



HAL
open science

Discontinuous generalized synchronization of chaos

Bastien Fernandez

► **To cite this version:**

| Bastien Fernandez. Discontinuous generalized synchronization of chaos. 2010. hal-00562485v1

HAL Id: hal-00562485

<https://hal.science/hal-00562485v1>

Preprint submitted on 3 Feb 2011 (v1), last revised 11 May 2011 (v2)

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Discontinuous generalized synchronization of chaos

Bastien Fernandez

October 4, 2010

Centre de Physique Théorique, UMR 6207
CNRS - Université Aix-Marseille II
CNRS Luminy Case 907
13288 Marseille CEDEX 9 France

Dedicated to the memory of Jaroslav Stark

Abstract

We analyze discontinuous synchronization functions in basic examples of forced systems with uniform contractive response and chaotic driving. The forcing is given by generalized (linear and nonlinear) baker's map and the response depends monotonically on the drive. The resulting synchronization graphs appear to be extremely wrinkled with dense sets of discontinuities. We provide a criteria on the contraction rate for the associated total variation to be bounded and we study in detail smoothness properties of the continuous component.

2010 *Mathematical Subject Classification.* 37A99, 26A45

Keywords: Synchronization graphs, skew-products, bounded variation

1 Introduction

Analyzing the asymptotic response to a random or erratic stimulus is an ubiquitous problem in Nonlinear Dynamics. The archetypical example is given by synchronization phenomena in directionally coupled systems [1]. (For convenience, focus will be on discrete time dynamics throughout this paper). When an autonomous forcing $x^{t+1} = f(x^t)$ compels iterations of a dissipative factor $z^{t+1} = g(x^t, z^t)$, the dynamics is known to be attracted by an invariant graph $z = h(x)$ [2, 3, 4]. In brief, the response z is asymptotically locked (*viz.* conjugated) to the drive x .

In this context, regularity and smoothness of the synchronization function h - which depend on f and g - prescribe those drive features (such as Lyapunov exponents, fractal dimensions, etc) that are conveyed to the factor. For instance, Lipschitz regularity guarantees identical attractor dimensions. Applications range from modeling of low-pass filters in signal analysis [5, 6, 7, 8] to damage detection in material science [9].

The mathematical analysis of synchronization function is part of the broader study of inertial manifolds in dynamical systems. It goes back to the seminal work of Hirsch, Pugh and Shub [10, 11] who proved existence and continuity under the assumption that f is a homeomorphism and g is contracting for z . They also showed Lipschitz regularity when g contraction rate is smaller than f contraction rate (and additional mild assumptions on f and g). These results were perfected later on [12, 13, 14], especially by Stark [15, 16] who established smoothness and extended to non uniform contractive responses.

Studies of continuity have been pursued beyond the homeomorphic case, not only when the response function g remains blind to drive discontinuities, but also when f is not invertible [17, 18]. However, continuity may not always hold in applications [6, 7, 8] and sensitive response functions also have to be considered.

To that aim, this paper considers discontinuous synchronization graphs in basic examples of skew products (f, g) with chaotic driving f and contractive response g *i.e.*

$$|g(x, z) - g(x, z')| \leq \lambda |z - z'| \tag{1}$$

where $0 < \lambda < 1$. As noticed in the literature [13, 15, 16], the response h lacks monotonicity and may have infinitely many discontinuity points that accumulate on a dense subset. Appropriate notions to investigate then are overall graph "length" (*i.e.* total variation of h) and continuous and discontinuous component properties. We proceed to a thorough analytic study in the case where f is an extension of the baker's map and $g(x, z) = \lambda z + x$ is linear [6, 8, 13]. The total variation is shown to possibly diverge depending on the contraction parameter λ . When this quantity remains finite, an analysis of regularity and of the derivative of the continuous component is given. The specific form of response function allows for results well beyond the standard theory of real functions. Finally, mention is proffered of those results that extend to more general nonlinear skew-product systems (f, g) .

