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Optimal control of direction of reflection for jump diffusions

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Abstract

We extend the optimal control of direction of reflection problem introduced in Bouchard [4] to the jump diffusion case. In a Brownian diffusion framework with jumps, the controlled process is defined as the solution of a stochastic differential equation (SDE) reflected at the boundary of a domain along oblique directions of reflection which are controlled by a predictable process which may have jumps. We also provide a version of the weak dynamic programming principle of Bouchard and Touzi [5] adapted to our context and which is sufficient to provide a viscosity characterization of the associated value function without requiring the usual heavy measurable selection arguments nor the a-priori continuity of the value function.

Key words : Optimal control, Dynamic programming, Skorokhod problem, discontinuous viscosity solutions.

MSC Classification (2000): 93E20, 60G99, 49L25, 91B28.

1 Introduction

The aim of this paper is to study a class of optimal control problem for reflected processes on the boundary of a bounded domain $\mathcal{O}$, whose direction of reflection can be controlled. When the direction of reflection γ is not controlled, the existence of a solution to reflected SDEs was studied in the case where the domain $\mathcal{O}$ is a half space by El Karoui and Marchan [6], and, Ikeda and Watanabe [10]. Tanaka [18] considered the case of convex sets. More general domains have been discussed by Dupuis and Ishii [8], where they proved the strong existence and uniqueness of solutions in two cases. In the first case, the direction of reflection γ at each point of the boundary is single valued and varies smoothly, even if the domain $\mathcal{O}$ may be non smooth. In the second case, the domain $\mathcal{O}$ is the intersection of a finite number of domains with relatively smooth boundaries. Motivated by applications in financial mathematics, Bouchard [4] then proved the existence of a solution to a class of reflected SDEs, in which the oblique direction of reflection is controlled. This result is restricted to Brownian SDEs and to the case where the control is a deterministic combination of an Itô process and a continuous process with bounded variation. In this paper, we extend Bouchard's result to the case of jump diffusion and allow the control to have discontinuous paths.

As a first step, we start with an associated deterministic Skhorokhod problem:

$$\phi(t) = \psi(t) + \int_0^t \gamma(\phi(s), \varepsilon(s)) 1_{\phi(s)\in\partial\mathcal{O}} d\eta(s),\ \phi(t) \in \mathcal{O}, \tag{1.1}$$

where η is a non decreasing function and γ is controlled by a control process ε taking values in a given compact set E of $\mathbb{R}^l$. Bouchard [4] proved the strong existence of a solution for such problems in the family of continuous functions when ε is a continuous function with bounded variation. Extending this result, we consider the Skhorokhod problem in the family of càdlàg functions with finite number of points of discontinuity. The difficulty comes from the way the solution map is defined at the jump times. In this paper, we will investigate on a particular class of solutions, which is parameterized through the choice of a projection operator π. If the value $\phi(s-) + \Delta\psi(s)$ is out of the closure of the domain at a jump time s, we simply project this value on the boundary $\partial\mathcal{O}$ of the domain along the direction γ. The value after the jump of ϕ is chosen as $\pi(\phi(s-) + \Delta\psi(s), \varepsilon(s))$, where the projection π along the oblique direction γ satisfies

$$y = \pi(y, e) - l(y, e)\gamma(\pi(y, e), e),\ \text{for all } y \notin \bar{\mathcal{O}} \text{ and } e \in E,$$

for some suitable positive function l. This leads to

$$\phi(s) = (\phi(s-) + \Delta\psi(s)) + \gamma(\phi(s), \varepsilon(s))\Delta\eta(s),$$

with $\Delta\eta(s) = l(\phi(s-) + \Delta\psi(s), \varepsilon(s))$. When the direction of reflection is not oblique and the domain $\mathcal{O}$ is convex, the function π is just the usual projection operator and $l(y)$ coincides with the distance to the closure of the domain $\bar{\mathcal{O}}$.
We next consider the stochastic case. Namely, we prove the existence of an unique pair formed by a reflected process X^ε and a non decreasing process L^ε satisfying

$$\begin{cases} X(r) = x + \int_t^r F(X(s-))dZ_s + \int_t^r \gamma(X(s), \varepsilon(s)) 1_{X(s)\in\partial\mathcal{O}} dL(s), \\ \qquad\qquad X(r) \in \bar{\mathcal{O}},\ \text{for all } r \in [t, T] \end{cases} \tag{1.2}$$

where Z is the sum of a drift term, a Brownian stochastic integral and an adapted compound Poisson process, and the control process ε belongs to the class $\mathcal{E}$ of E-valued càdlàg predictable processes with bounded variation and finite activity. As in the deterministic case, we only study a particular class of solutions, which is parameterized by π. This means that whenever X is not in the domain $\mathcal{O}$ because of a jump, it is projected on the boundary $\partial\mathcal{O}$ along the direction γ and the value after the jump is also chosen as $\pi\left(X(s-) + F(X(s-))\Delta Z_s,\ \varepsilon(s)\right)$.

In section 3, we then introduce an optimal control problem, which extends the framework of [4] to the jump diffusion case,

$$v(t, x) = \sup_{\varepsilon\in\mathcal{E}} J(t, x; \varepsilon) \tag{1.3}$$

where the cost function $J(t, x; \varepsilon)$ is defined as $\mathbb{E}\left[\beta^\varepsilon_{t,x}(T)g(X^\varepsilon_{t,x}(T)) + \int_t^T \beta^\varepsilon_{t,x}(s)f(X^\varepsilon_{t,x}(s))ds\right]$ with $\beta^\varepsilon_{t,x}(s) = e^{-\int_t^s \rho(X^\varepsilon_{t,x}(r-))dL^\varepsilon_{t,x}(r)}$, f, g, ρ are some given functions, and the subscript t, x means that the solution of (1.2) is considered from time t with the initial condition x. As usual, the technical key for deriving the associated PDEs is the dynamic programming principle (DPP). The formal statement of the DPP may be written as follows, for τ in the set $\mathcal{T}(t, T)$ of stopping times taking values in $[t, T]$,

$$v(t, x) = \tilde{v}(t, x), \tag{1.4}$$

where

$$\tilde{v}(t,x) := \sup_{\varepsilon\in\mathcal{E}} \mathbb{E}\left[\beta^{\varepsilon}_{t,x}(\tau)v(\tau, X^{\varepsilon}_{t,x}(\tau)) + \int_t^{\tau} \beta^{\varepsilon}_{t,x}(s)f(X^{\varepsilon}_{t,x}(s))ds\right],$$

see [4], Fleming and Soner [9] and Lions [13]. Bouchard and Touzi [5] recently discussed a weaker version of the classical DPP (1.4), which is sufficient to provide a viscosity characterization of the associated value function, without requiring the usual heavy measurable selection argument nor the a priori continuity on the associated value function. In this paper, we apply their result to our context:

$$v(t,x) \le \sup_{\varepsilon\in\mathcal{E}} \mathbb{E}\left[\beta^{\varepsilon}_{t,x}(\tau)[v^*, g](\tau, X^{\varepsilon}_{t,x}(\tau)) + \int_t^{\tau} \beta^{\varepsilon}_{t,x}(s)f(X^{\varepsilon}_{t,x}(s))ds\right], \tag{1.5}$$

and, for every upper semi-continuous function φ such that $\varphi \le v_*$,

$$v(t,x) \ge \sup_{\varepsilon\in\mathcal{E}} \mathbb{E}\left[\beta^{\varepsilon}_{t,x}(\tau)[\varphi, g](\tau, X^{\varepsilon}_{t,x}(\tau)) + \int_t^{\tau} \beta^{\varepsilon}_{t,x}(s)f(X^{\varepsilon}_{t,x}(s))ds\right], \tag{1.6}$$

where v^* (resp. v_*) is the upper (resp. lower) semi-continuous envelope of v, and $[w,g](s,x) := w(s,x)1_{s<T} + g(x)1_{s=T}$ for any map w define on $[0,T]\times\bar{\mathcal{O}}$. This allows us to provide a PDE characterization of the value function v in the viscosity sense. We finally extend the comparison principle of Bouchard [4] to our context.

Following are some notations that will be used through out this paper.

Notations. For $T>0$ and a Borel set K of $\mathbb{R}^d$, $D^f([0,T],K)$ is the set of càdlàg functions from $[0,T]$ into K with a finite number of discontinuous points, and $BV^f([0,T],K)$ is the subset of elements in $D^f([0,T],K)$ with bounded variation. For $\varepsilon \in BV^f([0,T],K)$, we set $|\varepsilon| := \sum_{i\le n}|\varepsilon^i|$, where $|\varepsilon^i|$ is the total variation of ε^i. We denote by $N^{\varepsilon}_{[t,T]}$ the number of jump times of ε on the interval $[t,T]$. In the space $\mathbb{R}^d$, we denote by $\langle\cdot,\cdot\rangle$ natural scalar product and by $\|\cdot\|$ the associated norm. Any element of $\mathbb{R}^d$ is viewed as a column vector. For $x\in\mathbb{R}^d$, we denote by $B(x,r)$ the open ball of radius $r>0$ and center x. $\mathbb{M}^d$ is the set of square matrices of dimension n, Trace$[M]$ is the trace of $M\in\mathbb{M}^d$ and M^* is its transposition. For a set $K\subset\mathbb{R}^d$, we note K^c its complement and ∂K its boundary. Given a smooth map φ on $[0,T]\times\mathbb{R}^d$, we denote by $\partial_t\varphi$ its partial derivatives with respect to its first variable, and by $D\varphi$ and $D^2\varphi$ the partial gradient and Hessian matrix with respect to its second variable. If nothing else is specified, identities involving random variables have to be taken in the a.s. sense.

2 The SDEs with controlled oblique reflection

2.1 The deterministic problem.

For sake of simplicity, we first explain how we construct a class of solutions for the deterministic Skorokhod problem (SP), given an open domain $\mathcal{O}\subset\mathbb{R}^d$ and a continuous deterministic map ψ:

$$\phi(t) = \psi(t) + \int_0^t \gamma(\phi(s),\varepsilon(s))1_{\phi(s)\in\partial\mathcal{O}}d\eta(s)\ ,\ \phi(t)\in\bar{\mathcal{O}}\ \ \forall\, t\le T\,. \tag{SP}$$

In the case where the direction of reflection γ is not controlled, i.e. γ is a smooth function from $\mathbb{R}^d$ to $\mathbb{R}^d$ satisfying $|\gamma|=1$ which does not depend on ε, Dupuis and Isshi [8] proved the strong existence of a solution to the SP when $\mathcal{O}$ is a bounded open set and there exists $r\in(0,1)$ such that

$$\bigcup_{0\le\lambda\le r} B(x-\lambda\gamma(x),\lambda r)\subset\mathcal{O}^c,\ \forall x\in\partial\mathcal{O}. \tag{2.1}$$

In the case of controlled directions of reflection, Bouchard [4] showed that the existence holds whenever the condition (2.1) is imposed uniformly in the control variable:
G1. *$\mathcal{O}$ is a bounded open set, γ is a smooth function from $\mathbb{R}^d \times E$ to $\mathbb{R}^d$ satisfying $|\gamma| = 1$, and there exists some $r \in (0,1)$ such that*

$$\bigcup_{0 \leq \lambda \leq r} B(x - \lambda\gamma(x,e), \lambda r) \subset \mathcal{O}^c, \ \forall (x,e) \in \partial\mathcal{O} \times E. \tag{2.2}$$

In all this paper, E denotes a given compact subset of $\mathbb{R}^l$ for some $l \geq 1$.

