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Reductions for branching coefficients

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Abstract

Let G be a connected reductive subgroup of a complex connected reductive group \hat{G} . We are interested in the branching problem. Fix maximal tori and Borel subgroups of G and \hat{G} . Consider the cone $\mathcal{LR}(G, \hat{G})$ generated by the pairs $(\nu, \hat{\nu})$ of dominant characters such that V_ν^* is a submodule of $V_{\hat{\nu}}$. It is known that $\mathcal{LR}(G, \hat{G})$ is a closed convex polyhedral cone. In this work, we show that every regular face of $\mathcal{LR}(G, \hat{G})$ gives rise to a *reduction rule* for multiplicities. More precisely, we prove that for $(\nu, \hat{\nu})$ on such a face, the multiplicity of V_ν^* in $V_{\hat{\nu}}$ equal to a similar multiplicity for representations of Levi subgroups of G and \hat{G} . This generalizes, by different methods, results obtained by Brion, Derksen-Weyman, Roth...

1 Introduction

Let G be a connected reductive subgroup of a complex connected reductive group \hat{G} . We are interested in the branching problem:

Decompose irreducible representations of \hat{G} as sum of irreducible G -modules.

We fix maximal tori $T \subset \hat{T}$ and Borel subgroups $B \supset T$ and $\hat{B} \supset \hat{T}$ of G and \hat{G} . Let $X(T)$ denote the group of characters of T and let $X(T)^+$ denote the set of dominant characters. For $\nu \in X(T)^+$, we denote by V_ν the irreducible representation of highest weight ν . Similarly, we use notation $X(\hat{T})$, $X(\hat{T})^+$, $V_{\hat{\nu}}$ relatively to \hat{G} . For any G -module V , we denote by V^G the subspace of G -fixed vectors. Consider the following integers

$$c_{\nu\hat{\nu}}(G, \hat{G}) = \dim(V_\nu \otimes V_{\hat{\nu}})^G. \quad (1)$$

Sometimes we simply write $c_{\nu\hat{\nu}}$ for $c_{\nu\hat{\nu}}(G, \hat{G})$. Let V_{ν}^* denote the dual representation of V_{ν} . The branching problem is equivalent to knowledge of these coefficients since we have

$$V_{\hat{\nu}} = \sum_{\nu \in X(T)^+} c_{\nu\hat{\nu}} V_{\nu}^*. \quad (2)$$

The set $\text{LR}(G, \hat{G})$ of pairs $(\nu, \hat{\nu}) \in X(T)^+ \times X(\hat{T})^+$ such that $c_{\nu\hat{\nu}} \neq 0$ is a finitely generated subsemigroup of the free abelian group $X(T) \times X(\hat{T})$. Consider the convex cone $\mathcal{LR}(G, \hat{G})$ generated in $(X(T) \times X(\hat{T})) \otimes \mathbb{Q}$ by $\text{LR}(G, \hat{G})$. It is a closed convex polyhedral cone in $(X(T) \times X(\hat{T})) \otimes \mathbb{Q}$.

Let \mathcal{F} be a face of $\mathcal{LR}(G, \hat{G})$. We assume that \mathcal{F} is regular that is, that it contains regular dominant weights $(\nu, \hat{\nu})$. Let \hat{W} be the Weyl group of \hat{G} and \hat{T} . If S is a torus in G and H is a subgroup of G containing S then we will denote by H^S the centralizer of S in H . By [Res10b], the regular face \mathcal{F} corresponds to a pair (S, \hat{w}) where S is a subtorus of T and $\hat{w} \in \hat{W}$ such that

$$\hat{G}^S \cap \hat{w}\hat{B}\hat{w}^{-1} = \hat{B}^S, \quad (3)$$

and the span of \mathcal{F} is the set of pairs $(\nu, \hat{\nu}) \in (X(T) \times X(\hat{T})) \otimes \mathbb{Q}$ such that

$$\nu|_S + \hat{w}\hat{\nu}|_S = 0 \in X(S) \otimes \mathbb{Q}. \quad (4)$$

Now, we can state our main result

Theorem 1 *Let $(\nu, \hat{\nu}) \in X(T)^+ \times X(\hat{T})^+$ be a pair of dominant weights. We assume that $(\nu, \hat{\nu})$ belongs to the span of \mathcal{F} (equivalently that it satisfies condition (4)). Then*