2 Piecewise linear skew-products and synchronization graphs

As announced before, autonomous forcing in this study materializes via the generalized baker's map $(x, y) \mapsto f(x, y) = (T_{a,b}(x, y), T_b(y))$ of the unit square $[0, 1]^2$ into itself [6, 8]. Here $0 < a, b < 1$ and the mapping components write

$$T_{a,b}(x, y) = \begin{cases} ax & \text{if } 0 \leq y < b \\ (1-a)x + a & \text{if } b \leq y \leq 1 \end{cases}$$

and

$$T_b(y) = \begin{cases} \frac{y}{b} & \text{if } 0 \leq y < b \\ \frac{y-b}{1-b} & \text{if } b \leq y \leq 1 \end{cases}$$

Due to the contraction $0 < \lambda < 1$ the response system $z^t \mapsto z^{t+1} = g(x^t, z^t) = \lambda z^t + x^t$ results to be asymptotically locked to the forcing term [2, 3, 4]. More precisely, the large time behavior of the sequence $\{z^t\}$ is independent of z^0 and approaches $\{h(x^t)\}$. This is a consequence of the equality $z^t - h(x^t) = \lambda^t(z^0 - h(x^0))$ where the synchronization function h is given by

$$h(x) = \sum_{t=0}^{\infty} \lambda^t T_a^{t+1}(x), \quad x \in [0, 1]. \quad (2)$$

The response h only depends on the first forcing variable because f turns out to be invertible and the first coordinate of the inverse $f^{-1}(x, y) = (T_a(x), T_{b,a}(y, x))$ only depends on x . (Besides, as a function of λ , the map h is smooth and strictly increasing and uniformly converges to T_a when $\lambda \rightarrow 0$.) Furthermore, this function solves the conjugacy equation

$$g(x, h(x)) = h \circ T_{a,b}(x, y)$$

(thanks to the property $T_a \circ T_{a,b}(x, y) = x$). When f is not invertible, h *a priori* depends on backward histories $\{x^t\}_{t \leq 0}$ [17, 18].

Exponential convergence of the series (2) guarantees that h is well-defined on $[0, 1]$. The function can be viewed as a uniform limit of piecewise affine maps obtained by truncating the series to finite order (see equation (5) below). This comment provides a convenient way to numerically compute the graph up to arbitrary accuracy, see Figure 1.

The map T_a is right continuous and piecewise increasing. The same properties hold for all maps T_a^t . By uniform convergence, right continuity therefore transfers to h , *viz.* we have $h(x+0) = h(x)$ for all x . In addition, the maps T_a^t are all piecewise increasing; thus the left limit $h(x-0)$ exists

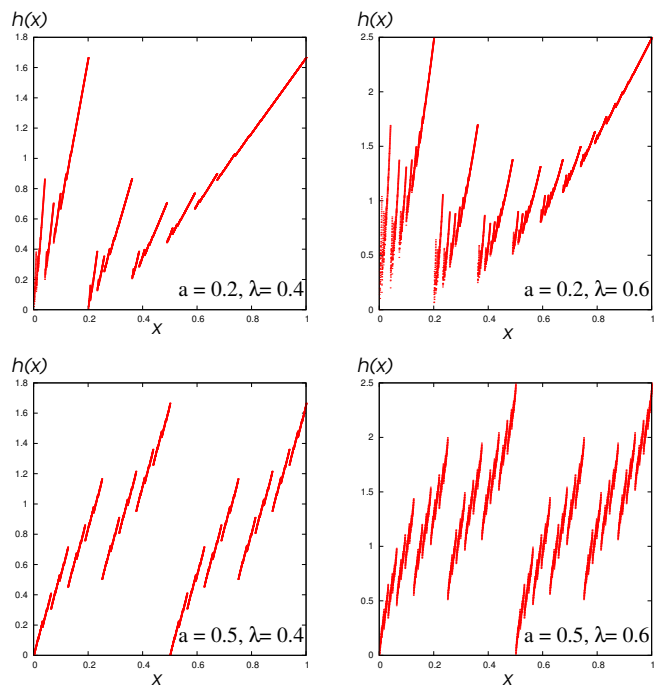


Figure 1: Examples of graph of h (of the function h_{14} indeed - see equation (5)). In the left pictures, h is of bounded variation; in the right ones, it has infinite variation. Notice the symmetry $h(x) = 1 - h(1 - x - 0)$ for $a = 0.5$. (Color online)

everywhere. This limit coincides with $h(x)$ unless x is a pre-image of the discontinuity point a . More precisely, we have $h(x-0) > h(x)$ iff $x \in D_a$ which is the dense subset of $[0, 1]$ defined by

$$D_a = \bigcup_{t=0}^{\infty} T_a^{-t}(a).$$

To prove that all jump discontinuities are negative, we start to notice that uniform convergence yields

$$h(x-0) = \sum_{t=0}^{\infty} \lambda^t T_a^{t+1}(x-0), \quad \forall x \in [0, 1].$$