In order to extend this result to the case where ψ is a deterministic càdlàg function with finite number of points of discontinuity, we focus on the definition of the solution value at the jump times. At each jump time s, the value after the jump of ϕ is chosen as $\pi(\phi(s-) + \Delta\psi(s), \varepsilon(s))$, where π is the image of a projection operator on the boundary $\partial\mathcal{O}$ along the direction γ satisfying following conditions:
G2. *For any $y \in \mathbb{R}^d$ and $e \in E$, there exists $(\pi(y,e),\ l(y,e)) \in \bar{\mathcal{O}} \times \mathbb{R}_+$ satisfying*

$$\begin{cases} \text{if } y \in \bar{\mathcal{O}}, & \pi(y,e) = y,\ l(y,e) = 0, \\ \text{if } y \notin \bar{\mathcal{O}}, & \pi(y,e) \in \partial\mathcal{O} \text{ and } y = \pi(y,e) - l(y,e)\gamma(\pi(y,e),e) \end{cases},$$

Moreover, π and l are Lipchitz continuous functions with respect to their first variable and uniformly in the second one.
This means that the value of ϕ just after the jump at time s is defined as

$$\phi(s) = (\phi(s-) + \Delta\psi(s)) + \gamma(\phi(s), \varepsilon(s))\Delta\eta(s),$$

where $\Delta\eta(s) = l(\phi(s-) + \Delta\psi(s), \varepsilon(s))$, or equivalently

$$\Delta\phi(s) = \Delta\psi(s) + \gamma(\phi(s), \varepsilon(s))\Delta\eta(s).$$

In view of the existence result of Bouchard [4], we already know that the existence of a solution to (SP) is guaranteed between the jump times and that the uniqueness between the jump times holds if $(\psi, \varepsilon) \in BV^f([0,T], \mathbb{R}^d) \times BV^f([0,T], \mathbb{R}^l)$. By pasting together the solutions at the jumps times according to the above rule, we clearly obtain an existence on the whole time interval $[0,T]$ when ψ and ε have only a finite number of discontinuous points.

Lemma 2.1 *Assume that* **G1** *and* **G2** *hold and fix $\psi, \varepsilon \in D^f([0,T], \mathbb{R}^d)$. Then, there exists a solution (ϕ, η) to (SP) associated to (π, l), i.e. there exists $(\phi, \eta) \in D^f([0,T], \mathbb{R}^d) \times D^f([0,T], \mathbb{R})$ such that*

(i) $\phi(t) = \psi(t) + \int_0^t \gamma(\phi(s), \varepsilon(s)) 1_{\phi(s) \in \partial\mathcal{O}} d\eta(s)$,
(ii) $\phi(t) \in \bar{\mathcal{O}}$ *for* $t \in [0,T]$,
(iii) η *is a non decreasing function*,
(iv) $\phi(s) = \pi(\phi(s-) + \Delta\psi(s), \varepsilon(s))$ *and* $\Delta\eta(s) = l(\phi(s-) + \Delta\psi(s), \varepsilon(s))$ *for* $s \in [0,T]$.

Moreover, the uniqueness holds if $(\psi, \varepsilon) \in BV^f([0,T], \mathbb{R}^d) \times BV^f([0,T], \mathbb{R}^l)$.

Remark 2.1 (i) *When the domain $\mathcal{O}$ is convex and the reflection is not oblique, i.e. $\gamma(x,e) \equiv n(x)$ for all $x \in \partial\mathcal{O}$ with $n(x)$ standing for the inward normal vector at the point $x \in \partial\mathcal{O}$, then the usual projection operator together with the distance function to the boundary $\partial\mathcal{O}$ satisfies assumption* **G2**.
(ii) *Clearly, the Lipschitz continuity conditions of π and l are not necessary for proving the existence of a solution to (SP). They will be used later to provide some technical estimates on the controlled processes, see Proposition 3.1 below.*

2.2 SDEs with oblique reflection.

We now consider the stochastic version of (SP). Let W be a standard n-dimensional Brownian motion and μ be a Poisson random measure on $\mathbb{R}^d$, which are defined on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$, such that W and μ are independent. We denote by $\mathbb{F} := \{\mathcal{F}_t\}_{0\le t\le T}$ the $\mathbb{P}$-complete filtration generated by (W, μ). We suppose that μ admits a deterministic $(\mathbb{P}, \mathbb{F})$-intensity kernel $\tilde{\mu}(dz)ds$ which satisfies

$$\int_0^T \int_{\mathbb{R}^d} \tilde{\mu}(dz)ds < \infty \, . \tag{2.3}$$

The aim of this section is to study the existence and uniqueness of a solution (X, L) to the class of reflected SDEs with controlled oblique reflection γ:

$$X(t) = x + \int_0^t F_s(X(s-))dZ_s + \int_0^t \gamma(X(s), \varepsilon(s))1_{X(s)\in\partial\mathcal{O}}dL(s), \tag{2.4}$$

where $(F_s)_{s\le T}$ is a predictable process with values in the set of Lipchitz functions from $\mathbb{R}^d$ to $\mathbb{M}^d$ such that $F(0)$ is essentially bounded, and Z is a $\mathbb{R}^d$-valued càdlàg Lévy process defined as

$$Z_t = \int_0^t b_s ds + \int_0^t \sigma_s dW_s + \int_0^t \int_{\mathbb{R}^d} \beta_s(z)\mu(dz, ds), \tag{2.5}$$

where $(t, z) \in [0, T] \times \mathbb{R}^d \mapsto (b_t, \sigma_t, \beta_t(z))$ is a deterministic bounded map with values in $\mathbb{R}^d \times \mathbb{M}^d \times \mathbb{R}^d$. As already mentioned, an existence of solutions was proved for Itô processes, i.e. $\beta = 0$, and the controls ε with continuous path and essentially bounded variation in [4]. In this paper, we extend this result to the case of jump diffusion and to the case where the controls ε can have discontinuous paths with a.s. finite activity. As in the deterministic case, we only consider a particular class of solutions which is parameterized by the projection operator π. Namely, X is projected on the boundary $\partial\mathcal{O}$ through the projection operator π whenever it is out of the domain because of a jump. The value after the jump of X is chosen as $\pi(X(s-) + F_s(X(s-))\Delta Z_s, \varepsilon(s))$.

In order to state rigorously the main result of this section, we first need to introduce some additional notations and definitions. For any Borel set K, we denote by $D_{\mathbb{F}}([0, T], K)$ the set of K-valued adapted càdlàg semimartingales with finite activity and $BV_{\mathbb{F}}([0, T], K)$ the set of processes in $D_{\mathbb{F}}([0, T], K)$ with a.s. bounded variation on $[0, T]$. We set $\mathcal{E} := BF_{\mathbb{F}}([0, T], E)$ for case of our notations.

Definition 2.1 *Given $x \in \mathcal{O}$ and $\varepsilon \in \mathcal{E}$, we say that $(X, L) \in D_{\mathbb{F}}([0, T], \mathbb{R}^d) \times D_{\mathbb{F}}([0, T], \mathbb{R})$ is a solution of the reflected SDEs with direction of reflection γ, projection operator (π, l) and initial condition x, if*

$$\begin{cases} X(t) = x + \int_0^t F_s(X(s-))dZ_s + \int_0^t \gamma(X(s), \varepsilon(s))1_{X(s)\in\partial\mathcal{O}}dL(s), \\ X(s) \in \bar{\mathcal{O}} \ \forall\ s \le T, \quad L \text{ is a non decreasing process}, \\ X(s) = \pi(X(s-) + F_s(X(s-))\Delta Z_s, \varepsilon(s)) \ , \ \Delta L(s) = l(X(s-) + F_s(X(s-))\Delta Z_s, \varepsilon(s)) \ \forall\ s \le T \, . \end{cases} \tag{2.6}$$

We now state the main result of this section.

Theorem 2.1 *Fix $x \in \mathcal{O}$ and $\varepsilon \in \mathcal{E}$, and assume that* **G1** *and* **G2** *hold. Then, there exists an unique solution $(X, L) \in D_{\mathbb{F}}([0, T], \mathbb{R}^d) \times D_{\mathbb{F}}([0, T], \mathbb{R})$ of the reflected SDEs* (2.6) *with oblique direction of reflection γ, projection operator (π, l) and initial condition x.*

Proof. Let $\{T_k\}_{k\geq 1}$ be the jump times of (Z,ε). Assume that (2.6) admits a solution on $[T_0, T_{k-1}]$ for $k \geq 1$, with $T_0 = 0$. It then follows from Theorem 2.2 in [4] that there exists an unique solution (X, L) of (2.6) on $[T_{k-1}, T_k)$. We set

$$X(T_k) = \pi(X(T_k-) + F_{T_k}(X(T_k-))\Delta Z_{T_k}, \varepsilon(T_k))$$

so that

$$X(T_k) = X(T_k-) + F_{T_k}(X(T_k-))\Delta Z_{T_k} + \gamma(X(T_k), \varepsilon(T_k))1_{X(T_k)\in\partial\mathcal{O}}\Delta L(T_k),$$

with $\Delta L(T_k) = l(X(T_k-) + F_{T_k}(X(T_k-))\Delta Z_{T_k}, \varepsilon(T_k))$. Since $N^{(Z,\varepsilon)}_{[0,T]} < \infty$ $\mathbb{P}-$ a.s, an induction leads to an existence result on $[0, T]$. Uniqueness follows from the uniqueness of the solution on each interval $[T_{k-1}, T_k)$, see Theorem 2.2 in [4]. □

3 The optimal control problem

3.1 Definitions.

We now introduce the optimal control problem which extends the one considered in [4]. The set of control processes $\zeta := (\alpha, \varepsilon)$ is defined as $\mathcal{A} \times \mathcal{E}$, where $\mathcal{A}$ is the set of predictable processes taking values in a given compact subset A of $\mathbb{R}^m$, for some $m \geq 1$. The family of controlled processes $(X^{\alpha,\varepsilon}_{t,x}, L^{\alpha,\varepsilon}_{t,x})$ is defined as follows. Let b, σ and χ be continuous maps on $\bar{\mathcal{O}} \times A$ and $\mathcal{O} \times A \times \mathbb{R}^d$ with values in $\mathbb{R}^d$, $\mathbb{M}^d$ and $\mathbb{R}^d$ respectively. We assume that they are Lipchitz continuous with respect to their first variable, uniformly in the others, and that χ is bounded with respect to its last component. It then follows from Theorem 2.1 that, for $(t, x) \in [0, T] \times \mathcal{O}$ and $(\alpha, \varepsilon) \in \mathcal{A} \times \mathcal{E}$, there exists an unique pair $(X^{\alpha,\varepsilon}_{t,x}, L^{\alpha,\varepsilon}_{t,x}) \in D_{\mathbb{F}}([0, T], \mathbb{R}^d) \times D_{\mathbb{F}}([0, T], \mathbb{R})$ which satisfies

$$\begin{aligned} X(r) = x &+ \int_t^r b(X(s), \alpha(s))ds + \int_t^r \sigma(X(s), \alpha(s))dW_s \\ &+ \int_t^r \int_{\mathbb{R}^d} \chi(X(s-), \alpha(s), z)\mu(dz, ds) + \int_t^r \gamma(X(s), \varepsilon(s))1_{X(s)\in\partial\mathcal{O}}dL(s), \forall t \leq r \leq T, \quad (3.1) \\ X(s) \in \bar{\mathcal{O}}\ \forall s \in [t, T],\ &L \text{ is non decreasing}, \quad (3.2) \\ (X(s), \Delta L(s)) = \int_{\mathbb{R}^d} &(\pi, l)\left(X(s-) + \chi(X(s-), \alpha(s), z), \varepsilon(s)\right)\mu(dz, \{s\}),\ \text{for } s \in [t, T]. \quad (3.3) \end{aligned}$$