$$c_{\nu\hat{\nu}}(G, \hat{G}) = c_{\nu\hat{\nu}}(G^S, \hat{G}^S).$$

Let $X = G/P \times \hat{G}/\hat{P}$ be a flag manifold of the group $G \times \hat{G}$. Let λ be a one-parameter subgroup of G and C be an irreducible component of the fixed point set X^λ of λ in X . Let G^λ be the centralizer of the image of λ in G . We assume that (C, λ) is a (well) covering pair in the sense of [Res10a, Definition 3.2.2] (see also Definition 1 below). Theorem 1 will be a direct consequence of the more geometric

Theorem 2 *Let \mathcal{L} be a G -linearized line bundle on X generated by its global sections such that λ acts trivially on the restriction $\mathcal{L}|_C$. Then the restriction map induces an isomorphism*

$$H^0(X, \mathcal{L})^G \longrightarrow H^0(C, \mathcal{L}|_C)^{G^\lambda}.$$

Several particular cases of Theorems 1 and 2 was known. If $G = T$ is a maximal torus of $G = \mathrm{GL}_n$, our theorem is equivalent to [KTT07, Theorem 5.8]. If $\hat{G} = G \times G$ (or more generally $\hat{G} = G^s$ for some integer $s \geq 2$) and G is diagonally embedded in \hat{G} then $c_{\nu, \hat{\nu}}(G, \hat{G})$ (resp. $c_{\nu, \hat{\nu}}(G^S, \hat{G}^S)$) are tensor product multiplicities for the group G (resp. G^S). This case was recently proved independently by Derksen and Weyman in [DW10, Theorem 7.4] and King, Tollu and Toumazet in [KTT09, Theorem 1.4] if $G = \mathrm{GL}_n$ and for any reductive group by Roth in [Rot11]. If ν is regular then Theorem 2 can be obtained applying [Bri99, Theorem 3] and [Res10a]. Similar reductions can be found in [Bri93, Man97, Mon96].

Note that our proof is new and uses strongly the normality of the Schubert varieties. For example, in Roth's proof (which may be the closest from our) the normality of Schubert varieties play no role. In [DW10], the case $\mathrm{GL}_n \subset \mathrm{GL}_n \times \mathrm{GL}_n$ is obtained as a consequence of a more general result on quivers. The Derksen-Weyman's theorem on quivers can be proved by the method used here.

In Section 4, we apply Theorem 2 to recover known results.

2 Proof of Theorem 2

Let us consider the variety $X = G/P \times \hat{G}/\hat{P}$ endowed with the diagonal G -action: $g'.(gP/P, \hat{g}\hat{P}/\hat{P}) = (g'gP/P, g'\hat{g}\hat{P}/\hat{P})$.

Let λ be a one-parameter subgroup of G . Let us consider the centralizer G^λ of λ in G . We associate to λ the parabolic subgroup (see [MFK94]):

$$P(\lambda) = \left\{ g \in G : \lim_{t \rightarrow 0} \lambda(t).g.\lambda(t)^{-1} \text{ exists in } G \right\}.$$

Let C be an irreducible component of the fixed point set X^λ of λ in X . We set:

$$C^+ := \{ x \in X : \lim_{t \rightarrow 0} \lambda(t)x \text{ belongs to } C \}. \quad (5)$$

Now, C^+ is $P(\lambda)$ -stable and locally closed in X .

Consider the subvariety Y of $G/P(\lambda) \times X$ defined by

$$Y = \{(gP(\lambda)/P(\lambda), x) : g^{-1}x \in C^+\}.$$

The morphism $\pi : G \times C^+ \rightarrow Y$, $(g, x) \mapsto (gP(\lambda)/P(\lambda), gx)$ identifies Y to the quotient of $G \times C^+$ by the action of $P(\lambda)$ given by $p.(g, x) = (gp^{-1}, px)$.

