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# Lognormal $\star$ -scale invariant random measures

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## Abstract

In this article, we consider the continuous analog of the celebrated Mandelbrot star equation with lognormal weights. We show existence and uniqueness of measures satisfying the aforementioned continuous equation; these measures fall under the scope of the Gaussian multiplicative chaos theory developed by J.P. Kahane in 1985 (or possibly extensions of this theory). As a by product, we also obtain an explicit characterization of the covariance structure of these measures.

AMS subject classification: primary 60G57; secondary 28A80,60H10,60G15.

Keywords: random measure, star equation, scale invariance, multiplicative chaos, uniqueness, multifractal processes.

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## 1. Introduction

Fractality and the related concept of scale invariance is nowadays well introduced in many fields of applications ranging from physics, finance, information or social sciences. The scale-invariance property of a stochastic process is usually quantified by the scaling exponents  $\xi(q)$  associated with the power-law behavior of the order  $q$  moments of the fluctuations at different scales. More

precisely, if  $X_t$  is a 1-d process with stationary increments, we can consider the  $q$ -th moments of its fluctuations at scale  $l$ :

$$\mathbb{E}[|X_{t+l} - X_t|^q].$$

The scaling exponents  $\xi(q)$  are defined through the following power-law scaling:

$$\mathbb{E}[|X_{t+l} - X_t|^q] = C_q l^{\xi(q)} \quad \forall l < T.$$

When  $\xi(q) = qH$  is linear, the process is said to be monofractal. Famous examples of such processes are (fractional) Brownian motion,  $\alpha$ -stable Lévy processes or Hermitte processes. When  $\xi$  is nonlinear, the process is said to be multifractal. The concept of nonlinear power-law scalings goes back to the Kolmogorov theory of fully developed turbulence in the sixties (see [5, 20, 22, 6, 11] and references therein), introduced to render the intermittency effects in turbulence. Mandelbrot [16] came up with the first mathematical discrete approach of multifractality, the now celebrated multiplicative cascades. Roughly speaking, a (dyadic) multiplicative cascade is a positive random measure  $M$  on the unit interval  $[0, 1]$  that obeys the following decomposition rule:

$$M(dt) \stackrel{\text{law}}{=} Z^0 \mathbf{1}_{[0, \frac{1}{2}]}(t) M^0(2dt) + Z^1 \mathbf{1}_{[\frac{1}{2}, 1]}(t) M^1(2dt - 1), \quad (1)$$

where  $M^0, M^1$  are two independent copies of  $M$  and  $(Z^0, Z^1)$  is a random vector with prescribed law and positive components of mean 1 independent from  $M^0, M^1$ . Such an equation (and its generalisations to  $b$ -adic trees for  $b \geq 2$ ), the celebrated star equation introduced by Mandelbrot in [15], uniquely determines the law of the multiplicative cascade. Despite the fact that multiplicative cascades have been widely used as reference models in many applications, they possess many drawbacks related to their discrete scale invariance, mainly they involve a particular scale ratio and they do not possess stationary fluctuations (this comes from the fact that they are constructed on a dyadic tree structure).

Much effort has been made to develop a continuous parameter theory of suitable stationary multifractal random measures ever since, stemming from the theory of multiplicative chaos introduced by Kahane [13, 3, 20, 1, 17, 19]. The construction of such measures is now well understood and they are largely used in mathematical modeling since they obey a so-called stochastic scale invariance property, namely the property of being equal in law at different scales up to an independent stochastic factor. This is some kind of continuous parameter generalization of (1). Stochastic scale invariance property is observed in many experimental and theoretical problems, like turbulence (see [11, 5] and many others), quantum gravity (see [14, 10, 18]), mathematical finance, etc... and this is the main motivation for introducing multifractal random measures. However, as far as we know, the following question has never been solved: are these measures the only existing stochastic scale invariant object? This is fundamental since a positive answer gives a full justification to their intensive use. In this paper, we characterize stochastic scale invariant measures when the stochastic factor is assumed to be Gaussian. We prove that the class of such objects is made up of Gaussian multiplicative chaos with a specific structure of the covariation kernel, which turns out to be larger than described in the literature.

## 2. Background

Let us first remind the reader of the main definitions we will use throughout the paper. We denote by  $\mathcal{B}(E)$  the Borelian sigma field on a topological space  $E$ . A random measure  $M$  is a random variable taking values into the set of positive Radon measures defined on  $\mathcal{B}(\mathbb{R})$  such that  $\mathbb{E}[M(K)] < +\infty$  for every compact set  $K$ . A random measure  $M$  is said to be stationary if for all  $y \in \mathbb{R}$  the random measures  $M(\cdot)$  and  $M(y + \cdot)$  have the same laws.

## 2.1 Gaussian multiplicative chaos

We remind the reader of the notion of Gaussian multiplicative chaos as introduced by Kahane [13]. Consider a sequence  $(X^n)_n$  of independent centered stationary Gaussian processes with associated nonnegative covariance kernel  $k_n(r) = \mathbb{E}[X_r^n X_0^n] \geq 0$ . For each  $N \geq 1$ , we can define a Radon measure  $M^N$  on the Borelian subsets of  $\mathbb{R}^d$  by

$$M^N(A) = \int_A e^{\sum_{n=0}^N X_r^n - \frac{1}{2}\mathbb{E}[(X_r^n)^2]} dr.$$

For each Borelian set  $A$ , the sequence  $(M^N(A))_N$  is a positive martingale. Thus it converges almost surely towards a random variable denoted by  $M(A)$ . One can deduce that the sequence of measures  $(M^N)_N$  weakly converges towards a Radon measure  $M$ , commonly denoted by

$$M(A) = \int_A e^{X_r - \frac{1}{2}\mathbb{E}[X_r^2]} dr \quad (2)$$

and called Gaussian multiplicative chaos associated to the kernel

$$K(r) = \sum_{n=0}^{+\infty} k_n(r). \quad (3)$$

Roughly speaking, (2) can be understood as a measure admitting as density the exponential of a Gaussian process  $X$  with covariance kernel  $K$ . Of course, this is purely formal because  $X$  can only be understood as a (random Gaussian) distribution (in the sense of Schwartz) because of the possible singularities of the kernel  $K$ .

Of special interest is the situation when the function  $K$  can be rewritten as (for some  $\lambda^2 > 0$ )

$$K(r) = \lambda^2 \ln_+ \frac{T}{|r|} + g(r) \quad (4)$$

for some bounded function  $g$  (and  $\ln_+(x) = \max(0, \ln(x))$ ). In that case, Kahane proved that the martingale  $(M^N(A))_N$ , for some Borelian set  $A$  with finite Lebesgue measure, is uniformly integrable if and only if  $\lambda^2 < 2$ . This condition is necessary and sufficient in order for the limiting measure  $M$  to be non identically null. For kernels of the form (4) which can not be written as a sum of nonnegative terms as (3), we refer to the extended Gaussian multiplicative theory developed in [17]. We remind that Gaussian multiplicative chaos with kernel given by (4) has found applications in many fields in science:

- In dimension 1, the measure  $M$  is called the lognormal Multifractal Random Measure (MRM). It is used to model the volatility of a financial asset (see [2], [7]).
- In dimension 2, the measure  $M$  is a probabilistic formulation of the quantum gravity measure (more precisely, the quantum gravity measure is defined as the exponential of the Gaussian Free Field and therefore is defined in a bounded domain). We refer to references [4], [10], [18] for probabilistic papers on this topic.
- In dimension 3, the measure  $M$  is called the Kolmogorov-Obhukov model (see textbook [11]): it is a model of energy dissipation in the statistical theory of fully developed turbulence.

## 3. Main results

### 3.1 Definitions

In this paper we are interested in stationary random measures in dimension 1 satisfying the following scale invariance property:

**Lognormal scale invariance.** A random measure  $M$  is said to be lognormal  $\star$ -scale invariant if there is a parameter  $T > 0$  such that for all  $T' < T$ ,  $M$  obeys the cascading rule

$$(M(A))_{A \in \mathcal{B}(\mathbb{R})} \stackrel{\text{law}}{=} \left( \int_A e^{\omega_{T'}(r)} M^{T'}(dr) \right)_{A \in \mathcal{B}(\mathbb{R})} \quad (5)$$

where  $\omega_{T'}$  is a Gaussian process and  $M^{T'}$  is a random measure independent from  $\omega_{T'}$  satisfying the relation

$$(M^{T'}(\frac{T'}{T}A))_{A \in \mathcal{B}(\mathbb{R})} \stackrel{\text{law}}{=} \frac{T'}{T} (M(A))_{A \in \mathcal{B}(\mathbb{R})}.$$

Intuitively, this relation means that when you zoom in the measure  $M$ , you should observe the same behaviour up to an independent Gaussian factor.

*Remark.* In order for a measure  $M$  satisfying (5) to be non trivial, it is obvious to check that the Gaussian process  $\omega_{T'}$  must be normalized so that  $\mathbb{E}[e^{\omega_{T'}(r)}] = 1$ .

**Definition 1.** We will say that a stationary random measure  $M$  satisfies the good lognormal  $\star$ -scale invariance if  $M$  is lognormal  $\star$ -scale invariant and for each  $T' < T$ , the covariance kernel  $k_{T'}$  of the process  $\omega_{T'}$  involved in (5) satisfies, for some positive constants  $C_{T'}, \alpha > 0$ :

$$\forall |r| \geq 1, \quad |k_{T'}(r)| \leq C_{T'} \frac{1}{1 + |\ln r|^{1+\alpha}}, \quad (6)$$

$$\forall r, r' \in \mathbb{R}, \quad |k_{T'}(r) - k_{T'}(r')| \leq \frac{C_{T'}}{1 + (|r| + |r'|)^{1+\alpha}} |r - r'|. \quad (7)$$

Let us make a few comments about the above definition, which is all about technical considerations. In order for equation (5) to be tractable, we need a decorrelation property at infinity: (6) just describes how fast the covariance function of the process  $\omega_{T'}$  decays at infinity. This is a rather weak assumption. (7) mainly expresses that the kernel  $k_{T'}$  is Lipschitzian with a local Lipschitz constant that at most polynomially decays to 0 when approaching infinity. Because of (7) and the Dudley criterion [9], the process  $\omega_{T'}$  possesses almost surely continuous sample paths.

