Adaptive wavelet regression in random design and general errors with weakly dependent data

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Abstract

We investigate function estimation in a nonparametric regression model having the following particularities: the design is random (with a known distribution), the errors admit finite moments of order 2 and the data are weakly dependent; the exponentially strongly mixing case is considered. In this general framework, we construct a new adaptive estimator. It is based on wavelets and the combination of two hard thresholding rules. We determine an upper bound of the associated mean integrated squared error and prove that it is sharp for a wide class of regression functions.

Keywords: Nonparametric regression, weakly dependent data, minimax estimation, Besov balls, wavelets, hard thresholding.

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1 Introduction

Let $(X_i, Y_i)_{i\in\mathbb{Z}}$ be a bivariate stationary random process where, for any $i \in \mathbb{Z}$,

$$Y_i = f(X_i) + \xi_i, \tag{1.1}$$

$f : [0,1] \to \mathbb{R}$ is an unknown function, $(X_i)_{i\in\mathbb{Z}}$ is a sequence of identically distributed random variables having the common known density $g : [0,1] \to [0,\infty)$ and $(\xi_i)_{i\in\mathbb{Z}}$ is a sequence of identically distributed variables independent of $(X_i)_{i\in\mathbb{Z}}$ satisfying $\mathbb{E}(\xi_1) = 0$ and $\mathbb{E}(\xi_1^2) < \infty$. We suppose that $(X_i, Y_i)_{i\in\mathbb{Z}}$ is strongly mixing (to be defined in Section 2). Given n observations $(X_1, Y_1), \ldots, (X_n, Y_n)$ drawn from $(X_i, Y_i)_{i\in\mathbb{Z}}$, we aim to estimate f globally on $[0,1]$.

To measure the performance of an estimator $\widehat{f}$ of f, we use the Mean Integrated Squared Error (MISE) defined by:

$$\mathbf{R}(\widehat{f}, f) = \mathbb{E}\left(\int_0^1 (\widehat{f}(x) - f(x))^2 dx\right).$$

Our goal is to construct $\widehat{f}$ such that the associated MISE is as small as possible. Many methods can be considered (kernel, spline, wavelets,...) (see e.g. [31]). In this study, we

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focus our attention on the wavelet methods. They are attractive for nonparametric function estimation because of their spatial adaptivity, computational efficiency and asymptotic optimality properties. Further details can be found in [1] and [22].

In the litterature, when $(X_1, Y_1), \ldots, (X_n, Y_n)$ are *i.i.d.*, various wavelet methods have been developed. See, e.g., [14–16], [17], [21], [2, 3], [4], [30], [5, 6] [7, 8], [10], [29], [13], [33], [23] and [9]. When $\xi_1, \ldots, \xi_n$ have some kind of dependence (long memory, ρ-mixing,...), see e.g. [18], [19], [25], [26] and [24]. To the best of our knowledge, the wavelet estimation of f when $(X_1, Y_1), \ldots, (X_n, Y_n)$ are weakly dependent has only been investigated by [26]. More precisely, [26] constructs a linear nonadaptive wavelet estimator of f which attained a sharp rate of convergence under the uniform risk over Besov balls. However, the adaptive wavelet estimation of f, more realistic, has never been addressed earlier and motivates this study. In addition to this new challenge, we relax some classical assumptions on the errors: the common distribution of $\xi_1, \ldots, \xi_n$ can be unknown; only $\mathbb{E}(\xi_1) = 0$ and $\mathbb{E}(\xi_1^2) < \infty$ are required.

We construct a new adaptive wavelet estimator based on the following steps: we estimate the unknown wavelet coefficients of f by a new thresholded versions of the empirical ones, we operate a term-by-term selection of these estimators via a hard thresholding rule, then we reconstruct the selected estimators by taking the initial wavelet basis and choosing appropriate levels. Naturally, the definitions of both thresholds take into account the dependence of the data and are chosen to minimize the associated MISE. Assuming that f belongs to a Besov balls $B^s_{p,q}(M)$ (to be defined in Section 3), we prove that our estimator $\widehat{f}$ satisfies

$$\mathbf{R}(\widehat{f}, f) \leq C \left(\frac{\ln n_\theta}{n_\theta} \right)^{2s/(2s+1)},$$

where $C > 0$ is a constant (independent of n), $n_\theta = n^{\theta/(\theta+1)}$ and θ refers to the exponentially strong mixing case.

The paper is organized as follows. Section 2 clarifies the assumptions on the model and introduces some notations. Section 3 describes the wavelet basis on $[0, 1]$ and the Besov balls $B^s_{p,q}(M)$. Our wavelet hard thresholding estimator is presented in Section 4. The results are set in Section 5. Section 6 is devoted to the proofs.

2 Assumptions and notations

Dependence assumption on $(X_i, Y_i)_{i\in\mathbb{Z}}$. Set, for any $i \in \mathbb{Z}$, $Z_i = (X_i, Y_i)$. We suppose that $Z = (Z_i)_{i\in\mathbb{Z}}$ is stationary and exponentially strongly mixing. Let us now clarify this kind of dependence.

For any $m \in \mathbb{Z}$, we define the m-th strongly mixing coefficient of Z by

$$a_m = \sup_{(A,B)\in\mathcal{F}^Z_{-\infty,0}\times\mathcal{F}^Z_{m,\infty}} |\mathbb{P}(A \cap B) - \mathbb{P}(A)\mathbb{P}(B)|,$$

where, for any $i \in \mathbb{Z}$, $\mathcal{F}^Z_{-\infty,i}$ is the σ-algebra generated by $\ldots, Z_{i-1}, Z_i$ and $\mathcal{F}^Z_{i,\infty}$ is the σ-algebra generated by $Z_i, Z_{i+1}, \ldots$. The bivariate random process Z is said to be strongly mixing if $\lim_{m\to\infty} a_m = 0$. The exponentially strongly mixing condition is characterized by the following inequality: there exists three known constants, $\gamma > 0$, $c > 0$ and $\theta > 0$ such that, for any $m \in \mathbb{Z}$,

$$a_m \leq \gamma exp\left(-c|m|^\theta\right). \tag{2.1}$$

This assumption is satisfied by a large class of processes (ARMA processes, ARCH process, ...). See e.g. [32], [20] and [28]. Remark that, when $\theta \to \infty$, $Z = (Z_i)_{i\in\mathbb{Z}}$ becomes a bivariate sequence of *i.i.d.* random variables.