Let ρ, f, g be bounded Borel measurable real valued maps on $\bar{\mathcal{O}} \times E$, $\bar{\mathcal{O}} \times A$ and $\bar{\mathcal{O}}$, respectively. We assume that g is Lipchitz continuous and $\rho \geq 0$. The functions ρ and f are also assumed to be Lipchitz continuous in their first variable, uniformly in their second one. We then define the cost function

$$J(t, x; \zeta) := \mathbb{E}\left[\beta^\zeta_{t,x}(T)g(X^\zeta_{t,x}(T)) + \int_t^T \beta^\zeta_{t,x}(s)f(X^\zeta_{t,x}(s), \alpha(s))ds\right], \quad (3.4)$$

where $\beta^\zeta_{t,x}(s) := e^{-\int_t^s \rho(X^\zeta_{t,x}(r-), \varepsilon(r))dL^\zeta_{t,x}(r)}$, for $\zeta = (\alpha, \varepsilon) \in \mathcal{A} \times \mathcal{E}$.
The aim of the controller is to maximize $J(t, x; \zeta)$ over the set $\mathcal{A}_t \times \mathcal{E}_t$ of controls in $\mathcal{A} \times \mathcal{E}$ which are independent on $\mathcal{F}_t$, compare with [5]. The associated value function is then defined as

$$v(t, x) := \sup_{\zeta\in\mathcal{A}_t\times\mathcal{E}_t} J(t, x; \zeta).$$

3.2 Dynamic programming.

In order to provide a PDE characterization of the value function v, we shall appeal as usual to the dynamic programming principle. The classical DPP (1.4) relates the time-t value function $v(t,\cdot)$ to the later time-τ value $v(\tau,\cdot)$, for any stopping time $\tau \in \mathcal{T}(t,T)$. Recently Bouchard and Touzi [5] provided a weaker version of the DPP, which is sufficient to provide a viscosity characterization of v. This version allows us to avoid the technical difficulties related to the use of non-trivial measurable selection arguments or the a-priori continuity of v.
From now, for $t \leq T$, we denote by $\mathcal{T}_t(\tau_1,\tau_2)$ the set of elements in $\mathcal{T}(\tau_1,\tau_2)$ which are independent on $\mathcal{F}_t$. The weak version of the DPP reads as follows:

Theorem 3.1 *Fix $(t,x) \in [0,T] \times \bar{\mathcal{O}}$ and $\tau \in \mathcal{T}_t(t,T)$, then*

$$v(t,x) \leq \sup_{\zeta \in \mathcal{A}_t \times \mathcal{E}_t} \mathbb{E}\left[\beta^\zeta_{t,x}(\tau)[v^*,g](\tau, X^\zeta_{t,x}(\tau)) + \int_t^\tau \beta^\zeta_{t,x}(s) f(X^\zeta_{t,x}(s))ds\right], \tag{3.5}$$

and

$$v(t,x) \geq \sup_{\zeta \in \mathcal{A}_t \times \mathcal{E}_t} \mathbb{E}\left[\beta^\zeta_{t,x}(\tau)[\varphi,g](\tau, X^\zeta_{t,x}(\tau)) + \int_t^\tau \beta^\zeta_{t,x}(s) f(X^\zeta_{t,x}(s))ds\right], \tag{3.6}$$

for any upper semi continuous function φ such that $v \geq \varphi$.

Arguing as in [5], the result follows once $J(\cdot;\zeta)$ is proved to be lower semicontinuous for all $\zeta \in \mathcal{A} \times \mathcal{E}$. In our setting, one can actually prove the continuity of the above map.

Proposition 3.1 *Fix $(t_0,x_0) \in [0,T] \times \bar{\mathcal{O}}$ and $\zeta \in \mathcal{A} \times \mathcal{E}$. Then, we have*

$$\lim_{(t,x)\to(t_0,x_0)} J(t,x;\zeta) = J(t_0,x_0;\zeta). \tag{3.7}$$

Proposition 3.1 will be proved later in the Subsection 3.3. Before providing the proof of the DPP, we verify the consistency with deterministic initial data assumption, see Assumption A4 in [5].

Lemma 3.1 (i) *Fix $(t,x) \in [0,T] \times \bar{\mathcal{O}}$, $(\zeta,\theta) \in \mathcal{A}_t \times \mathcal{E}_t \times \mathcal{T}_t(t,T)$. For $\mathbb{P}$-a.e $\omega \in \Omega$, there exists $\tilde{\zeta}_\omega \in \mathcal{A}_{\theta(\omega)} \times \mathcal{E}_{\theta(\omega)}$ such that*

$$\begin{aligned} &\mathbb{E}\left[\beta^\zeta_{t,x}(T) g(X^\zeta_{t,x}(T)) + \int_t^T \beta^\zeta_{t,x}(s) f(X^\zeta_{t,x}(s),\alpha(s))ds | \mathcal{F}_\theta\right](\omega) \\ = \;& \beta^\zeta_{t,x}(\theta(\omega)) J(\theta(\omega), X^\zeta_{t,x}(\theta)(\omega); \tilde{\zeta}_\omega) + \int_t^{\theta(\omega)} \beta^\zeta_{t,x}(s)(\omega) f(X^\zeta_{t,x}(s)(\omega), \alpha(s)(\omega))ds. \end{aligned}$$

(ii) *For $t \leq s \leq T$, $\theta \in \mathcal{T}_t(t,s)$, $\tilde{\zeta} \in \mathcal{A}_s \times \mathcal{E}_s$ and $\bar{\zeta} := \zeta 1_{[t,\theta)} + \tilde{\zeta} 1_{[\theta,T]}$, we have, for $\mathbb{P}$-a.e. $\omega \in \Omega$,*

$$\begin{aligned} &\mathbb{E}\left[\beta^{\bar{\zeta}}_{t,x}(T) g(X^{\bar{\zeta}}_{t,x}(T)) + \int_t^T \beta^{\bar{\zeta}}_{t,x}(s) f(X^{\bar{\zeta}}_{t,x}(s),\alpha(s))ds | \mathcal{F}_\theta\right](\omega) \\ = \;& \beta^\zeta_{t,x}(\theta(\omega)) J(\theta(\omega), X^\zeta_{t,x}(\theta)(\omega); \tilde{\zeta}) + \int_t^{\theta(\omega)} \beta^\zeta_{t,x}(s)(\omega) f(X^\zeta_{t,x}(s)(\omega), \alpha(s)(\omega))ds. \end{aligned}$$

Proof. In this proof, we consider the space $(\Omega,\mathbb{F},\mathbb{P})$ as being the product space $C([0,T],\mathbb{R}^d) \times S([0,T],\mathbb{R}^d)$, where $S([0,T],\mathbb{R}^d) := \{(t_i,z_i)_{i\geq 1} : \; t_i \uparrow T, \; z_i \in \mathbb{R}^d\}$, equipped with the product measure $\mathbb{P}$ induced by the Wiener measure and the Poisson random measure μ. We denote by ω or $\tilde{\omega}$ a generic

point. We also define the stopping operator $\omega^r_\cdot := \omega_{r\wedge\cdot}$ and the translation operator $T_r(\omega) := \omega_{\cdot+r} - \omega_r$. We then obtain from direct computations that, for $\zeta = (\alpha, \varepsilon) \in \mathcal{A}_t \times \mathcal{E}_t$:

$$\begin{aligned}
&\mathbb{E}\left[\beta^\zeta_{t,x}(T)g(X^\zeta_{t,x}(T)) + \int_t^T \beta^\zeta_{t,x}(s)f(X^\zeta_{t,x}(s),\alpha(s))ds|\mathcal{F}_\theta\right](\omega)\\
=\;& \beta^\zeta_{t,x}(\theta)(\omega)\int\left[\beta^{\zeta(\omega^{\theta(\omega)}+T_{\theta(\omega)}(\omega))}_{\theta(\omega),X^\zeta_{t,x}(\theta)(\omega)}(T)\,g\left(X^{\zeta(\omega^{\theta(\omega)}+T_{\theta(\omega)}(\omega))}_{\theta(\omega),X^\zeta_{t,x}(\theta)(\omega)}(T)\right)\right.\\
+\;& \left.\int_{\theta(\omega)}^T \beta^{\zeta(\omega^{\theta(\omega)}+T_{\theta(\omega)}(\omega))}_{\theta(\omega),X^\zeta_{t,x}(\theta)(\omega)}(s)f\left(X^{\zeta(\omega^{\theta(\omega)}+T_{\theta(\omega)}(\omega))}_{\theta(\omega),X^\zeta_{t,x}(\theta)(\omega)}(s),\alpha(\omega^{\theta(\omega)}+T_{\theta(\omega)}(\omega))(s)\right)ds\right]d\mathbb{P}(T_{\theta(\omega)}(\omega))\\
+\;& \int_t^{\theta(\omega)}\beta^\zeta_{t,x}(s)(\omega)f(X^\zeta_{t,x}(s)(\omega),\alpha(s)(\omega))ds\\
=\;& \beta^\zeta_{t,x}(\theta)(\omega)J(\theta(\omega),X^\zeta_{t,x}(\theta)(\omega);\tilde\zeta_\omega) + \int_t^{\theta(\omega)}\beta^\zeta_{t,x}(s)(\omega)f(X^\zeta_{t,x}(s)(\omega),\alpha(s)(\omega))ds,
\end{aligned}$$

where $\tilde\omega \in \Omega \mapsto \tilde\zeta_\omega(\tilde\omega) := \zeta(\omega^{\theta(\omega)} + T_{\theta(\omega)}(\tilde\omega)) \in \mathcal{A}_{\theta(\omega)} \times \mathcal{E}_{\theta(\omega)}$.
This leads proves (i). The assertion (ii) is proved similarly by using the fact that $\theta(\omega) \in [t, s]$ for $\mathbb{P}$-a.e. $\omega \in \Omega$. □

Proof of Theorem 3.1
The inequality (3.5) is clearly a consequence of (i) in Lemma 3.1. So it remains to prove the inequality (3.6). Fix $\epsilon > 0$. In view of definition of J and v, there exists a family $\{\zeta(s,y)\}_{(s,y)\in[0,T]\times\bar{\mathcal{O}}}$ such that

$$J(s,y;\zeta(s,y)) \geq v(s,y) - \epsilon/3, \text{ for all } (s,y) \in [0,T]\times\bar{\mathcal{O}}.$$

Using Proposition 3.1 and the upper semi continuity of φ, we can choose a family $\{r(s,y)\}_{(s,y)\in[0,T]\times\bar{\mathcal{O}}} \subset (0,\infty)$ such that

$$J(s,y;\zeta(s,y)) - J(\cdot;\zeta(s,y)) \leq \epsilon/3 \text{ and } \varphi - \varphi(s,y) \leq \epsilon/3 \text{ on } U(s,y;r(s,y)), \tag{3.8}$$

where $U(s,y;r) := [(s-r)\vee 0, s]\times B(y,r)$. Hence,

$$J(\cdot;\zeta(s,y)) \geq \varphi - \epsilon \text{ on } U(s,y;r(s,y)).$$

Note that $\{U(s,y;r) : (s,y) \in [0,T]\times\bar{\mathcal{O}}, 0 < r < r(s,y)\}$ is a Vitali covering of $[0,T]\times\bar{\mathcal{O}}$. It then follows from the Vitali's covering Theorem that there exists a countable sequence $\{t_i,x_i\}_{i\in\mathbb{N}}$ so that $[0,T]\times\bar{\mathcal{O}} \subset \cup_{i\in\mathbb{N}}U(t_i,x_i;r_i)$ with $r_i := r(t_i,x_i)$. We can then extract a partition $\{B_i\}_{i\in\mathbb{N}}$ of $[0,T]\times\bar{\mathcal{O}}$ and a sequence $\{t_i,x_i\}_{i\geq1}$ satisfying $(t_i,x_i) \in B_i$ for each $i \in \mathbb{N}$, such that $(t,x) \in B_i$ implies $t \leq t_i$, and

$$J(\cdot;\zeta_i) - \varphi \geq \varepsilon \text{ on } B_i, \text{for some } \zeta_i \in \mathcal{A}_{t_i}\times\mathcal{E}_{t_i}. \tag{3.9}$$

