Unique continuation and extensions of Killing vectors at boundaries for stationary vacuum space-times

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UNIQUE CONTINUATION AND EXTENSIONS OF KILLING VECTORS AT BOUNDARIES FOR STATIONARY VACUUM SPACE-TIMES

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ABSTRACT. Generalizing Riemannian theorems of Anderson-Herzlich and Biquard, we show that two $(n + 1)$-dimensional stationary vacuum space-times (possibly with cosmological constant $\Lambda \in \mathbb{R}$) that coincide up to order one along a timelike hypersurface $\mathscr{T}$ are isometric in a neighbourhood of $\mathscr{T}$. We further prove that KIDS of ∂M extend to Killing vectors near ∂M. In the AdS type setting, we show unique continuation near conformal infinity if the metrics have the same conformal infinity and the same undetermined term. Extension near ∂M of conformal Killing vectors of conformal infinity which leave the undetermined Fefferman-Graham term invariant is also established.

CONTENTS

1. INTRODUCTION

Unique continuation theorems for initial data sets with Killing Initial Data (KIDs) are of current interest (see, e.g., [2,13] and references therein). Such theorems are relevant for uniqueness theorems for stationary solutions [1,4]. In a recent paper [8] (see also [5]) unique continuation theorems for the Riemannian Einstein equations have been established. The aim of this work is to point out that Biquard's arguments [8] generalize in a rather straightforward manner to initial data sets with a timelike KID. More precisely, we prove that the stationary Einstein equations, viewed as equations satisfied by the initial data on a space-like surface, have the unique continuation property at timelike hypersurfaces. While this is hardly surprising, since

stationary solutions are analytic in the interior in an appropriate atlas [16], it should be borne in mind that the last reference does not apply to solutions with boundary, as considered here. In particular the solution need not to be analytic up to the boundary. See Section 6 for further comments on this issue.

The precise result is as follows (compare [8, Theorem 4]). Consider a space-time $\mathscr{M} = \mathbb{R} \times M$, where M is an n-dimensional manifold with smooth compact boundary ∂M; we denote by t the coordinate along the $\mathbb{R}$ factor. Let g_- be a smooth Lorentzian metric on $\mathscr{M}$, with Killing vector $X = \partial/\partial t$. In adapted coordinates such a metric can be written as

$$(1.1) \quad g_- = -V^2(dt + \underbrace{\theta_i dx^i}_{=\theta})^2 + \underbrace{g_{ij} dx^i dx^j}_{=g_+}, \quad \text{with} \quad \partial_t V = \partial_t \theta = \partial_t g_+ = 0 \ .$$

We assume that the metric is Einstein,

$$(1.2) \qquad\qquad \text{Ric}(g_-) = \frac{2}{n-1}\Lambda g_- \ ,$$

where Λ is a constant. In Section 4 we prove:

THEOREM 1.1. *Let $n = \dim M \geq 2$, and consider two C^∞ stationary Lorentzian Einstein metric of the form (1.1), with strictly positive V near ∂M, inducing the same metric on ∂M. If the second fundamental forms of $\mathbb{R} \times \partial M$ coincide, the metrics are pull-backs of each other near $\mathbb{R} \times \partial M$.*

An infinitesimal version of Theorem 1.1 is also valid (compare [8, Theorem 2]):

THEOREM 1.2. *Let h be a smooth t–independent solution of the linearization of the equation (1.2) at a stationary solution as (1.1), defined near the boundary, and smooth up to the boundary. Assume that h has no dx components[1] in a Gauss coordinate system near the boundary, where x is the distance to the boundary on M, and that $h = o(x^2)$. Then $h \equiv 0$ near ∂M.*

In a manner analogous to the Lorentzian case [6], couples (α, ν) on ∂M that satisfy equations (5.8)-(5.9) below (which would be satisfied at ∂M by normal and tangential parts of a Killing form on M) will be called *Killing Initial Data* (KIDs) on ∂M. In the special case $\alpha = 0$, this reduces to the condition that $\nu^\#$ is a Killing vector of the metric induced on ∂M, the flow of which leaves the extrinsic curvature of ∂M invariant. As a Corollary of Theorem 1.2, in Section 5.3 we prove:

THEOREM 1.3. *Let $n = \dim M \geq 2$, and let g_- by a C^∞ stationary Lorentzian Einstein metric of the form (1.1), with strictly positive V near ∂M. Then any time-independent KID on $\mathbb{R} \times \partial M$ arises from the restriction to $\mathbb{R} \times \partial M$ of a time-independent Killing vector defined on a neighbourhood of $\mathbb{R} \times \partial M$.*

There exist topological obstructions for global extensions, compare Remark 5.2 below.