Boundedness assumption on f. We suppose that there exists a known constant $C_* > 0$ such that

$$\sup_{x\in[0,1]} |f(x)| \leq C_*. \tag{2.2}$$

Boundedness ssumption on g. We suppose that there exists a known constant $c_* > 0$ such that

$$\inf_{x\in[0,1]} g(x) \geq c_*. \tag{2.3}$$

3 Wavelet bases and Besov balls

3.1 Wavelet basis

Let $N \in \mathbb{N}^*$ (the set of positive integers), and ϕ and ψ be the initial wavelet functions of the Daubechies wavelets dbN. Set

$$\phi_{j,k}(x) = 2^{j/2}\phi(2^j x - k), \qquad \psi_{j,k}(x) = 2^{j/2}\psi(2^j x - k).$$

Then, with an appropriate treatments at the boundaries, there exists an integer τ satisfying $2^\tau \geq 2N$ such that, for any integer $\ell \geq \tau$,

$$\mathcal{B} = \{\phi_{\ell,k}(.),\ k \in \{0,\ldots,2^\ell - 1\};\ \psi_{j,k}(.);\ j \in \mathbb{N} - \{0,\ldots,\ell-1\},\ k \in \{0,\ldots,2^j-1\}\},$$

is an orthonormal basis of $\mathbb{L}^2([0,1]) = \{h : [0,1] \to \mathbb{R};\ \int_0^1 h^2(x)dx < \infty\}$. See [11].

For any integer $\ell \geq \tau$, any $h \in \mathbb{L}^2([0,1])$ can be expanded on $\mathcal{B}$ as

$$h(x) = \sum_{k=0}^{2^\ell-1} \alpha_{\ell,k}\phi_{\ell,k}(x) + \sum_{j=\ell}^{\infty}\sum_{k=0}^{2^j-1} \beta_{j,k}\psi_{j,k}(x),$$

where $\alpha_{j,k}$ and $\beta_{j,k}$ are the wavelet coefficients of h defined by

$$\alpha_{j,k} = \int_0^1 h(x)\phi_{j,k}(x)dx, \qquad \beta_{j,k} = \int_0^1 h(x)\psi_{j,k}(x)dx. \tag{3.1}$$

3.2 Besov balls

Let $M > 0$, $s > 0$, $p \geq 1$ and $r \geq 1$. A function h belongs to $B^s_{p,r}(M)$ if, and only if, there exists a constant $M^* > 0$ (depending on M) such that the associated wavelet coefficients (3.1) satisfy

$$2^{\tau(1/2-1/p)}\left(\sum_{k=0}^{2^\tau-1}|\alpha_{\tau,k}|^p\right)^{1/p} + \left(\sum_{j=\tau}^{\infty}\left(2^{j(s+1/2-1/p)}\left(\sum_{k=0}^{2^j-1}|\beta_{j,k}|^p\right)^{1/p}\right)^r\right)^{1/r} \leq M^*.$$

In this expression, s is a smoothness parameter and p and r are norm parameters. For a particular choice of s, p and r, the Besov balls contain the standard Hölder and Sobolev balls. See [27].

4 Estimators

4.1 Wavelet coefficient estimators and some of their statistical properties

The first step to estimate f in (1.1) consists in expanding f on the wavelet basis $\mathcal{B}$. Then we aim to estimate the unknown wavelet coefficients: $\alpha_{j,k} = \int_0^1 f(x)\phi_{j,k}(x)dx$ and $\beta_{j,k} = \int_0^1 f(x)\psi_{j,k}(x)dx$. The considered estimators are described below.

Set

$$\gamma_n = \mu\sqrt{\frac{n_\theta}{\ln n_\theta}}$$

where $n_\theta = n^{\theta/(\theta+1)}$, θ is the one in (2.1) and $\mu = \sqrt{(C_*^2 + \mathbb{E}(\xi_1^2))/c_*}$.

For any integer $j \geq \tau$ and any $k \in \{0, \ldots, 2^j - 1\}$,

- we estimate $\alpha_{j,k}$ by

$$\widehat{\alpha}_{j,k} = \frac{1}{n}\sum_{i=1}^{n}\frac{Y_i}{g(X_i)}\phi_{j,k}(X_i)\mathbb{1}_{\left\{\left|\frac{Y_i}{g(X_i)}\phi_{j,k}(X_i)\right|\leq\gamma_n\right\}}, \tag{4.1}$$

  where, for any random event $\mathcal{A}$, $\mathbb{1}_{\mathcal{A}}$ is the indicator function on $\mathcal{A}$.

- we estimate $\beta_{j,k}$ by

$$\widehat{\beta}_{j,k} = \frac{1}{n}\sum_{i=1}^{n}\frac{Y_i}{g(X_i)}\psi_{j,k}(X_i)\mathbb{1}_{\left\{\left|\frac{Y_i}{g(X_i)}\psi_{j,k}(X_i)\right|\leq\gamma_n\right\}}. \tag{4.2}$$

Remark that $\widehat{\alpha}_{j,k}$ and $\widehat{\beta}_{j,k}$ are thresholded versions of the standard empirical wavelet estimators for (1.1) (see, e.g., [14–16]). Such an observation thresholding has been introduced by [13] for (1.1) when $(X_1, Y_1), \ldots, (X_n, Y_n)$ are *i.i.d.*. In our study, it allows

us to have non restrictive assumptions on $\xi_1, \ldots, \xi_n$ and treat the weak dependence of $(X_1, Y_1), \ldots, (X_n, Y_n)$.

Propositions 4.1 and 4.2 below show moments properties for (4.1) and (4.2).

Proposition 4.1. *Consider* (1.1) *under the assumptions of Section* 2. *For any integer* $j \geq \tau$ *and any* $k \in \{0, \ldots, 2^j - 1\}$, *let* $\beta_{j,k} = \int_0^1 f(x)\psi_{j,k}(x)dx$ *and* $\widehat{\beta}_{j,k}$ *be* (4.2). *Then there exists a constant* $C > 0$ *such that*

$$\mathbb{E}\left((\widehat{\beta}_{j,k} - \beta_{j,k})^2\right) \leq C\frac{\ln n_\theta}{n_\theta}.$$

This inequality holds with $\alpha_{j,k} = \int_0^1 f(x)\phi_{j,k}(x)dx$ *instead of* $\beta_{j,k}$ *and* $\widehat{\alpha}_{j,k}$ *defined by* (4.1) *instead of* $\widehat{\beta}_{j,k}$.

Proposition 4.2. *Consider* (1.1) *under the assumptions of Section* 2. *For any integer* $j \geq \tau$ *and any* $k \in \{0, \ldots, 2^j - 1\}$, *let* $\beta_{j,k} = \int_0^1 f(x)\psi_{j,k}(x)dx$ *and* $\widehat{\beta}_{j,k}$ *be* (4.2). *Then there exists a constant* $C > 0$ *such that*

$$\mathbb{E}\left((\widehat{\beta}_{j,k} - \beta_{j,k})^4\right) \leq C.$$

Proposition 4.3 below determines a concentration inequality for (4.2).