By definition, for every point $x \in D_a$ there is a unique $t_0 \geq 0$ such that $T_a^{t_0}(x) = a$ and $T_a^t(x) \neq a$ when $0 \leq t < t_0$. The map T_a is continuous everywhere but at the point a ; hence we have

$$T_a^{t+1}(x-0) = T_a^{t+1}(x), \quad \forall 0 \leq t < t_0.$$

Using again $T_a^{t_0}(x) = a$, we get $T_a^{t_0+1}(x-0) = 1$ and $T_a^{t_0+1}(x) = 0$. However, these points 0 and 1 are fixed points; hence the same values hold for subsequent iterates. Altogether we obtain explicit estimates for jump discontinuities at every point of D_a

$$h(x-0) - h(x) = \sum_{t=t_0}^{\infty} \lambda^t > 0.$$

On the other hand, if x lies outside D_a , we have $\lim_{y \rightarrow x} T^t(y) = T^t(x)$ for all $t \geq 0$. Uniform convergence then implies that $\lim_{y \rightarrow x} h(y) = h(x)$ as claimed.

Finally, uniform convergence and right continuity allow to prove that the range of h is an entire interval, namely $h([0, 1]) = \frac{1}{1-\lambda}$, a property clearly suggested by the pictures in Figure 1.

3 Total variation and component regularity of the response function

The graph of the response function h is extremely wrinkled because monotonicity fails and discontinuity points are dense. The appropriate characteristic to evaluate in this case is the total variation [19] which gives the mean value of the modulus of derivative. As formally claimed in the next statement, the finiteness of this quantity depends on the contraction factor λ .

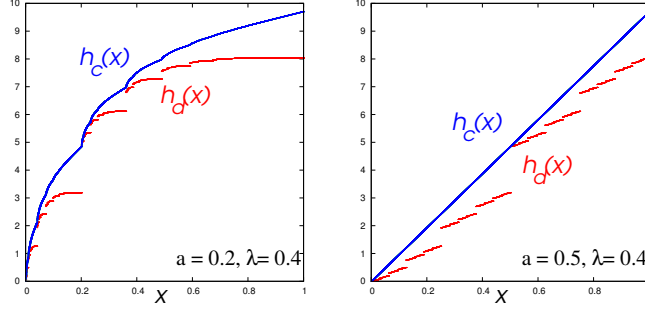


Figure 2: Graphs of h_c (blue dots - upper continuous curve) and h_d (red dots - lower discontinuous curve); fluctuations of h_c are barely visible for $a = 0.5$ (Color online)

Theorem 3.1 h is of bounded variation iff $\lambda < \frac{1}{2}$. Under this condition, the function h_d given by

$$h_d(x) = \sum_{y \in D_a : y \leq x} h(y-0) - h(y)$$

is well-defined (and is increasing) on $[0, 1]$ and the function h_c defined by $h_c = h + h_d$ is continuous and strictly increasing.

Some illustrations are given in Figure 2. All proofs are postponed to the Appendix. A classical theorem states that every function of bounded variation can be written as the difference of two increasing functions [19]. Another non less standard statement claims that bounded variation implies unique decomposition into the sum of a step function and of a continuous one. Interestingly, the difference $h_c - h_d$ here simultaneously proffers these alternative breakdowns for h .

We now study in detail smoothness properties of the continuous component h_c . In a general setting, Kolmogorov's theorem tells that monotonicity implies the existence of a finite derivative almost everywhere. In our case, the derivative $h'_c(x)$ turns out to exist everywhere out of the countable set D_a , provided that λ is further restricted.

Proposition 3.2 If $\lambda \leq \min\{a, 1-a\} < \frac{1}{2}$, then the map h_c is differentiable with bounded derivative at any point in the complement D_a^c . The derivative can be written

$$h'_c(x) = \sum_{t=0}^{\infty} \lambda^t (T_a^{t+1}(x))', \quad \forall x \in D_a^c.$$

In addition for any $\lambda \geq \min\{a, 1 - a\}$, there are points in D_a^c where $h'_c(x)$ diverges.