We will denote Y by $G \times_{P(\lambda)} C^+$ and we set $[g : x] = \pi(g, x)$. Consider now the G -equivariant map

$$\begin{aligned} \eta : G \times_{P(\lambda)} C^+ &\longrightarrow X \\ [g : x] &\longmapsto g.x. \end{aligned}$$

We now recall from [Res10a] the following

Definition 1 *The pair (C, λ) is said to be covering if η is birational. The pair (C, λ) is said to be well covering if there exists a $P(\lambda)$ -stable open subset Ω of C^+ intersecting C such that η induces an isomorphism from $G \times_{P(\lambda)} \Omega$ onto an open subset of X .*

Proof.[of Theorem 2] Consider the closure $\overline{C^+}$ of C^+ in X . Since (C, λ) is covering the map

$$\begin{aligned} \overline{\eta} : G \times_{P(\lambda)} \overline{C^+} &\longrightarrow X \\ [g : x] &\longmapsto gx. \end{aligned}$$

is proper and birational. Hence it induces a G -equivariant isomorphism

$$H^0(X, \mathcal{L}) \simeq H^0(G \times_{P(\lambda)} \overline{C^+}, \overline{\eta}^*(\mathcal{L})).$$

In particular, we have

$$H^0(X, \mathcal{L})^G \simeq H^0(G \times_{P(\lambda)} \overline{C^+}, \overline{\eta}^*(\mathcal{L}))^G.$$

We embed $\overline{C^+}$ in $G \times_{P(\lambda)} \overline{C^+}$, by $x \mapsto [e : x]$. Note that the composition of the immersion of $\overline{C^+}$ in $G \times_{P(\lambda)} \overline{C^+}$ with $\overline{\eta}$ is the immersion of $\overline{C^+}$ in X . In particular, $\overline{\eta}^*(\mathcal{L})|_{\overline{C^+}} = \mathcal{L}|_{\overline{C^+}}$. Now, the restriction induces the following isomorphism (see for example [Res10a, Lemma 4])

$$H^0(G \times_{P(\lambda)} \overline{C^+}, \overline{\eta}^*(\mathcal{L}))^G \simeq H^0(\overline{C^+}, \mathcal{L}|_{\overline{C^+}})^{P(\lambda)}.$$

Since once more, the composition of the immersion of $\overline{C^+}$ in $G \times_{P(\lambda)} \overline{C^+}$ with $\overline{\eta}$ is the immersion of $\overline{C^+}$ in X , we just proved that the restriction induces the following isomorphism

$$H^0(X, \mathcal{L})^G \simeq H^0(\overline{C^+}, \mathcal{L}|_{\overline{C^+}})^{P(\lambda)}. \quad (6)$$

On the other hand, it is proved in [Res10a, Lemma 5] that since λ acts trivially on $\mathcal{L}|_C$, the restriction induces the following isomorphism

$$H^0(C^+, \mathcal{L}|_{C^+})^{P(\lambda)} \simeq H^0(C, \mathcal{L}|_C)^{G^\lambda}. \quad (7)$$

By isomorphisms (6) and (7), it remains to prove that the restriction induces the following isomorphism

$$H^0(\overline{C^+}, \mathcal{L}|_{\overline{C^+}})^{P(\lambda)} \simeq H^0(C^+, \mathcal{L}|_{C^+})^{P(\lambda)};$$

that is, that any regular $P(\lambda)$ -invariant section σ of \mathcal{L} on C^+ extends to $\overline{C^+}$.

Note that, λ is also a one-parameter subgroup of \hat{G} and that we can define $\hat{P}(\lambda)$. Let us fix a maximal torus T of G containing the image of λ and a maximal torus \hat{T} of \hat{G} containing T . Note that P and \hat{P} have not been fixed up to now; we have only considered the $G \times \hat{G}$ -variety X . In other words, we can change P and \hat{P} by conjugated subgroups. Let us fix a $T \times \hat{T}$ -fixed point x_0 in C , and let us denote by $P \times \hat{P}$ its stabilizer in $G \times \hat{G}$.

It is well known that $C^+ = P(\lambda)P/P \times \hat{P}(\lambda)\hat{P}/\hat{P}$. In particular, $\overline{C^+}$ is a product of Schubert varieties and is normal. So, it is sufficient to prove that σ has no pole. Since σ is regular on C^+ , we have to prove that σ has no pole along any irreducible component D of codimension one of $\overline{C^+} - C^+$. We are going to compute the order of the pole of σ along D by a quite explicit computation in a neighborhood of D in $\overline{C^+}$.