Proposition 4.3. *Consider* (1.1) *under the assumptions of Section* 2. *For any integer* $j \geq \tau$ *and any* $k \in \{0, \ldots, 2^j - 1\}$, *let* $\beta_{j,k} = \int_0^1 f(x)\psi_{j,k}(x)dx$, $\widehat{\beta}_{j,k}$ *be* (4.2) *and*

$$\lambda_n = \mu\sqrt{\frac{\ln n_\theta}{n_\theta}}.$$

Then, for any $\kappa \geq 2 + 16/(3u) + 4\sqrt{(1/u)(16/9u^2 + 2)}$ *with* $u = (1/2)(c/8)^{1/(\theta+1)}$, *we have*

$$\mathbb{P}\left(|\widehat{\beta}_{j,k} - \beta_{j,k}| \geq \kappa\lambda_n/2\right) \leq 2(1 + 4e^{-2}\gamma)\frac{1}{n_\theta^4}.$$

4.2 Hard thresholding estimator

We define the hard thresholding estimator $\widehat{f}$ by

$$\widehat{f}(x) = \sum_{k=0}^{2^\tau - 1} \widehat{\alpha}_{\tau,k}\phi_{\tau,k}(x) + \sum_{j=\tau}^{j_1}\sum_{k=0}^{2^j-1} \widehat{\beta}_{j,k}\mathbb{I}_{\left\{|\widehat{\beta}_{j,k}| \geq \kappa\lambda_n\right\}}\psi_{j,k}(x), \tag{4.3}$$

where $\widehat{\alpha}_{\tau,k}$ is defined by (4.1) with $j = \tau$, $\widehat{\beta}_{j,k}$ by (4.2), j_1 is the integer satisfying

$$\frac{1}{2}n_\theta < 2^{j_1} \leq n_\theta,$$

$\kappa \geq 2 + 16/(3u) + 4\sqrt{(1/u)(16/9u^2 + 2)}$ with $u = (1/2)(c/8)^{1/(\theta+1)}$ and

$$\lambda_n = \mu\sqrt{\frac{\ln n_\theta}{n_\theta}}.$$

The feature of $\widehat{f}$ is to only estimate the "large" unknown wavelet coefficients of f because they are those which contain his main characteristics.

5 Results

Theorem 5.1. *Consider* (1.1) *under the assumptions of Section* 2*. Let* $\widehat{f}$ *be* (4.3)*. Suppose that* $f \in B^s_{p,r}(M)$ *with* $r \geq 1$*,* $\{p \geq 2$ *and* $s > 0\}$ *or* $\{p \in [1,2)$ *and* $s > 1/p\}$*. Then there exists a constant* $C > 0$ *such that*

$$\mathbf{R}(\widehat{f}, f) \leq C \left(\frac{\ln n_\theta}{n_\theta} \right)^{2s/(2s+1)},$$

where $n_\theta = n^{\theta/(\theta+1)}$.

The proof of Theorem 5.1 uses a suitable decomposition of the MISE with the results in Propositions 4.1, 4.2 and 4.3.

If we restrict our study to independent $(X_1, Y_1), \ldots, (X_n, Y_n)$ i.e. $\theta \to \infty$, our rate of convergence becomes $(\ln n/n)^{2s/(2s+1)}$ which is the standard "near optimal" one in the minimax sense. See e.g. [22] and [13].

Note that the rate of convergence $(\ln n_\theta/n_\theta)^{2s/(2s+1)}$ is also achieved by the abstract minimum complexity regression estimator in [28, Theorem 2.1] but for a slightly different regression problem with more restrictions on $(X_1, Y_1), \ldots, (X_n, Y_n)$, $\xi_1, \ldots, \xi_n$ and f (see [28, Section II. B.]).

6 Proofs

In this section, C represents a positive constant which may differ from one term to another.

Proof of Proposition 4.1. For any $i \in \{1, \ldots, n\}$, set

$$W_{i,j,k} = \frac{Y_i}{g(X_i)} \psi_{j,k}(X_i).$$

Since X_1 and ξ_1 are independent and $\mathbb{E}(\xi_1) = 0$, we have

$$\begin{aligned}
\mathbb{E}(W_{1,j,k}) &= \mathbb{E}\left(\frac{Y_1}{g(X_1)} \psi_{j,k}(X_1) \right) \\
&= \mathbb{E}\left(\frac{f(X_1)}{g(X_1)} \psi_{j,k}(X_1) \right) + \mathbb{E}(\xi_1)\mathbb{E}\left(\frac{1}{g(X_1)} \psi_{j,k}(X_1)\right) \\
&= \mathbb{E}\left(\frac{f(X_1)}{g(X_1)} \psi_{j,k}(X_1) \right) = \int_0^1 \frac{f(x)}{g(x)} \psi_{j,k}(x) g(x) dx \\
&= \int_0^1 f(x) \psi_{j,k}(x) dx = \beta_{j,k}.
\end{aligned}$$

Hence, since $W_{1,j,k}, \ldots, W_{n,j,k}$ are identically distributed,

$$\begin{aligned}
\beta_{j,k} &= \mathbb{E}\left(W_{1,j,k}\right) = \mathbb{E}\left(W_{1,j,k} \mathbb{1}_{\{|W_{1,j,k}| \leq \gamma_n\}}\right) + \mathbb{E}\left(W_{1,j,k} \mathbb{1}_{\{|W_{1,j,k}| > \gamma_n\}}\right) \\
&= \mathbb{E}\left(\frac{1}{n} \sum_{i=1}^n W_{i,j,k} \mathbb{1}_{\{|W_{i,j,k}| \leq \gamma_n\}} \right) + \mathbb{E}\left(W_{1,j,k} \mathbb{1}_{\{|W_{1,j,k}| > \gamma_n\}}\right). \qquad (6.1)
\end{aligned}$$

This with the elementary inequality $(x+y)^2 \leq 2(x^2+y^2)$, $(x,y) \in \mathbb{R}^2$, imply that

$$\mathbb{E}\left((\widehat{\beta}_{j,k} - \beta_{j,k})^2\right) \leq 2(A+B), \tag{6.2}$$

where

$$A = \mathbb{E}\left(\left(\frac{1}{n}\sum_{i=1}^{n}\left(W_{i,j,k}\mathbb{I}_{\{|W_{i,j,k}|\leq\gamma_n\}} - \mathbb{E}(W_{i,j,k}\mathbb{I}_{\{|W_{i,j,k}|\leq\gamma_n\}})\right)\right)^2\right)$$

and

$$B = \left(\mathbb{E}\left(|W_{1,j,k}|\mathbb{I}_{\{|W_{1,j,k}|>\gamma_n\}}\right)\right)^2.$$

Let us bound B and A.