The well known example of good lognormal  $\star$ -scale invariant random measure was first explicitly described in [3] (see also [1]). Let us consider the Gaussian multiplicative chaos  $M$  with associated kernel

$$K(r) = \int_{|r|}^{+\infty} \frac{k(u)}{u} du \quad \text{with} \quad k(u) = \lambda^2 \left(1 - \frac{u}{T}\right) \mathbf{1}_{[0, T]}(u). \quad (8)$$

For  $T' < T$ , the measure  $M$  can be decomposed as  $M(dr) = e^{\omega_{T'}(r)} M^{T'}(dr)$  where both measures  $M^{T'}(\frac{T'}{T}dr)$  and  $\frac{T'}{T}M(dr)$  have the same law and the covariance kernel of the process  $\omega_{T'}$  is given by

$$k_{T'}(r) = \int_{|r|}^{|\frac{r}{T'}|} \frac{k(u)}{u} du.$$

As far as we know, this is the only known example of random measures with such property in the literature. We further stress that this measure admits a cut-off in the sense described just below.

## 3.2 Results

It turns out that the set of random measures satisfying (5) is much larger than the above example. Actually, we have the following description of the solutions to (5), which is the main result of the paper:

**Theorem 2.** Let  $M$  be a good lognormal  $\star$ -scale invariant random measure. Assume that

$$\mathbb{E}[M([0, 1])^{1+\delta}] < +\infty$$

for some  $\delta > 0$ . Then  $M$  is the product of a nonnegative random variable  $Y \in L^{1+\delta}$  and an independent Gaussian multiplicative chaos

$$\forall A \subset \mathbb{B}(\mathbb{R}), \quad M(A) = Y \int_A e^{X_r - \frac{1}{2}\mathbb{E}[X_r^2]} dr \quad (9)$$

with associated covariance kernel given by the improper integral

$$K(r) = \int_{|r|}^{+\infty} \frac{k(u)}{u} du \quad (10)$$

for some continuous covariance function  $k$  such that  $k(0) \leq \frac{2}{1+\delta}$ .

Conversely, given some data  $k$  and  $Y$  as above, the relation (9) defines a good lognormal  $\star$ -scale invariant random measure with finite moments of order  $1 + \gamma$  for every  $\gamma \in [0, \delta)$ .

Let us comment on Theorem 2. First we point out that  $Y$  is deterministic as soon as the measure  $M$  in Theorem 2 is ergodic. Second, good lognormal  $\star$ -scale invariant measures exhibit a multifractal behaviour. More precisely, if we consider a measure  $M$  as in Theorem 2, we define its structure exponent

$$\forall q > 0, \quad \xi(q) = \left(1 + \frac{k(0)}{2}\right)q - \frac{k(0)}{2}q^2.$$

Then we have the following asymptotic power-law spectrum, for  $q \leq 1 + \delta$ :

$$\mathbb{E}[M([0, t])^q] \simeq C_q t^{\xi(q)} \quad \text{as } t \rightarrow 0,$$

for some positive constant  $C_q$ .

Finally, Theorem 2 has the following consequence about the regularity of good lognormal  $\star$ -scale invariant measures:

**Corollary 3.** *Almost surely, the measure  $M$  does not possess any atom on  $\mathbb{R}$ , that is:*

$$\text{almost surely, } \forall x \in \mathbb{R}, \quad M(\{x\}) = 0.$$

Now we investigate the long-range independence for good lognormal  $\star$ -scale invariant measures. So we introduce the related notion of cut-off:

**Definition 4.** *We will say that a stationary random measure  $M$  admits a cut-off  $d > 0$  if, for  $t < s$ , the  $\sigma$ -algebras  $\mathcal{H}_{-\infty}^t = \sigma\{M(A); A \in \mathbb{B}(\mathbb{R}), A \subset (-\infty, t]\}$  and  $\mathcal{H}_s^{+\infty} = \sigma\{M(A); A \in \mathbb{B}(\mathbb{R}), A \subset [s, +\infty)\}$  are independent, conditionally to the asymptotic  $\sigma$ -algebra of  $M$ , as soon as  $s - t > d$ .*

Of course, if the measure  $M$  is ergodic then the asymptotic  $\sigma$ -algebra of  $M$  is trivial and we can remove the sentence "conditionally to the asymptotic  $\sigma$ -algebra of  $M$ " from the definition. For instance the measure constructed in (8) admits a cut-off  $T$  and is ergodic. It results from the proof of Theorem 2 that the cut-off property can be read off the cascading rule (5):

**Proposition 5.** *Let  $M$  be a good lognormal  $\star$ -scale invariant random measure with finite  $1 + \delta$  moment. Then  $M$  admits a cutoff if and only if, for some  $T' < T$  (or equivalently for all  $T' < T$ ), the covariance kernel  $k_{T'}$  of the process  $\omega_{T'}$  in (5) reduces to 0 outside a compact set.*

Finally, we mention that another notion of exact stochastic scale invariance is given in [1, 5, 19]. More precisely, if the Gaussian multiplicative chaos  $M$  admits a covariance kernel  $K$  such that  $K(x) = \lambda^2 \ln \frac{T}{|x|} + C$  for some constant  $C$  and for all  $x$  in a ball  $B(0, R)$  then  $M$  satisfies the "exact stochastic scale invariance":

$$\forall \alpha \in (0, 1), \quad (M(\alpha A))_{A \subset B(0, R)} \stackrel{law}{=} \alpha^d e^{Y_\alpha - \frac{1}{2}\mathbb{E}[Y_\alpha^2]} (M(A))_{A \subset B(0, R)}$$

where  $Y_\alpha$  is a centered Gaussian random variable with variance  $\lambda^2 \ln \frac{1}{\alpha}$ . The reader may wonder if we can construct random measures that are both exactly stochastically scale invariant and good lognormal  $\star$ -scale invariant. Theorem 2 shows that this is impossible because, otherwise, the kernel  $k$  in (10) must be constant in the vicinity of 0 and therefore constant on the whole space  $\mathbb{R}$ . This is impossible because  $K$  would then be infinite or because the function  $\ln \frac{T}{|x|}$  is not positive definite.

### 3.3 Multidimensional results

We stress that our results remain true in higher dimensions without changes in the proofs. For the sake of completeness, we state the main result:

**Theorem 6.** *Let  $M$  be a good lognormal  $\star$ -scale invariant random measure such that for each  $T' < T$ , the covariance kernel  $k_{T'}$  of the process  $\omega_{T'}$  is differentiable. Assume that*

$$\mathbb{E}[M([0, 1])^{1+\delta}] < +\infty$$

for some  $\delta > 0$ . Then  $M$  is the product of a nonnegative random variable  $Y \in L^{1+\delta}$  and an independent Gaussian multiplicative chaos:

$$\forall A \subset \mathcal{B}(\mathbb{R}), \quad M(A) = Y \int_A e^{X_r - \frac{1}{2}\mathbb{E}[X_r^2]} dr \quad (11)$$

with associated covariance kernel given by the improper integral

$$\forall x \in \mathbb{R}^d \setminus \{0\}, \quad K(x) = \int_1^{+\infty} \frac{k(xu)}{u} du \quad (12)$$

for some continuous covariance function  $k$  such that  $k(0) \leq \frac{2d}{1+\delta}$ .

Conversely, given some data  $k$  and  $Y$  as above, the relation (9) defines a good lognormal  $\star$ -scale invariant random measure with finite moments of order  $1 + \gamma$  for every  $\gamma \in [0, \delta)$ .

In dimension greater than 1, it may be interesting to focus on the isotropy properties. In the same spirit as Proposition 5, for a good lognormal  $\star$ -scale invariant measure  $M$  with a finite moment of order  $1 + \delta$ , the following assertions are equivalent:

1.  $M$  is isotropic,
2. its covariance kernel  $K$  (or equivalently  $k$  in (12)) is isotropic,
3. the covariance kernel  $k_{T'}$  is isotropic for some  $T' < T$ ,
4. the covariance kernels  $k_{T'}$  are isotropic for all  $T' < T$ .

## 4. Proofs of the main theorem

This section is devoted to the proof of Theorem 2. For the sake of readability, some proofs of auxiliary results are gathered in the appendix.

In the following, we suppose, without loss of generality, that  $T = 1$ . Let  $M$  be a good lognormal scale invariant random measure defined on a probability space  $(\Omega, \mathcal{F}, \mathbb{P})$ . We introduce as usually the spaces  $L^p$  on  $(\Omega, \mathcal{F}, \mathbb{P})$  for  $1 \leq p \leq \infty$ . The measure  $M$  satisfies, for all  $\epsilon \in (0, 1)$

$$(M(A))_{A \in \mathcal{B}(\mathbb{R})} \stackrel{\text{law}}{=} \left( \int_A e^{\omega_\epsilon(r)} M^\epsilon(dr) \right)_{A \in \mathcal{B}(\mathbb{R})} \quad (13)$$

where  $\omega_\epsilon$  is a Gaussian process independent from  $M^\epsilon$ , with  $M^\epsilon(dr) = \epsilon M\left(\frac{dr}{\epsilon}\right)$  in law. We denote by  $k_\epsilon$  the covariation kernel of the process  $\omega_\epsilon$ . Furthermore, we assume that the measure  $M$  is non trivial ( $M \neq 0$ ) so that the process  $\omega_\epsilon$  is necessarily normalized, that is  $\mathbb{E}[e^{\omega_\epsilon}] = 1$ .

Now we introduce some definitions and tools that will be used throughout this section. For each  $\epsilon \in (0, 1)$ , define

$$\forall r \neq 0, \quad K^\epsilon(r) = \sum_{n=0}^{+\infty} k_\epsilon\left(\frac{r}{\epsilon^n}\right). \quad (14)$$

The pointwise convergence of the series is assured by (6). Actually, because of (6) again, the convergence is uniform on the sets  $\{z \in \mathbb{R}; |z| > \rho\}$  for any  $\rho > 0$ . Furthermore, (7) also ensures that  $K^\epsilon$  is Lipschitzian over each set  $\{z \in \mathbb{R}; |z| > \rho\}$  for any  $\rho > 0$ .