We now fix $\zeta \in \mathcal{E}_t \times \mathcal{A}_t$ and $\tau \in \mathcal{T}_t(t,T)$ and set

$$\bar\zeta := \zeta 1_{[t,\tau)} + 1_{[\tau,T]}\sum_{n\geq1}\zeta_n 1_{\{(\tau,X^\zeta_{t,x}(\tau))\in B_n\}}.$$

Note that $\bar\zeta \in \mathcal{A}_t \times \mathcal{E}_t$ since τ, $(\zeta_n)_{n\geq1}$ and $X^\zeta_{t,x}$ are independent of $\mathcal{F}_t$. It then follows from (ii) of Lemma 3.1 and (3.9) that

$$\begin{aligned}
&J(t,x,\bar\zeta) - \mathbb{E}[g(X^\zeta_{t,x}(T))1_{\{\tau=T\}}] - \mathbb{E}[\int_t^\tau \beta^\zeta_{t,x}(s)f(X^\zeta_{t,x}(s))ds]\\
\geq\;& \mathbb{E}\left[\beta^\zeta_{t,x}(\tau)J(\tau,X^\zeta_{t,x}(\tau);\bar\zeta)1_{\{\tau<T\}}\right]\\
\geq\;& \mathbb{E}\left[\beta^\zeta_{t,x}(\tau)\varphi(\tau,X^\zeta_{t,x}(\tau))1_{\{\tau<T\}}\right] - \epsilon.
\end{aligned}$$

This implies that

$$v(t,x) \geq \mathbb{E}\left[\beta^{\zeta}_{t,x}(\tau)[\varphi,g](\tau, X^{\zeta}_{t,x}(\tau)) + \int_t^{\tau} \beta^{\zeta}_{t,x}(s) f(X^{\zeta}_{t,x}(s))ds\right].$$

□

3.3 Proof of Proposition 3.1.

In this section, we prove the continuity of the cost function in the (t,x)-variable as follows.
1. We first show that the map $J(\cdot;\alpha,\varepsilon)$ is continuous if μ and ε are such that

$$N^{\mu}_{[t,T]} \leq m \ \mathbb{P}-\text{a.s and } \varepsilon \in \mathcal{E}^b_k, \text{for some } m,k \geq 1,$$

where $\mathcal{E}^b_k$ is defined as the set of $\varepsilon \in \mathcal{E}$ such that $|\varepsilon| \leq k$ and the number of jump times of ε is a.s smaller than k, and $N^{\mu}_{[t,T]} := \mu(\mathbb{R}^d, [t,T])$. This result is proved as a consequence of the estimates of X and β in Lemma 3.2 presented below together with the Lipchitz continuity conditions on f, g and ρ.

Lemma 3.2 *Fix $k,m \in \mathbb{N}$. Assume that* **G1** *and* **G2** *hold, $N^{\mu}_{[t,T]} \leq m$ $\mathbb{P}-$a.e and $\varepsilon \in \mathcal{E}^b_k$. Then, there exist a constant $M>0$ and a function λ so that, for all $t \leq t' \leq T$ and $x,x' \in \bar{\mathcal{O}}$, we have*

$$\mathbb{E}\left[\sup_{t' \leq s \leq T} |X^{\alpha,\varepsilon}_{t,x}(s) - X^{\alpha,\varepsilon}_{t',x'}(s)|^4\right] \leq M|x-x'|^4 + \lambda(|t-t'|), \tag{3.10}$$

$$\mathbb{E}\left[\sup_{t' \leq s \leq T} |\ln \beta^{\alpha,\varepsilon}_{t,x}(s) - \ln \beta^{\alpha,\varepsilon}_{t',x'}(s)|\right] \leq M|x-x'| + \lambda(|t-t'|), \tag{3.11}$$

where $\lim_{a \to 0} \lambda(a) = 0$.

Proof. In order to prove the result, we use a similar argument as in Proposition 3.1 [4] on the time intervals where (Z,ε) is continuous. We focus on the differences which come from the points of discontinuity of (Z,ε). From now, we denote $(X,L) := (X^{\alpha,\varepsilon}_{t,x}, L^{\alpha,\varepsilon}_{t,x})$ and $(X',L') := (X^{\alpha,\varepsilon}_{t',x'}, L^{\alpha,\varepsilon}_{t',x'})$. We only prove the first assertion, the second one follows from the same line of arguments.
Let $\{T_i\}_{i \geq 1}$ be the sequence of jump times on $[t',T]$ of (Z,ε) and $T_0 := t'$. By the same argument as in Proposition 3.1 of [4] we obtain that

$$\mathbb{E}[\sup_{r \in [T_i, T_{i+1})} |X(r) - X'(r)|^4] \leq C_1 \mathbb{E}[|X(T_i) - X'(T_i)|^4], \text{ for some } C_1 > 0. \tag{3.12}$$

It follows from (3.3) that

$$\begin{aligned} & \mathbb{E}[|X(T_{i+1}) - X'(T_{i+1})|^4] \\ \leq \ & \mathbb{E}[|\int_{\mathbb{R}^d} [\pi(\chi(X(T_{i+1}-), \alpha(T_{i+1}), z), \varepsilon(T_{i+1})) - \pi(\chi(X'(T_{i+1}-), \alpha(T_{i+1}), z), \varepsilon(T_{i+1}))]\mu(dz, \{T_{i+1}\})|^4] \end{aligned}$$

This, together with (2.3) and the Lipschitz continuity assumption on χ and π, implies that

$$\mathbb{E}[|X(T_{i+1}) - X'(T_{i+1})|^4] \leq C_2 \mathbb{E}[|X(T_{i+1}-) - X'(T_{i+1}-)|^4], \text{ for some } C_2 > 0\,. \tag{3.13}$$

Using the previous inequality and (3.12), we deduce that there exists $C>0$ s.t

$$\mathbb{E}[\sup_{r \in [T_i, T_{i+1}]} |X(r) - X'(r)|^4] \leq C \mathbb{E}[|X(T_i) - X'(T_i)|^4].$$

Applying an induction argument, it implies to

$$\mathbb{E}[\sup_{r\in[T_i,T_{i+1}]}|X(r)-X'(r)|^4]\leq C^i\mathbb{E}[|X(t')-X'(t')|^4].$$

Since $N^{\mu}_{[t,T]}\leq m$ and $N^{\varepsilon}_{[t,T]}\leq k$, we have

$$\mathbb{E}[\sup_{t'\leq s\leq T}|X(s)-X'(s)|^4]\leq C^{m+k}\mathbb{E}[|X(t')-x'|^4],\quad \text{for some } C>1. \tag{3.14}$$

It remains to estimate $\mathbb{E}[|X(t')-x'|^4]$. Let $\{\bar{T}_i\}_{i\geq 1}$ be the sequence of jump times on $[t,t']$ and set $\bar{T}_0:=t$. Using a similar argument as in Proposition 3.1 [4], we deduce that there exists $\bar{C}_1>0$ such that

$$\mathbb{E}[\sup_{r\in[\bar{T}_i,\bar{T}_{i+1})}|X(r)-x'|^4]\leq \bar{C}_1\mathbb{E}[|X(\bar{T}_i)-x'|^4+|\bar{T}_{i+1}-\bar{T}_i|]. \tag{3.15}$$

Recalling (3.3) and the Lipschitz continuity assumption on π, we deduce that there exists $\bar{C}_2>0$ so that

$$\begin{aligned}
&\mathbb{E}[|X(\bar{T}_{i+1})-x'|^4]\\
&=\mathbb{E}\left[|\int_{\mathbb{R}^d}[\pi(X(\bar{T}_{i+1}-),\chi(X(\bar{T}_{i+1}-),\alpha(\bar{T}_{i+1}),z),\varepsilon(\bar{T}_{i+1}))-x]\mu(dz,\{\bar{T}_{i+1}\})|^4\right]\\
&=\mathbb{E}\left[|\int_{\mathbb{R}^d}[\pi(X(\bar{T}_{i+1}-),\chi(X(\bar{T}_{i+1}-),\alpha(\bar{T}_{i+1}),z),\varepsilon(\bar{T}_{i+1}))-\pi(x,\varepsilon(\bar{T}_{i+1}))]\mu(dz,\{\bar{T}_{i+1}\})|^4\right]\\
&\leq \bar{C}_2\mathbb{E}\left[|X(\bar{T}_{i+1}-)-x'|^4+\int_{\mathbb{R}^d}|\chi(X(\bar{T}_{i+1}-),\alpha(\bar{T}_{i+1}),z)|^4\mu(dz,\{\bar{T}_{i+1}\})\right].
\end{aligned}$$

The previous inequality together with (3.15) leads to

$$\mathbb{E}[\sup_{r\in[\bar{T}_i,\bar{T}_{i+1}]}|X(r)-x'|^4]\leq C\mathbb{E}\left[|X(\bar{T}_i)-x'|^4+|\bar{T}_{i+1}-\bar{T}_i|+\int_{\mathbb{R}^d}|\chi(X(\bar{T}_{i+1}-),\alpha(\bar{T}_{i+1}),z)|^4\mu(dz,\{\bar{T}_{i+1}\})\right],$$

for some $C>0$.
Then,

$$\mathbb{E}[\sup_{t\leq r\leq t'}|X(r)-x'|^4]\leq C^{m+k}|x-x'|^4+C|t'-t|+C\mathbb{E}[\int_t^{t'}\int_{\mathbb{R}^d}|\chi(X(s-),\alpha(s),z)|^4\tilde{\mu}(dz)ds],$$

for some $C>1$.
Since χ is bounded with respect to z, we then conclude that

$$\mathbb{E}[\sup_{t\leq r\leq t'}|X(r)-x'|^4]\leq M'|x-x'|^4+\lambda(|t'-t|),$$

for some $M'>0$ and a function λ satisfying $\lim_{a\to 0}\lambda(a)=0$. □

2. We now provide the proof of Proposition 3.1 in the general case. Fix $(\alpha,\varepsilon)\in\mathcal{A}\times\mathcal{E}$. We denote by $\{T_m\}_{m\geq 1}$ the sequence of jump times of μ on the interval $]t,T]$ of (Z,ε), $T_0:=t$ and

$$\mu^{(m)}(A,B):=\mu(A,B\cap[t,T_m])\text{ for }A\in\mathcal{B}_{\mathbb{R}^d},B\in\mathcal{B}_{[0,T]}$$

where $\mathcal{B}_{\mathbb{R}^d}$ and $\mathcal{B}_{[0,T]}$ denote the Borel tribes of $\mathbb{R}^d$ and $[0,T]$ respectively. Let $(X^{(m)}_{t,x},L^{(m)}_{t,x},\beta^{(m)}_{t,x},J_m(t,x;\alpha,\varepsilon^{(m)}))$ be defined as $(X^{\alpha,\varepsilon}_{t,x},L^{\alpha,\varepsilon}_{t,x},\beta^{\alpha,\varepsilon}_{t,x},J(t,x;\alpha,\varepsilon))$ in (3.1), (3.1),(3.3) and (3.4) with $\mu^{(m)}$ in place of μ and $\varepsilon^{(m)}:=\varepsilon 1_{[t,T'_m)}+\varepsilon(T'_m)1_{[T'_m,T]}$ in place of ε, where

$$T'_m:=\sup\{s\geq t:\quad N^{\varepsilon}_{[t,s]}\leq m,\ |\varepsilon|(s)\leq m\}\wedge T_m.$$