Upper bound for B. Since $\mathbb{I}_{\{|W_{1,j,k}|>\gamma_n\}} \leq |W_{1,j,k}|/\gamma_n$, we have

$$\mathbb{E}\left(|W_{1,j,k}|\mathbb{I}_{\{|W_{1,j,k}|>\gamma_n\}}\right) \leq \frac{\mathbb{E}(W_{1,j,k}^2)}{\gamma_n}.$$

Let us now bound $\mathbb{E}(W_{1,j,k}^2)$. Since X_1 and ξ_1 are independent and $\mathbb{E}(\xi_1)=0$, we have

$$\begin{aligned}
\mathbb{E}(W_{1,j,k}^2) &= \mathbb{E}\left(\frac{Y_1^2}{g^2(X_1)}\psi_{j,k}^2(X_1)\right) = \mathbb{E}\left(\frac{(f(X_1)+\xi_1)^2}{g^2(X_1)}\psi_{j,k}^2(X_1)\right) \\
&= \mathbb{E}\left(\frac{f^2(X_1)}{g^2(X_1)}\psi_{j,k}^2(X_1)\right) + 2\mathbb{E}(\xi_1)\mathbb{E}\left(\frac{f(X_1)}{g^2(X_1)}\psi_{j,k}^2(X_1)\right) \\
&+ \mathbb{E}\left(\xi_1^2\right)\mathbb{E}\left(\frac{1}{g^2(X_1)}\psi_{j,k}^2(X_1)\right) \\
&= \mathbb{E}\left(\frac{f^2(X_1)}{g^2(X_1)}\psi_{j,k}^2(X_1)\right) + \mathbb{E}(\xi_1^2)\mathbb{E}\left(\frac{1}{g^2(X_1)}\psi_{j,k}^2(X_1)\right).
\end{aligned}$$

Using (2.2), (2.3) and $\int_0^1 \psi_{j,k}^2(x)dx = 1$, we obtain

$$\begin{aligned}
\mathbb{E}\left(\frac{f^2(X_1)}{g^2(X_1)}\psi_{j,k}^2(X_1)\right) &\leq C_*^2\mathbb{E}\left(\frac{1}{g^2(X_1)}\psi_{j,k}^2(X_1)\right) \\
&= C_*^2\int_0^1 \frac{1}{g^2(x)}\psi_{j,k}^2(x)g(x)dx = C_*^2\int_0^1 \frac{1}{g(x)}\psi_{j,k}^2(x)dx \\
&\leq \frac{C_*^2}{c_*}\int_0^1 \psi_{j,k}^2(x)dx = \frac{C_*^2}{c_*}.
\end{aligned}$$

And, in a similar way,

$$\begin{aligned}
\mathbb{E}\left(\frac{1}{g^2(X_1)}\psi_{j,k}^2(X_1)\right) &= \int_0^1 \frac{1}{g^2(x)}\psi_{j,k}^2(x)g(x)dx = \int_0^1 \frac{1}{g(x)}\psi_{j,k}^2(x)dx \\
&\leq \frac{1}{c_*}\int_0^1 \psi_{j,k}^2(x)dx = \frac{1}{c_*}.
\end{aligned}$$

So

$$\mathbb{E}(W_{1,j,k}^2) \leq \frac{1}{c_*}(C_*^2 + \mathbb{E}(\xi_1^2)) = \mu^2 = C. \tag{6.3}$$

Therefore

$$B = \left(\mathbb{E}\left(|W_{1,j,k}| \mathbb{I}_{\{|W_{1,j,k}|>\gamma_n\}}\right)\right)^2 \leq \left(\frac{\mathbb{E}(W_{1,j,k}^2)}{\gamma_n}\right)^2 \leq C\frac{1}{\gamma_n^2} \leq C\frac{\ln n_\theta}{n_\theta}. \tag{6.4}$$

Upper bound for A. We have

$$\begin{aligned}
A &= \mathbb{V}\left(\frac{1}{n}\sum_{i=1}^n W_{i,j,k}\mathbb{I}_{\{|W_{i,j,k}|\leq\gamma_n\}}\right) \\
&= \frac{1}{n^2}\sum_{v=1}^n\sum_{\ell=1}^n \mathbb{C}\left(W_{v,j,k}\mathbb{I}_{\{|W_{v,j,k}|\leq\gamma_n\}}, W_{\ell,j,k}\mathbb{I}_{\{|W_{\ell,j,k}|\leq\gamma_n\}}\right) \\
&= \frac{1}{n}\mathbb{V}(W_{1,j,k}\mathbb{I}_{\{|W_{1,j,k}|\leq\gamma_n\}}) + \frac{2}{n^2}\sum_{v=2}^n\sum_{\ell=1}^{v-1} \mathbb{C}\left(W_{v,j,k}\mathbb{I}_{\{|W_{v,j,k}|\leq\gamma_n\}}, W_{\ell,j,k}\mathbb{I}_{\{|W_{\ell,j,k}|\leq\gamma_n\}}\right) \\
&\leq \frac{1}{n}\mathbb{V}(W_{1,j,k}\mathbb{I}_{\{|W_{1,j,k}|\leq\gamma_n\}}) + \frac{2}{n^2}\left|\sum_{v=2}^n\sum_{\ell=1}^{v-1} \mathbb{C}\left(W_{v,j,k}\mathbb{I}_{\{|W_{v,j,k}|\leq\gamma_n\}}, W_{\ell,j,k}\mathbb{I}_{\{|W_{\ell,j,k}|\leq\gamma_n\}}\right)\right|.
\end{aligned} \tag{6.5}$$

Using (6.3), we obtain

$$\mathbb{V}(W_{1,j,k}\mathbb{I}_{\{|W_{1,j,k}|\leq\gamma_n\}}) \leq \mathbb{E}(W_{1,j,k}^2\mathbb{I}_{\{|W_{1,j,k}|\leq\gamma_n\}}) \leq \mathbb{E}(W_{1,j,k}^2) \leq \mu^2 = C. \tag{6.6}$$

It follows from the stationarity of $(X_i, Y_i)_{i\in\mathbb{Z}}$ that

$$\begin{aligned}
&\left|\sum_{v=2}^n\sum_{\ell=1}^{v-1} \mathbb{C}\left(W_{v,j,k}\mathbb{I}_{\{|W_{v,j,k}|\leq\gamma_n\}}, W_{\ell,j,k}\mathbb{I}_{\{|W_{\ell,j,k}|\leq\gamma_n\}}\right)\right| \\
&\quad= \left|\sum_{m=1}^n (n-m)\mathbb{C}\left(W_{0,j,k}\mathbb{I}_{\{|W_{0,j,k}|\leq\gamma_n\}}, W_{m,j,k}\mathbb{I}_{\{|W_{m,j,k}|\leq\gamma_n\}}\right)\right| \\
&\quad\leq n\sum_{m=1}^n \left|\mathbb{C}\left(W_{0,j,k}\mathbb{I}_{\{|W_{0,j,k}|\leq\gamma_n\}}, W_{m,j,k}\mathbb{I}_{\{|W_{m,j,k}|\leq\gamma_n\}}\right)\right|.
\end{aligned}$$

By the Davydov inequality for strongly mixing processes (see [12]), the inequality $|W_{0,j,k}|\mathbb{I}_{\{|W_{0,j,k}|\leq\gamma_n\}} \leq \max(\gamma_n, |W_{0,j,k}|)$ and (6.3), for any $q \in (0,1)$, we have

$$\begin{aligned}
&\left|\mathbb{C}\left(W_{0,j,k}\mathbb{I}_{\{|W_{0,j,k}|\leq\gamma_n\}}, W_{m,j,k}\mathbb{I}_{\{|W_{m,j,k}|\leq\gamma_n\}}\right)\right| \\
&\quad\leq 10a_m^q\left(\mathbb{E}\left(|W_{0,j,k}|^{2/(1-q)}\mathbb{I}_{\{|W_{0,j,k}|\leq\gamma_n\}}\right)\right)^{1-q} \\
&\quad= 10a_m^q\left(\mathbb{E}\left(|W_{0,j,k}|^{2q/(1-q)}\mathbb{I}_{\{|W_{0,j,k}|\leq\gamma_n\}}W_{0,j,k}^2\mathbb{I}_{\{|W_{0,j,k}|\leq\gamma_n\}}\right)\right)^{1-q} \\
&\quad\leq Ca_m^q\left(\gamma_n^{2q/(1-q)}\right)^{1-q}\left(\mathbb{E}\left(W_{0,j,k}^2\right)\right)^{1-q} \leq Ca_m^q\left(\frac{n_\theta}{\ln n_\theta}\right)^q.
\end{aligned}$$