If β is a root of (T, G) , we denote by s_β the associated reflection in the Weyl group. Now, D is the closure of $P(\lambda).s_\beta P/P \times \hat{P}(\lambda)\hat{P}/\hat{P}$ for some root β or of $P(\lambda)P/P \times \hat{P}(\lambda)s_{\hat{\beta}}\hat{P}/\hat{P}$ for some root $\hat{\beta}$. Consider the first case. The second one works similarly.

Set $y = (s_\beta P/P, \hat{P}/\hat{P})$; it is a point in D . Consider be the unipotent radical U^- of the parabolic subgroup of G containing T and opposite to P . Similarly, we define \hat{U}^- . Consider the groups $U_y = P(\lambda) \cap s_\beta U^- s_\beta$ and $\hat{U}_y = \hat{P}(\lambda) \cap \hat{U}^-$. Let δ be the T -stable line in G/P containing P/P and $s_\beta P/P$. Consider the map

$$\begin{aligned} \theta : U_y \times \hat{U}_y \times (\delta - \{P/P\}) &\longrightarrow X \\ (u, \hat{u}, x) &\longmapsto (ux, \hat{u}\hat{P}/\hat{P}). \end{aligned}$$

The map θ is an immersion and its image Ω is open in $\overline{C^+}$. But Ω intersects D ; so, it is sufficient to prove that σ extends on Ω . Equivalently, we are going to prove that $\theta^*(\sigma)$ extends to a regular section of $\theta^*(\mathcal{L})$.

The torus T acts on $U_y \times \hat{U}_y \times (\delta - \{x_0\})$ by $t.(u, \hat{u}, x) = (tut^{-1}, t\hat{u}t^{-1}, tx)$. This action makes θ equivariant. The curve $(\delta - \{x_0\})$ is isomorphic to \mathbb{C} . The group U_y is unipotent and so isomorphic to its Lie algebra. It follows that $U_y \times \hat{U}_y \times (\delta - \{x_0\})$ is isomorphic as a T -variety to an affine space V with linear action of T .

Fix root (for the action of $T \times \hat{T}$) coordinates ξ_i on the Lie algebra of $U_y \times \hat{U}_y$. Fix a T -equivariant coordinate ζ on $\delta - \{P/P\}$. So that (ξ_i, ζ) are coordinates on V . Let (a_i, a) be the opposite of the weights of the variables for the action of λ . The weights of T corresponding to the part U_y are roots of $P(\lambda)$ and the weights of \hat{T} corresponding to the part \hat{U}_y are roots of $\hat{P}(\lambda)$. The weight of the action of T on $T_{s_\beta P/P} \delta$ is a root of G but not of $P(\lambda)$. Then we have

$$a_i \geq 0 \text{ and } a < 0. \quad (8)$$

Note that $(\iota \circ \theta)^{-1}(D)$ is the divisor $\zeta = 0$ on V .

Consider now, the \mathbb{C}^* -linearized line bundle $\theta^*(\mathcal{L})$ on V . It is trivial as a line bundle (the Picard group of V is trivial) and so, it is isomorphic to $V \times \mathbb{C}$ linearized by

$$t.(v, \tau) = (\lambda(t)v, t^\mu \tau) \quad \forall t \in \mathbb{C}^*,$$

for some integer μ .

We first admit that

$$\mu \leq 0 \quad (9)$$

and end the proof. The section $\theta^*(\sigma)$ corresponds to a polynomial in the variables ξ_i, ζ and ζ^{-1} ; that is, a linear combination of monomials $m = \prod_i \xi_i^{j_i} \cdot \zeta^j$ for some $j_i \in \mathbb{Z}_{\geq 0}$ and $j \in \mathbb{Z}$. The opposite of the weight of m for the action of \mathbb{C}^* is $\sum_i j_i a_i + ja$. The fact that σ is \mathbb{C}^* -invariant implies that the monomials occurring in the expression of $(\iota \circ \theta)^*(\sigma)$ satisfy

$$\sum_i j_i a_i + ja = \mu.$$

So, we have:

$$j = \frac{-1}{a} \left(\sum_i j_i a_i - \mu \right).$$

Now, inequalities (8) and (9) imply that $j \geq 0$. In particular, $(\iota \circ \theta)^*(\sigma)$ extends to a regular function on V . It follows that σ has no pole along D .