We let  $(X^n)_n$  denote a sequence of independent centered stationary Gaussian processes with respective covariance kernels

$$\mathbb{E}[X_r^n X_s^n] = k_\epsilon\left(\frac{r-s}{\epsilon^n}\right) \stackrel{\text{def}}{=} \bar{k}_n(r-s).$$

Clearly  $X^n$  depends on  $\epsilon$  but this parameter is omitted from the notations for the sake of readability. We assume that the whole sequence  $(X^n)_n$  and the measure  $M$  are constructed on the same probability space and are mutually independent. We further define the measure  $M^N$  for  $N \geq 0$  by

$$\forall A \in \mathcal{B}(\mathbb{R}), \quad M^N(A) = \epsilon^{N+1} M\left(\frac{1}{\epsilon^{N+1}} A\right).$$

By iterating the scale invariance relation (5), it is plain to see that, for each  $N \geq 0$ , the measure  $\widetilde{M}^N$  defined by

$$\widetilde{M}^N(A) = \int_A \exp\left(\sum_{n=0}^N X_r^n - \frac{1}{2} \mathbb{E}[(X_r^n)^2]\right) M^N(dr) \quad (15)$$

has the same law as the measure  $M$ .

## 4.1 Ergodic properties

First we investigate the immediate properties of  $M$  resulting from the definitions.

**Lemma 7.** *Let  $M$  be a stationary random measure on  $\mathbb{R}$  admitting a moment of order  $1 + \delta$ . There is a nonnegative integrable random variable  $Y \in L^{1+\delta}$  such that, for every bounded interval  $I \subset \mathbb{R}$ ,*

$$\lim_{T \rightarrow \infty} \frac{1}{T} M(TI) = Y|I| \quad \text{almost surely and in } L^{1+\delta},$$

where  $|\cdot|$  stands for the Lebesgue measure on  $\mathbb{R}$ . As a consequence, almost surely the random measure

$$A \in \mathcal{B}(\mathbb{R}) \mapsto \frac{1}{T} M(TA)$$

weakly converges towards  $Y|\cdot|$  and  $\mathbb{E}_Y[M(A)] = Y|A|$  ( $\mathbb{E}_Y[\cdot]$  denotes the conditional expectation with respect to  $Y$ ).

*Proof.* If  $M$  is a stationary random measure, the Birkhoff ergodic theorem implies the following convergence, for  $n \in \mathbb{N}, n \rightarrow \infty$ ,

$$\frac{1}{n} M([0, n]) = \frac{1}{n} \sum_{i=1}^n M([i-1, i]) \rightarrow Y \quad \text{almost surely and in } L^{1+\delta} \quad (16)$$

where  $Y \in L^{1+\delta}$  is a nonnegative random variable. Using monotonicity of the mapping  $t \mapsto M([0, t])$ , one can show that  $\frac{1}{T} M([0, T]) \rightarrow Y$  almost surely and in  $L^{1+\delta}$ . For  $a > 0, b > a$ , it is clear that  $\frac{1}{T} M(T[0, a]) \rightarrow aY$  and that  $\frac{1}{T} M(T[a, b]) \rightarrow (b-a)Y$  almost surely and in  $L^{1+\delta}$ . So, for every bounded interval  $I \subset \mathbb{R}_+$ , the following convergence holds  $\frac{1}{T} M(TI) \rightarrow |I|Y$  almost surely and in  $L^{1+\delta}$ . Along the same lines, one can show the same convergence for every bounded interval  $I \subset \mathbb{R}_-$  involving some nonnegative random variable  $Y' \in L^{1+\delta}$ . Stationarity implies that  $\frac{1}{T} M(T[-1, 1])$  has the same law as  $\frac{1}{T} M(T[0, 2])$ . By letting  $T$  go to  $\infty$ , we find that  $Y + Y'$  has the same law as  $2Y$ . Stationarity also implies that  $Y'$  has the same law as  $Y$ . Let  $0 < \alpha < 1$ . We prove

$$\mathbb{E}[Y^\alpha] = \mathbb{E}\left[\left(\frac{Y + Y'}{2}\right)^\alpha\right] \geq \frac{1}{2} (\mathbb{E}[Y^\alpha] + \mathbb{E}[Y'^\alpha]) = \mathbb{E}[Y^\alpha] \quad (17)$$



by using the Jensen inequality for the concave function  $x \mapsto x^\alpha$ . So the above inequality turns out to be an equality and thus  $Y = Y'$  almost surely. We have shown that  $\frac{1}{T}M(TI) \rightarrow |I|Y$  almost surely and in  $L^{1+\delta}$  when  $T \rightarrow \infty$  for every bounded interval  $I \subset \mathbb{R}$ .

Finally, by the portemanteau theorem, the convergence of the measure  $A \in \mathcal{B}(\mathbb{R}) \mapsto \frac{1}{T}M(TA)$  on the intervals towards  $Y|\cdot|$  is enough to ensure the weak convergence.  $\square$

## 4.2 Mixing properties

This section is devoted to the mixing properties of the measure  $M$ , which can be read off the structure of the kernel  $K^\epsilon$ .

We first draw attention to the following relation, which will be used throughout the paper:

$$\mathbb{E}_Y[F(M(A_1), \dots, M(A_n))] = \mathbb{E}_Y[F(\widetilde{M}^N(A_1), \dots, \widetilde{M}^N(A_n))] \quad a.s.$$

for every positive measurable function  $F : \mathbb{R}^n \rightarrow \mathbb{R}$ . The proof is deferred to the appendix A (see Lemma 20).

**Lemma 8.** *Let  $A, B$  be two disjoint sets such that  $\text{dist}(A, B) > 0$ . Then the random variable  $M(A)M(B)$  is integrable under  $\mathbb{E}_Y[\cdot]$  and*

$$\mathbb{E}_Y[M(A)M(B)] = Y^2 \int_{A \times B} e^{K^\epsilon(r-u)} dr du.$$

*Proof.* We fix  $R > 0$  and denote by  $\mathcal{G}$  the  $\sigma$ -field generated by  $M$ . Because the function  $x \in \mathbb{R}_+ \mapsto \min(R, x)$  is concave, we have

$$\begin{aligned} \mathbb{E}_Y[\min(R, M(A)M(B))] &= \mathbb{E}_Y[\min(R, \widetilde{M}^N(A)\widetilde{M}^N(B))] \\ &= \mathbb{E}_Y[\mathbb{E}[\min(R, \widetilde{M}^N(A)\widetilde{M}^N(B))|\mathcal{G}]] \\ &\leq \mathbb{E}_Y\left[\min\left(R, \mathbb{E}[\widetilde{M}^N(A)\widetilde{M}^N(B)|\mathcal{G}]\right)\right]. \end{aligned}$$

Since  $\widetilde{M}^N$  is given by (15), it is straightforward to compute:

$$\mathbb{E}[\widetilde{M}^N(A)\widetilde{M}^N(B)|\mathcal{G}] = \int_{A \times B} e^{\sum_{n=0}^N \bar{k}_n(r-u)} M^N(dr)M^N(du). \quad (18)$$

Because of the uniform convergence of the series  $(\sum_{n=0}^N \bar{k}_n(r-u))_N$  on the set  $\{(r, u) \in \mathbb{R}^2; |r-u| \geq d\}$  towards  $K^\epsilon$  and the weak convergence of the measure  $M^N$  towards  $Y|\cdot|$  (cf. Lemma 7), the random variable

$$\int_{A \times B} e^{\sum_{n=0}^N \bar{k}_n(r-u)} M^N(dr)M^N(du)$$

almost surely converges towards

$$Y^2 \int_{A \times B} e^{K^\epsilon(r-u)} dr du.$$

The dominated convergence theorem then yields:

$$\mathbb{E}_Y[\min(R, M(A)M(B))] \leq \mathbb{E}_Y\left[\min\left(R, Y^2 \int_{A \times B} e^{K^\epsilon(r-u)} dr du\right)\right].$$

By letting  $R \rightarrow \infty$ , the monotone convergence theorem yields

$$\mathbb{E}_Y[M(A)M(B)] \leq Y^2 \int_{A \times B} e^{K^\epsilon(r-u)} dr du.$$

On the other hand, we also have

$$\mathbb{E}_Y [M(A)M(B)] = \mathbb{E}_Y [\widetilde{M}^N(A)\widetilde{M}^N(B)] = \mathbb{E}_Y [\mathbb{E}[\widetilde{M}^N(A)\widetilde{M}^N(B)|\mathcal{G}]]. \quad (19)$$

By gathering (18) and (19) and by using the Fatou's lemma, we deduce

$$\mathbb{E}_Y [M(A)M(B)] \geq Y^2 \int_{A \times B} e^{K^\epsilon(r-u)} dr du.$$

This completes the proof.  $\square$

**Lemma 9.** *We have*

$$\sup_{|r| \geq d} |K^\epsilon(r)| \rightarrow 0 \quad \text{as } d \rightarrow \infty.$$

*Proof.* From (6), we have for  $r \geq 1$ :

$$|K^\epsilon(r)| \leq \sum_{n=0}^{+\infty} \frac{C_\epsilon}{1 + |\ln r + n \ln \frac{1}{\epsilon}|^{1+\alpha}}.$$

Hence the result is an easy consequence of the Lebesgue dominated convergence theorem. Details are left to the reader.  $\square$

**Proposition 10.** *The measure  $M$  possesses the following mixing property: given two disjoint sets  $A, B$  such that  $\text{dist}(A, B) = d > 0$  we have:*

$$|\mathbb{E}_Y [M(A)M(B)] - Y^2|A||B|| \leq Y^2 \xi(d)|A||B| \quad (20)$$

for some function  $\xi : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  such that  $\lim_{d \rightarrow \infty} \xi(d) = 0$ .