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[1]More precisely, h as neither $dxdy$ nor $dydx$ components whatever $y = t, x$, or any coordinate on ∂M.

In the same vein, a unique continuation result is established for a stationary metric g_-, satisfying the vacuum Einstein equations with a negative cosmological constant and admitting a C^2 conformal completion at infinity, with a smooth conformal class at the conformal boundary. From [9, Theorem 7.1] the metric g_- is then polyhomogeneous.

Then, using a suitable coordinate ρ near ∂M that vanishes at ∂M, the metric takes the form
$$g_- = \rho^{-2}(d\rho^2 + G(\rho)),$$
where $G(\rho)$ is a family of Lorentzian stationary metrics on $\mathbb{R} \times \partial M$, which are Einstein and thus admit the Fefferman-Graham expansions [11, 12][2] :

$$G(\rho) = \begin{cases} G_0 + G_2\rho^2 + ... + G_n\rho^n + ... & n \text{ odd }, \\ G_0 + G_2\rho^2 + ... + \mathcal{O}\rho^n \ln\rho + G_n\rho^n + ... & n \text{ even }, \end{cases}$$

where all the coefficients G_i are determined by G_0, the conformal infinity and the undetermined term G_n. In particular if two metrics as above have the same G_0 and G_n in a common coordinate system, then they coincide to infinite order.

In the conformally compact setting our result, proved in Section 3, reads:

THEOREM 1.4. *Let $n = \dim M \geq 2$, and consider two C^2 compactifiable stationary Lorentzian Einstein metrics, with negative cosmological constant, of the form* (1.1) *which define the same class $[(G_0, G_n)]$. Then the metrics are diffeomorphic near infinity.*

REMARK 1.5. *Theorem 1.4 has also its infinitesimal version similar to Theorem 2 in [8], which we do not spell out in detail here.*

Note that large families of stationary Lorentzian Einstein metrics as above have been constructed in [9].

Consider, next, the associated problem of conformal isometries extension from a conformal boundary at infinity. Such boundary maps naturally decouple into conformal isometries of the boundary which can be made into isometries by an appropriate choice of the conformal factor, and those that cannot. Here we only consider the former, and in Section 5.3 we prove (compare [3, 9, 17] for results under different conditions):

THEOREM 1.6. *Let $n = \dim M \geq 2$, and let g be C^2 Lorentzian Einstein metric of the form* (1.1) *on $\mathbb{R} \times M$, with negative cosmological constant, and with a smooth conformal boundary at infinity. Let X be a conformal Killing vector of the conformal boundary which is a Killing vector for some choice of the conformal factor at the boundary. Then X extends to a Killing vector field defined near conformal infinity if and only if X leaves the associated undetermined term invariant.*

The proofs are an adaptation of our context of a related analysis of Biquard in the Riemannian setting [8]. Here one needs to control some new terms related to the stationary Einstein equations, which did not occur in Biquard's problem.

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[2]The original references assume a Riemannian signature at the boundary, but the expansions are independent of this.

2. Definitions, notations and conventions

Let $\overline{M}$ be a smooth, compact n-dimensional manifold with boundary ∂M, thus $M := \overline{M} \backslash \partial M$ is a non-compact manifold without boundary. As already mentioned, we consider both the case where the boundary is at finite distance, and the case where the boundary ∂M is a *conformal boundary at infinity*: we say that a Riemannian manifold (M, g) is *conformally compact* if there exists on $\overline{M}$ a smooth defining function ρ for ∂M (that is $\rho \in C^\infty(\overline{M})$, $\rho > 0$ on M, $\rho = 0$ on ∂M and $d\rho$ nowhere vanishing on ∂M) such that $\overline{g} := \rho^2 g$ is a $C^{2,\alpha}(\overline{M}) \cap C_0^\infty(M)$ Riemannian metric on $\overline{M}$. We will denote by $\widehat{g}$ the metric induced on ∂M. Our definitions of function spaces follow [14]. Now if $|d\rho|_{\overline{g}} = 1$ on ∂M, it is well known (see [15] for instance) that g has asymptotically sectional curvature -1 near its boundary at infinity, in that case we say that (M, g) is *asymptotically hyperbolic*.