It remains to prove inequality (9). Consider the restriction of \mathcal{L} to δ . Note that δ is isomorphic to \mathbb{P}^1 and $\mathcal{L}|_\delta$ to $\mathcal{O}(d)$ as a line bundle for some integer d . Since \mathcal{L} is semiample, d is nonnegative. The group \mathbb{C}^* acts on $T_{x_0}\delta$ by the weight $-a$ and on $T_y\delta$ by the weight a . By assumption, the group \mathbb{C}^* acts trivially on the fiber \mathcal{L}_{x_0} (recall that x_0 belongs to C). It acts on the fiber \mathcal{L}_y by the weight μ . Now, the theory of \mathbb{P}^1 implies that:

$$d = \frac{\mu - 0}{a}.$$

But, $d \geq 0$ and $a < 0$. It follows that $\mu \leq 0$. □

3 Proof of Theorem 1

Let T , B , \hat{T} and \hat{B} be like in the introduction. To any character ν of B we associate a G -linearized line bundle \mathcal{L}_ν on G/B such that B acts on the fiber in \mathcal{L}_ν over B/B with the weight $-\nu$. By Borel-Weil's theorem, the line bundle \mathcal{L}_ν is generated by its global sections if and only if ν is dominant and in this case $H^0(G/B, \mathcal{L}_\nu)$ is isomorphic to the dual V_ν^* of the simple G -module V_ν with highest weight ν .

Consider the complete flag variety $X = G/B \times \hat{G}/\hat{B}$ of the group $G \times \hat{G}$. Let now ν and $\hat{\nu}$ be like in Theorem 1. Let \mathcal{L} be the exterior product on X of \mathcal{L}_ν and $\mathcal{L}_{\hat{\nu}}$. By Borel-Weil's theorem, we have

$$V_\nu^* \otimes V_{\hat{\nu}}^* = H^0(X, \mathcal{L}).$$

In particular, $c_{\nu, \hat{\nu}}(G, \hat{G})$ is the dimension of $H^0(X, \mathcal{L})^G$.

Let $C = G^S B/B \times \hat{G}^S \hat{w} \hat{B}/\hat{B}$. By [Res10b], there exists a one-parameter subgroup λ of S such that (C, λ) is well covering and $G^S = G^\lambda$. Moreover, assumption (4) implies that λ acts trivially on $\mathcal{L}|_C$. So, we can apply Theorem 2 to get

$$H^0(X, \mathcal{L})^G \simeq H^0(C, \mathcal{L}|_C)^{G^S}.$$

However, C is isomorphic to the complete flag manifold of the group $G^S \times \hat{G}^S$. By condition (3), $\mathcal{L}|_C$ is the line bundle $\mathcal{L}_\nu \otimes \mathcal{L}_{\hat{w}\hat{\nu}}$. Hence Borel-Weil's theorem implies that $H^0(C, \mathcal{L}|_C)$ is isomorphic to $V_\nu^*(G^S) \otimes V_{\hat{w}\hat{\nu}}^*(\hat{G}^S)$. In particular, $c_{\nu, \hat{w}\hat{\nu}}(G^S, \hat{G}^S)$ is the dimension of $H^0(C, \mathcal{L}|_C)^{G^S}$. The theorem is proved.

4 Examples

4.1 Tensor product decomposition

In this subsection, we consider the case when $\hat{G} = G \times G$ and G is diagonally embedded in \hat{G} . Let us also assume that $\hat{B} = B \times B$ and $\hat{T} = T \times T$. Then a dominant weight $\hat{\nu}$ of \hat{T} is a pair (λ, μ) of dominant weights of T and $V_{\hat{\nu}} = V_{\lambda} \otimes V_{\mu}$. For short, we denote by $c_{\lambda\mu\nu}(G)$ the coefficient $c_{\nu\hat{\nu}}(G, \hat{G})$. We have

$$V_{\lambda} \otimes V_{\mu} = \sum_{\nu} c_{\lambda\mu\nu}(G) V_{\nu}^*, \quad (10)$$

and $c_{\lambda\mu\nu}(G)$ is a tensor product multiplicity for G . Using the notation of Theorem 1, we have $\hat{G}^S = G^S \times G^S$. In particular, the coefficient $c_{\nu\hat{\nu}}(G^S, \hat{G}^S)$ is a tensor product multiplicity for the Levi subgroup G^S of G . In this case, Theorem 1 is equivalent to the main result of [Rot11].