Due to (2.1), we have $\sum_{m=1}^{n} a_m^q \leq \gamma \sum_{m=1}^{\infty} exp\left(-cq|m|^\theta\right) = C$. So

$$\left|\sum_{v=2}^{n}\sum_{\ell=1}^{v-1} \mathbb{C}\left(W_{v,j,k}\mathbb{I}_{\{|W_{v,j,k}|\leq\gamma_n\}}, W_{\ell,j,k}\mathbb{I}_{\{|W_{\ell,j,k}|\leq\gamma_n\}}\right)\right| \leq Cn\left(\frac{n_\theta}{\ln n_\theta}\right)^q. \tag{6.7}$$

Taking $q = 1/\theta$, we have $n_\theta^q/n = 1/n_\theta$. It follows from (6.5), (6.6) and (6.7) that

$$A \leq C\left(\frac{1}{n} + \frac{1}{n^2}n\left(\frac{n_\theta}{\ln n_\theta}\right)^q\right) \leq C\frac{n_\theta^q}{n} = C\frac{1}{n_\theta} \leq C\frac{\ln n_\theta}{n_\theta}. \tag{6.8}$$

It follows from (6.2), (6.4) and (6.8) that

$$\mathbb{E}\left((\widehat{\beta}_{j,k} - \beta_{j,k})^2\right) \leq C\frac{\ln n_\theta}{n_\theta}. \tag{6.9}$$

Replacing ϕ instead of ψ in the previous proof, one can show that (6.9) holds with $\alpha_{j,k} = \int_0^1 f(x)\phi_{j,k}(x)dx$ instead of $\beta_{j,k}$ and $\widehat{\alpha}_{j,k}$ defined by (4.1) instead of $\widehat{\beta}_{j,k}$.

The proof of Proposition 4.1 is complete.

□

Proof of Proposition 4.2. We have

$$|\widehat{\beta}_{j,k} - \beta_{j,k}| \leq |\widehat{\beta}_{j,k}| + |\beta_{j,k}|.$$

We have

$$|\widehat{\beta}_{j,k}| \leq \frac{1}{n}\sum_{i=1}^{n}\left|\frac{Y_i}{g(X_i)}\psi_{j,k}(X_i)\right| \mathbb{I}_{\left\{\left|\frac{Y_i}{g(X_i)}\psi_{j,k}(X_i)\right|\leq\gamma_n\right\}} \leq \gamma_n = \mu\sqrt{\frac{n_\theta}{\ln n_\theta}}.$$

It follows from (2.2), the Cauchy-Schwarz inequality and $\int_0^1 \psi_{j,k}^2(x)dx = 1$ that

$$\begin{aligned} |\beta_{j,k}| &\leq \int_0^1 |f(x)||\psi_{j,k}(x)|dx \leq C_* \int_0^1 |\psi_{j,k}(x)|dx \\ &\leq C_*\left(\int_0^1 \psi_{j,k}^2(x)dx\right)^{1/2} = C_* \leq C\sqrt{\frac{n_\theta}{\ln n_\theta}}. \end{aligned}$$

Therefore

$$|\widehat{\beta}_{j,k} - \beta_{j,k}| \leq C\sqrt{\frac{n_\theta}{\ln n_\theta}}. \tag{6.10}$$

By (6.10) and Proposition 4.1, we have

$$\mathbb{E}\left((\widehat{\beta}_{j,k} - \beta_{j,k})^4\right) \leq C\frac{n_\theta}{\ln n_\theta}\mathbb{E}\left((\widehat{\beta}_{j,k} - \beta_{j,k})^2\right) \leq C\frac{n_\theta}{\ln n_\theta}\frac{\ln n_\theta}{n_\theta} = C.$$

□

Proof of Proposition 4.3. For any $i \in \{1,\ldots,n\}$, set

$$W_{i,j,k} = \frac{Y_i}{g(X_i)}\psi_{j,k}(X_i).$$

By (6.1), we have

$$\begin{aligned}
&|\widehat{\beta}_{j,k} - \beta_{j,k}| \\
&= \left|\frac{1}{n}\sum_{i=1}^{n}\left(W_{i,j,k}\mathbb{I}_{\{|W_{i,j,k}|\leq\gamma_n\}} - \mathbb{E}\left(W_{i,j,k}\mathbb{I}_{\{|W_{i,j,k}|\leq\gamma_n\}}\right)\right) - \mathbb{E}\left(W_{1,j,k}\mathbb{I}_{\{|W_{1,j,k}|>\gamma_n\}}\right)\right| \\
&\leq \left|\frac{1}{n}\sum_{i=1}^{n}\left(W_{i,j,k}\mathbb{I}_{\{|W_{i,j,k}|\leq\gamma_n\}} - \mathbb{E}\left(W_{i,j,k}\mathbb{I}_{\{|W_{i,j,k}|\leq\gamma_n\}}\right)\right)\right| + \mathbb{E}\left(|W_{1,j,k}|\mathbb{I}_{\{|W_{1,j,k}|>\gamma_n\}}\right).
\end{aligned}$$

Using (6.3), we obtain

$$\mathbb{E}\left(|W_{1,j,k}|\mathbb{I}_{\{|W_{1,j,k}|>\gamma_n\}}\right) \leq \frac{\mathbb{E}(W_{1,j,k}^2)}{\gamma_n} \leq \mu^2\frac{1}{\mu}\sqrt{\frac{\ln n_\theta}{n_\theta}} = \lambda_n.$$

Hence

$$\begin{aligned}
&\mathbb{P}\left(|\widehat{\beta}_{j,k} - \beta_{j,k}| \geq \kappa\lambda_n/2\right) \\
&\leq \mathbb{P}\left(\left|\frac{1}{w_n}\sum_{i=1}^{n}\left(W_{i,j,k}\mathbb{I}_{\{|W_{i,j,k}|\leq\gamma_n\}} - \mathbb{E}\left(W_{i,j,k}\mathbb{I}_{\{|W_{i,j,k}|\leq\gamma_n\}}\right)\right)\right| \geq (\kappa/2-1)\lambda_n\right).
\end{aligned} \tag{6.11}$$

Let us now present a Bernstein inequality for exponentially strongly mixing process. This is a slightly modified version of [28, Theorem 4.2].

