}$$

□

**Proposition 3.1.** *Let  $(v(z, t), w(z, t))$  be an exponentially localized soliton of (1.1) in the sense (1.7). Let  $b(\lambda, t)$  be the scattering data for  $v(z, t)$  for some  $E = E_{fixed} < 0$ . Then  $b(\lambda, t) \equiv 0$  in the domain where it is well-defined, i.e. in  $\mathbb{C} \setminus \mathcal{E}$ , where  $\mathcal{E}$  is defined by (2.4).*

*Proof.* In virtue of (2.28) and Statement 2.3 it is sufficient to prove that  $b(\lambda, 0) \equiv 0$  in some neighborhoods of 0 and  $\infty$ .

Let  $U_0, U_\infty$  be the neighborhoods of 0 and  $\infty$ , respectively, such that  $\Delta \neq 0$  in  $U_0, U_\infty$  (such neighborhoods exist in virtue of item 2 of Statement 2.2). For  $\lambda \in U_0 \cup U_\infty$  the function  $b(\lambda, 0)$  is well-defined and continuous. As  $(v(z, t), w(z, t))$  is a soliton, the dynamics of the function  $b(\lambda, t)$  can be written as

$$b(\lambda, t) = \exp\left(-\frac{1}{2} \left( \left(\lambda - \frac{1}{\lambda}\right) \bar{c} - \left(\bar{\lambda} - \frac{1}{\bar{\lambda}}\right) c \right) t\right) b(\lambda, 0) \quad (3.3)$$

(see Lemma 3.1).

Combining this with formula (2.28), we obtain

$$\begin{aligned}
\exp\left\{-\frac{1}{2} \left( \left(\lambda - \frac{1}{\lambda}\right) \bar{c} - \left(\bar{\lambda} - \frac{1}{\bar{\lambda}}\right) c \right) t\right\} b(\lambda, 0) &= \\
&= \exp\left\{\left(\lambda^3 + \frac{1}{\lambda^3} - \bar{\lambda}^3 - \frac{1}{\bar{\lambda}^3}\right) t\right\} b(\lambda, 0).
\end{aligned}$$

As functions  $\lambda, \bar{\lambda}, \frac{1}{\lambda}, \frac{1}{\bar{\lambda}}, \lambda^3, \bar{\lambda}^3, \frac{1}{\lambda^3}, \frac{1}{\bar{\lambda}^3}, 1$  are linearly independent in any neighborhood of 0 and  $\infty$ , we obtain that  $b(\lambda, 0) \equiv 0$  in  $U_0 \cup U_\infty$ . □

**Proposition 3.2.** *Let  $v(z)$  satisfy (2.2)–(2.3) and  $b(\lambda)$  be its scattering data for some  $E = E_{fixed} < 0$ . If  $b(\lambda) \equiv 0$  in the domain where it is well-defined, i.e. in  $\mathbb{C} \setminus \mathcal{E}$ , where  $\mathcal{E}$  is defined by (2.4), then  $v \equiv 0$ .*

Note that Proposition 3.2 can be considered as an analog of Corollary 3 of [GN1].

*Proof of Proposition 3.2.* 1. First we will prove that from the assumptions of this proposition it follows that  $a(\lambda) \equiv 0$  in  $U_0 \cup U_\infty$ , where  $U_0$  and  $U_\infty$  are such neighborhoods of 0 and  $\infty$ , respectively, that  $\Delta(\lambda) \neq 0$  for  $\lambda \in U_0 \cup U_\infty$ . We note that from (2.13), (2.14), (2.21a), (2.22) it follows that

$$\frac{\partial a(\lambda)}{\partial \bar{\lambda}} = \frac{\pi \operatorname{sgn}(1 - \lambda \bar{\lambda})}{\bar{\lambda}} b(\lambda) \overline{b(\lambda)}, \quad \lambda \in \mathbb{C} \setminus (\mathcal{E} \cup 0), \quad (3.4)$$

$$a(\lambda) \rightarrow \hat{v}(0) \text{ as } \lambda \rightarrow \infty \text{ or } \lambda \rightarrow 0, \text{ where} \quad (3.5)$$

$$\hat{v}(p) = \left(\frac{1}{2\pi}\right)^2 \iint_{z \in \mathbb{C}} e^{\frac{i}{2}(\bar{p}z + p\bar{z})} v(z) dz_R dz_I, \quad p \in \mathbb{C}. \quad (3.6)$$