We recall that the Lichnerowicz Laplacian acting on a symmetric two-tensor field is defined as [7, § 1.143]

$$\Delta_L h_{ij} = -\nabla^k \nabla_k h_{ij} + R_{ik} h^k{}_j + R_{jk} h^k{}_j - 2R_{ikjl} h^{kl} .$$

The operator Δ_L arises naturally when linearizing the Ricci operator. Let us define the divergence of a covariant two tensor h:

$$(\delta h)_j := -\nabla^i h_{ij},$$

with symmetrized adjoint

$$(\delta^* \omega)_{ij} := \frac{1}{2}(\nabla_i \omega_j + \nabla_j \omega_i).$$

We use the following sign convention for the divergence of a one form:

$$d^* \omega := -\nabla^i \omega_i.$$

We denote by $\mathcal{T}_p^q$ the set of rank p covariant and rank q contravariant tensors. When $p = 2$ and $q = 0$, we denote by $\mathcal{S}_2$ the subset of symmetric tensors. We use the summation convention, indices are lowered and raised with g_{ij} and its inverse g^{ij}.

3. Proof in the AdS type setting

In space-time dimension $n + 1$ we consider a Lorentzian metric g_- of the form

$$g_- = -V^2(dt + \theta)^2 + g_+ = \rho^{-2}(d\rho^2 + \widetilde{G}(\rho)),$$

where g_+ is Riemannian. Thus $\widetilde{G}(\rho)$ is a family of Lorentzian metrics, parameterized by ρ, of the form

$$\rho^{-2}\widetilde{G}(\rho) = -V^2(dt + \theta)^2 + \rho^{-2}\widetilde{g}(\rho) .$$

Here $\widetilde{g}(\rho)$ can be thought of as a family of Riemannian metrics defined on the $(n - 1)$-dimensional level sets of ρ, and note that

$$g_+ = \rho^{-2}(d\rho^2 + \widetilde{g}(\rho)) .$$

It is convenient to introduce a coordinate $r = -\ln \rho$, and to write $\rho^{-2}\widetilde{G}(\rho) = G(r) = G$ and $\rho^{-2}\widetilde{g}(\rho) = g(r) = g$. Thus

$$g_- = dr^2 + G,$$

where
$$G = -V^2(dt + \theta)^2 + g,$$
so
$$G^{-1} = (|\theta|^2 - V^{-2})\partial_t^2 - (\theta^\# \otimes \partial_t + \partial_t \otimes \theta^\#) + g^{-1}.$$
The second fundamental forms of the level sets of r are
$$\mathbb{I}_- = \frac{1}{2}G' = -VV'(dt + \theta)^2 - V^2\frac{1}{2}[(dt + \theta) \otimes \theta' + \theta' \otimes (dt + \theta)] + \mathbb{I},$$
where primes denote partial r-derivatives and[3]
$$\mathbb{I} = \frac{1}{2}g'.$$
Let also define the mean curvature
$$H_- = \mathrm{Tr}_G\, \mathbb{I}_- = V^{-1}V' + H.$$
Rescaling the metric to achieve a convenient normalization of the constant $\Lambda = -n(n-1)/2$, the vacuum Einstein equations for a metric satisfying (1.1)-(1.2) read (see, e.g., [10])

$$(3.1) \qquad\qquad V(\nabla_{g_+}^* \partial V + nV) = \frac{1}{4}|{}^+\lambda|_g^2 ,$$

$$(3.2) \qquad\qquad \mathrm{Ric}(g_+) + ng_+ - V^{-1}\mathrm{Hess}_{g_+} V = \frac{1}{2V^2}{}^+\lambda \circ {}^+\lambda ,$$

$$(3.3) \qquad\qquad \delta_{g_+}(V^+\lambda) = 0 ,$$

where
$$({}^+\lambda)_{ij} = -V^2(\partial_i\theta_j - \partial_j\theta_i) , \quad ({}^+\lambda \circ{}^+ \lambda)_{ij} = ({}^+\lambda)_i{}^k({}^+\lambda)_{kj} .$$
As $g_+ = dr^2 + g$, the non trivial Christoffel symbols of g_+ are
$${}^+\Gamma_{AB}^r = -\mathbb{I}_{AB} , \quad {}^+\Gamma_{rB}^A = g^{AC}\mathbb{I}_{CB} , \quad {}^+\Gamma_{BC}^A = \Gamma_{BC}^A,$$
in particular, the Hessian of V is given by
$${}^+\mathrm{Hess}_{rr} V = V'' , \quad {}^+\mathrm{Hess}_{rA} V = d_A V' - g^{BC}\mathbb{I}_{AB}d_C V ,$$
$${}^+\mathrm{Hess}_{AB} V = (\mathrm{Hess}_g)_{AB}V + V'\mathbb{I}_{AB}.$$
Let us recall that θ is purely tangential:
$$\theta = 0\, dr + \xi .$$
Then ${}^+\lambda = -V^2 d_+\theta$ (here d_+ means the differential on M), which gives
$${}^+\lambda_{rr} = 0 , \quad {}^+\lambda_{rA} = -V^2\xi_A' ,$$
$${}^+\lambda_{AB} = -V^2(\partial_A\xi_B - \partial_B\xi_A) = -V^2(d\xi)_{AB} =: \lambda_{AB} .$$
So
$$({}^+\lambda \circ {}^+\lambda)_{rr} = V^4|\xi'|^2,$$
$$({}^+\lambda \circ {}^+\lambda)_{rA} = -V^2(\xi_C')g^{CD}(r)\lambda_{AD},$$
$$({}^+\lambda \circ {}^+\lambda)_{AB} = V^4(\xi_A')(\xi_B') + g^{CD}(r)\lambda_{AD}\lambda_{BC},$$
$$|{}^+\lambda|^2 = \mathrm{Tr}_g({}^+\lambda \circ {}^+\lambda) = 2V^4|\xi'|^2 + |\lambda|^2,$$