Lemma 6.1 ([28]). *Let $\gamma > 0$, $c > 0$, $\theta > 1$ and $(S_i)_{i\in\mathbb{Z}}$ be a stationary process such that, for any $m \in \mathbb{Z}$, the associated m-th strongly mixing coefficient satisfies*

$$a_m \leq \gamma exp(-c|m|^\theta).$$

Let $n \in \mathbb{N}^*$, $h : \mathbb{R} \to \mathbb{R}$ be a measurable function and, for any $i \in \mathbb{Z}$, $U_i = h(S_i)$. We assume that $\mathbb{E}(U_1) = 0$ and there exists a constant $M > 0$ satisfying $|U_1| \leq M < \infty$. Then, for any $\lambda > 0$, we have

$$\mathbb{P}\left(\left|\frac{1}{n}\sum_{i=1}^{n}U_i\right| \geq \lambda\right) \leq 2(1+4e^{-2}\gamma)\exp\left(-\frac{u\lambda^2 n^{\theta/(\theta+1)}}{2\left(\mathbb{E}\left(U_1^2\right)+\lambda M/3\right)}\right),$$

where $u = (1/2)(c/8)^{1/(\theta+1)}$.

Set, for any $i \in \{1,\ldots,n\}$,

$$U_{i,j,k} = W_{i,j,k}\mathbb{I}_{\{|W_{i,j,k}|\leq\gamma_n\}} - \mathbb{E}\left(W_{i,j,k}\mathbb{I}_{\{|W_{i,j,k}|\leq\gamma_n\}}\right).$$

Then $U_{1,j,k},\dots,U_{n,j,k}$ are identically distributed, depend on the stationary strongly mixing process $(X_i,Y_i)_{i\in\mathbb{Z}}$ which satisfies (2.1), $\mathbb{E}(U_{1,j,k})=0$,

$$|U_{1,j,k}|\le |W_{1,j,k}|\,\mathbb{1}_{\{|W_{1,j,k}|\le\gamma_n\}}+\mathbb{E}\left(|W_{1,j,k}|\mathbb{1}_{\{|W_{1,j,k}|\le\gamma_n\}}\right)\le 2\gamma_n$$

and, by (6.3),

$$\mathbb{E}(U_{1,j,k}^2)=\mathbb{V}\left(W_{1,j,k}\mathbb{1}_{\{|W_{1,j,k}|\le\gamma_n\}}\right)\le\mathbb{E}(W_{1,j,k}^2)\le\mu^2.$$

It follows from Lemma 6.1 that

$$\begin{aligned}&\mathbb{P}\left(\left|\frac{1}{n}\sum_{i=1}^n U_{i,j,k}\right|\ge(\kappa/2-1)\lambda_n\right)\\ &\quad\le\quad 2(1+4e^{-2}\gamma)\exp\left(-\frac{u(\kappa/2-1)^2\lambda_n^2 n_\theta}{2\left(\mu^2+2(\kappa/2-1)\lambda_n\gamma_n/3\right)}\right).\end{aligned}\tag{6.12}$$

We have

$$\lambda_n\gamma_n=\mu\sqrt{\frac{\ln n_\theta}{n_\theta}}\mu\sqrt{\frac{n_\theta}{\ln n_\theta}}=\mu^2,\qquad \lambda_n^2=\mu^2\frac{\ln n_\theta}{n_\theta}.$$

Combining (6.11) and (6.12), for any $\kappa\ge 2+16/(3u)+4\sqrt{(1/u)(16/9u^2+2)}$, we have

$$\begin{aligned}&\mathbb{P}\left(|\widehat{\beta}_{j,k}-\beta_{j,k}|\ge\kappa\lambda_n/2\right)\\ &\quad\le\quad 2(1+4e^{-2}\gamma)\exp\left(-\frac{u(\kappa/2-1)^2\ln n_\theta}{2\left(1+2(\kappa/2-1)/3\right)}\right)\\ &\quad=\quad 2(1+4e^{-2}\gamma)n_\theta^{-\frac{u(\kappa/2-1)^2}{2(1+2(\kappa/2-1)/3)}}\le 2(1+4e^{-2}\gamma)\frac{1}{n_\theta^4}.\end{aligned}$$

This ends the proof of Proposition 4.3.

□

Proof of Theorem 5.1. We expand the function f on $\mathcal{B}$ as

$$f(x)=\sum_{k=0}^{2^\tau-1}\alpha_{\tau,k}\phi_{\tau,k}(x)+\sum_{j=\tau}^{\infty}\sum_{k=0}^{2^j-1}\beta_{j,k}\psi_{j,k}(x),\quad x\in[0,1],$$

where

$$\alpha_{\tau,k}=\int_0^1 f(x)\phi_{\tau,k}(x)dx,\qquad \beta_{j,k}=\int_0^1 f(x)\psi_{j,k}(x)dx.$$

We have, for any $x\in[0,1]$,

$$\begin{aligned}&\widehat{f}(x)-f(x)\\ &\quad=\quad\sum_{k=0}^{2^\tau-1}(\widehat{\alpha}_{\tau,k}-\alpha_{\tau,k})\phi_{\tau,k}(x)+\sum_{j=\tau}^{j_1}\sum_{k=0}^{2^j-1}\left(\widehat{\beta}_{j,k}\mathbb{1}_{\left\{|\widehat{\beta}_{j,k}|\ge\kappa\lambda_n\right\}}-\beta_{j,k}\right)\psi_{j,k}(x)\\ &\quad-\quad\sum_{j=j_1+1}^{\infty}\sum_{k=0}^{2^j-1}\beta_{j,k}\psi_{j,k}(x).\end{aligned}$$

Since $\mathcal{B}$ is an orthonormal basis of $\mathbb{L}^2([0,1])$, we can write

$$\mathbf{R}(\widehat{f},f)=\mathbb{E}\left(\int_0^1(\widehat{f}(x)-f(x))^2dx\right)=R+S+T, \tag{6.13}$$

where

$$R=\sum_{k=0}^{2^\tau-1}\mathbb{E}\left((\widehat{\alpha}_{\tau,k}-\alpha_{\tau,k})^2\right),\qquad S=\sum_{j=\tau}^{j_1}\sum_{k=0}^{2^j-1}\mathbb{E}\left((\widehat{\beta}_{j,k}\mathbb{I}_{\left\{|\widehat{\beta}_{j,k}|\geq\kappa\lambda_n\right\}}-\beta_{j,k})^2\right)$$

and

$$T=\sum_{j=j_1+1}^{\infty}\sum_{k=0}^{2^j-1}\beta_{j,k}^2.$$

Let us bound R, T and S.