It means that

$$a(\lambda) \text{ is holomorphic in } \mathbb{C} \setminus (\mathcal{E}). \quad (3.7)$$

According to item 3 of Statement 2.2,  $\Delta(\lambda) \equiv \text{const}$  for  $\lambda \in T$ . We will consider separately two cases:  $\Delta \equiv C \neq 0$  on  $T$  and  $\Delta \equiv 0$  on  $T$ .

(a)  $\Delta(\lambda) \equiv C \neq 0$  on  $T$ :

From item 1 of Statement 2.2 it follows that there exists  $U_T$ , a neighborhood of  $T$ , such that  $\Delta(\lambda) \neq 0$  in  $U_T$ . Thus  $a(\lambda)$  is holomorphic in  $U_T$ . From Statement 2.1 we obtain that  $a(\lambda) = b(\lambda) = 0$  on  $T$ . It follows then that  $a(\lambda) \equiv 0$  in  $U_T$ . Using statement 2.3, we obtain that  $a(\lambda) \equiv 0$  in  $U_0 \cup U_\infty$ .

(b)  $\Delta(\lambda) \equiv 0$  on  $T$ :

In [HN] the  $\bar{\partial}$ -equation for  $\Delta$  was derived. In variables  $\lambda, \bar{\lambda}$  it is written as

$$\frac{\partial \ln \Delta(\lambda)}{\partial \bar{\lambda}} = -\frac{\pi \operatorname{sgn}(\lambda \bar{\lambda} - 1)}{\bar{\lambda}} \left( a \left( \frac{1}{\lambda} \right) - \hat{v}(0) \right). \quad (3.8)$$

Equation (3.8) and properties (3.5), (3.7) imply that  $\frac{\partial \ln \Delta}{\partial \bar{\lambda}}$  is an antiholomorphic function in  $U_0 \cup U_\infty$ , where  $\Delta$  is close to 1 and, thus,  $\ln \Delta$  is a well-defined one-valued function. As  $\Delta$  is a real-valued real analytic function, it follows that

$$\ln \Delta = f(\lambda) + \overline{f(\lambda)} \quad (3.9)$$

for some holomorphic function  $f(\lambda)$ , or

$$\Delta = F(\lambda) \overline{F(\lambda)} \quad (3.10)$$

for some holomorphic function  $F(\lambda)$  on  $U_0 \cup U_\infty$ . Now we will use the following lemma (the proof of this lemma is given in Section 4):

**Lemma 3.2.** *Let  $\Delta(\lambda)$  be real-analytic in  $D_+ = \{\lambda \in \mathbb{C} : 0 < |\lambda| \leq 1\}$ . Suppose that  $\Delta(\lambda)$  can be represented as*

$$\Delta(\lambda) = F(\lambda)\overline{F(\lambda)}, \quad \lambda \in U_0, \quad (3.11)$$

for some function  $F(\lambda)$  holomorphic on  $U_0$ , a neighborhood of zero. Then the representation (3.11) holds on  $D_+$ , i.e.  $F(\lambda)$  can be extended analytically to  $D_+$ .

Thus, the representation (3.10) is valid separately on  $D_+$  and on  $D_-$ , where we used also item 5 of Statement 2.2. As  $\Delta(\lambda) \equiv 0$  on  $T$ , it follows that  $F(\lambda) \equiv 0$  on  $T$  and, further,  $F(\lambda) \equiv 0$  on  $\mathbb{C}$ . This contradicts with item 2 of Statement 2.2. Thus we have shown that under the assumptions of Proposition 3.2 the case  $\Delta(\lambda) \equiv 0$  on  $T$  cannot hold.