We now consider the case when $G = \mathrm{GL}_n(\mathbb{C})$, T consists in diagonal matrices and B in upper triangular matrices. In this case a dominant weight λ is a nonincreasing sequence $(\lambda_1, \dots, \lambda_n)$ of n integers and $c_{\lambda\mu\nu}(G)$ is a Littlewood-Richardson coefficient denoted by $c_{\lambda\mu\nu}^n$.

Let us introduce notation to describe $LR(G, \hat{G})$ in this case. Let $\mathbb{G}(r, n)$ be the Grassmann variety of r -dimensional subspaces of \mathbb{C}^n . Let $F_{\bullet}: \{0\} = F_0 \subset F_1 \subset F_2 \subset \dots \subset F_n = V$ be the standard flag of \mathbb{C}^n . Let $\mathcal{P}(r, n)$ denote the set of parts of $\{1, \dots, n\}$ with r elements. Let $I = \{i_1 < \dots < i_r\} \in \mathcal{P}(r, n)$. The Schubert variety $\Omega_I(F_{\bullet})$ in $\mathbb{G}(r, n)$ is defined by

$$\Omega_I(F_{\bullet}) = \{L \in \mathbb{G}(r, n) : \dim(L \cap F_{i_j}) \geq j \text{ for } 1 \leq j \leq r\}.$$

The Poincaré dual of the homology class of $\Omega_I(F_{\bullet})$ is denoted by σ_I . The σ_I form a \mathbb{Z} -basis for the cohomology ring of $\mathbb{G}(r, n)$. The class associated to $[1; r]$ is the class of the point; it will be denoted by [pt].

By [Kly98], [KT99] and finally [Bel01], we have the following statement.

Theorem 3 *Let (λ, μ, ν) be a triple of nonincreasing sequences of n integers. Then $c_{\lambda\mu\nu}^n \neq 0$ if and only if*

$$\sum_i \lambda_i + \sum_j \mu_j + \sum_k \nu_k = 0 \quad (11)$$

and for any $r = 1, \dots, n-1$, for any $(I, J, K) \in \mathcal{P}(r, n)^3$ such that

$$\sigma_I \cdot \sigma_J \cdot \sigma_K = [\text{pt}] \in \mathbb{H}^*(\mathbb{G}(r, n), \mathbb{Z}), \quad (12)$$

we have

$$\sum_{i \in I} \lambda_i + \sum_{j \in J} \mu_j + \sum_{k \in K} \nu_k \leq 0. \quad (13)$$

Knutson, Tao and Woodward proved in [KTW04] that this statement is optimal in the following sense:

Theorem 4 *In Theorem 3, no inequality can be omitted.*

In other words, each inequality (13) corresponds to a regular face \mathcal{F}_{IJK} of the cone $\mathcal{LR}(G, \hat{G})$. For $I = \{i_1 < \dots < i_r\} \in \mathcal{P}(r, n)$ and λ a sequence of n integers, we set $\lambda_I = (\lambda_{i_1}, \dots, \lambda_{i_r}) \in \mathbb{Z}^r$. We also denote by $I^c \in \mathcal{P}(n-r, n)$ the complement of I in $\{1, \dots, n\}$. It is easy to check that Theorem 1 gives in this case the following

Theorem 5 *Let (λ, μ, ν) be a triple of nonincreasing sequences of n integers. Let $(I, J, K) \in \mathcal{P}(r, n)$ such that*

$$\sigma_I \cdot \sigma_J \cdot \sigma_K = [\text{pt}]. \quad (14)$$

If

$$\sum_{i \in I} \lambda_i + \sum_{j \in J} \mu_j + \sum_{k \in K} \nu_k = 0, \quad (15)$$

then

$$c_{\lambda \mu \nu}^n = c_{\lambda_I \mu_J \nu_K}^r \cdot c_{\lambda_{I^c} \mu_{J^c} \nu_{K^c}}^{n-r}. \quad (16)$$

Theorem 5 has been proved independently in [KTT09] and [DW10]. Note that if equation (15) does not hold then $c_{\lambda_I \mu_J \nu_K}^r = 0$.