Upper bound for R. Using Proposition 4.1 and $2s/(2s+1)<1$, we obtain

$$R\leq C2^\tau\frac{\ln n_\theta}{n_\theta}\leq C\left(\frac{\ln n_\theta}{n_\theta}\right)^{2s/(2s+1)}. \tag{6.14}$$

Upper bound for T. For $r\geq 1$ and $p\geq 2$, we have $B^s_{p,r}(M)\subseteq B^s_{2,\infty}(M)$. Since $2s/(2s+1)<2s$, we have

$$T\leq C\sum_{j=j_1+1}^{\infty}2^{-2js}\leq C2^{-2j_1s}\leq Cn_\theta^{-2s}\leq C\left(\frac{\ln n_\theta}{n_\theta}\right)^{2s}\leq C\left(\frac{\ln n_\theta}{n_\theta}\right)^{2s/(2s+1)}.$$

For $r\geq 1$ and $p\in[1,2)$, we have $B^s_{p,r}(M)\subseteq B^{s+1/2-1/p}_{2,\infty}(M)$. Since $s>1/p$, we have $s+1/2-1/p>s/(2s+1)$. So

$$\begin{aligned}T&\leq C\sum_{j=j_1+1}^{\infty}2^{-2j(s+1/2-1/p)}\leq C2^{-2j_1(s+1/2-1/p)}\\&\leq Cn_\theta^{-2(s+1/2-1/p)}\leq C\left(\frac{\ln n_\theta}{n_\theta}\right)^{2(s+1/2-1/p)}\leq C\left(\frac{\ln n_\theta}{n_\theta}\right)^{2s/(2s+1)}.\end{aligned}$$

Hence, for $r\geq 1$, $\{p\geq 2$ and $s>0\}$ or $\{p\in[1,2)$ and $s>1/p\}$, we have

$$T\leq C\left(\frac{\ln n_\theta}{n_\theta}\right)^{2s/(2s+1)}. \tag{6.15}$$

Upper bound for S. We have

$$S=S_1+S_2+S_3+S_4, \tag{6.16}$$

where

$$S_1=\sum_{j=\tau}^{j_1}\sum_{k=0}^{2^j-1}\mathbb{E}\left((\widehat{\beta}_{j,k}-\beta_{j,k})^2\mathbb{I}_{\left\{|\widehat{\beta}_{j,k}|\geq\kappa\lambda_n\right\}}\mathbb{I}_{\{|\beta_{j,k}|<\kappa\lambda_n/2\}}\right),$$

$$S_2 = \sum_{j=\tau}^{j_1} \sum_{k=0}^{2^j-1} \mathbb{E}\left((\widehat{\beta}_{j,k} - \beta_{j,k})^2 \mathbb{I}_{\left\{|\widehat{\beta}_{j,k}| \geq \kappa\lambda_n\right\}} \mathbb{I}_{\{|\beta_{j,k}| \geq \kappa\lambda_n/2\}} \right),$$

$$S_3 = \sum_{j=\tau}^{j_1} \sum_{k=0}^{2^j-1} \mathbb{E}\left(\beta_{j,k}^2 \mathbb{I}_{\left\{|\widehat{\beta}_{j,k}| < \kappa\lambda_n\right\}} \mathbb{I}_{\{|\beta_{j,k}| \geq 2\kappa\lambda_n\}} \right)$$

and

$$S_4 = \sum_{j=\tau}^{j_1} \sum_{k=0}^{2^j-1} \mathbb{E}\left(\beta_{j,k}^2 \mathbb{I}_{\left\{|\widehat{\beta}_{j,k}| < \kappa\lambda_n\right\}} \mathbb{I}_{\{|\beta_{j,k}| < 2\kappa\lambda_n\}} \right).$$

Let us investigate the bounds of S_1, S_2, S_3 and S_4.

Upper bounds for S_1 and S_3. We have

$$\left\{ |\widehat{\beta}_{j,k}| < \kappa\lambda_n,\ |\beta_{j,k}| \geq 2\kappa\lambda_n \right\} \subseteq \left\{ |\widehat{\beta}_{j,k} - \beta_{j,k}| > \kappa\lambda_n/2 \right\},$$

$$\left\{ |\widehat{\beta}_{j,k}| \geq \kappa\lambda_n,\ |\beta_{j,k}| < \kappa\lambda_n/2 \right\} \subseteq \left\{ |\widehat{\beta}_{j,k} - \beta_{j,k}| > \kappa\lambda_n/2 \right\}$$

and

$$\left\{ |\widehat{\beta}_{j,k}| < \kappa\lambda_n,\ |\beta_{j,k}| \geq 2\kappa\lambda_n \right\} \subseteq \left\{ |\beta_{j,k}| \leq 2|\widehat{\beta}_{j,k} - \beta_{j,k}| \right\}.$$

So

$$\max(S_1, S_3) \leq C \sum_{j=\tau}^{j_1} \sum_{k=0}^{2^j-1} \mathbb{E}\left((\widehat{\beta}_{j,k} - \beta_{j,k})^2 \mathbb{I}_{\left\{|\widehat{\beta}_{j,k} - \beta_{j,k}| > \kappa\lambda_n/2\right\}} \right).$$

It follows from the Cauchy-Schwarz inequality, Propositions 4.2 and 4.3 that

$$\begin{aligned}
&\mathbb{E}\left((\widehat{\beta}_{j,k} - \beta_{j,k})^2 \mathbb{I}_{\left\{|\widehat{\beta}_{j,k} - \beta_{j,k}| > \kappa\lambda_n/2\right\}} \right) \\
&\leq \left(\mathbb{E}\left((\widehat{\beta}_{j,k} - \beta_{j,k})^4 \right) \right)^{1/2} \left(\mathbb{P}\left(|\widehat{\beta}_{j,k} - \beta_{j,k}| > \kappa\lambda_n/2 \right) \right)^{1/2} \\
&\leq C \left(\frac{1}{n_\theta^4} \right)^{1/2} = C \frac{1}{n_\theta^2}.
\end{aligned}$$

Since $2s/(2s+1) < 1$, we have

$$\max(S_1, S_3) \leq C \frac{1}{n_\theta^2} \sum_{j=\tau}^{j_1} 2^j \leq C \frac{1}{n_\theta^2} 2^{j_1} \leq C \frac{1}{n_\theta} \leq C \left(\frac{\ln n_\theta}{n_\theta} \right)^{2s/(2s+1)}. \tag{6.17}$$

Upper bound for S_2. Using again Proposition 4.1, we obtain

$$\mathbb{E}\left((\widehat{\beta}_{j,k} - \beta_{j,k})^2 \right) \leq C \frac{\ln n_\theta}{n_\theta}.$$

Hence

$$S_2 \leq C \frac{\ln n_\theta}{n_\theta} \sum_{j=\tau}^{j_1} \sum_{k=0}^{2^j-1} \mathbb{I}_{\{|\beta_{j,k}| > \kappa\lambda_n/2\}}.$$

Let j_2 be the integer defined by

$$\frac{1}{2}\left(\frac{n_\theta}{\ln n_\theta}\right)^{1/(2s+1)} < 2^{j_2} \leq \left(\frac{n_\theta}{\ln n_\theta}\right)^{1/(2s+1)}. \tag{6.18}$$