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[3]The reader is warned that our definition is the negative of that in [8].

where λ is the restriction of $^{+}\lambda$ to the level sets of r. Equation (3.2) is equivalent to

$$(3.4)\ \mathrm{Ric}(g) - H\,\mathbb{I} - \mathbb{I}' + 2\mathbb{I} \circ \mathbb{I} + ng - V^{-1}\,\mathrm{Hess}_g\,V - V^{-1}V'\mathbb{I}$$
$$= \frac{1}{2}V^2(\xi') \otimes (\xi') + \frac{1}{2}V^{-2}\lambda \circ \lambda\ ,$$

$$(3.5)\qquad -\delta_g\mathbb{I} - dH - V^{-1}[dV' - \mathbb{I}(\nabla V, \cdot)] = \frac{1}{2}\lambda(\cdot, (-\xi'))\ ,$$

$$(3.6)\qquad -H' - |\mathbb{I}|^2 + n - V^{-1}V'' = \frac{1}{2}V^2|\xi'|^2\ .$$

Equations (3.5) and (3.6) can be rewritten, respectively, as

$$(3.7)\qquad \delta_g\mathbb{I} = -dH_- - V^{-2}V'dV + V^{-1}\mathbb{I}(\nabla V, \cdot) - \frac{1}{2}\lambda(\cdot, -\xi')\ ,$$

$$(3.8)\qquad H'_- = n - |\mathbb{I}|^2 - (V^{-1}V')^2 - \frac{1}{2}V^2|\xi'|^2\ .$$

The system (3.3) is equivalent to

$$(3.9)\qquad\qquad\qquad d^*(V^3\xi') = 0\ ,$$

$$(3.10)\qquad (V^3\xi')' + \delta(V\lambda) - HV^3\xi' - 2V^2\mathbb{I}(\xi', \cdot) = 0\ .$$

We want to show how Biquard's reduction of the Riemannian vacuum Einstein equations to an elliptic system generalizes to the problem at hand. From the linearization of the Ricci curvature operator (see eg. [7]) we have

$$\mathrm{Ric}(g)' = \Delta_L(\mathbb{I}) - 2\delta^*\delta\mathbb{I} - \delta^*dH\ ,$$

and so we obtain that the r-derivative of equation (3.4) reads

$$(3.11)\qquad \Delta_L(\mathbb{I}) - \mathbb{I}'' - 2\delta^*\delta\mathbb{I} - \delta^*dH - V^{-1}\,\mathrm{Hess}\,V'$$
$$+V^{-2}V'\,\mathrm{Hess}\,V + \text{ first order in } (\mathbb{I}, \xi', V') = 0\ .$$

Using (3.7), and the fact that

$$\delta^*dH_- = \delta^*dH + V^{-1}\,\mathrm{Hess}\,V' - V^{-2}V'\,\mathrm{Hess}\,V$$
$$+2V^{-3}V'dV \otimes dV - V^{-2}(dV \otimes dV' + dV' \otimes dV),$$

together with (3.11), one is led to an equation which is elliptic for $\mathbb{I}$ if one disregards the fact that H_- is related to $\mathbb{I}$:

$$(3.12)\qquad \nabla^*\nabla\mathbb{I} - \mathbb{I}'' + \text{ first order in } (\mathbb{I}, \xi', V') = -\delta^*dH_-\ .$$

Next, the r-derivative of (3.10) reads

$$(3.13)\qquad V^3(\xi''' - \delta d\xi') + \text{ first order in } (\mathbb{I}, \xi', V') = 0\ .$$