It is known that Theorem 3 also holds if condition (12) is replaced by

$$\sigma_I \cdot \sigma_J \cdot \sigma_K = d[\text{pt}] \in \mathbb{H}^*(\mathbb{G}(r, n), \mathbb{Z}), \quad (17)$$

for some positive integer d . The following example shows that condition (14) cannot be replaced by condition (17) in Theorem 5.

Example. Here, $n = 6$, $r = 3$ and $I = J = K = \{1, 3, 5\}$. Set $\lambda = \mu = \nu = (1100 - 1 - 1)$. We have $\lambda_I = \mu_J = \nu_K = \lambda_{I^c} = \mu_{J^c} = \nu_{K^c} = (10 - 1)$. We have $c_{\lambda \mu \nu}^n = 3$ and $c_{\lambda_I \mu_J \nu_K}^r = c_{\lambda_{I^c} \mu_{J^c} \nu_{K^c}}^{n-r} = 2$. In this case, we have $\sigma_I \cdot \sigma_J \cdot \sigma_K = 2[\text{pt}]$.

Note that Knutson and Purbhoo proved in [KP10] some equalities (16) with assumptions different from Theorem 5.

4.2 Kronecker coefficients

Let $\alpha = (\alpha_1 \geq \alpha_2 \geq \dots)$ be a partition. We set $|\alpha| = \sum_i \alpha_i$, we say that α is a partition of $|\alpha|$. Consider the symmetric group S_n acting on n letters. The irreducible representations of S_n are parametrized by the partitions of n , let $[\alpha]$ denote the representation corresponding to α . The Kronecker coefficients $k_{\alpha\beta\gamma}$, depending on three partitions α , β and γ of the same integer n , are defined by the identity

$$[\alpha] \otimes [\beta] = \sum_{\gamma} k_{\alpha\beta\gamma} [\gamma]. \quad (18)$$

We will prove the following well known result due to Murnaghan and Littlewood (see [Mur55]).

Corollary 1 (i) *If $k_{\alpha\beta\gamma} \neq 0$ then we have*

$$(n - \alpha_1) + (n - \beta_1) \geq n - \gamma_1. \quad (19)$$

(ii) *We now assume that equality holds in formula (19) but not necessarily that $k_{\alpha\beta\gamma} \neq 0$. Let us define $\bar{\alpha} = (\alpha_2 \geq \alpha_3 \dots)$ and similarly $\bar{\beta}$ and $\bar{\gamma}$. Then we have*

$$k_{\alpha\beta\gamma} = c_{\bar{\alpha}\bar{\beta}}^{\bar{\gamma}}, \quad (20)$$

where $c_{\bar{\alpha}\bar{\beta}}^{\bar{\gamma}}$ is the Littlewood-Richardson coefficient.

Proof. Let us first introduce some notation on linear group. Let V be a complex finite dimensional vector space and let $GL(V)$ be the corresponding linear group. If α is a partition with at most $\dim(V)$ parts, $S^\alpha V$ denotes the Schur power of V ; it is an irreducible $GL(V)$ -module. Let $\mathcal{Fl}(V)$ denote the variety of complete flags of V . Given integers a_i such that $1 \leq a_1 < \dots < a_s \leq \dim(V) - 1$, we denote by $\mathcal{Fl}(a_1, \dots, a_s; V)$ the set of flags $V_1 \subset \dots \subset V_s \subset V$ such that $\dim(V_i) = a_i$ for any i .