We have

$$S_2 \leq S_{2,1} + S_{2,2},$$

where

$$S_{2,1} = C\frac{\ln n_\theta}{n_\theta}\sum_{j=\tau}^{j_2}\sum_{k=0}^{2^j-1} \mathbb{I}_{\{|\beta_{j,k}|>\kappa\lambda_n/2\}}$$

and

$$S_{2,2} = C\frac{\ln n_\theta}{n_\theta}\sum_{j=j_2+1}^{j_1}\sum_{k=0}^{2^j-1} \mathbb{I}_{\{|\beta_{j,k}|>\kappa\lambda_n/2\}}.$$

We have

$$S_{2,1} \leq C\frac{\ln n_\theta}{n_\theta}\sum_{j=\tau}^{j_2} 2^j \leq C\frac{\ln n_\theta}{n_\theta}2^{j_2} \leq C\left(\frac{\ln n_\theta}{n_\theta}\right)^{2s/(2s+1)}.$$

For $r \geq 1$ and $p \geq 2$, since $B^s_{p,r}(M) \subseteq B^s_{2,\infty}(M)$, we have

$$\begin{aligned} S_{2,2} &\leq C\frac{\ln n_\theta}{n_\theta\lambda_n^2}\sum_{j=j_2+1}^{j_1}\sum_{k=0}^{2^j-1}\beta_{j,k}^2 \leq C\sum_{j=j_2+1}^{\infty}\sum_{k=0}^{2^j-1}\beta_{j,k}^2 \leq C2^{-2j_2s} \\ &\leq C\left(\frac{\ln n_\theta}{n_\theta}\right)^{2s/(2s+1)}. \end{aligned}$$

For $r \geq 1$, $p \in [1,2)$ and $s > 1/p$, using $\mathbb{I}_{\{|\beta_{j,k}|>\kappa\lambda_n/2\}} \leq C|\beta_{j,k}|^p/\lambda_n^p$, $B^s_{p,r}(M) \subseteq B^{s+1/2-1/p}_{2,\infty}(M)$ and $(2s+1)(2-p)/2 + (s+1/2-1/p)p = 2s$, we have

$$\begin{aligned} S_{2,2} &\leq C\frac{\ln n_\theta}{n_\theta\lambda_n^p}\sum_{j=j_2+1}^{j_1}\sum_{k=0}^{2^j-1}|\beta_{j,k}|^p \leq C\left(\frac{\ln n_\theta}{n_\theta}\right)^{(2-p)/2}\sum_{j=j_2+1}^{\infty}2^{-j(s+1/2-1/p)p} \\ &\leq C\left(\frac{\ln n_\theta}{n_\theta}\right)^{(2-p)/2}2^{-j_2(s+1/2-1/p)p} \leq C\left(\frac{\ln n_\theta}{n_\theta}\right)^{2s/(2s+1)}. \end{aligned}$$

So, for $r \geq 1$, $\{p \geq 2 \text{ and } s > 0\}$ or $\{p \in [1,2) \text{ and } s > 1/p\}$, we have

$$S_2 \leq C\left(\frac{\ln n_\theta}{n_\theta}\right)^{2s/(2s+1)}. \tag{6.19}$$

Upper bound for S_4. We have

$$S_4 \leq \sum_{j=\tau}^{j_1}\sum_{k=0}^{2^j-1}\beta_{j,k}^2\mathbb{I}_{\{|\beta_{j,k}|<2\kappa\lambda_n\}}.$$

Let j_2 be the integer (6.18). Then

$$S_4 \leq S_{4,1} + S_{4,2},$$

where

$$S_{4,1} = \sum_{j=\tau}^{j_2} \sum_{k=0}^{2^j-1} \beta_{j,k}^2 \mathbb{I}_{\{|\beta_{j,k}|<2\kappa\lambda_n\}}, \qquad S_{4,2} = \sum_{j=j_2+1}^{j_1} \sum_{k=0}^{2^j-1} \beta_{j,k}^2 \mathbb{I}_{\{|\beta_{j,k}|<2\kappa\lambda_n\}}.$$

We have

$$S_{4,1} \leq C \sum_{j=\tau}^{j_2} 2^j \lambda_n^2 = C \frac{\ln n_\theta}{n_\theta} \sum_{j=\tau}^{j_2} 2^j \leq C \frac{\ln n_\theta}{n_\theta} 2^{j_2} \leq C \left(\frac{\ln n_\theta}{n_\theta} \right)^{2s/(2s+1)}.$$

For $r \geq 1$ and $p \geq 2$, since $B_{p,r}^s(M) \subseteq B_{2,\infty}^s(M)$, we have

$$S_{4,2} \leq \sum_{j=j_2+1}^{\infty} \sum_{k=0}^{2^j-1} \beta_{j,k}^2 \leq C 2^{-2j_2 s} \leq C \left(\frac{\ln n_\theta}{n_\theta} \right)^{2s/(2s+1)}.$$

For $r \geq 1$, $p \in [1,2)$ and $s > 1/p$, using $\beta_{j,k}^2 \mathbb{I}_{\{|\beta_{j,k}|<2\kappa\lambda_n\}} \leq C\lambda_n^{2-p}|\beta_{j,k}|^p$, $B_{p,r}^s(M) \subseteq B_{2,\infty}^{s+1/2-1/p}(M)$ and $(2s+1)(2-p)/2 + (s+1/2-1/p)p = 2s$, we have

$$\begin{aligned} S_{4,2} &\leq C\lambda_n^{2-p} \sum_{j=j_2+1}^{j_1} \sum_{k=0}^{2^j-1} |\beta_{j,k}|^p = C \left(\frac{\ln n_\theta}{n_\theta} \right)^{(2-p)/2} \sum_{j=j_2+1}^{j_1} \sum_{k=0}^{2^j-1} |\beta_{j,k}|^p \\ &\leq C \left(\frac{\ln n_\theta}{n_\theta} \right)^{(2-p)/2} \sum_{j=j_2+1}^{\infty} 2^{-j(s+1/2-1/p)p} \leq C \left(\frac{\ln n_\theta}{n_\theta} \right)^{(2-p)/2} 2^{-j_2(s+1/2-1/p)p} \\ &\leq C \left(\frac{\ln n_\theta}{n_\theta} \right)^{2s/(2s+1)}. \end{aligned}$$

So, for $r \geq 1$, $\{p \geq 2 \text{ and } s > 0\}$ or $\{p \in [1,2) \text{ and } s > 1/p\}$, we have

$$S_4 \leq C \left(\frac{\ln n_\theta}{n_\theta} \right)^{2s/(2s+1)}. \tag{6.20}$$

It follows from (6.16), (6.17), (6.19) and (6.20) that

$$S \leq C \left(\frac{\ln n_\theta}{n_\theta} \right)^{2s/(2s+1)}. \tag{6.21}$$

Combining (6.13), (6.14), (6.15) and (6.21), we have, for $r \geq 1$, $\{p \geq 2 \text{ and } s > 0\}$ or $\{p \in [1,2) \text{ and } s > 1/p\}$,

$$\mathbf{R}(\widehat{f}, f) \leq C \left(\frac{\ln n_\theta}{n_\theta} \right)^{2s/(2s+1)}.$$

The proof of Theorem 5.1 is complete.

□

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