The function h_c can not be differentiable in D_a when $a \neq \frac{1}{2}$ because the right and left derivatives of T_a are unequal at $x = a$. However, by artificially setting $T'_a(a) = \frac{1}{1-a}$ for the derivative at the discontinuity point, the domain of h'_c extends to the entire interval $[0, 1]$ when $\lambda \leq \min\{a, 1 - a\}$. Alternatively, the map h'_c can be regarded as the (well-defined) right derivative on $[0, 1]$. The extended map h'_c intriguingly shares several properties with the original function h . Assuming $a \neq \frac{1}{2}$, this map is right continuous with jump discontinuities at every point in D_a . (This map is constant when $a = \frac{1}{2}$.) Unlike for h however, h'_c jump signs do depend on parameters. The analysis reveals that all signs $h'_c(x-0) - h'_c(x)$ are positive if $(a - \frac{1}{2})(\lambda - a(1 - a)) > 0$ and negative otherwise. Moreover we have the following statement that in particular provides an example of a strictly increasing function with derivative well-defined and vanishing everywhere excepted in a countable set.

Proposition 3.3 *Assume $a \neq \frac{1}{2}$. The function h'_c is of bounded variation iff $\lambda < a(1 - a)$. Under this condition, the map $(h'_c)_d$ defined by*

$$(h'_c)_d(x) = \sum_{y \in D_a : y \leq x} h'_c(y-0) - h'_c(y)$$

is well-defined on $[0, 1]$ (and non-increasing or non-decreasing depending on the sign of $a - \frac{1}{2}$).

In addition the function defined by $(h'_c)_c = h'_c + (h'_c)_d$ is continuous and strictly increasing if $a < \frac{1}{2}$ (resp. strictly decreasing if $a > \frac{1}{2}$). It is differentiable at any $x \in D_a^c$ and the derivative identically vanishes.

4 Extensions to Nonlinear Systems

Many results in this paper do not rely on the piecewise affine assumption. They extend to more general nonlinear systems. To present extensions, we start by introducing nonlinear generalized baker's maps with finitely many pieces.

Let $N > 1$ be an arbitrary integer and consider finite collections $\{I_i\}_{i=1}^N$ and $\{J_i\}_{i=1}^N$ of intervals defined by

$$I_i = [x_i, x_{i+1}) \text{ where } 0 = x_1 < x_2 < \dots < x_{N+1} = 1$$

and

$$J_i = [y_i, y_{i+1}) \text{ where } 0 = y_1 < y_2 < \dots < y_{N+1} = 1$$

for $i = 1, \dots, N - 1$ by $I_N = [x_N, 1]$ and $J_N = [y_N, 1]$. Consider two mappings T and S on $[0, 1]$ defined by

$$T|_{I_i} \equiv T_i \quad \text{and} \quad S|_{J_i} \equiv S_i, \quad i = 1, \dots, N$$

where each $T_i : I_i \rightarrow [0, 1]$ (and $T_N : I_N \rightarrow [0, 1]$) is a C^1 increasing, one-to-one and onto function and similarly for S_i . Now define the map f on $[0, 1]^2$ by $f(x, y) = (T(x, y), S(y))$ where

$$T(x, y) = T_i^{-1}(x) \text{ if } y \in J_i$$

The map f is easily checked to be invertible with inverse given by $f^{-1}(x, y) = (T(x), \mathcal{S}(y, x))$ where

$$\mathcal{S}(y, x) = S_i^{-1}(y) \text{ if } x \in I_i$$

As in the piecewise affine case, the dynamics of the skew-product (f, g) (where $g(x, z) = \lambda z + x$ still remains unchanged) is attracted by the graph of a function h defined by equation (2) where T_a is replaced by T .

The new synchronization function h shares several properties with the original one. It is piecewise increasing, right continuous with negative jump discontinuities at every point of the set D defined by

$$D = \bigcup_{t=0}^{\infty} T^{-t}(\{x_2, \dots, x_N\}) \quad (3)$$

and $h([0, 1]) = [0, \frac{1}{1-\lambda}]$. Moreover, the conclusions of the Theorem equally repeat in this case provided that $\frac{1}{2}$ is replaced by $\frac{1}{N}$ in the condition on the contraction parameter λ . Indeed, the only novelty sits in the number of t -preimages of discontinuity point $\{x_2, \dots, x_N\}$ (where $h(x-0) - h(x) = \frac{\lambda^t}{1-\lambda}$) which is now given by $(N - 1)N^t$.

(Of note, analogous to statements of Propositions 3.2 and 3.3 hold for maps f with $N > 2$ when one assumes that T and S are piecewise affine maps.)

Next, we consider the case where the response g is also nonlinear, *i.e.* we assume

- relation (1) holds for some $0 < \lambda < 1$,
- the maps $g(\cdot, z)$ are all increasing and the family is equi-continuous.

In such cases, the existence of a globally attracting synchronization function has previously been established [13, 16]. We complete this result here by specifying some properties of this function.