As a consequence, for any Lebesgue integrable function  $\phi$  on  $\mathbb{R}^2$  and  $d > 0$ , we have:

$$\left| \mathbb{E}_Y \left[ \int_{|u-r|>d} \phi(u, r) M(dr) M(du) \right] - Y^2 \int_{|u-r|>d} \phi(u, r) du dr \right| \leq Y^2 \xi(d) \int_{|u-r|>d} |\phi(u, r)| du dr. \quad (21)$$

*Proof.* From Lemma 8, we have

$$\begin{aligned} |\mathbb{E}_Y [M(A)M(B)] - Y^2|A||B|| &= Y^2 \int_{A \times B} (e^{K^\epsilon(r-u)} - 1) dr du \\ &\leq Y^2 \varepsilon(d) |A||B| \end{aligned}$$

where we have set  $\xi(d) = \sup_{|r| \geq d} |e^{K^\epsilon(r)} - 1|$ . From Lemma 9, we have  $\lim_{d \rightarrow \infty} \xi(d) = 0$ . It is then plain to derive (21).  $\square$

As a direct consequence, we obtain:

**Corollary 11.** *For any Lebesgue integrable function  $\phi$  on  $\mathbb{R}^2$  and  $d > 0$ , we have for all  $N \in \mathbb{N} \setminus \{0\}$ :*

$$\begin{aligned} \left| \int_{|u-r|>d} \phi(u, r) \mathbb{E}_Y [M^N(dr) M^N(du)] - Y^2 \int_{|u-r|>d} \phi(u, r) du dr \right| \\ \leq Y^2 \xi\left(\frac{d}{\epsilon^N}\right) \int_{|u-r|>d} |\phi(u, r)| du dr. \end{aligned}$$

### 4.3 Characterization of the measure $M$

Having in mind that the measure  $M^N$  weakly converges towards  $Y|\cdot|$  as  $N$  goes to infinity, it is very tantalizing to think that the solution of our problem reduces to taking the limit in (15) as  $N \rightarrow \infty$ . However, multiplicative chaos badly behaves with respect to weak convergence of measures. So we want to get rid of the measure  $M^N$  and have the Lebesgue measure instead in order to deal with a multiplicative chaos in the sense of Kahane. This is the main difficulty of the proof. For that purpose, it is appropriate to take the conditional expectation of  $\widetilde{M}^N$  with respect to the  $\sigma$ -algebra  $\mathcal{F}_N = \sigma(X^0, \dots, X^N, Y)$ . Therefore, for any Borelian subset  $A$  of  $\mathbb{R}$ , we define

$$G_N(A) = \mathbb{E}[\widetilde{M}^N(A)|\mathcal{F}_N]$$

and we claim

**Lemma 12.** *The following relation holds for each  $N \geq 0$ :*

$$G_N(A) = Y \int_A \exp\left(\sum_{n=0}^N X_r^n - \frac{1}{2}\mathbb{E}[(X_r^n)^2]\right) dr. \quad (22)$$

Furthermore, for each bounded Borelian set  $A$ , the sequence  $(G_N(A))_N$  is a positive martingale bounded in  $L^{1+\delta}$ .

*Proof.* If  $A$  has infinite Lebesgue measure, both sides of (22) are infinite. So we focus on the case when  $A$  has finite Lebesgue measure. First observe that for each  $s < t$  and  $A \in \mathcal{F}_N$ , we have from Lemma 7

$$\mathbb{E}\left[\int_{\mathbb{R}} \mathbf{1}_{[s,t]}(r) \mathbf{1}_A M^N(dr) | \mathcal{F}_N\right] = \mathbf{1}_A \mathbb{E}_Y[M^N([s,t])] = \mathbf{1}_A Y(t-s).$$

By using density arguments and Fatou's lemma, we establish that, for each positive  $\mathcal{F}_N \otimes \mathcal{B}(\mathbb{R})$ -measurable function  $\varphi \in L^1(\Omega \times \mathbb{R}; \mathbb{P} \otimes dt)$ , we have

$$\mathbb{E}\left[\int_{\mathbb{R}} \varphi(\omega, r) M^N(dr) | \mathcal{F}_N\right] = \int_{\mathbb{R}} \varphi(\omega, r) Y dr.$$

So (22) is proved. Finally, for each bounded set  $A$  we have  $\mathbb{E}[M(A)^{1+\delta}] < +\infty$  for some  $\delta > 0$ . The Jensen inequality then yields

$$\mathbb{E}[(G_N(A))^{1+\delta}] = \mathbb{E}[(\mathbb{E}[\widetilde{M}^N(A)|\mathcal{F}_N])^{1+\delta}] \leq \mathbb{E}[(\widetilde{M}^N(A))^{1+\delta}] = \mathbb{E}[M(A)^{1+\delta}] < +\infty.$$

The martingale  $(G_N(A))_N$  is thus bounded in  $L^{1+\delta}$ .  $\square$

Being bounded in  $L^{1+\delta}$ , the martingale converges almost surely and in  $L^{1+\delta}$  towards a random variable  $Q(A)$ , which can be formally thought of as

$$Q(A) = Y \int_A \exp\left(X_r - \frac{1}{2}\mathbb{E}[X_r^2]\right) dr$$

where  $(X_r)_{r \in \mathbb{R}}$  is a "Gaussian process" with covariance kernel  $K^\epsilon(r)$ , that is a Gaussian multiplicative chaos. The remaining part of our argument can be roughly summed up as follows. First, we obtain estimates on the kernel  $K^\epsilon$  derived from the fact that the Gaussian multiplicative chaos  $Q$  admits a moment of order  $1 + \delta$ . Second, we use these estimates to prove that  $Q$  has the same law as  $M$ . Finally, since  $Q$  has the same law as  $M$ , which does not depend on  $\epsilon$ , the kernel  $K^\epsilon$  should not depend on  $\epsilon$  either. This is a strong constraint on  $K^\epsilon$ , from which we derive the specific structure of  $K^\epsilon$  given by (10).

So we claim

**Proposition 13.** *For each  $0 < \gamma < \delta$ , we can find  $\rho > 0$  such that:*

$$\sup_n n^{1+\rho} \mathbb{E}[M([0, \frac{1}{n}])^{1+\gamma}] < +\infty. \quad (23)$$

*Proof.* Since the martingale  $(G_N(A))_N$  is bounded in  $L^{1+\delta}$ , the Gaussian multiplicative chaos  $Q(A) = Y \int_A \exp(X_r - \frac{1}{2}\mathbb{E}[X_r^2]) dr$  with associated kernel  $K^\epsilon$  admits a moment of order  $1 + \delta$ . Then we prove that  $K^\epsilon$  can be decomposed as (see the proof in the appendix A):

**Lemma 14.** *The kernel  $K^\epsilon$  can be written as:*

$$K^\epsilon(r) = \frac{\ln_+ r}{\ln \epsilon} k_\epsilon(0) + g_\epsilon(r) \quad (24)$$

for some function  $g_\epsilon$  that is continuous and bounded on  $\mathbb{R} \setminus \{0\}$ .

Multiplicative chaos with such covariance kernels have been intensively studied. For instance in [13], it is proved that the condition of moment of order  $1+\delta$  implies the relation  $k_\epsilon(0) < \frac{2}{1+\delta} \ln \frac{1}{\epsilon}$  via a rather complicated connexion with discrete Mandelbrot's cascades. For the sake of completeness, we give a direct proof in appendix A:

**Lemma 15.** *The existence of a moment of order  $1 + \delta$  for the multiplicative chaos with associated kernel  $K^\epsilon$  implies the following bound:*

$$k_\epsilon(0) \leq \frac{2}{1+\delta} \ln \frac{1}{\epsilon}.$$

Since we have for all  $r \in \mathbb{R}$ :  $k_\epsilon(r) \leq k_\epsilon(0)$ , the covariance kernel of the process  $\omega_\epsilon$  is dominated by that of the constant process  $\omega_\epsilon(0)$ . Hence, by using (5) and Lemma 21, it is plain to see that, for each  $\gamma > 0$ :

$$\begin{aligned} \mathbb{E}[M([0, \frac{1}{n}])^{1+\gamma}] &= \mathbb{E} \left[ \left( \int_0^{1/n} e^{\omega_{1/n}(r)} M^{1/n}(dr) \right)^{1+\gamma} \right] \\ &\leq \mathbb{E} \left[ \left( \int_0^{1/n} e^{\omega_{1/n}(0)} M^{1/n}(dr) \right)^{1+\gamma} \right] \\ &\leq \mathbb{E} \left[ e^{(1+\gamma)\omega_{1/n}(0)} \right] \mathbb{E} \left[ \left( M^{1/n}([0, \frac{1}{n}]) \right)^{1+\gamma} \right] \\ &= e^{\frac{(1+\gamma)^2}{2} k_{1/n}(0) - \frac{1+\gamma}{2} k_{1/n}(0)} \mathbb{E} \left[ \left( M([0, 1]) \right)^{1+\gamma} \right] \frac{1}{n^{1+\gamma}}. \end{aligned}$$

Since  $k_{1/n}(0) \leq \frac{2}{1+\delta} \ln n$ , we deduce

$$\begin{aligned} \mathbb{E}[M([0, \frac{1}{n}])^{1+\gamma}] &\leq e^{(\frac{\gamma^2 + \gamma}{1+\delta} - \gamma - 1) \ln n} \mathbb{E} \left[ \left( M([0, 1]) \right)^{1+\gamma} \right] \\ &= \frac{1}{n^{1+\rho}} \mathbb{E} \left[ \left( M([0, 1]) \right)^{1+\gamma} \right] \end{aligned}$$

where we have set

$$\rho \stackrel{def}{=} -\frac{\gamma^2 + \gamma}{1 + \delta} + \gamma.$$

Clearly, we have  $\rho > 0$  provided that  $0 < \gamma < \delta$ . The proof of Proposition 13 is complete.  $\square$

We are now in position to tackle the main step of the proof:

**Proposition 16.** *The random measures  $(Q(A))_{A \in \mathcal{B}(\mathbb{R})}$  and  $(M(A))_{A \in \mathcal{B}(\mathbb{R})}$  have the same law.*

*Proof.* Let  $F$  be some function defined on  $\mathbb{R}_+$  such that:

- $F$  is convex,
- $F(x) \leq Cx^{1+\gamma}$  for some constants  $C > 0$  and  $0 < \gamma < \delta$ ,
- $F \circ \sqrt{\cdot}$  is concave, nondecreasing and sub-additive.