2. Our next step is to prove that  $\Delta(\lambda) \equiv 1$  for  $\lambda \in \mathbb{C}$ .

Formula (3.5) states that  $a(0) = a(\infty) = \hat{v}(0)$ . Thus from equation (3.8) it follows that  $\frac{\partial \ln \Delta}{\partial \lambda} = 0$ , and  $\ln \Delta$  is holomorphic in some neighborhood of 0 and  $\infty$ . As  $\Delta(\lambda)$  is a real-valued function and item 2 of Statement 2.2 holds, we conclude that  $\Delta \equiv 1$  in some neighborhood of 0 and  $\infty$ . Now using Statement 2.3, we obtain that  $\Delta \equiv 1$  in  $\mathbb{C}$  and, as a corollary,  $\mathcal{E} = \emptyset$ .

3. From the previous item it follows that equation (2.21a) holds for  $\forall \lambda : \lambda \in \mathbb{C} \setminus 0$ . Due to the assumptions of Proposition 3.2 and the property that  $\mathcal{E} = \emptyset$ , we have that  $b \equiv 0$  on  $\mathbb{C}$  which means that  $\mu(z, \lambda)$  is holomorphic on  $D_+, D_-$ . As it is also continuous on  $\mathbb{C}$  and property (2.22) holds, we conclude that  $\mu(z, \lambda) \equiv 1$  and  $v(z) \equiv 0$ .

□

*Proof of Theorem 1.1 for  $E = E_{fixed} < 0$ .* The result follows immediately from Propositions 3.1, 3.2. □

## 4 Proof of Lemma 3.2

As  $F(\lambda)$  is analytic in  $U_0$ , it can be represented in this domain by a Taylor series. Let us consider the radius of convergence  $R$  of this Taylor series. Suppose that the statement of Lemma 3.2 is not true and  $R < 1$ .

Let us take a point  $\lambda_0$ , such that  $|\lambda_0| = R$ . In this point  $\Delta(\lambda)$  can be represented by the following series

$$\Delta(\lambda) = \sum_{k,j=0}^{\infty} b_{k,j}(\lambda - \lambda_0)^k(\bar{\lambda} - \bar{\lambda}_0)^j \quad (4.1)$$

uniformly convergent in  $U_{\lambda_0}$ , some neighborhood of  $\lambda_0$ . We will prove that the coefficients  $b_{j,k}$  satisfy the following properties:

$$\begin{aligned} b_{k,k} &\in \mathbb{R}, \quad b_{k,k} \geq 0; & (a) \\ b_{k,j} &= \overline{b_{j,k}}; & (b) \\ b_{k,j}b_{m,l} &= b_{k,l}b_{m,j}. & (c) \end{aligned}$$

Indeed,

$$\begin{aligned} (a): b_{k,k} &= \frac{1}{(k!)^2} \partial_{\lambda}^k \overline{\partial_{\lambda}^k \Delta(\lambda)} \Big|_{\lambda=\lambda_0} = \lim_{\lambda \rightarrow \lambda_0} \frac{1}{(k!)^2} \partial_{\lambda}^k \overline{\partial_{\lambda}^k \Delta(\lambda)} = \lim_{\lambda \rightarrow \lambda_0} \frac{1}{(k!)^2} |\partial_{\lambda}^k F(\lambda)|^2 \in \mathbb{R}, \geq 0. \\ (b): b_{k,j} &= \lim_{\lambda \rightarrow \lambda_0} \frac{1}{k!j!} \partial_{\lambda}^k F(\lambda) \overline{\partial_{\lambda}^j F(\lambda)} = \lim_{\lambda \rightarrow \lambda_0} \frac{1}{k!j!} \overline{\partial_{\lambda}^k F(\lambda) \partial_{\lambda}^j F(\lambda)} = \overline{b_{j,k}}. \\ (c): b_{k,j}b_{m,l} &= \lim_{\lambda \rightarrow \lambda_0} \frac{1}{k!j!m!l!} \partial_{\lambda}^k F(\lambda) \overline{\partial_{\lambda}^j F(\lambda) \partial_{\lambda}^m F(\lambda) \partial_{\lambda}^l F(\lambda)} = b_{k,l}b_{m,j}. \end{aligned}$$