Since f, g and β are bounded on the domain $\mathcal{O}$, there exists a constant $M > 0$ such that, for all $(t,x) \in [0,T] \times \mathbb{R}^d$, we have

$$\begin{aligned} |J_m(t,x;\alpha,\varepsilon^{(m)}) - J(t,x,y;\alpha,\varepsilon)| &\leq \mathbb{E}\left[|\beta^{(m)}(T)g(X^{(m)}(T)) - \beta^{\alpha,\varepsilon}_{t,x}(T)g(X^{\alpha,\varepsilon}_{t,x,y}(T))|1_{A_t^{(m)\ c}}\right] \\ &\quad + \mathbb{E}\left[|\int_t^T \beta^{(m)}(u)f(X^{(m)}(u)) - \beta^{\alpha,\varepsilon}_{t,x}(u)f(X^{\alpha,\varepsilon}_{t,x}(u))du|1_{A_t^{(m)\ c}}\right] \\ &\leq M\mathbb{P}\left\{A_t^{(m)\ c}\right\} \leq M\mathbb{P}\left\{A_0^{(m)\ c}\right\}. \end{aligned}$$

where $A_t^{(m)} := \{\omega \in \Omega : \ N^{\mu}_{[t,T]} \leq m, \ N^{\varepsilon}_{[t,T]} \leq m, |\varepsilon|(T) \leq m\}$. Since $N^{\mu}_{[0,T]} \leq \infty$ $\mathbb{P}-$a.s and ε is a process with bounded variation and finite activity, then $\mathbb{P}\{A_0^{(m)\ c}\}$ converges to 0 when m goes to ∞. Hence,

$$\sup_{(t,x)\in[0,T]\times\bar{\mathcal{O}}} |J_m(\cdot;\alpha,\varepsilon^{(m)}) - J(\cdot;\alpha,\varepsilon)| \to 0, \text{ when } m \to \infty. \tag{3.16}$$

It then follows the first part of proof that $J_m(\cdot;\alpha,\varepsilon^{(m)})$ is continuous map. This leads to the required result (3.7). □

3.4 The PDE characterization.

We are now ready to provide a PDE characterization for v. Note that the fact that the process $X^{\zeta}_{t,x}$ is projected through the projection operator π whenever it exists the domain $\mathcal{O}$ because of a jump implies that the associated Dynkin operator is given, for values (a,e) of the control process, by

$$\begin{aligned} \mathcal{L}^{a,e}\varphi(s,x) &:= \partial_t\varphi(s,x) + \langle b(x,a), D\varphi(s,x)\rangle + \frac{1}{2}\text{Trace}\left[\sigma(x,a)\sigma^*(x,a)D^2\varphi(s,x)\right] \\ &+ \int_{\mathbb{R}^d}[e^{-\rho(x,e)l^{a,e}_x(z)}\varphi(s,\pi^{a,e}_x(z)) - \varphi(s,x)]\tilde{\mu}(dz), \end{aligned}$$

for smooth functions φ, where

$$(\pi^{a,e}_x(z), l^{a,e}_x(z)) := (\pi,l)(x + \chi(x,a,z),e).$$

Also note that the probability of having a jump at the time where the boundary is reached is 0. It follows that the reflection terms does not play the same role, from the PDE point of view, depending whenever the reflection operate at a point of continuity or at a jump time. In the first case, it corresponds to a Neumman type boundary condition, while, in the second case, it only appears in the Dynkin operator which drives the evolution of the value function in the domain as described above. This formally implies that v should be a solution of

$$\mathcal{B}\varphi = 0, \tag{3.17}$$

where

$$\mathcal{B}\varphi := \begin{cases} \min\limits_{(a,e)\in A\times E}\{-\mathcal{L}^{a,e}\varphi - f(\cdot,a)\} & \text{on} \quad [0,T)\times\mathcal{O} \\ \min\limits_{e\in E}\mathcal{H}^e\varphi & \text{on} \quad [0,T)\times\partial\mathcal{O} \\ \varphi - g & \text{on} \quad \{T\}\times\bar{\mathcal{O}} \end{cases},$$

and, for a smooth function φ on $[0,T]\times\bar{\mathcal{O}}$ and $(a,e) \in A \times E$,

$$\mathcal{H}^e(x,y,p) := \rho(x,e)y - \langle\gamma(x,e),p\rangle \text{ and } \mathcal{H}^e\varphi(t,x) := \mathcal{H}^e(x,\varphi(t,x),D\varphi(t,x)).$$

Since v is not known to be smooth a-priori, we shall appeal as usual to the notion of viscosity solutions. Also note that the above operator $\mathcal{B}$ is not continuous, so that we have to consider its upper- and lower-semicontinuous envelopes to properly define the PDE. From now, given a function w on $[0,T]\times\bar{\mathcal{O}}$, we set

$$\left\{\begin{array}{rcl} w_*(t,x) & = & \liminf\limits_{(t',x')\to(t,x),\ (t',x')\in[0,T)\times\mathcal{O}} w(t',x') \\ w^*(t,x) & = & \limsup\limits_{(t',x')\to(t,x),\ (t',x')\in[0,T)\times\mathcal{O}} w(t',x') \end{array}\right. , \text{for } (t,x)\in[0,T]\times\bar{\mathcal{O}}.$$

Definition 3.1 *A lower semicontinuous (resp. upper semicontinuous) function w on $[0,T]\times\bar{\mathcal{O}}$ is a viscosity super-solution (resp. sub-solution)of* (3.17) *if, for any test function $\varphi\in C^{1,2}([0,T]\times\bar{\mathcal{O}})$ and $(t_0,x_0)\in[0,T]\times\bar{\mathcal{O}}$ that achieves a local minimum (resp. maximum) of $w-\varphi$ so that $(w-\varphi)(t_0,x_0)=0$, we have $\mathcal{B}_+\varphi\geq 0$ (resp. $\mathcal{B}_-\varphi\leq 0$), where*

$$\mathcal{B}_+\varphi \ := \ \left\{\begin{array}{lll} \mathcal{B}\varphi & on & [0,T]\times\mathcal{O} \\ \min\limits_{(a,e)\in A\times E}\max\left\{-\mathcal{L}^{a,e}\varphi - f(\cdot,a),\ \mathcal{H}^e\varphi\right\} & on & [0,T)\times\partial\mathcal{O} \\ \varphi - g & on & \{T\}\times\partial\mathcal{O} \end{array}\right. ,$$

$$\mathcal{B}_-\varphi \ := \ \left\{\begin{array}{lll} \mathcal{B}\varphi & on & [0,T]\times\mathcal{O} \\ \min\left\{\min\limits_{(a,e)\in A\times E}\{-\mathcal{L}^{a,e}\varphi - f(\cdot,a)\},\ \min\limits_{e\in E}\mathcal{H}^e\varphi\right\} & on & [0,T)\times\partial\mathcal{O} \\ \min\{\varphi-g,\ \min\limits_{e\in E}\mathcal{H}^e\varphi\} & on & \{T\}\times\partial\mathcal{O} \end{array}\right. ,$$

A local bounded function w is a discontinuous viscosity solution of (3.17) *if w_* (resp. w^*) is a super-solution (resp. sub-solution) of* (3.17).

We can now state our main result.

Theorem 3.2 *Assume that* **G1** *and* **G2** *hold. Then, v is a discontinuous viscosity solution of* (3.17).

The proof of this result is reported in the subsequent sections.

3.4.1 The super-solution property.

Let $\varphi\in C^{1,2}([0,T]\times\bar{\mathcal{O}})$ and $(t_0,x_0)\in[0,T]\times\bar{\mathcal{O}}$ be such that

$$\min(\text{strict})_{[0,T]\times\bar{\mathcal{O}}}(v_*-\varphi)=(v_*-\varphi)(t_0,x_0)=0.$$

1. We first prove the required result in the case where $(t_0,x_0)\in[0,T)\times\partial\mathcal{O}$.
Then, arguing by contradiction, we suppose that

$$\min_{(a,e)\in A\times E}\max\{-\mathcal{L}^{a,e}\varphi(t_0,x_0)-f(x_0,a),\ \mathcal{H}^e\varphi(t_0,x_0)\}\leq -2\epsilon<0.$$

Define $\phi(t,x):=\varphi(t,x)-|t-t_0|^2-\eta|x-x_0|^4$, so that, for $\eta>0$ small enough, we have

$$\min_{(a,e)\in A\times E}\max\{-\mathcal{L}^{a,e}\phi(t_0,x_0)-f(x_0,a),\ \mathcal{H}^e\phi(t_0,x_0)\}\leq -2\epsilon,$$

recall (2.3). This implies that there exists $(a_0,e_0)\in A\times E$ and $\delta>0$ such that $t_0+\delta<T$ and

$$\max\{-\mathcal{L}^{a_0,e_0}\phi(t,x)-f(x,a_0),\ \mathcal{H}^{e_0}\phi(t,x)\}\leq -\epsilon \text{ on } \bar{B}\cap[0,T]\times\bar{\mathcal{O}}, \tag{3.18}$$

where $B := [t_0 - \delta, t_0 + \delta) \times B(x_0, \delta)$.

Let $(t_n, x_n)_n$ be a sequence in $[0, T) \times \mathcal{O}$ converging to (t_0, x_0) such that

$$v(t_n, x_n) \to v_*(t_0, x_0) \ ,$$

and set $(X_n, L_n, \beta_n) := (X^{a_0,e_0}_{t_n,x_n}, L^{a_0,e_0}_{t_n,x_n}, \beta^{a_0,e_0}_{t_n,x_n})$. Obviously, we can assume that $(t_n, x_n) \in B$. Let τ_n and θ_n be the first exit times of $(s, X_n(s))_{s \geq t_n}$ and $(X_n(s))_{s \geq t_n}$ from B and $\mathcal{O}$, respectively.