Let  $f$  be a lower semi-continuous positive function on  $\mathbb{R}$  with compact support. We have by Jensen's inequality:

$$\begin{aligned} \mathbb{E}\left[F\left(\int_{\mathbb{R}} f(x) M(dx)\right)\right] &= \mathbb{E}\left[F\left(\int_{\mathbb{R}} f(x) \widetilde{M}^N(dx)\right)\right] \\ &= \mathbb{E}\left[\mathbb{E}\left[F\left(\int_{\mathbb{R}} f(x) \widetilde{M}^N(dx)\right) \middle| \mathcal{F}_N\right]\right] \\ &\geq \mathbb{E}\left[F\left(\int_{\mathbb{R}} f(x) G_N(dx)\right)\right]. \end{aligned}$$

We let  $N$  go to  $+\infty$ . By using the weak convergence of  $G_N(dr)$  towards  $Q(dr)$ , we obtain:

$$\mathbb{E}\left[F\left(\int_{\mathbb{R}} f(r) M(dr)\right)\right] \geq \mathbb{E}\left[F\left(\int_{\mathbb{R}} f(r) Q(dr)\right)\right]. \quad (25)$$

Now we want to establish the converse inequality. We set  $\widetilde{F} = F \circ \sqrt{\cdot}$ . For any  $\tau > 0$ , we have by using the sub-additivity of  $\widetilde{F}$ :

$$\begin{aligned} \mathbb{E}\left[F\left(\int_{\mathbb{R}} f(r) M(dr)\right)\right] &= \mathbb{E}\left[\widetilde{F}\left(\left(\int_{\mathbb{R}} f(r) \widetilde{M}^N(dr)\right)^2\right)\right] \\ &= \mathbb{E}\left[\widetilde{F}\left(\int_{\mathbb{R}} \int_{\mathbb{R}} f(r) f(u) \widetilde{M}^N(dr) \widetilde{M}^N(du)\right)\right] \\ &\leq \mathbb{E}\left[\widetilde{F}\left(\int_{|r-u| \leq \tau} f(r) f(u) \widetilde{M}^N(dr) \widetilde{M}^N(du)\right)\right] \\ &\quad + \mathbb{E}\left[\widetilde{F}\left(\int_{|r-u| > \tau} f(r) f(u) \widetilde{M}^N(dr) \widetilde{M}^N(du)\right)\right]. \end{aligned}$$

Then, by conditioning with respect to  $\mathcal{F}_N$  and by using the Jensen inequality in the second term of the latter inequality, we deduce:

$$\mathbb{E}\left[F\left(\int_{\mathbb{R}} f(r) M(dr)\right)\right] \quad (26)$$

$$\begin{aligned} &\leq \mathbb{E}\left[\widetilde{F}\left(\int_{|r-u| \leq \tau} f(r) f(u) \widetilde{M}^N(dr) \widetilde{M}^N(du)\right)\right] \\ &\quad + \mathbb{E}\left[\widetilde{F}\left(\int_{|r-u| > \tau} f(r) f(u) \exp\left(\sum_{k=0}^N X_r^k + X_u^k - k_n(0)\right) \mathbb{E}_Y[M^N(dr) M^N(du)]\right)\right] \\ &\stackrel{def}{=} C(1, \tau, N) + C(2, \tau, N). \end{aligned} \quad (27)$$

We claim:

**Lemma 17.** *For each fixed  $\tau > 0$ ,  $C(2, \tau, N)$  converges as  $N \rightarrow \infty$  towards*

$$\mathbb{E}\left[\widetilde{F}\left(\int_{|r-u| > \tau} f(r) f(u) Q(dr) Q(du)\right)\right].$$

Furthermore, this latter quantity converges, as  $\tau \rightarrow 0$ , towards

$$\mathbb{E}\left[F\left(\int f(r) Q(dr)\right)\right].$$

Finally, the quantity  $C(1, \tau, N)$  converges to 0 as  $\tau \rightarrow 0$  uniformly with respect to  $N \in \mathbb{N}^*$ .

Let us admit for a while the above lemma to finish the proof of Proposition 16. By gathering (27) and Lemma 17, we deduce

$$\begin{aligned} \mathbb{E}\left[F\left(\int_{\mathbb{R}} f(r) M(dr)\right)\right] &\leq \liminf_{\tau \rightarrow 0} \mathbb{E}\left[\tilde{F}\left(\int_{|r-u|>\tau} f(r)f(u) Q(dr)Q(du)\right)\right] \\ &= \mathbb{E}\left[F\left(\int f(r) Q(dr)\right)\right]. \end{aligned}$$

Hence we have proved

$$\mathbb{E}\left[F\left(\int_{\mathbb{R}} f(r) M(dr)\right)\right] = \mathbb{E}\left[F\left(\int_{\mathbb{R}} f(r) Q(dr)\right)\right]. \quad (28)$$

The basic choice for  $F$  is the function  $x \mapsto x^{1+\gamma}$  with  $0 < \gamma < \delta$ . Thus we have proved that the mappings

$$\mathbb{E}\left[\exp\left(z \ln \int_{\mathbb{R}} f(r) M(dr)\right)\right] \quad \text{and} \quad \mathbb{E}\left[\exp\left(z \int_{\mathbb{R}} f(r) Q(dr)\right)\right]$$

coincide for  $z \in ]1, 1+\delta[$ . By analyticity arguments, we deduce that  $\int_{\mathbb{R}} f(x) M(dx)$  and  $\int_{\mathbb{R}} f(x) Q(dx)$  have the same law. This is enough to prove that the random measures  $M$  and  $Q$  have the same law. Indeed, if we consider two families  $(\lambda_i)_{1 \leq i \leq n}$  of positive real numbers and  $(A_i)_{1 \leq i \leq n}$  of bounded open subsets of  $\mathbb{R}$ , we define the lower semi-continuous function

$$f(x) = \sum_{i=1}^n \lambda_i \mathbf{1}_{A_i}(x)$$

and we obtain

$$\sum_{i=1}^n \lambda_i M(A_i) \stackrel{\text{law}}{=} \sum_{i=1}^n \lambda_i Q(A_i).$$

It turns out that the law of a random vector  $(Y_1, \dots, Y_n)$  made up of positive random variables is characterized by the combinations

$$\sum_{i=1}^n \lambda_i Y_i$$

where  $(\lambda_i)_{1 \leq i \leq n}$  is a family of positive real numbers. The proof of Proposition 16 is complete.  $\square$

*Proof of Lemma 17.* Let us first investigate the quantity  $C(1, \tau, N)$ . Assume the function  $f$  has its support included in the ball  $B(0, R)$  for some  $R > 0$ . We can cover the set

$$\{(x, y) \in \mathbb{R}^2; |x - y| \leq \tau \text{ and } \max(|x|, |y|) \leq R\}$$

by the squares

$$A_j^n = [t_j^n, t_{j+2}^n] \times [t_j^n, t_{j+2}^n] \quad \text{where } t_j^n = -R + 2\tau j, \quad \text{for } j = 0, \dots, E\left(\frac{R}{\tau}\right).$$

We set  $S = \sup_{\mathbb{R}} f$ . Because  $\tilde{F}$  is sub-additive and increasing, we have:

$$\begin{aligned}
C(1, \tau, N) &\leq \mathbb{E} \left[ \tilde{F} \left( \sum_{0 \leq j \leq E(\frac{R}{\tau})} \int_{A_j^n} f(r) f(u) \tilde{M}^N(dr) \tilde{M}^N(du) \right) \right] \\
&\leq \sum_{0 \leq j \leq E(\frac{R}{\tau})} \mathbb{E} \left[ \tilde{F} \left( \int_{A_j^n} f(r) f(u) \tilde{M}^N(dr) \tilde{M}^N(du) \right) \right] \\
&\leq \sum_{0 \leq j \leq E(\frac{R}{\tau})} \mathbb{E} \left[ \tilde{F} \left( S^2 \int_{A_j^n} \tilde{M}^N(dr) \tilde{M}^N(du) \right) \right] \\
&= \sum_{0 \leq j \leq E(\frac{R}{\tau})} \mathbb{E} \left[ \tilde{F} \left( S^2 (\tilde{M}^N([t_j^n, t_{j+2}^n]))^2 \right) \right] \\
&= \sum_{0 \leq j \leq E(\frac{R}{\tau})} \mathbb{E} \left[ F \left( SM([t_j^n, t_{j+2}^n]) \right) \right].
\end{aligned}$$

By stationarity, we deduce

$$\begin{aligned}
C(1, \tau, N) &\leq \frac{2R}{\tau} \mathbb{E} \left[ F \left( SM([0, 2\tau]) \right) \right] \\
&\leq \frac{2R}{\tau} S^{1+\gamma} \mathbb{E} \left[ M([0, 2\tau])^{1+\gamma} \right].
\end{aligned}$$

It results from Proposition 13 that the last quantity converges towards 0 as  $\tau$  goes to 0 uniformly with respect to  $N$ .

Now we investigate the quantity  $C(2, \tau, N)$ . Since  $\tilde{F}$  is sub-additive and increasing, we have  $|\tilde{F}(a) - \tilde{F}(b)| \leq \tilde{F}(|b - a|)$  for all positive real numbers  $a, b$ . This together with Corollary 11 yields

$$\begin{aligned}
&\left| C(2, \tau, N) - \mathbb{E} \left[ \tilde{F} \left( \int_{|r-u|>\tau} f(r) f(u) \exp \left( \sum_{k=0}^N X_r^n + X_u^n - k_n(0) \right) Y^2 dr du \right) \right] \right| \\
&\leq \mathbb{E} \left[ \tilde{F} \left( Y^2 \xi \left( \frac{\tau}{\epsilon^N} \right) \int_{|r-u|>\tau} f(r) f(u) \exp \left( \sum_{k=0}^N X_r^n + X_u^n - k_n(0) \right) dr du \right) \right] \\
&\leq \mathbb{E} \left[ \tilde{F} \left( \xi \left( \frac{\tau}{\epsilon^N} \right) S^2 G_N([-R, R])^2 \right) \right] \\
&\leq \mathbb{E} \left[ F \left( S \xi \left( \frac{\tau}{\epsilon^N} \right)^{1/2} G_N([-R, R]) \right) \right] \\
&\leq \xi \left( \frac{\tau}{\epsilon^N} \right)^{\frac{1+\gamma}{2}} S^{1+\gamma} \mathbb{E} \left[ G_N([-R, R])^{1+\gamma} \right].
\end{aligned}$$