Inserting equality (3.9) in the last equation and dividing by $-V^3$ gives an elliptic equation for ξ':

$$(3.14)\qquad \nabla^*\nabla\xi' - (\xi')'' + \text{ first order in } (\mathbb{I}, \xi', V') = 0\ .$$

Finally the r-derivative of (3.1) divided by V gives an elliptic equation for V':

$$(3.15)\qquad \nabla^*\nabla V' - (V')'' + \text{ first order in } (\mathbb{I}, \xi', V') = 0\ .$$

Combining the three equations (3.12), (3.14) and (3.15), and momentarily ignoring that H_- is not an independent field, gives an elliptic system for $(V', \xi', \mathbb{I})$ or, more geometrically, an elliptic system of equations for

$$\mathbb{A} := VV'dt^2 + \frac{1}{2}V^2(\xi' \otimes dt + dt \otimes \xi') + \mathbb{I}.$$

Define now the metric

$$\mathcal{G} := V^2 dt^2 + g.$$

So the $\mathcal{G}$-norm of $\mathbb{A}$ is:

$$|\mathbb{A}|^2 = |\mathbb{I}|^2 + (V^{-1}V')^2 + \frac{1}{2}V^2|\xi'|^2,$$

and the $\mathcal{G}$-trace of $\mathbb{A}$ is H_-. In particular, (3.8) becomes

(3.16) $$H'_- = n - |\mathbb{A}|^2 .$$

Assume we have two stationary Einstein metrics g_- and $(g_0)_-$, and suppose that there exists a choice of conformal factors at the boundary so that the metrics coincide to order n at infinity in their respective coordinates (ρ, x^A) as above. Then they coincide to infinite order. We wish to show they are equal near infinity. We will put a subscript 0 for all quantities relative to $(g_0)_-$. First, we show that all quantities relative to the difference between the metrics are controlled by quantities relative to the difference of the second fundamental form of the level set of r, and the same is true for the difference between the mean curvature.

As in [8], the simplest control comes from the fact that

(3.17) $$\mathbb{I} - \mathbb{I}_0 = \frac{1}{2}(g - g_0)' ,$$

so from Equation (10) in [8] for $s > 2$,

(3.18) $$\int_{r_0}^{\infty} |\mathbb{I} - \mathbb{I}_0|_{g_0}^2 e^{2sr} dr \geq C^{-1}s^2 \int_{r_0}^{\infty} |g - g_0|_{g_0}^2 e^{2sr} dr .$$

Commuting (3.17) with derivatives, it is standard to obtain

(3.19) $$\int_{r_0}^{\infty} \sum_{i=0}^{k} |\nabla_0^{(i)}(\mathbb{I} - \mathbb{I}_0)|_{g_0}^2 e^{2sr} dr$$

$$\geq C^{-1}s^2 \int_{r_0}^{\infty} \sum_{i=0}^{k} |\nabla_0^{(i)}(g - g_0)|_{g_0}^2 e^{2sr} dr .$$

REMARK 3.1. Replacing (3.17) by $(V^2)' - (V_0^2)' = [(V^2) - (V_0^2)]'$, we will obtain the same kind of integral inequality comparing $(V^2)' - (V_0^2)'$ with $(V^2) - (V_0^2)$.

We also have (see (3.16))

(3.20) $$[H_- - (H_0)_-]' = |\mathbb{A}_0|_{\mathcal{G}_0}^2 - |\mathbb{A}|_{\mathcal{G}}^2 .$$

Now recall that $\mathcal{G}$ and all of its derivatives are bounded relatively to $\mathcal{G}_0$, so (see Lemma A.1, Appendix A for details)

$$\left||\mathbb{A}_0|_{\mathcal{G}_0}^2 - |\mathbb{A}|_{\mathcal{G}}^2\right| \leq C(|\mathbb{A}_0 - \mathbb{A}|_{\mathcal{G}_0} + |\mathcal{G}_0 - \mathcal{G}|_{\mathcal{G}_0}).$$

Combining this with equation (3.19), Remark 3.1 and (derivatives of) equation (3.20), shows that

$$(3.21) \qquad \int_{r_0}^{\infty} \sum_{i=0}^{k} |D_0^{(i)}(\mathbb{A} - \mathbb{A}_0)|^2_{\mathcal{G}_0^+} e^{2sr} dr$$

$$\geq C^{-1} s^2 \int_{r_0}^{\infty} \sum_{i=0}^{k} |D_0^{(i)}[H_- - (H_0)_-]|^2_{\mathcal{G}_0^+} e^{2sr} dr \ ,$$

where D_0 is the covariant derivative relative to $dr^2 + \mathcal{G}_0$, $\mathbb{A}$ is identified with $0 dr^2 + \mathbb{A}$ and the same for $\mathbb{A}_0$.