Proposition 4.1 For any $f(x, y) = (\mathcal{T}(x, y), S(y))$ and $g(x, z)$ as above, there exists a function h whose graph $z = h(x)$ attracts all sequences $\{(x^t, z^t)\}$ of the skew-product (f, g) . Moreover, h is right continuous, with negative jump discontinuities if any and, when $\lambda < \frac{1}{N}$, is of bounded variation.

Acknowledgments I am grateful to G. Mantica for inspiring discussions and especially to A. Quas for pointing out crucial errors in preliminary versions of the manuscript. I thank the Courant Institute (New York University) for hospitality. Support was partly provided by CNRS and by the EU Marie Curie fellowship PIOF-GA-2009-235741.

References

- [1] L.M. Pecora and T.L. Carroll, *Synchronization in chaotic systems*, Phys. Rev. Lett. **64** (1990) 821–824
- [2] V.S. Afraimovich, N.N. Verichev, and M.I. Rabinovich, *Stochastic synchronization of oscillations in dissipative systems*, Radiophys. Quantum Electron. **29** (1986) 795–803
- [3] L. Kocarev and U. Parlitz, *Generalized synchronization, predictability, and equivalence of unidirectionally coupled dynamical systems*, Phys. Rev. Lett. **76** (1996) 1816–1819
- [4] N.F. Rulkov, M.M. Sushchik, L.S. Tsimring, and H.D.I. Abarbanel, *Generalized synchronization of chaos in directionally coupled chaotic systems*, Phys. Rev. E **51** (1996) 980–94
- [5] J.L. Kaplan, J. Mallet-Paret and J.A. Yorke, *The Lyapunov dimension of a nowhere differentiable attracting torus*, Ergod. Th. Dynam. Sys. **4** (1984) 261–281
- [6] R. Badii and A. Politi, *Dimension function and phase transition-like behavior in strange attractors*, Physica Scripta **35** (1987) 243–246
- [7] D.S. Broomhead, J.P. Huke and M.R. Muldoon, *Linear filters and non-linear systems*, J. Roy. Stat. Soc. B **54** 373–382 (1992)
- [8] B.R. Hunt, E. Ott, and J.A. Yorke, *Differentiable generalized synchronization of chaos*, Phys. Rev. E **55** (1997) 4029–4034

- [9] L. Moniz, J. Nichols, S. Trickey, M. Seaver, D. Pecora and L. Pecora, *Using chaotic forcing to detect damage in a structure* Chaos **15** (2005) 023106
- [10] M. Hirsch and C. Pugh, *Stable Manifolds and Hyperbolic Sets* in *Global Analysis*, Amer. Math. Soc. Proc. Symp. Pure Math. **14** (1970) 133–164
- [11] M. Hirsch, C. Pugh and M. Shub, *Invariant Manifolds*, Lec. Notes Math. **583** (1977)
- [12] K.M. Campbell and M.E. Davies, *The existence of inertial functions in skew product systems* Nonlinearity **9** (1996) 801–817
- [13] V.S. Afraimovich, J.-R. Chazottes and A. Cordonet, *Synchronization in unidirectionally coupled systems: some rigorous results*, Discrete & Cont. Dyn. Sys. Ser. B **1** (2001) 421–442
- [14] J. Urias, *Filters display inverse limit spaces*, Chaos **14** (2004) 963–968
- [15] J. Stark, *Invariant graphs for forced systems*, Physica D **109** (1997) 163–179
- [16] J. Stark, *Regularity of invariant graphs for forced systems* Ergod. Th. Dynam. Sys. **19** (1999) 155–199
- [17] N.F. Rulkov and V.S. Afraimovich, *Detectability of nondifferentiable generalized synchrony*, Phys. Rev. E **67** (2003) 066218
- [18] E. Barreto, K. Josic, C.J. Morales, E. Sander, and P. So, *The geometry of chaos synchronization*, Chaos **13** 151–164 (2003)
- [19] A.N. Kolmogorov and S.V. Fomin, *Elements of the theory of functions and functional analysis*, Dover (1999)
- [20] see e.g. Theorem 6.1, Chapter X in S. Lang, *Undergraduate Mathematics*, 2nd ed. Springer (1997)

A Proofs

Proof that $\text{Ran}(h) = [0, \frac{1}{1-\lambda}]$. The crucial point is to show that for every n we have $h_n([0, 1]) = [0, \frac{1-\lambda^{n+1}}{1-\lambda}]$ where the approximations h_n are defined in the relation (5) below.