Using Itô's Lemma, we have

$$\begin{aligned} \mathbb{E}\left[\beta_n(\tau_n)\phi\left(\tau_n, X_n(\tau_n)\right)\right] &= \phi(t_n, x_n) + \mathbb{E}\left[\int_{t_n}^{\tau_n} \beta_n(s-)\mathcal{L}^{a_0,e_0}\phi(s, X_n(s-))ds\right] \\ &\quad -\mathbb{E}\left[\int_{t_n}^{\tau_n} \beta_n(s)\mathcal{H}^{e_0}\phi(s, X_n(s))dL^c_n(s)\right], \end{aligned}$$

where L^c_n denotes the continuous part of L_n. It follows from (3.18) that

$$\mathbb{E}[\beta_n(\tau_n)\phi\left(\tau_n, X_n(\tau_n)\right)] \geq \phi(t_n, x_n) - \mathbb{E}\left[\int_{t_n}^{\tau_n} \beta_n(s) f(X_n(s), a_0)ds\right] + \mathbb{E}\left[\int_{t_n}^{\tau_n} \epsilon\beta_n(s)dL^c_n(s)\right].$$

Since $\rho \geq 0$, the function $\beta_n(\cdot)$ is non increasing. Hence

$$\mathbb{E}\left[\beta_n(\tau_n)\phi\left(\tau_n, X_n(\tau_n)\right) + \int_{t_n}^{\tau_n} \beta_n(s) f(X_n(s), a_0)ds\right] \geq \phi(t_n, x_n) + \mathbb{E}[\epsilon\beta_n(\tau_n)L^c_n(\tau_n)].$$

Since $\bar{\mathcal{O}}$ is bounded,

$$\varphi - \phi \geq \zeta \text{ on } \partial^c_p B \text{ for some } \zeta > 0,$$

where $\partial^c_p B := [t_0 - \delta, t_0 + \delta] \times (B(x_0, \delta)^c \cap \bar{\mathcal{O}}) \cup \{t_0 + \delta\} \times \bar{\mathcal{O}}$.
This, together with the fact that $\beta_n(\tau_n) = 1,\ L_n(\tau_n) = 0$ on $\{\tau_n < \theta_n\}$, leads to

$$\begin{aligned} &\mathbb{E}\left[\beta_n(\tau_n)\varphi\left(\tau_n, X_n(\tau_n)\right) + \int_{t_n}^{\tau_n} \beta_n(s) f(X_n(s), a_0)ds\right] \\ \geq\ & \phi(t_n, x_n) + \mathbb{E}\left[\zeta 1_{\tau_n < \theta_n} + \beta_n(\tau_n)(\zeta + \epsilon L^c_n(\tau_n))1_{\tau_n \geq \theta_n}\right]. \end{aligned}$$

Let $c > 0$ be a positive constant satisfying $|\rho| \leq c$, we have

$$\begin{aligned} &\mathbb{E}\left[\beta_n(\tau_n)\varphi\left(\tau_n, X_n(\tau_n)\right) + \int_{t_n}^{\tau_n} \beta_n(s) f(X_n(s), a_0)ds\right] \\ \geq\ & \phi(t_n, x_n) + \mathbb{E}[\zeta 1_{\tau_n < \theta_n} + e^{-cL_n(\tau_n)}(\zeta + \epsilon L^c_n(\tau_n))1_{\tau_n \geq \theta_n}] \\ \geq\ & \phi(t_n, x_n) + \min\{\zeta; \mathbb{E}[e^{-c\Sigma_{s \leq \tau_n}\Delta L_n(s)}] \inf_{k \geq 0}\left(e^{-ck}(\zeta + \epsilon k)\right)\}. \end{aligned}$$

It follows from Jensen's inequality, the Lipschitz property of l, (2.3) and the fact that $l(\cdot, e_0) = 0$ on $\mathcal{O}$

that

$$\begin{aligned}
\ln \mathbb{E}\left[e^{-c\sum_{s\le T}\Delta L_n(s)}\right] &\geq -c\mathbb{E}\left[\sum_{s\le T}\Delta L_n(s)\right] \\
&\geq -c\mathbb{E}\left[\sum_{s\le T} l(X_n(s-)+\chi(X_n(s-),a_0,\Delta Z(s)),e_0)\right] \\
&\geq -c\mathbb{E}\left[\int_0^T\int_{\mathbb{R}^d} l(X_n(s-)+\chi(X_n(s-),a_0,z),e_0)\tilde{\mu}(dz)ds\right] \\
&\geq -c\mathbb{E}\left[\int_0^T\int_{\mathbb{R}^d} [l(X_n(s-)+\chi(X_n(s-),a_0,z),e_0)-l(X_n(s-),e_0)]\tilde{\mu}(dz)ds\right] \\
&\geq -c'\int_0^T \sup_{\bar{\mathcal{O}}\times E}|\chi|\int_{\mathbb{R}^d}\tilde{\mu}(dz)ds > -\infty.
\end{aligned}$$

Then, there exists $\epsilon_0 > 0$ so that

$$\mathbb{E}\left[\beta_n(\tau_n)\varphi(\tau_n,X_n(\tau_n))+\int_{t_n}^{\tau_n}\beta_n(s)f(X_n(s),a_0)ds\right] \geq \phi(t_n,x_n)+\epsilon_0.$$

For n large enough, this leads to a contradiction to the statement (3.6) of Theorem 3.1.

2. The proof is similar for $(t_0,x_0)\in[0,T)\times\mathcal{O}$. Indeed, by a similar localization as above, we can restrict to the case where X_n does not escape the domain $\mathcal{O}$, expect possible by a jump, i.e. $L_n^c(\tau_n)=0$. It follows that a contradiction can be obtained by exactly the same argument as step **1.** by only assuming

$$\min_{(a,e)\in A\times E}(-\mathcal{L}^{a,e}\varphi(t_0,x_0)-f(x_0,a)) < 0.$$

When $(t_0,x_0)\in\{T\}\times\bar{\mathcal{O}}$, it follows from Proposition 3.1 and the fact that $\mathcal{A}_t\times\mathcal{E}_t\supset\mathcal{A}_T\times\mathcal{E}_T$ for all $t\le T$ that v is lower-semicontinuous at the points of $\{T\}\times\bar{\mathcal{O}}$. This leads clearly to $v_*\geq g$ on $\{T\}\times\bar{\mathcal{O}}$. □

3.4.2 The sub-solution property.

Let $(t_0,x_0)\in[t,T]\times\bar{\mathcal{O}}$ and $\varphi\in C^{1,2}([0,T]\times\bar{\mathcal{O}})$ such that

$$\max(\text{strict})_{[0,T]\times\bar{\mathcal{O}}}(v^*-\varphi) = (v^*-\varphi)(t_0,x_0) = 0.$$

1. We first consider the case where $(t_0,x_0)\in\{T\}\times\partial\mathcal{O}$.
We argue by contradiction and suppose that

$$\min\{\min_{e\in E}\mathcal{H}^e\varphi(t_0,x_0),(\varphi-g)(t_0,x_0)\} =: 2\epsilon > 0.$$

Since A is compact, after replacing φ by $(t,x)\mapsto\varphi(t,x)+\sqrt{T-t+\iota}$, with $\iota>0$ small, we can assume that $\lim_{t\to T}\partial_t\varphi(t,x) = -\infty$. Then, there exists $\delta>0$ such that

$$\min\{\min_{(a,e)\in A\times E}\{-\mathcal{L}^{a,e}\varphi-f(\cdot,a)\},\ \min_{e\in E}\mathcal{H}^e\varphi,\varphi-g\}\geq\epsilon \text{ on } \bar{B}\cap[T-\delta,T]\times\bar{\mathcal{O}}, \tag{3.19}$$

where $B := [T-\delta,T)\times B(x_0,\delta)$.
It follows from the fact that $v^*-\varphi$ achieves a strict local maximum at (t_0,x_0) and the fact that the domain $\mathcal{O}$ is bounded that

$$\sup_{\partial_p^c B}(v^*-\varphi) =: -\eta < 0, \tag{3.20}$$

where $\partial_p^c B := [T-\delta, T] \times (B(x_0,\delta)^c \cap \bar{\mathcal{O}}) \cup \{T\} \times B(x_0,\delta)$.
Let $(t_n, x_n)_n$ be a sequence in $[0,T) \times \mathcal{O}$ converging to (t_0, x_0) such that

$$v(t_n, x_n) \to v^*(t_0, x_0) \ .$$

Obviously, we can assume that $(t_n, x_n) \in B$. Fix $(\varepsilon, \alpha) \in \mathcal{E}_{t_n} \times \mathcal{A}_{t_n}$ and define

$$(X_n, L_n, \beta_n) := (X^{\varepsilon,\alpha}_{t_n,x_n}, L^{\varepsilon,\alpha}_{t_n,x_n}, \beta^{\varepsilon,\alpha}_{t_n,x_n}),$$

together with τ_n and θ_n, the first exit times of $(s, X_n(s))_{s\geq t_n}$ and $(X_n(s))_{s\geq t_n}$ from B and $\mathcal{O}$, respectively.

Using Itô's Lemma and (3.19), we deduce that

$$\begin{aligned}
& \mathbb{E}\left[\beta_n(\tau_n)\varphi\left(\tau_n, X_n(\tau_n)\right) 1_{\{\tau_n<T\}}\right] + \mathbb{E}[\beta_n(T)\left(g(X_n(T)) + \epsilon\right) 1_{\{\tau_n=T\}}] \\
\leq\ & \varphi(t_n,x_n) - \mathbb{E}\left[\int_{t_n}^{\tau_n} \beta_n(s) f(X_n(s),\alpha_s) ds\right] - \mathbb{E}\left[\int_{t_n}^{\tau_n} \epsilon\beta_n(s) dL_n^c(s)\right] \\
\leq\ & \varphi(t_n,x_n) - \mathbb{E}\left[\int_{t_n}^{\tau_n} \beta_n(s) f(X_n(s),\alpha_s) ds\right] - \mathbb{E}[\epsilon\beta_n(\tau_n) L_n^c(\tau_n)].
\end{aligned}$$

This, together with (3.20), implies that

$$\begin{aligned}
& \mathbb{E}[\beta_n(\tau_n)[v^*, g]\left(\tau_n, X_n(\tau_n)\right)] \\
\leq\ & \varphi(t_n,x_n) - \mathbb{E}\left[\int_{t_n}^{\tau_n} \beta_n(s) f(X_n(s),\alpha_s) ds\right] - \mathbb{E}[(\epsilon\wedge\eta)\beta_n(\tau_n)] - \mathbb{E}[\epsilon\beta_n(\tau_n) L_n^c(\tau_n)].
\end{aligned}$$

Recalling that $|\rho| \leq c$ for some $c > 0$, we deduce that

$$\begin{aligned}
\mathbb{E}[\beta_n(\tau_n)[v^*, g]\left(\tau_n, X_n(\tau_n)\right)] \ \leq\ & \varphi(t_n,x_n) - \mathbb{E}\left[\int_{t_n}^{\tau_n} \beta_n(s) f(X_n(s),\alpha_s) ds\right] \\
& - \mathbb{E}[(\epsilon\wedge\eta) 1_{\tau_n\leq\theta_n} + e^{-cL_n(\tau_n)}(\epsilon L_n^c(\tau_n) + \epsilon\wedge\eta) 1_{\tau_n>\theta_n}].
\end{aligned}$$

Then,

$$\mathbb{E}\left[\beta_n(\tau_n)[v^*, g]\left(\tau_n, X_n(\tau_n)\right) + \int_{t_n}^{\tau_n} \beta_n(s) f(X_n(s),\alpha_s) ds\right] \leq \varphi(t_n,x_n) - \min\{\epsilon, \eta, \nu\mathbb{E}[e^{-c\Sigma_{s\leq\tau_n}\Delta L(s)}]\},$$

where $\nu := \inf_{k\geq 0}(e^{-ck}(\varepsilon k + \zeta\wedge\epsilon))$. Note that the same argument as in the proof of section 3.4.1 shows that $\mathbb{E}[e^{-c\Sigma_{s\leq\tau_n}\Delta L(s)}] \geq \kappa > 0$, for some κ independent on n and ζ.

Using the fact that $\lim_{n\to\infty}(v-\varphi)(t_n,x_n) = 0$ and (2.3), we may then find $\eta' > 0$, which is independent on $\varepsilon,\ \alpha$ and n, such that

$$v(t_n,x_n) - \eta' \geq \mathbb{E}\left[\beta_n(\tau_n)[v^*, g]\left(\tau_n, X_n(\tau_n)\right) + \int_{t_n}^{\tau_n} \beta_n(s) f(X_n(s),\alpha_s) ds\right],$$

for n large enough. This leads to a contradiction to (3.5) in Theorem 3.1.