The rest of the proof is a straightforward adaptation of Section 3 of [8] where $\mathbb{I}$ there correspond to $\mathbb{A}$ here, g there correspond to $dr^2 + \mathcal{G}$ here, etc. The argument there shows that $\mathbb{A} = \mathbb{A}_0$ so $g_- = (g_0)_-$. This concludes the proof of Theorem 1.4.

4. Boundary at finite distance

We are interested now in stationary Lorentzian metrics (see (1.1)), solutions of

$$(4.1) \qquad \qquad \mathrm{Ric}(g_-) = \lambda g_- \ ,$$

where λ is a constant, and where the $\{t = 0\}$ slice is a compact manifold M with smooth boundary ∂M. All quantities $(V, \theta, g_+, ...)$ are then assumed to be smooth up to the boundary. If two metrics g_- and $(g_0)_-$ as above coincide on the boundary together with their second fundamental form, then (up to a diffeomorphism) they coincide to order one and then to infinite order. We will show they are in fact equal near the boundary. The proof proceeds exactly as in Section 3, where we replace the parameter r by x, the distance to the boundary. Henceforth we write

$$g_- = -V^2(dt + \theta)^2 + g_+ = dx^2 + G(x),$$

thus

$$G(x) = -V^2(dt + \theta)^2 + g(x),$$

where $g_+ = dx^2 + g(x) = dx^2 + g$. So we can write

$$g_- = dx^2 + G,$$

where

$$G = -V^2(dt + \theta)^2 + g \ .$$

Then, for example, (3.19) is replaced by
$$(4.2)$$
$$\int_0^{x_0} |\nabla_0^{(2)}(\mathbb{I} - \mathbb{I}_0)|^2_{g_0} x^{-2s} dx \geq C^{-1} s^2 \int_0^{x_0} \sum_{i=0}^{2} s^{4-2i} x^{2i-4} |\nabla_0^{(i)}(g - g_0)|^2_{g_0} x^{-2s} dx \ ,$$

with similar other obvious changes in the remainder of the argument. This then gives the result for a boundary at finite distance.

5. KID extensions

5.1. **The Riemannian case.** We first give the result in the Riemannian setting, which does not seem to have appeared in the literature before. Let (M, g_+) be smooth n-dimensional Riemannian manifold with smooth compact boundary ∂M. Take x to be the geodesic distance from the boundary. Near ∂M the metric takes the form

$$(5.1) \qquad g_+ = dx^2 + g,$$

where $g = g(x) = g_{AB} dx^A dx^B$ is a family of metrics, parameterized by x, on ∂M. Let us define $\mathbb{I} = \frac{1}{2} g'$, $H = Tr \mathbb{I}$ and $(\delta h)_j = -\nabla^i h_{ij}$ then, the non trivial Christoffel symbols are

$$^+\Gamma_{AB}^x = -\mathbb{I}_{AB}, \quad {}^+\Gamma_{Ax}^C = \mathbb{I}_A^C, \quad {}^+\Gamma_{AB}^C = \Gamma(g)_{AB}^C,$$

in particular one has (see the Gauss and Codazzi equations in [7] for instance)

$$(5.2) \qquad \mathrm{Ric}(g_+)_{AB} = (\mathrm{Ric}(g) - H\,\mathbb{I} - \mathbb{I}' + 2\mathbb{I} \circ \mathbb{I})_{AB},$$

$$(5.3) \qquad \mathrm{Ric}(g_+)_{xA} = -\delta_g \mathbb{I} - dH,$$

$$(5.4) \qquad \mathrm{Ric}(g_+)_{xx} = -H' - |\mathbb{I}|^2.$$

Given $\omega = \alpha dx + \nu$, a one form decomposed in normal and tangential parts, let $h = (\mathcal{L}_{\omega^\#} g_+)$, then

$$(5.5) \qquad h_{xx} = 2\alpha',$$

$$(5.6) \qquad h_{xA} = (\nu_A' - 2\mathbb{I}_A^C \nu_C + d_A \alpha) = g_{AB}(\nu^B)' + d_A \alpha,$$

$$(5.7) \qquad h_{AB} = (\mathcal{L}_{\nu^\#} g + 2\alpha \mathbb{I})_{AB}.$$

We then see that if $\omega^\#$ is a Killing vector field of M, then $\alpha' = 0$.