From properties (a)–(c) it follows that there exist such  $a_k \in \mathbb{C}$ ,  $k = 0, 1, \dots$ , that

$$b_{k,j} = a_k \bar{a}_j. \quad (4.2)$$

We will prove this statement by considering two different cases:

1.  $b_{k,k} = 0 \quad \forall k \in \mathbb{N} \cup 0$

In this case from properties (b), (c) it follows that  $b_{k,j} = 0 \quad \forall k, j \in \mathbb{N} \cup 0$ , and we can take  $a_k = 0 \quad \forall k \in \mathbb{N} \cup 0$ .

2.  $b_{k,k} \neq 0$  for some  $k \in \mathbb{N} \cup 0$ .

In this case we take  $l$  to be the minimal number such that  $b_{l,l} \neq 0$ . Then we set  $a_0 = a_1 = \dots = a_{l-1} = 0$  and we take an arbitrary

complex number  $a_l$  satisfying  $|a_l|^2 = b_{l,l}$ . For the rest of the coefficients we set

$$a_n = \frac{b_{n,l}}{\bar{a}_l}, \quad (4.3)$$

where  $n = l + 1, l + 2, \dots$

Now let us prove property (4.2). Let us suppose that  $k < l$ . Then  $a_k = 0$ ,  $b_{k,k} = 0$  and from properties (b), (c) it follows that  $b_{k,j} = 0 \forall j \in \mathbb{N} \cup 0$ . Thus property (4.2) holds when  $k < l$ . A similar reasoning can be carried out when  $j < l$ . Now let us suppose that  $k \geq l$ ,  $j \geq l$ . Then

$$a_k \bar{a}_j = \frac{b_{k,l} \bar{b}_{j,l}}{\bar{a}_l a_l} = \frac{b_{k,l} b_{l,j}}{b_{l,l}} = b_{k,j}. \quad (4.4)$$

Representation (4.2) is proved.

From convergence of series (4.1) it follows that the following series

$$F_1(\lambda) = \sum_{k=0}^{\infty} a_k (\lambda - \lambda_0)^k \quad (4.5)$$

converges uniformly in  $U_{\lambda_0}$  (indeed, the case when  $b_{k,k} = 0 \forall k \in \mathbb{N} \cup 0$  is trivial, and in the case when  $\exists l: b_{l,l} \neq 0$  we take the sum of the members of series (4.1) with coefficients  $b_{k,l}$ ,  $k = 0, 1, \dots$ , and obtain series (4.5) multiplied by  $\bar{a}_l (\bar{\lambda} - \bar{\lambda}_0)^l$ ). Thus there exists the function  $F_1(\lambda)$  analytic in  $U_0$  such that  $\Delta(\lambda) = F_1(\lambda) \overline{F_1(\bar{\lambda})}$ . Consequently, we have two functions  $F(\lambda)$  and  $F_1(\lambda)$  analytic in a common domain lying in  $\{\lambda \in \mathbb{C}: |\lambda| \leq R\} \cap U_{\lambda_0}$  and such that  $|F(\lambda)| = |F_1(\lambda)|$ . It means that  $F(\lambda)$  and  $F_1(\lambda)$  are equal up to a constant factor:  $F(\lambda) = \mu F_1(\lambda)$ ,  $|\mu| = 1$ . It follows then that  $\mu F_1(\lambda)$  is an analytic continuation of  $F(\lambda)$  to  $U_{\lambda_0}$ .

The same reasoning can be applied to any point  $\lambda_0$  on the boundary of the ball  $B_R = \{\lambda \in \mathbb{C}: |\lambda| \leq R\}$ , i.e.  $F(\lambda)$  can be continued analytically to some larger domain. Hence we obtain a contradiction to the assumption that  $R < 1$  is the radius of convergence of the Taylor series for  $F(\lambda)$ . Thus  $R = 1$  and  $F(\lambda)$  can be extended analytically to  $D_+$ .  $\square$

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