Consider the cylinder sets $[\theta^0 \dots \theta^n]$ that are associated with the usual symbolic dynamics of $(T_a, [0, 1])$, i.e. $\theta^t \equiv H(T_a^t(x) - a)$ where H is the Heaviside function. Cylinder sets are intervals and their union (with length fixed) covers $[0, 1]$.

The iterates T_a^{t+1} are piecewise bijections and we have $T_a^{t+1}([\theta^0 \dots \theta^n]) = [\theta^{t+1} \dots \theta^n]$ for $0 \leq t \leq n$ (where $[\theta^{n+1} \dots \theta^n]$ should be understood as $[0, 1]$). This yields

$$h_n([\theta^0 \dots \theta^n]) = \sum_{t=0}^n \lambda^t [\theta^{t+1} \dots \theta^n]$$

which is an interval. Moreover, given two adjacent cylinders $[\theta^0 \dots \theta^n] \leq [\bar{\theta}^0 \dots \bar{\theta}^n]$, their images $[\theta^{t+1} \dots \theta^n]$ and $[\bar{\theta}^{t+1} \dots \bar{\theta}^n]$ are either equal or adjacent. As a result, the intervals $h_n([\theta^0 \dots \theta^n])$ and $h_n([\bar{\theta}^0 \dots \bar{\theta}^n])$ must intersect; hence $h_n([0, 1])$ must be an interval. Computing its extrema gives $h_n([0, 1]) = [0, \frac{1-\lambda^{n+1}}{1-\lambda}]$.

The previous property immediately implies that $\text{Ran}(h)$ is dense in $\frac{1}{1-\lambda}$. Indeed for any strict sub-interval I in this set, there must be at least one point of $h([0, 1])$. To see this, it suffices to take n such that $|h - h_n|$ is (uniformly) smaller than $|I|/2$. Assume also that n is sufficiently large so that I is contained in $[0, \frac{1-\lambda^{n+1}}{1-\lambda}]$. Choose x such that $h_n(x)$ is the middle of I . Then $h(x)$ must belong to I .

Now, by continuity we know that every $h(x)$ for $x \in D_a^c$ can be realized as a limit $\lim_{y \rightarrow x} h(y)$. For points in D_a , we similarly take right limits. This shows that $h([0, 1])$ is the right closure of a dense subset in $[0, \frac{1}{1-\lambda}]$, namely it consists of this entire interval excepted the right boundary $\frac{1}{1-\lambda}$. But $h(1) = \frac{1}{1-\lambda}$, thus the proof is complete.

Proof of the Theorem. We begin to prove that the variation of h is infinite when $\lambda \geq \frac{1}{2}$. Since all discontinuities of h are negative jumps, the total variation of h is at least $\sum_{x \in D_a} h(x-0) - h(x)$. There are 2^n points in D_a for which $T_a^n(x) = a$ and the corresponding difference $h(x-0) - h(x)$ is equal to $\sum_{t=n}^{\infty} \lambda^t$. Therefore the variation of h is at least

$$\sum_{x \in D_a} h(x-0) - h(x) = \sum_{n=0}^{\infty} \frac{(2\lambda)^n}{1-\lambda}. \quad (4)$$

It easily follows that the total variation is infinite when $\lambda \geq \frac{1}{2}$.

To continue, we assume that $\lambda < \frac{1}{2}$. Relation (4) implies that the function h_d is well-defined (increasing and bounded) on $[0, 1]$. Moreover $h_d(x-0) = \sum_{y \in D_a : y < x} h(y-0) - h(y)$, hence $h_d(x) - h_d(x-0) = h(x-0) - h(x)$. All discontinuities of h are contained in h_d and thus the map h_c defined by $h_c = h + h_d$ is continuous.

We now show that h_c is strictly increasing and bounded, and as a consequence, that h is of bounded variation. Assume that $x < y$ and let

$$t_0 = \min\{t \geq 0 : \exists n \ x < x_t^n \leq y\},$$

where x_t^n is defined by $T_a^t(x_t^n) = a$. The number t_0 exists because D_a is a dense subset. By definition of t_0 all iterates of x and y up to $t_0 - 1$ must lie on the same side of the discontinuity point. By monotonicity of T_a^t this implies

$$T_a^t(x) < T_a^t(x_{t_0}^n) \leq T_a^t(y), \quad 0 \leq t \leq t_0.$$

Hence

$$h(x) < \sum_{t=0}^{t_0-1} \lambda^t T_a^{t+1}(x_{t_0}^n) + \sum_{t=t_0}^{\infty} \lambda^t = h(x_{t_0}^n - 0)$$