Let us choose integers e and f such that

$$\begin{cases} l(\alpha) \leq e, \\ l(\beta) \leq f, \\ l(\gamma) \leq e + f - 1. \end{cases} \quad (21)$$

Let E and F be two complex vector spaces of dimension e and f . Consider the group $G = \mathrm{GL}(E) \times \mathrm{GL}(F)$. The Kronecker coefficient $k_{\alpha\beta\gamma}$ can be interpreted in terms of representations of G . Namely (see for example [Mac95, Ful91]) $k_{\alpha\beta\gamma}$ is the multiplicity of $S^\alpha E \otimes S^\beta E$ in $S^\gamma(E \otimes F)$. To interpret this multiplicity geometrically, we consider the variety

$$X = \mathcal{F}l(E) \times \mathcal{F}l(F) \times \mathcal{F}l(1, \dots, e+f-1; E \otimes F)$$

endowed with its natural G -action. Consider the $\mathrm{GL}(E)$ -linearized line bundle \mathcal{L}_α on $\mathcal{F}l(E)$ associated to α , and respectively \mathcal{L}_β on $\mathcal{F}l(F)$. Because of assumption (21), to γ we can associate a $\mathrm{GL}(E \otimes F)$ -linearized line bundle \mathcal{L}_γ on $\mathcal{F}l(1, \dots, e+f-1; E \otimes F)$. Consider the line bundle $\mathcal{L} = \mathcal{L}_\alpha \otimes \mathcal{L}_\beta \otimes \mathcal{L}_\gamma$ on X endowed with its natural G -action. Then

$$k_{\alpha\beta\gamma} = \dim(H^0(X, \mathcal{L})^G). \quad (22)$$

Let H_E, H_F, l_E and l_F be hyperplanes and lines respectively in E and F such that $E = H_E \oplus l_E$ and $F = H_F \oplus l_F$. Let λ be the one-parameter subgroup of G acting on H_E and H_F with weight 1 and on l_E and l_F with weight 0. Let C_E be the set of complete flags of E whose the hyperplane is H_E . Note that, C_E is an irreducible component of $\mathcal{F}l(E)^\lambda$. Similarly, we define C_F . Let $C_{E \otimes F}$ be the set of points $V_1 \subset \dots \subset V_{e+f-1}$ in $\mathcal{F}l(1, \dots, e+f-1; E \otimes F)$ such that $V_1 = l_E \otimes l_F$ and $V_{e+f-1} = (l_E \otimes l_F) \oplus (H_E \otimes l_F) \oplus (l_E \otimes H_F)$. Note that, $C_{E \otimes F}$ is an irreducible component of $\mathcal{F}l(1, \dots, e+f-1; E \otimes F)^\lambda$ isomorphic to $\mathcal{F}l(H_E \oplus H_F)$. Finally, set $C = C_E \times C_F \times C_{E \otimes F}$.

Note that $C_{E \otimes F}^+$ is open in $\mathcal{F}l(1, \dots, e+f-1; E \otimes F)$, (C_E, λ) and (C_F, λ) are covering in $\mathcal{F}l(E)$ and $\mathcal{F}l(F)$ for the actions of $\mathrm{GL}(E)$ and $\mathrm{GL}(F)$. It follows that (C, λ) is covering.

Let x be a point in C . Let $\mu^\mathcal{L}(x, \lambda)$ be the opposite of the weight of the action of λ on the fiber of \mathcal{L} over x . Now, [Res10a, Lemma 3] implies that if $\dim(H^0(X, \mathcal{L})^G) > 0$ then $\mu^\mathcal{L}(x, \lambda) \leq 0$ which is the inequality of the corollary. We now assume that $\mu^\mathcal{L}(x, \lambda) = 0$, that is that λ acts trivially on $\mathcal{L}|_C$. Theorem 1 shows that

$$\dim(H^0(X, \mathcal{L})^G) = \dim(H^0(C, \mathcal{L}|_C)^{G^\lambda}).$$

Moreover, $\dim(H^0(C, \mathcal{L}|_C)^{G^\lambda})$ is the multiplicity of the simple $\mathrm{GL}(H_E) \times \mathrm{GL}(H_F)$ -module $S^{\tilde{\alpha}} H_E \otimes S^{\tilde{\beta}} H_F$ in the $\mathrm{GL}(H_E \oplus H_F)$ -module $S^{\tilde{\gamma}}(H_E \oplus H_F)$. By for example [Mac95, Chapter I, 5.9], this multiplicity is precisely $c_{\tilde{\alpha}\tilde{\beta}}^{\tilde{\gamma}}$. \square

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