2. The case where $(t_0, x_0) \in [0,T) \times \partial\mathcal{O}$ can be treated similarly by similar argument as in previous step, see (3.19). We can indeed use a localization in order to assume that $\tau_n \leq T - \epsilon$ for some $\epsilon > 0$. Therefore, we do not need to compare the values of v^* with g.
The case $(t_0, x_0) \in [0,T] \times \mathcal{O}$ is also treated similarly by using a localization argument as in step 2 of the proof in section 3.4.1. □

3.5 The comparison theorem.

We now prove a comparison result for $\mathcal{B}_+\varphi \geq 0$ and $\mathcal{B}_-\varphi \leq 0$ on $[0,T]\times\bar{\mathcal{O}}$, which implies that v is continuous on $[0,T)\times\mathcal{O}$, admits a continuous extension to $[0,T]\times\bar{\mathcal{O}}$, and is the unique viscosity super-solution (resp. sub-solution) of $\mathcal{B}_+\varphi \geq 0$ (resp. $\mathcal{B}_-\varphi \leq 0$) on $[0,T]\times\bar{\mathcal{O}}$.
We first introduce an equivalent definition of viscosity solutions, which eliminates the appearance of test function in the integral associated to the measure of jumps. Let us denote by $G_{a,e}$ the operator from $[0,T]\times\bar{\mathcal{O}}\times\mathbb{R}\times\mathbb{R}^d\times\mathbb{M}^d$ to $\mathbb{R}$ parameterized by a smooth function φ as

$$\begin{aligned} G_{a,e}(x,q,p,M;\varphi) &:= q+\langle b(x,a),p\rangle + f(x,a) + \frac{1}{2}\mathrm{Trace}\left[\sigma(x,a)\sigma^*(x,a)M\right] \\ &\quad + \int_{\mathbb{R}^d}[e^{-\rho(x,e)l_x^{a,e}(z)}\varphi(\pi_x^{a,e}(z)) - \varphi(x)]\tilde{\mu}(dz) \end{aligned}$$

so that

$$\mathcal{L}^{a,e}\varphi(t,x) + f(x,a) = G_{a,e}(x,\partial_t\varphi(t,x),D\varphi(t,x),D^2\varphi(t,x);\varphi(t,\cdot)).$$

We also define $F_\pm$ as the operator associated to $\mathcal{B}_\pm$ by the implicit relation:

$$F_\pm(x,\partial_t\varphi(t,x),D\varphi(t,x),D^2\varphi(t,x);\varphi(t,\cdot)) = \mathcal{B}_\pm\varphi(t,x).$$

Note that, for $(a,e)\in A\times E$, $G_{a,e}$ is a continuous function satisfying the elliptical condition, i.e. it is non increasing with respect to $M\in\mathbb{M}^d$ and I. In view of Definition 4 in [1] and the fact that A and E are compact, we can provide the following equivalent definition of viscosity solutions:

Definition 3.2 *A lower semicontinuous (resp. upper semicontinuous) function w on $[0,T]\times\bar{\mathcal{O}}$ is a viscosity super-solution (resp. sub-solution) of (3.17) if, for $(t_0,x_0)\in[0,T]\times\bar{\mathcal{O}}$, $(q_0,p_0,M_0)\in\bar{\mathcal{P}}^-_{\bar{\mathcal{O}}}w(t,x)$ (resp. $\bar{\mathcal{P}}^+_{\bar{\mathcal{O}}}w(t,x)$) and $\varphi\in C^{1,2}([0,T]\times\bar{\mathcal{O}})$ so that*

$$(t_0,x_0) \text{ is a maximum (resp. minimum) point of } w-\varphi,\ w(t_0,x_0)=\varphi(t_0,x_0),$$

and

$$q_0=\partial_t\varphi(t_0,x_0),\ p_0=D\varphi(t_0,x_0),\ M_0\geq D^2\varphi(t_0,x_0)\ \ (resp.\ M_0\leq D^2\varphi(t_0,x_0)),$$

we have

$$F_+(x_0,q_0,p_0,M_0;w(t_0,\cdot))\geq 0\ \ (\ resp.\ F_-(x_0,q_0,p_0,M_0;w(t_0,\cdot))\leq 0).$$

See [7] for the standard notations $\bar{\mathcal{P}}^+_{\bar{\mathcal{O}}}$ and $\bar{\mathcal{P}}^-_{\bar{\mathcal{O}}}$.

Motivated by the comparison result of Proposition 3.4 in [4], we add some assumptions:

G3.
(i) *There exists $b>0$ such that*

$$B(x-b\gamma(x,e),b)\cap\mathcal{O}=\emptyset,\ \ for\ all\ (x,e)\in\partial\mathcal{O}\times E. \tag{3.21}$$

(ii) *There exists a $C^2(\bar{\mathcal{O}})$ function $\tilde{h}$ such that*

$$\langle\gamma(x,e),D\tilde{h}(x)\rangle\geq 1,\ \ for\ all\ x\in\partial\mathcal{O}\ and\ e\in E. \tag{3.22}$$

(iii) *For all $x\in\partial\mathcal{O}$, we have*

$$\inf_{e\in E}\langle\gamma(x,e),\gamma(x,\bar{e}_x)\rangle>0,$$

where $\bar{e}_x\in argmax\{\rho(x,e):\ e\in E\}$. Note that **G3**(iii) is weaker than the corresponding assumption (3.17) in [4]. We now prove a comparison result:

Theorem 3.3 *Suppose that* **G3** *holds. Let V (resp. U) be a lower-semicontinuous (resp. upper-semicontinuous) locally bounded map on $[0,T]\times\bar{\mathcal{O}}$. Assume that V is a viscosity supersolution of $\mathcal{B}_+\varphi \geq 0$ on $[0,T]\times\bar{\mathcal{O}}$ and U is a viscosity subsolution of $\mathcal{B}_-\varphi \leq 0$ on $[0,T]\times\bar{\mathcal{O}}$. Then, $V \geq U$ on $[0,T]\times\bar{\mathcal{O}}$.*

Proof. Fix $\kappa > 0$ and set $\tilde{U}(t,x) := e^{\kappa t}U(t,x)$, $\tilde{V}(t,x) := e^{\kappa t}V(t,x)$, $\tilde{f}(t,x) := e^{\kappa t}f(t,x)$ and $\tilde{g}(t,x) := e^{\kappa t}g(x)$. Here, κ is chosen such that

$$-\tilde{f}(\cdot,a) - \mathcal{L}^{a,e}H \geq 0, \text{ for all } (a,e)\in A\times E, \tag{3.23}$$

where $H(t,x) := e^{-\kappa t - \tilde{h}(x)}$.

We argue by contradiction, and therefore assume that

$$\sup_{[0,T]\times\bar{\mathcal{O}}}\left(\tilde{U}-\tilde{V}\right) > 0. \tag{3.24}$$

It follows from the fact that the domain $\mathcal{O}$ is bounded that $\Phi^\eta := \tilde{U}-\tilde{V}-2\eta H$ achieves its maximum at (t^η,x^η) on $[0,T]\times\bar{\mathcal{O}}$ and satisfies

$$\Phi^\eta(t^\eta,x^\eta) =: m > 0, \text{ for } \eta > 0 \text{ small enough }. \tag{3.25}$$

1. We first study the case $U(t^\eta,x^\eta)\geq 0$, up to a subsequence. We define the function Ψ_n^η on $[0,T]\times\bar{\mathcal{O}}^2$ as

$$\Psi_n^\eta(t,x,y) := \Theta(t,x,y) - |x-x^\eta|^4 - |t-t^\eta|^2 - \frac{n}{2}|x-y|^2 - \rho(x^\eta,\underline{e}_\eta)\tilde{U}(t^\eta,x^\eta)\langle\gamma(x^\eta,\underline{e}_\eta),x-y\rangle\ ,$$

where

$$\Theta(t,x,y) \;:=\; \tilde{U}(t,x) - \tilde{V}(t,y) - \eta\left(H(t,x)+H(t,y)\right)\ ,$$

and $\underline{e}_\eta \in \operatorname{argmin}\{\rho(x^\eta,e):\ e\in E\}$.
Assume that Ψ_n^η achieves its maximum at some $(t_n^\eta,x_n^\eta,y_n^\eta)\in[0,T]\times\bar{\mathcal{O}}^2$. The inequality $\Psi_n^\eta(t_n^\eta,x_n^\eta,y_n^\eta) \geq \Psi_n^\eta(t^\eta,x^\eta,x^\eta)$ implies that

$$\begin{aligned}\Theta(t_n^\eta,x_n^\eta,y_n^\eta) \;\geq\; & \Theta(t^\eta,x^\eta,x^\eta) + \rho(x^\eta,\underline{e}_\eta)\tilde{U}(t^\eta,x^\eta)\langle\gamma(x^\eta,\underline{e}_\eta),x_n^\eta-y_n^\eta\rangle \\ & + |x_n^\eta-x^\eta|^4 + |t_n^\eta-t^\eta|^2 + \frac{n}{2}|x_n^\eta-y_n^\eta|^2.\end{aligned}$$

We deduce that the term on the second line is bounded in n so that, up to a subsequence, $x_n^\eta, y_n^\eta \xrightarrow[n\to\infty]{} \bar{x}^\eta \in \bar{\mathcal{O}}$ and $t_n^\eta \xrightarrow[n\to\infty]{} \bar{t}^\eta \in [0,T]$. Sending $n\to\infty$ in the previous inequality and using the maximum property of Φ^η at (t^η,x^η), we obtain

$$\begin{aligned}0 \;\geq\; & \Phi^\eta(\bar{t}^\eta,\bar{x}^\eta) - \Phi^\eta(t^\eta,x^\eta) \\ \geq\; & \limsup_{n\to\infty}\left(|x_n^\eta-x^\eta|^4 + |t_n^\eta-t^\eta|^2 + \frac{n}{2}|x_n^\eta-y_n^\eta|^2\right),\end{aligned}$$

This, together with (3.25), implies that

$$\begin{aligned}&\text{(a)} \quad x_n^\eta, y_n^\eta \xrightarrow[n\to\infty]{} x^\eta \ \text{ and } t_n^\eta \xrightarrow[n\to\infty]{} t^\eta\ , \\ &\text{(b)} \quad |x_n^\eta-x^\eta|^4 + |t_n^\eta-t^\eta|^2 + \frac{n}{2}|x_n^\eta-y_n^\eta|^2 \xrightarrow[n\to\infty]{} 0\ , \\ &\text{(c)} \quad \tilde{U}(t_n^\eta,x_n^\eta) - \tilde{V}(t_n^\eta,y_n^\eta) \xrightarrow[n\to\infty]{} \left(\tilde{U}-\tilde{V}\right)(t^\eta,x^\eta) \;\geq\; m + 2\eta H(t^\eta,x^\eta) > 0\ .\end{aligned}$$

In view of Ishii's Lemma, see Theorem 8.3 in [7], we deduce that, for each $\lambda > 0$, there are real coefficients $b^\eta_{1,n}$, $b^\eta_{2,n}$ and symmetric matrices $\mathcal{X}^{\eta,\lambda}_n$ and $\mathcal{Y}^{\eta,\lambda}_n$ such that

$$\big(b^\eta_{1,n}, p^\eta_n, \mathcal{X}^{\eta,\lambda}_n\big) \in \bar{\mathcal{P}}^+_{\bar{\mathcal{O}}}\tilde{U}(t^\eta_n, x^\eta_n) \quad \text{and} \quad \big(-b^\eta_{2,n}, q^\eta_n, \mathcal{Y}^{\eta,\lambda}_n\big) \in \bar{\mathcal{P}}^-_{\bar{\mathcal{O}}}\tilde{V}(t^\eta_n, y^\eta_n)\,,$$

where

$$\begin{aligned} p^\eta_n &:= 4|x^\eta_n - x^\eta|^2(x^\eta_n - x^\eta) + n(x^\eta_n - y^\eta_n) + \rho(x^\eta, \underline{e}_\eta)\tilde{U}(t^\eta, x^\eta)\gamma(x^\eta, \underline{e}_\eta) + \eta DH(t^\eta_n, x^\eta_n) \\ q^\eta_n &:= n(x^\eta_n - y^\eta_n) + \rho(x^\eta, \underline{e}_\eta)\tilde{U}(t^\eta, x^\eta)\gamma(x^\eta, \underline{e}_\eta) - \eta DH(t^\eta_n, y^\eta_n)\,, \end{aligned}$$