We are interested in the Riemannian equivalent of *Killing initial data*, which we continue to call KIDs. By definition, these are couples (α, ν) on ∂M that satisfy equations which would be satisfied by normal and tangential parts of a Killing form ω on M.

We first find necessary conditions on (α, ν) on ∂M assuming $h \equiv 0$. Using $h_{AB}(0) = h_{AB}'(0) = 0$ (also using (5.6) to replace $(\nu^\#)'$ and (5.2), assuming g_+ is Einstein, to replace $\mathbb{I}'$ when calculating $h_{AB}'(0)$), we obtain the Riemannian KID equations:

$$(5.8) \qquad \mathcal{L}_{\nu^\#} g + 2\alpha \mathbb{I} = 0,$$

$$(5.9) \qquad 2\mathcal{L}_{\nu^\#} \mathbb{I} - \mathcal{L}_{\nabla \alpha} g + 2\alpha [\mathrm{Ric}(g) - \lambda - H\mathbb{I} + 2\mathbb{I} \circ \mathbb{I}] = 0.$$

Reciprocally, assuming (5.8) and (5.9) on ∂M, one can define $\alpha(x) = \alpha(0)$ and $\nu(x)$ such that $h_{xA} = 0$ in (5.6), that is $\nu^\#$ solves $(\nu^\#)' = -\mathrm{grad}_g \alpha$. Then h is purely tangential (i.e., h has no dx components) and $h(0) = h'(0) = 0$. Now as g_+ is Einstein:

$$\mathrm{Ein}(g_+) := \mathrm{Ric}(g_+) - \lambda g_+ = 0,$$

then $D\mathrm{Ein}(g_+) h = 0$; this can be seen by a direct calculation, or by considering the Einstein metric $g_+^t = \Phi_t^* g_+$, where Φ_t is the local flow of $\omega^\#$. From [8, Theorem 2] we can conclude that $h \equiv 0$ near ∂M so $\omega^\#$ is a Killing

near ∂M. We have thus proved the following Riemannian equivalent of Theorem 1.3:

THEOREM 5.1. *Let $n = \dim M \geq 2$, and let g_+ by a C^∞ Einstein metric on M. Then any KID on ∂M arises from the restriction to ∂M of a Killing vector defined on a neighborhood of ∂M.*

REMARK 5.2. The extensions above are not necessarily global. For example, consider a sufficiently small ball $B(p, \varepsilon)$ in a flat torus $\mathbb{T}^n$. If $\partial B(p, \varepsilon)$ is viewed as the boundary of the ball, then every KID of the boundary extends to a globally defined Killing vector in the interior. However, if $\partial B(p, \varepsilon)$ is viewed as the boundary of $\mathbb{T}^n \setminus B(p, \varepsilon)$, then only those KIDs which correspond to translations of the torus extend globally.

5.2. **The conformally compact Riemannian case.** We now treat the case where (M, g_+) is conformally compact, Einstein, with a smooth conformal boundary at infinity. Let ρ be a defining function such that $|d\rho|_{\overline{g}_+} = 1$ near ∂M. We have

$$\overline{g}_+ = d\rho^2 + \overline{g},$$

where $\overline{g} = \overline{g}(\rho) = \overline{g}_{AB} dx^A dx^B$ is a family of metrics on ∂M.

Let us define $r = -\ln \rho$. Near ∂M, the metric g_+ takes the form

$$g_+ = dr^2 + g,$$

where $g = g(r) = g_{AB} dx^A dx^B$ is a family of metrics on ∂M. We then recover the form given for finite distance boundary with x replaced by r.

To establish the "only if" part of Theorem 5.1, consider a Killing field $\omega^\#$ for g_+. Denoting by $\nu^\#(0)$ the part of $\omega^\#(0)$ tangent to ∂M, we further suppose that $\nu^\#(0)$ is a Killing vector field for $\overline{g}(0)$. In the notation of [4, Appendix A], we then see by the equations there that $\alpha \equiv 0$ and $\nu^\#(x) = \nu^\#(0)$. We thus find the necessary condition that

$$(5.10) \qquad\qquad \mathcal{L}_{\nu^\#(0)} \overline{g}_{(n-1)} = 0,$$

where $\overline{g}_{(n-1)}$ is the undetermined term in the Fefferman-Graham expansions [11, 12].