and

$$\sum_{t=0}^{t_0-1} \lambda^t T_a^t(x_{t_0}^n) = h(x_{t_0}^n) \leq h(y)$$

(The sum from 0 to $t_0 - 1$ equals 0 if $t_0 = 0$.) Using also the definition of h_d and h_c , we obtain

$$\begin{aligned} h_c(x) &= h(x) + h_d(x) \\ &< h(x_{t_0}^n - 0) + h_d(x) \\ &\leq h(x_{t_0}^n - 0) + h_d(x_{t_0}^n - 0) = h_c(x_{t_0}^n), \end{aligned}$$

and

$$\begin{aligned} h_c(y) &= h(y) + h_d(y) \\ &\geq h(x_{t_0}^n) + h_d(y) \\ &\geq h(x_{t_0}^n) + h_d(x_{t_0}^n) = h_c(x_{t_0}^n), \end{aligned}$$

i.e. $h_c(x) < h_c(y)$. Now we have $h_c(0) = 0$ and $h_c(1) = \sum_{t=0}^{\infty} \lambda^t + h_d(1) < +\infty$ and the proof is complete.

Proof of Proposition 3.2. The map h can be regarded as the uniform limit of the sequence $\{h_n\}$ where

$$h_n(x) = \sum_{t=0}^n \lambda^t T_a^{t+1}(x), \quad x \in [0, 1]. \quad (5)$$

Each map h_n is right continuous and with negative jump discontinuities in the set $D_a^n = \bigcup_{t=0}^n T_a^{-t}(a)$. As expected, the map $h_{n,d}$ defined by

$$h_{n,d}(x) = \sum_{y \in D_a^n : y \leq x} h_n(y-0) - h_n(y)$$

is an increasing step function that contains all discontinuities of h_n . Consequently, the subsequent map $h_{n,c} = h_n + h_{n,d}$ is a piecewise affine continuous map with finitely many affine parts. As such, it is absolutely continuous and hence differentiable almost everywhere in $[0, 1]$ [19]. The derivative $h'_{n,c}$ is a summable function. The fundamental theorem of calculus then yields

$$h_{n,c}(x) - h_{n,c}(0) = \int_0^x h'_{n,c}(y) dy. \quad (6)$$

The map $h_{n,c}$ is actually differentiable on $[0, 1] \setminus D_a^n$. Letting $T'_a(a) = \frac{1}{1-a}$, the derivative $h'_{n,c}$ can be uniquely continued to the following step function on $[0, 1]$

$$h'_{n,c}(x) = \sum_{t=0}^n \lambda^t (T_a^{t+1}(x))'$$

Since $T'_a(x) \in \{\frac{1}{a}, \frac{1}{1-a}\}$, when $\lambda < \min\{a, 1-a\}$, the sequence $\{h'_{n,c}\}$ uniformly converges to a bounded map, say h'_c . Applying Lebesgue's dominated convergence theorem, one can take the limit $n \rightarrow \infty$ in (6) to obtain the following equality

$$h_c(x) - h_c(0) = \int_0^x h'_c(y) dy, \quad \forall x \in [0, 1].$$

Now, the map h'_c is uniformly approximated by step functions and is continuous at every point of $[0, 1] \setminus D_a$. A standard results [20] states that the map $x \mapsto \int_0^x h'_c(y) dy$ is differentiable at every point of $[0, 1] \setminus D_a$ and with derivative h'_c . We conclude from the previous equality that h_c is differentiable on $[0, 1] \setminus D_a$ with derivative h'_c .

Proof of Proposition 3.3. To begin, each map $x \mapsto (T_a^{t+1}(x))'$ is piecewise constant and right continuous; hence h'_c is also right continuous. As before, the set D_a collects all discontinuity points. A similar calculation to as in the proof of (2) shows that if $x \in D_a$ is such that $T_a^{t_0}(x) = a$ for some (minimal) $t_0 \geq 0$, then we have

$$h'_c(x-0) - h'_c(x) = C_{a,\lambda} \lambda^{t_0} (T_a^{t_0}(x))'$$

where

$$C_{a,\lambda} = \left(\frac{1}{a(1 - \frac{\lambda}{1-a})} - \frac{1}{(1-a)(1 - \frac{\lambda}{a})} \right).$$

In particular, the sign of $h'_c(x-0) - h'_c(x)$ is independent of x . However, it depends on parameters via the constant $C_{a,\lambda}$. The latter is positive iff $(a - \frac{1}{2})(\lambda - a(1-a)) > 0$.