$b^\eta_{1,n}$, $b^\eta_{2,n}$, $\mathcal{X}^{\eta,\lambda}_n$ and $\mathcal{Y}^{\eta,\lambda}_n$ satisfy

$$\left\{ \begin{array}{rcl} b^\eta_{1,n} + b^\eta_{2,n} & = & 2(t^\eta_n - t^\eta) - \kappa\eta\,(H(t^\eta_n, x^\eta_n) + H(t^\eta_n, y^\eta_n)) \\ \begin{pmatrix} \mathcal{X}^{\eta,\lambda}_n & 0 \\ 0 & -\mathcal{Y}^{\eta,\lambda}_n \end{pmatrix} & \leq & (A^\eta_n + B^\eta_n) + \lambda(A^\eta_n + B^\eta_n)^2 \end{array} \right. \tag{3.26}$$

with

$$\begin{aligned} A^\eta_n &:= \eta \begin{pmatrix} D^2H(t^\eta_n, x^\eta_n) & 0 \\ 0 & D^2H(t^\eta_n, y^\eta_n) \end{pmatrix} + \begin{pmatrix} 12(x^\eta_n - x^\eta)\otimes(x^\eta_n - x^\eta) & 0 \\ 0 & 0 \end{pmatrix}, \\ B^\eta_n &:= n \begin{pmatrix} I & -I \\ -I & I \end{pmatrix}, \end{aligned}$$

and I stands for the identical matrix with dimension $d \times d$.

1.1. We first suppose that, up to a subsequence, $x^\eta_n \in \partial\mathcal{O}$ for all n. Fix $e \in E$. It follows from (3.21) that

$$|x^\eta_n - b\gamma(x^\eta_n, e) - y^\eta_n|^2 \geq b^2.$$

Since $|\gamma| = 1$, this implies that

$$2\langle\gamma(x^\eta_n, e), y^\eta_n - x^\eta_n\rangle \leq -b^{-1}|y^\eta_n - x^\eta_n|^2. \tag{3.27}$$

Since $x^\eta_n \xrightarrow[n\to\infty]{} x^\eta$, we have

$$\begin{aligned} & \rho(x^\eta_n, e)\tilde{U}(t^\eta_n, x^\eta_n) - \langle\gamma(x^\eta_n, e), p^\eta_n\rangle \\ = \;& (\rho(x^\eta, e) - \rho(x^\eta, \underline{e}_\eta))\tilde{U}(t^\eta, x^\eta) + \rho(x^\eta, \underline{e}_\eta)\tilde{U}(t^\eta, x^\eta)(1 - \langle\gamma(x^\eta, e), \gamma(x^\eta, \underline{e}_\eta)\rangle) \\ & + n\langle\gamma(x^\eta_n, e), y^\eta_n - x^\eta_n\rangle + \eta\langle\gamma(x^\eta_n, e), D\tilde{h}(x^\eta_n)\rangle H(t^\eta_n, x^\eta_n) + \lambda_n, \end{aligned}$$

where λ_n is independent on e and comes to 0 when $n \to \infty$. This, together with (3.22), (b), (3.27) and the fact that $\langle\gamma(x^\eta, e), \gamma(x^\eta, \underline{e}_\eta)\rangle \leq 1$ since $|\gamma| \leq 1$, implies that

$$\mathcal{H}^e(x^\eta_n, \tilde{U}(t^\eta_n, x^\eta_n), p^\eta_n) > \eta H(t^\eta, x^\eta) > 0,$$

when n is large enough.
Using similar argument as above, if, up to a subsequence, $y^\eta_n \in \partial\mathcal{O}$, we then have

$$\mathcal{H}^{\underline{e}_\eta}(y^\eta_n, \tilde{V}(t^\eta_n, y^\eta_n), q^\eta_n) < -\eta H(t^\eta, x^\eta) < 0, \text{ for all } e \in E,$$

when n is large enough.

1.2. We now suppose that, up to a subsequence, $t^\eta_n = T$, for all $n \geq 1$. In view of step **1.1**, we have $\tilde{U}(T, x^\eta_n) \leq \tilde{g}(T, x^\eta_n)$ and $\tilde{V}(T, y^\eta_n) \geq \tilde{g}(T, y^\eta_n)$, for all $n \geq 1$. Then, passing to the limit, recalling (a)

and the fact that g is continuous, it implies that $\tilde{U}(T,x^\eta) \le \tilde{g}(T,x^\eta) \le \tilde{V}(T,x^\eta)$. This leads to a contradiction of (c).

1.3. It follows from **1.1** and **1.2** that $(t_n^\eta, x_n^\eta, y_n^\eta) \in [0,T) \times \mathcal{O}^2$ for all n, after possibly passing to a subsequence. Then, there exists $(a_n^\eta, e_n^\eta)_{n\ge1} \subset A \times E$ such that

$$\begin{aligned}
0 &\ge \kappa\tilde{U}(t_n^\eta,x_n^\eta) - b_{1,n}^\eta - \langle b(x_n^\eta,a_n^\eta),p_n^\eta\rangle - \frac{1}{2}\mathrm{Trace}\left[\sigma(x_n^\eta,a_n^\eta)\sigma^*(x_n^\eta,a_n^\eta)\mathcal{X}_n^{\eta,\lambda}\right] \\
&- \int_{\mathbb{R}^d}\left[e^{-\rho(x_n^\eta,e_n^\eta)l_{x_n^\eta}^{a_n^\eta,e_n^\eta}(z)}[\tilde{U}(t_n^\eta,\pi_{x_n^\eta}^{a_n^\eta,e_n^\eta}(z)) - \tilde{U}(t_n^\eta,x_n^\eta)]\right]\tilde{\mu}(dz) - \tilde{f}(x_n^\eta,a_n^\eta) \\
0 &\le \kappa\tilde{V}(t_n^\eta,y_n^\eta) + b_{1,n}^\eta - \langle b(y_n^\eta,a_n^\eta),q_n^\eta\rangle - \frac{1}{2}\mathrm{Trace}\left[\sigma(y_n^\eta,a_n^\eta)\sigma^*(y_n^\eta,a_n^\eta)\mathcal{Y}_n^{\eta,\lambda}\right] \\
&- \int_{\mathbb{R}^d}\left[e^{-\rho(y_n^\eta,e_n^\eta)l_{y_n^\eta}^{a_n^\eta,e_n^\eta}(z)}[\tilde{V}(t_n^\eta,\pi_{y_n^\eta}^{a_n^\eta,e_n^\eta}(z)) - \tilde{V}(t_n^\eta,y_n^\eta)]\right]\tilde{\mu}(dz) - \tilde{f}(y_n^\eta,a_n^\eta).
\end{aligned}$$

It follows from (b), (3.26) and the Lipchitz continuity of our coefficients that

$$\begin{aligned}
&\kappa[\tilde{U}(t_n^\eta,x_n^\eta) - \tilde{V}(t_n^\eta,y_n^\eta) - 2\eta H(t^\eta,x^\eta)] + 2\eta[\tilde{f}(x^\eta,a_n^\eta) + \mathcal{L}^{a_n^\eta,e_n^\eta}H(t^\eta,x^\eta)] + C(n,\lambda) \\
&\le \int_{\mathbb{R}^d}\left[e^{-\rho(x^\eta,e_n^\eta)l_{x_n^\eta}^{a_n^\eta,e_n^\eta}(z)}[(\tilde{U}-\tilde{V}-2\eta H)(t_n^\eta,\pi_{x_n^\eta}^{a_n^\eta,e_n^\eta}(z)) - (\tilde{U}-\tilde{V}-2\eta H)(t^\eta,y^\eta)]\right]\tilde{\mu}(dz),
\end{aligned}$$

where $C(n,\lambda)$ goes to 0 when $\lambda \to 0$ and then $n \to \infty$. This, together with the fact that

$$(\tilde{U}-\tilde{V}-2\eta H) \le (\tilde{U}-\tilde{V}-2\eta H)(t^\eta,x^\eta) = m > 0,$$

and $\rho, l \ge 0$, implies that

$$\kappa[\tilde{U}(t_n^\eta,x_n^\eta) - \tilde{V}(t_n^\eta,y_n^\eta)) - 2\eta H(t^\eta,x^\eta)] \le -2\eta[\tilde{f}(x^\eta,a_n^\eta) + \mathcal{L}^{a_n^\eta,e_n^\eta}H(t^\eta,x^\eta)] + C(n,\lambda).$$

Finally, using (c) and (3.23), we deduce by sending $\lambda \to 0$ and then $n \to \infty$ that

$$\kappa m \le 0,$$

which is the required contradiction.

2. The case where $U(t^\eta,x^\eta) < 0$, up to a subsequence, is treated similarly. The difference comes from the test function which is chosen as follows

$$\Psi_n^\eta(t,x,y) := \Theta(t,x,y) - |x-x^\eta|^4 - |t-t^\eta|^2 - \frac{n}{2}|x-y|^2 - \tilde{b}_\eta^{-1}\rho(x^\eta,\bar{e}_\eta)\tilde{U}(t^\eta,x^\eta)\langle\gamma(x^\eta,\tilde{e}_\eta),x-y\rangle\ ,$$

where $\bar{e}_\eta = \bar{e}_{x^\eta}$, $\tilde{b}_\eta$ and $\tilde{e}_\eta \in E$ satisfy

$$\min_{e\in E}\langle\gamma(x^\eta,e),\gamma(x^\eta,\bar{e}_x)\rangle = \langle\gamma(x^\eta,\tilde{e}_\eta),\gamma(x^\eta,\bar{e}_x)\rangle = \tilde{b}_\eta > 0.$$

Those variables are well defined, when the condition (iii) of assumption **G3** holds. Then, if, up to a subsequence, $x_n^\eta \in \partial\mathcal{O}$ for all n, we have

$$\begin{aligned}
&\rho(x_n^\eta,e)\tilde{U}(t_n^\eta,x_n^\eta) - \langle\gamma(x_n^\eta,e),p_n^\eta\rangle \\
= \ &(\rho(x^\eta,e)-\rho(x^\eta,\bar{e}_\eta))\tilde{U}(t^\eta,x^\eta) + \rho(x^\eta,\bar{e}_\eta)\tilde{U}(t^\eta,x^\eta)(1-\tilde{b}_\eta^{-1}\langle\gamma(x^\eta,e),\gamma(x^\eta,\tilde{e}_\eta)\rangle) \\
+ \ &n\langle\gamma(x_n^\eta,e),y_n^\eta-x_n^\eta\rangle + \eta\langle\gamma(x_n^\eta,e),D\tilde{h}(x_n^\eta)\rangle H(t_n^\eta,x_n^\eta) + \lambda_n,
\end{aligned}$$

where λ_n goes to 0 as $n \to \infty$. This, together with (a), (b), (3.22) and (3.27), implies that

$$\mathcal{H}^e(x_n^\eta,\tilde{U}(t_n^\eta,x_n^\eta),p_n^\eta) > \eta H(t^\eta,x^\eta) > 0,$$

when n is large enough. The other cases are treated similarly as in step **1** above. □

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