Obviously, the last quantity converges to 0 as  $N$  goes to  $\infty$ . Furthermore, the quantity

$$\tilde{F} \left( \int_{|r-u|>\tau} f(r) f(u) \exp \left( \sum_{k=0}^N X_r^n + X_u^n - k_n(0) \right) Y^2 dr du \right)$$

almost surely converges towards

$$\tilde{F} \left( \int_{|r-u|>\tau} f(r) f(u) Q(dr) Q(du) \right)$$

and is uniformly integrable because  $F(x) \leq Cx^{1+\gamma}$  and  $Q$  is a multiplicative chaos admitting a

moment of order  $1 + \delta$  with  $\delta > \gamma$ . The Lebesgue convergence theorem then yields:

$$\begin{aligned} \mathbb{E} \left[ \tilde{F} \left( \int_{|r-u|>\tau} f(r)f(u) \exp \left( \sum_{k=0}^N X_r^k + X_u^k - k_n(0) \right) Y^2 dr du \right) \right] \\ \rightarrow \mathbb{E} \left[ \tilde{F} \left( \int_{|r-u|>\tau} f(r)f(u) Q(dr) Q(du) \right) \right] \quad \text{as } N \rightarrow \infty. \end{aligned}$$

Gathering the above relations yields

$$C(2, \tau, N) \rightarrow \mathbb{E} \left[ \tilde{F} \left( \int_{|r-u|>\tau} f(r)f(u) Q(dr) Q(du) \right) \right] \quad \text{as } N \rightarrow \infty.$$

Similar arguments as those used above allow to establish that

$$\begin{aligned} \liminf_{\tau \rightarrow 0} \mathbb{E} \left[ \tilde{F} \left( \int_{|r-u|>\tau} f(r)f(u) Q(dr) Q(du) \right) \right] &= \mathbb{E} \left[ \tilde{F} \left( \int_{\mathbb{R}^2} f(r)f(u) Q(dr) Q(du) \right) \right] \\ &= \mathbb{E} \left[ F \left( \int_{\mathbb{R}} f(r) Q(dr) \right) \right]. \end{aligned}$$

Indeed, by proceeding as for  $C(1, \tau, N)$ , we can prove that the "diagonal contribution" goes to 0 as  $\tau \rightarrow 0$ . Details are left to the reader. The proof of the Lemma is complete.  $\square$

The final step of our argument is now to prove that the kernel  $K^\epsilon$  defined by (14) does not depend on  $\epsilon$ . Expressing the kernel  $K^\epsilon$  as a function of the marginals of the measure  $M$  is enough for that purpose. So we remind the reader of Lemma 8, which states

$$\mathbb{E}_Y [M(A)M(B)] = Y^2 \int_{A \times B} e^{K^\epsilon(r-u)} dr du.$$

We deduce that, for any  $s \neq 0$  and on the set  $\{Y > 0\}$ ,

$$K^\epsilon(s) = \lim_{h \rightarrow 0} \ln \left( \frac{1}{h^2} \mathbb{E}_Y [M([0, h])M([s, s+h])] \right) - 2 \ln Y. \quad (29)$$

As a straightforward consequence, the kernel  $K^\epsilon$  defined by (14) does not depend on  $\epsilon$  since the left-hand side in (29) does not either. So we can define the quantity

$$\forall r \neq 0, \quad K(r) = K^\epsilon(r)$$

for some  $\epsilon \in (0, 1)$  and this relation is also valid for any  $\epsilon \in (0, 1)$ . It is also plain to see that for each  $\epsilon \in (0, 1)$  we have:

$$\forall r \neq 0, \quad K(r) = k_\epsilon(r) + K\left(\frac{r}{\epsilon}\right) \quad (30)$$

since  $K^\epsilon$  satisfies such a relation. Such a specific functional equation implies a precise structure for the function  $K$ :

**Proposition 18.** *For  $r > 0$ , we have*

$$K(r) = \int_r^{+\infty} \frac{k(u)}{u} du \quad (31)$$

where  $k(u)$  is a continuous function  $\mathbb{R}_+ \rightarrow \mathbb{R}$ .

*Proof.* Because  $K$  is Lipschitzian on the compact subsets of  $\mathbb{R} \setminus \{0\}$ , there exists a locally bounded measurable function  $f$  on  $(0; +\infty)$  such that for all  $r, s > 0$ ,

$$K(s) - K(r) = \int_r^s f(t) dt.$$



Define, for  $r \in \mathbb{R}$ ,

$$\phi(r) = K(e^r)$$

It is straightforward to derive from (30) that, for all  $r \in \mathbb{R}, \alpha \geq 0$ ,

$$\phi(r + \alpha) - \phi(r) = -k_{e^{-\alpha}}(e^r) \quad (32)$$

Note that  $k_1(e^r) = 0$ . From equation (32), one obtains :

$$\frac{1}{\alpha} \int_r^{r+\alpha} e^u f(e^u) du = -\frac{k_{e^{-\alpha}}(e^r)}{\alpha} \quad (33)$$

For almost every  $r$ , the left-hand side of equation (33) tends to  $e^r f(e^r)$  when  $\alpha$  goes to 0. Thus, the right-hand side of (33) converges also for almost every  $r$  to  $e^r f(e^r)$  when  $\alpha$  goes to 0. We define the function  $g$  by the following limit for almost every  $r$ :

$$g(r) = \lim_{\alpha \rightarrow 0} -\frac{1}{\alpha} \int_r^{r+\alpha} e^u f(e^u) du = \lim_{\alpha \rightarrow 0} \frac{k_{e^{-\alpha}}(e^r)}{\alpha} \quad (34)$$

As defined, the function  $g$  is measurable with respect to the Borelian  $\sigma$ -field of  $\mathbb{R}$ . For almost every  $x \in (0, +\infty)$ , define

$$h(x) = g(\ln(x)),$$

and  $h(0)$  by  $h(0) = \frac{k_{e^{-\alpha}}(0)}{\alpha}$  for some  $\alpha > 0$ . Note that the definition of  $h(0)$  does not depend on  $\alpha$  in view of lemma 14 because we have

$$K(r) \simeq \frac{k_\epsilon(0)}{\ln \epsilon} \ln r \quad \text{as } r \rightarrow 0.$$

**Lemma 19.** *The function  $h(|\cdot|)$  is positive definite (as a tempered distribution in the sense of Schwartz, see [12] or [21]). One can also find a symmetric positive measure  $\mu$  on  $\mathbb{R}$  (with  $\mu(\mathbb{R}) < \infty$ ) such that for almost every  $x \in \mathbb{R}$ :*

$$h(|x|) = \int_{\mathbb{R}} e^{ix\xi} \mu(d\xi)$$

*Proof.* For almost every  $x \in \mathbb{R}$ ,  $h(|x|) = \lim_{\alpha \rightarrow 0} \frac{k_{e^{-\alpha}}(|x|)}{\alpha}$  and  $\frac{k_{e^{-\alpha}}(|x|)}{\alpha} \leq h(0)$  uniformly in  $\alpha$ . Thus, if  $\varphi$  is a smooth function with compact support, we get using the dominated convergence theorem:

$$\int_{\mathbb{R}} \int_{\mathbb{R}} h(|y-x|) \varphi(x) \overline{\varphi}(y) dx dx = \lim_{\alpha \rightarrow 0} \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{k_{e^{-\alpha}}(|y-x|)}{\alpha} \varphi(x) \overline{\varphi}(y) dx dx \geq 0.$$

We conclude that  $h(|\cdot|)$  is positive definite. By the Bochner-Schwartz theorem, the Fourier transform of  $h(|\cdot|)$  is a symmetric positive measure  $\mu(d\xi)$  such that there exists  $p \geq 0$  with:

$$\int_{\mathbb{R}} \frac{\mu(d\xi)}{(1+|\xi|)^p} < \infty.$$

In order to conclude, it is sufficient to prove that  $\mu(\mathbb{R}) < \infty$ . We note  $\theta(x) = \frac{e^{-x^2/2}}{\sqrt{2\pi}}$  and  $\theta^\epsilon = \frac{1}{\epsilon} \theta(\cdot/\epsilon)$  for  $\epsilon > 0$ . By the inverse Fourier theorem, we get:

$$(\theta^\epsilon * h)(0) = \int_{\mathbb{R}} e^{-\epsilon^2 \xi^2 / 2} \mu(d\xi).$$

Thus the right hand side of the above equality is bounded by  $h(0)$  and we conclude by letting  $\epsilon$  go to 0.  $\square$

Integrating with respect to the Lebesgue measure the relation  $g(t) = -e^t f(e^t)$  which is true for almost every  $t \in \mathbb{R}$ , one gets

$$K(s) - K(r) = - \int_r^s \frac{h(u)}{u} du.$$

Because  $K(s) \rightarrow 0$  as  $s \rightarrow +\infty$ , the function  $u \mapsto \frac{h(u)}{u}$  is integrable at the vicinity of  $+\infty$  in the generalized sense. We deduce:

$$K(r) = \int_r^{+\infty} \frac{h(u)}{u} du.$$

By the previous lemma, there exists a finite symmetric positive measure  $\mu$  on  $\mathbb{R}$  such that, for almost every  $x \in \mathbb{R}$ ,

$$h(x) = \int_{\mathbb{R}} e^{ix\xi} \mu(d\xi)$$

For simplicity, define for all  $x \in \mathbb{R}$ ,  $k(x) = \int_{\mathbb{R}} e^{ix\xi} \mu(d\xi)$ . The function  $k$  is continuous on  $\mathbb{R}$ . We get finally,

$$K(r) = \int_r^{+\infty} \frac{k(u)}{u} du. \quad (35)$$

The proof of Proposition 18 is complete.  $\square$

*Proof of Proposition 5.* This is just a direct consequence of Theorem 2 and equation (29).  $\square$

#### 4.4 Construction of lognormal $\star$ -scale invariant random measures

We are given a positive random variable  $Y \in L^{1+\delta}$  (for some  $\delta > 0$ ) and a continuous covariation kernel  $k$  such that  $k(0) < 2$ . Let  $F$  be the (symmetric) spectral measure associated to  $K$ , that is

$$k(t) = \int_{\mathbb{R}} e^{i\lambda t} F(d\lambda),$$

and we assume that the improper integral

$$K(r) = \int_r^{+\infty} \frac{k(u)}{u} du$$

converges for  $r > 0$ .