Now in order to estimate the total variation, as before, we have to sum up all contributions from discontinuities. Given x such that $T_a^{t_0}(x) = a$ for some $t_0 \geq 0$, the derivative $(T_a^{t_0}(x))'$ can be written $a^{-k}(1-a)^{-(t_0-k)}$ where k is the number of those iterates $\{T_a^t(x)\}_{t=0}^{t_0-1}$ that are smaller than a . Up to few exceptional sequences, the symbolic dynamics of T_a is the full shift on two symbols. Thus, for every $0 \leq k \leq t_0$, there are $\binom{t_0}{k}$ points with k iterates lying below a . Consequently, the total variation of $h'_c/C_{a,\lambda}$ is bounded below by

$$\sum_{t=0}^{\infty} \lambda^t \sum_{k=0}^t \binom{t}{k} \frac{1}{a^k(1-a)^{t-k}} = \frac{1}{\lambda} \sum_{t=0}^{\infty} \left(\frac{\lambda}{a(1-a)} \right)^{t+1}$$

which is infinite when $\lambda \geq a(1-a)$.

To prove bounded variation when $\lambda < a(1-a)$, we again proceed as before. Assume that $a < \frac{1}{2}$ and consider the non-decreasing step function $(h'_c)_d$ associated with h'_c , viz.

$$(h'_c)_d(x) = \sum_{y \in D_a : y \leq x} h'_c(y-0) - h'_c(y), \quad x \in [0, 1].$$

and also the continuous component defined by $(h'_c)_c = h'_c + (h'_c)_d$. Similar arguments to as in the proof of the Theorem show that the map $(h'_c)_c$ is strictly increasing and bounded on $[0, 1]$. It results that the variation of h'_c is finite when $\lambda < a(1-a)$. An analogous construction applies when $a > \frac{1}{2}$.

As in the proof of Proposition (3.2), the proof that $(h'_c)_c$ is differentiable proceeds by considering a sequence of uniform approximations of h'_c by piecewise affine functions with finitely many branches.

Proof of Proposition 4.1. Existence proofs in the nonlinear case have already been published in the literature [13, 16]. We provide another proof here that is more suitable to the present framework.

Let $M = \sup_{x \in [0,1]} |g_x(0)| < \infty$. Relation (1) results in $|g_x(y)| \leq |g_x(0)| + |y|$ which implies

$$g_x([-M_\lambda, M_\lambda]) \subset [-M_\lambda, M_\lambda], \quad \forall x \in [0, 1]$$

where $M_\lambda = \frac{M}{1-\lambda}$.

Given an arbitrary pair (x, z) and $t \geq 0$, define the t th iterate $h_t(x, z)$ as follows

$$h_t(x, z) = g_{T(x)} \circ g_{T^2(x)} \circ \cdots \circ g_{T^t(x)}(z).$$

By refining the arguments above, one shows that the interval $[-M_\lambda, M_\lambda]$ is not only invariant for h_t but it is also absorbing. In particular every sequence $\{h_t(x, z)\}$ must be bounded.

Next choose two integers $t > s$. We have

$$|h_t(x, z) - h_s(x, z)| < \lambda^s |g_{T^{s+1}(x)} \circ \cdots \circ g_{T^t(x)}(z) - z|$$

which implies that $\{h_t(x, z)\}$ is a Cauchy sequence; hence the following limit exists for all $x \in [0, 1]$

$$h(x) \equiv \lim_{t \rightarrow \infty} h_t(x, z)$$

and is independent of z . Moreover, the continuous dependence on z implies the following conjugacy equation, *i.e.*

$$\begin{aligned} g(x, h(x)) &= g_x\left(\lim_{t \rightarrow \infty} h_t(x, z)\right) \\ &= \lim_{t \rightarrow \infty} g_x(h_t(x, z)) = h \circ T(x, y) \end{aligned}$$

for some $y \in [0, 1]$. Global attraction easily follows by using contraction once again.

The arguments for properties of h are very similar to as in the linear case. By uniform convergence, and the fact that all $h_t(x, y)$ are right continuous, the same property hold for h . That the limit $h(x - 0)$ exists everywhere and jump discontinuities are all negative follow from properties of the map T^t and monotonicity of the $g(\cdot, z)$. The sufficient condition for bounded variation is obtained in the same way as before. The details are left to the reader.