We choose a constructive approach. Let  $\mu, \nu$  be two i.i.d. independently scattered Gaussian measures (independent from  $Y$ ) distributed on the half plane  $\mathbb{R} \times \mathbb{R}_+^*$  such that:

$$\forall A \in \mathcal{B}(\mathbb{R} \times \mathbb{R}_+^*), \quad \mathbb{E}[e^{q\mu(A)}] = e^{\frac{1}{2}q^2\theta(A)}$$

where

$$\theta(A) = \int_{\lambda \in \mathbb{R}} \int_{y \in \mathbb{R}_+^*} \mathbf{1}_A(\lambda, y) \frac{1}{y} dy F(d\lambda).$$

Fix  $T > 0$  and  $T' < T$ , we define the centered Gaussian process

$$\forall t \in \mathbb{R}, \quad X_{T'}(t) = \int_{\lambda \in \mathbb{R}} \int_{y \in [1, \frac{T}{T'}[} \cos(\lambda ty) \mu(d\lambda, dy) + \int_{\lambda \in \mathbb{R}} \int_{y \in [1, \frac{T}{T'}[} \sin(\lambda ty) \nu(d\lambda, dy).$$

It is plain to compute its covariation kernel, call it  $k_{T'}$ , by using the symmetry of the spectral measure  $F(d\lambda)$ :

$$\begin{aligned}
k_{T'}(t-s) &= \mathbb{E}[X_{T'}(s)X_{T'}(t)] \\
&= \int_{\lambda \in \mathbb{R}} \int_{y \in [1, \frac{T'}{T'}[} \cos(\lambda ty) \cos(\lambda ts) \frac{1}{y} dy F(d\lambda) + \int_{\lambda \in \mathbb{R}} \int_{y \in [1, \frac{T'}{T'}[} \sin(\lambda ty) \sin(\lambda sy) \frac{1}{y} dy F(d\lambda) \\
&= \int_{\lambda \in \mathbb{R}} \int_{y \in [1, \frac{T'}{T'}[} \cos(\lambda(t-s)y) \frac{1}{y} dy F(d\lambda) \\
&= \int_{y \in [1, \frac{T'}{T'}[} \int_{\lambda \in \mathbb{R}} e^{i\lambda(t-s)y} F(d\lambda) \frac{1}{y} dy \\
&= \int_{y \in [1, \frac{T'}{T'}[} \frac{k(|t-s|y)}{y} dy \\
&= \int_{|t-s|}^{\frac{T'}{T'}|t-s|} \frac{k(y)}{y} dy.
\end{aligned}$$

It is plain to see that for all  $A \in \mathcal{B}(\mathbb{R})$ , the process

$$M_{1/l}(A) = Y \int_A \exp(X_{1/l}(r) - \frac{1}{2}\mathbb{E}[X_{1/l}^2(r)]) dr$$

is a positive martingale and thus converges as  $l \rightarrow \infty$  towards a random variable  $M(A)$ . The stationary random measure  $(M(A))_{A \in \mathcal{B}(\mathbb{R})}$  is a Gaussian multiplicative chaos in the sense of [17] with associated covariance  $K$ .

Fix  $T' < T$ . Note that for  $l > 1/T'$ , we have  $\forall t \in \mathbb{R}$ :

$$\begin{aligned}
X_{1/l}(t) &= X_{T'}(t) + \int_{\lambda \in \mathbb{R}} \int_{y \in [\frac{T'}{T'}, lT'[} \cos(\lambda ty) \mu(d\lambda, dy) + \int_{\lambda \in \mathbb{R}} \int_{y \in [\frac{T'}{T'}, lT'[} \sin(\lambda ty) \nu(d\lambda, dy) \\
&\stackrel{def}{=} X_{T'}(t) + \bar{X}_{T', 1/l}(t),
\end{aligned} \tag{36}$$

where  $\bar{X}_{T', 1/l}$  is a centered stationary Gaussian process independent from  $X_{T'}$  with covariance kernel given by:

$$\bar{k}_{T', 1/l}(t-s) = \mathbb{E}[\bar{X}_{T', 1/l}(s)\bar{X}_{T', 1/l}(t)] = \int_{\frac{T'}{T'}|t-s|}^{lT'|t-s|} \frac{k(y)}{y} dy.$$

As above, we can define the random measure  $M^{T'}$  as the limit as  $l \rightarrow +\infty$  of the random measures

$$\forall A \in \mathcal{B}(\mathbb{R}), \quad M_{1/l}^{T'}(A) = Y \int_A \exp(\bar{X}_{T', 1/l}(r) - \frac{1}{2}\mathbb{E}[\bar{X}_{T', 1/l}^2(r)]) dr.$$

The stationary random measure  $(M^{T'}(A))_{A \in \mathcal{B}(\mathbb{R})}$  is a Gaussian multiplicative chaos in the sense of [17] with associated covariance  $K(\cdot \frac{T'}{T})$ . We deduce that  $\frac{T'}{T} M^{T'}(\frac{T'}{T} \cdot)$  is a Gaussian multiplicative chaos in the sense of [17] with associated covariance  $K(\cdot)$ . The measure  $\frac{T'}{T} M^{T'}(\frac{T'}{T} \cdot)$  thus has the same law as  $M$ . From (36), we obviously have:

$$M(A) = \int_A \exp(X_{T'}(r) - \frac{1}{2}\mathbb{E}[X_{T'}^2(r)]) M^{T'}(dr)$$

in such a way that (5) holds. Finally we point out that  $k(0) < 2$  implies that  $M$  admits a moment of order  $1 + \delta$  for some  $\delta > 0$  (see [13]).  $\square$

## A. Proofs of some auxiliary lemmas

**Lemma 20.** *Let  $F : \mathbb{R}^n \mapsto \mathbb{R}$  be a measurable function. Then, for all bounded Borelian sets  $A_1, \dots, A_n \subset \mathbb{R}$ , the following relation holds almost surely:*

$$\mathbb{E}_Y [F(M(A_1), \dots, M(A_n))] = \mathbb{E}_Y [F(\widetilde{M}^N(A_1), \dots, \widetilde{M}^N(A_n))]$$

*Proof.* By using the Jensen inequality, we have by

$$\begin{aligned} & \mathbb{E} \left[ \left| \frac{1}{T} \widetilde{M}^N [0; T] - \frac{1}{T} M^N [0; T] \right| \right] \\ &= \mathbb{E} \left[ \left( \left| \frac{1}{T} \widetilde{M}^N [0; T] - \frac{1}{T} M^N [0; T] \right|^2 \right)^{1/2} \right] \\ &\leq \mathbb{E} \left[ \left( \mathbb{E} \left[ \left| \frac{1}{T} \widetilde{M}^N [0; T] - \frac{1}{T} M^N [0; T] \right|^2 \middle| M \right] \right)^{1/2} \right] \\ &= \mathbb{E} \left[ \left( \frac{1}{T^2} \int_0^T \int_0^T \mathbb{E} \left[ \left( e^{\sum_{n=0}^N X_r^n - \frac{1}{2} \mathbb{E}[(X_r^n)^2]} - 1 \right) \left( e^{\sum_{n=0}^N X_u^n - \frac{1}{2} \mathbb{E}[(X_u^n)^2]} - 1 \right) M^N(dr) M^N(du) \right)^{1/2} \right] \right] \\ &= \mathbb{E} \left[ \left( \frac{1}{T^2} \int_0^T \int_0^T \left( e^{\sum_{n=0}^N \bar{k}_n(r-u)} - 1 \right) M^N(dr) M^N(du) \right)^{1/2} \right] \end{aligned}$$

The integrand in the above expectation converges almost surely towards 0 because, for each  $0 \leq n \leq N$ ,  $\bar{k}_n$  is bounded and converges to 0 in the vicinity of  $\infty$ . Furthermore, it is uniformly integrable because

$$\sup_T \mathbb{E} \left[ \left( \frac{1}{T} M^N([0; T]) \right)^{1+\delta} \right] < +\infty.$$

We deduce that

$$\mathbb{E} \left[ \left| \frac{1}{T} \widetilde{M}^N [0; T] - \frac{1}{T} M^N [0; T] \right| \right] \rightarrow 0 \quad \text{as } T \rightarrow +\infty.$$

As a consequence,  $\frac{1}{T} \widetilde{M}^N [0; T]$  converges almost surely along a subsequence towards  $Y$ .

One has, for all function  $h$  bounded and continuous,

$$\mathbb{E} \left[ F(M(A_1), \dots, M(A_n)) h \left( \frac{1}{T} M [0; T] \right) \right] = \mathbb{E} \left[ F(\widetilde{M}(A_1), \dots, \widetilde{M}(A_n)) h \left( \frac{1}{T} \widetilde{M}^N [0; T] \right) \right]$$

Sending  $T$  to  $+\infty$  along the subsequence, we get by the bounded convergence theorem

$$\mathbb{E} [F(M(A_1), \dots, M(A_n)) h(Y)] = \mathbb{E} [F(\widetilde{M}(A_1), \dots, \widetilde{M}(A_n)) h(Y)]$$

and the lemma is proved. □

**Lemma 21.** *Let  $F : \mathbb{R}_+ \rightarrow \mathbb{R}$  be some convex function such that*

$$\forall x \in \mathbb{R}_+, \quad |F(x)| \leq M(1 + |x|^\beta),$$

*for some positive constants  $M, \beta$ , and  $\sigma$  be a Radon measure on the Borelian subsets of  $\mathbb{R}$ . Given  $a < b$ , let  $(X_r)_{a \leq r \leq b}$ ,  $(Y_r)_{a \leq r \leq b}$  be two continuous centered Gaussian processes with continuous covariance kernels  $k_X$  and  $k_Y$  such that*

$$\forall u, v \in [a, b], \quad k_X(u, v) \leq k_Y(u, v).$$

*Then*

$$\mathbb{E} \left[ F \left( \int_a^b e^{X_r - \frac{1}{2} \mathbb{E}[X_r^2]} \sigma(dr) \right) \right] \leq \mathbb{E} \left[ F \left( \int_a^b e^{Y_r - \frac{1}{2} \mathbb{E}[Y_r^2]} \sigma(dr) \right) \right].$$

*Proof.* For each  $N \in \mathbb{N}$ , we define the smooth subdivision  $t_p^N = a + p \frac{b-a}{N}$ ,  $p = 0, \dots, N$ , of the interval  $[a, b]$ . We also introduce the random variables

$$S_N^X = \sum_{p=0}^{N-1} e^{X_{t_p^N} - \frac{1}{2}\mathbb{E}[X_{t_p^N}^2]} \sigma([t_p^N, t_{p+1}^N]) \quad \text{and} \quad S_N^Y = \sum_{p=0}^{N-1} e^{Y_{t_p^N} - \frac{1}{2}\mathbb{E}[Y_{t_p^N}^2]} \sigma([t_p^N, t_{p+1}^N]).$$

By classical Gaussian inequalities (see [17, corollary 6.2] for instance), we have

$$\forall N \geq 1, \quad \mathbb{E}\left[F\left(S_N^X\right)\right] \leq \mathbb{E}\left[F\left(S_N^Y\right)\right].$$

So it just remains to pass to the limit as  $N \rightarrow \infty$  by using the dominated convergence theorem. By continuity of the processes  $X, Y$  the random variables  $S_N^X, S_N^Y$  converge almost surely respectively towards  $\int_a^b e^{X_r - \frac{1}{2}\mathbb{E}[X_r^2]} \sigma(dr)$ ,  $\int_a^b e^{Y_r - \frac{1}{2}\mathbb{E}[Y_r^2]} \sigma(dr)$ . Clearly, we have:

$$|F(S_N^X)| \leq M(1 + |S_N^X|^\beta),$$

so that we just have to prove that  $|S_N^X|^\beta$  is uniformly integrable (the same argument holds for  $|S_N^Y|^\beta$ ). It is enough to establish that for each  $d \in \mathbb{N}$ ,

$$\sup_N \mathbb{E}[(S_N^X)^d] < +\infty.$$

We have

$$\begin{aligned} \mathbb{E}[(S_N^X)^d] &= \mathbb{E}\left[\left(\sum_{p=0}^{N-1} e^{X_{t_p^N} - \frac{1}{2}\mathbb{E}[X_{t_p^N}^2]} \sigma([t_p^N, t_{p+1}^N])\right)^d\right] \\ &= \sum_{p_1, \dots, p_d=0}^{N-1} \mathbb{E}\left[e^{X_{t_{p_1}^N} + \dots + X_{t_{p_d}^N}} e^{-\frac{1}{2}(\mathbb{E}[X_{t_{p_1}^N}^2] + \dots + \mathbb{E}[X_{t_{p_d}^N}^2])} \sigma([t_{p_1}^N, t_{p_1+1}^N]) \times \dots \times \sigma([t_{p_d}^N, t_{p_d+1}^N])\right] \\ &= \sum_{p_1, \dots, p_d=0}^{N-1} e^{\frac{1}{2}\sum_{i,j=1}^d k_X(t_{p_i}^N, t_{p_j}^N)} e^{-\frac{1}{2}(\mathbb{E}[X_{t_{p_1}^N}^2] + \dots + \mathbb{E}[X_{t_{p_d}^N}^2])} \sigma([t_{p_1}^N, t_{p_1+1}^N]) \times \dots \times \sigma([t_{p_d}^N, t_{p_d+1}^N]) \\ &\rightarrow \int_a^b \dots \int_a^b e^{\frac{1}{2}\sum_{i \neq j} k_X(u_i, u_j)} \sigma(du_1) \dots \sigma(du_d) \end{aligned}$$

as  $N \rightarrow \infty$ . This completes the proof.  $\square$

*Proof of Lemma 14.* First observe that we have:

$$K^\epsilon(r) = \frac{\ln_+ r}{\ln \epsilon} k_\epsilon(0) + g_\epsilon(r),$$

where we have set

$$g_\epsilon(r) = K^\epsilon(r) - \frac{\ln_+ r}{\ln \epsilon} k_\epsilon(0).$$

Because of (7), it is readily seen that  $g_\epsilon$  is continuous on  $\mathbb{R} \setminus \{0\}$  (or even locally lipschitz). Then proving the boundedness of  $g_\epsilon$  over  $\mathbb{R} \setminus \{0\}$  boils down to proving the boundedness of  $g_\epsilon$  over a

neighbourhood of 0. By using (6) and (7), we have for any  $|r| \leq 1$ :

$$\begin{aligned}
|g_\epsilon(r)| &= \left| K^\epsilon(r) - \frac{\ln r}{\ln \epsilon} k_\epsilon(0) \right| \\
&\leq \left| \sum_{n=0}^{\frac{\ln r}{\ln \epsilon} - 1} k_\epsilon\left(\frac{r}{\epsilon^n}\right) - k_\epsilon(0) \right| + \left| \frac{\ln r}{\ln \epsilon} k_\epsilon(0) - \sum_{n=0}^{\frac{\ln r}{\ln \epsilon} - 1} k_\epsilon(0) \right| + \left| \sum_{\frac{\ln r}{\ln \epsilon}}^{+\infty} k_\epsilon\left(\frac{r}{\epsilon^n}\right) \right| \\
&\leq \sum_{n=0}^{\frac{\ln r}{\ln \epsilon} - 1} \frac{C_\epsilon}{1 + \left|\frac{r}{\epsilon^n}\right|^{1+\alpha}} \left|\frac{r}{\epsilon^n}\right| + |k_\epsilon(0)| + \sum_{\frac{\ln r}{\ln \epsilon}}^{+\infty} \frac{C_\epsilon}{1 + |\ln r - n \ln \epsilon|^{1+\alpha}} \\
&\leq \frac{C_\epsilon}{1 - \epsilon^\alpha} + |k_\epsilon(0)| + C_\epsilon \int_{\frac{\ln r}{\ln \epsilon}}^{+\infty} \frac{1}{1 + |\ln r - u \ln \epsilon|^{1+\alpha}} du.
\end{aligned}$$

Note that the last integral matches  $\frac{C_\epsilon}{|\ln \epsilon|} \int_0^{+\infty} \frac{|\ln r|}{1 + |y \ln r|^{1+\alpha}} dy$  and is thus easily seen to be bounded in a neighbourhood of 0.  $\square$

*Proof of Lemma 15.* Let  $n \in \mathbb{N}$ .

$$\mathbb{E} [M[0; t]^{1+\delta}] = \mathbb{E} \left[ \left( M[0; \frac{t}{n}] + M[\frac{t}{n}; \frac{2t}{n}] + \dots + M[\frac{(n-1)t}{n}; t] \right)^{1+\delta} \right] \quad (37)$$

$$\geq \mathbb{E} \left[ \left( M[0; \frac{t}{n}] \right)^{1+\delta} + \left( M[\frac{t}{n}; \frac{2t}{n}] \right)^{1+\delta} + \dots + \left( M[\frac{(n-1)t}{n}; t] \right)^{1+\delta} \right] \quad (38)$$

$$= n \mathbb{E} \left[ \left( M[0; \frac{t}{n}] \right)^{1+\delta} \right] \quad (39)$$

We used the stationarity of the measure  $M$  in the second line. Now write, for  $h > 0$ :

$$g(h) = \sup_{r \leq h} |k_{1/n}(0) - k_{1/n}(r)| \quad (40)$$

We have, for every  $r \in (0, t/n]$  and  $n$  large enough:

$$|k^{1/n}(0) - g(t/n)| \leq k^{1/n}(r).$$

So, using classical gaussian inequality (see Lemma 21):

$$\begin{aligned}
\mathbb{E} \left[ M[0; \frac{t}{n}]^{1+\delta} \right] &= \mathbb{E} \left[ \left( \int_0^{t/n} e^{\omega_{1/n}(r)} M^{1/n}(dr) \right)^{1+\delta} \right] \\
&\geq \mathbb{E} \left[ \left( \int_0^{t/n} e^{\sqrt{|k_{1/n}(0) - g(t/n)|} Z_n - \frac{1}{2} |k_{1/n}(0) - g(t/n)|} M^{1/n}(dr) \right)^{1+\delta} \right] \\
&= \mathbb{E} \left[ \left( e^{\sqrt{|k_{1/n}(0) - g(t/n)|} Z_n - \frac{1}{2} |k_{1/n}(0) - g(t/n)|} \right)^{1+\delta} \right] \mathbb{E} \left[ \left( M^{1/n}[0; \frac{t}{n}] \right)^{1+\delta} \right] \\
&= e^{-\frac{1+\delta}{2} |k_{1/n}(0) - g(t/n)|} e^{\frac{(1+\delta)^2}{2} |k_{1/n}(0) - g(t/n)|} \frac{1}{n^{1+\delta}} \mathbb{E} \left[ (M[0; t])^{1+\delta} \right] \quad (41)
\end{aligned}$$

We used Lemma 13 in the second line. Using equations (39) and (41), one gets

$$e^{-\frac{1+\delta}{2} |k_{1/n}(0) - g(t/n)|} e^{\frac{(1+\delta)^2}{2} |k_{1/n}(0) - g(t/n)|} \frac{1}{n^\delta} \leq 1 \quad (42)$$

As  $h$  goes to 0,  $g(h)$  goes to 0 (the function  $k_{1/n}$  is continuous). Letting  $t$  goes to 0 in (42), one gets

$$k_{1/n}(0) \leq \frac{2}{1+\delta} \ln n.$$

and the lemma is proved.  $\square$

*Proof of Corollary 3.* By stationarity, it is enough to prove that, almost surely, the measure  $M$  does not possess any atom on the segment  $[0, 1]$ . From [8, Corollary 9.3 VI], it is enough to check that for each  $\alpha > 0$ :

$$\sum_{k=1}^n \mathbb{P}\left(M\left[\frac{k-1}{n}; \frac{k}{n}\right] > \alpha\right) = n\mathbb{P}\left(M\left[\frac{0}{n}; \frac{1}{n}\right] > \alpha\right) \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

This is a direct consequence of the Markov inequality and Lemma 13:

$$n\mathbb{P}\left(M\left[\frac{0}{n}; \frac{1}{n}\right] > \alpha\right) \leq \frac{n}{\alpha^{1+\gamma}} \mathbb{E}[M([0, \frac{1}{n}])^{1+\gamma}] \rightarrow 0 \text{ as } n \rightarrow \infty. \quad \square$$

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