Reciprocally, if $X \equiv \nu^\#(0)$ is a Killing field on ∂M and condition (5.10) holds, we set $\alpha = 0$ and $\nu^\#(x) = \nu^\#(0)$. The tensor $h := \mathcal{L}_{\omega^\#} g_+$ is then purely tangential and $h = o(\rho^n)$. Also, as before, h is in the kernel of $D\mathrm{Ein}(g_+)$. We conclude by [8] that $h \equiv 0$ near ∂M so $\omega^\#$ is a Killing for g_+.

5.3. **The Lorentzian case.** In the stationary Lorentzian setting, we start by introducing Gauss coordinates for the space-time metric near $\mathbb{R} \times \partial M$. All calculations of Section 5.1 remain valid, except that now the metric g of (5.1) is Lorentzian instead of Riemannian. This does not affect the argument, since the time-derivatives of all fields involved drop out, and so the Biquard method leads again to elliptic equations in the space variables. From there the proof proceeds exactly in the same fashion, using our unique continuation Theorem 1.2 in place of the Biquard one. This readily proves Theorem 1.3.

The proof of Theorem 1.6 proceeds similarly, using the linearized equivalent of Theorem 1.4 (compare Remark 1.5). □

6. Concluding remarks

Some readers might be tempted to think that our finite-boundary-unique-continuation results are a trivial consequence of the usual analyticity results for stationary solutions of vacuum Einstein equations [16], for if the metric can be extended across the boundary in the stationary class, then it is analytic at the boundary, and unique continuation is straightforward. The following example shows that this is not the case: Recall that Weyl metrics, which are static axi-symmetric solutions of the vacuum Einstein equations, are uniquely described by axisymmetric solutions u of the flat-space Laplace equation. So choose some axi-symmetric simply-connected domain Ω with analytic boundary in $\mathbb{R}^3$, and let u be a harmonic function on Ω with *non-analytic* boundary values on $\partial\Omega$. We can choose Ω and u so that the Killing vector is uniformly timelike on Ω. Then u cannot be extended to a harmonic function defined on a set larger than Ω, otherwise its trace on $\partial\Omega$ would have been analytic. Thus the corresponding metric g cannot be extended in the Weyl class.

Now, if g could be extended in the stationary class, then the associated stationary Killing vector would be timelike in a neighborhood of $\partial\Omega$, and thus the metric would be analytic across Ω. By analyticity the extension would be static and axi-symmetric, and therefore in the Weyl class. But then u would be analytic on $\partial\Omega$. So g cannot be extended across $\partial\Omega$ in the stationary vacuum class, and an extension within the vacuum class, if any exist, cannot be analytic at $\partial\Omega$. In particular the metrics in this example admit no stationary vacuum extensions away from Ω. Nevertheless, by our results above, any KID on the boundary arises from a space-time Killing vector defined on a one-sided neighborhood of $\mathbb{R} \times \partial\Omega$.

Appendix A.

Lemma A.1.
$$\left| |\mathbb{A}_0|^2_{\mathcal{G}_0} - |\mathbb{A}|^2_{\mathcal{G}} \right| \leq C(|\mathbb{A}_0 - \mathbb{A}|_{\mathcal{G}_0} + |\mathcal{G}_0 - \mathcal{G}|_{\mathcal{G}_0}).$$

Proof. We first recall that, in view of our assumptions on the metric, we clearly have, for coordinates smooth up to the boundary:

$$V = O(e^r),\ g = O(e^{2r}),\ V' - V = O(e^{-r}),\ \zeta' = O(e^{-2r}),\ \mathbb{I} - g = O(1),$$

in particular, near the boundary, one has

$$\mathbb{I} \sim g,\ \mathbb{A} \sim \mathcal{G},$$

and the same is true for the V_0, g_0, etc. Let us compute

$$|\mathbb{A}|^2_{\mathcal{G}} = |\mathbb{I}|^2 + (V^{-1}V')^2 + \frac{1}{2}V^2|\xi'|^2,$$

$$|\mathbb{A}_0|^2_{\mathcal{G}_0} = |\mathbb{I}_0|^2_0 + (V_0^{-1}V_0')^2 + \frac{1}{2}V_0^2|\xi_0'|^2_0,$$

$$|\mathbb{A} - \mathbb{A}_0|^2_{\mathcal{G}_0} = |\mathbb{I} - \mathbb{I}_0|^2_0 + V_0^{-4}(VV' - V_0V_0')^2 + \frac{1}{2}V_0^{-2}|V^2\xi' - V_0^2\xi_0'|^2_0,$$

$$|\mathcal{G}_0 - \mathcal{G}|^2_{\mathcal{G}_0} = V_0^{-4}(V^2 - V_0^2)^2 + |g - g_0|^2_0.$$

We will show that each term appearing in $||\mathbb{A}|^2_{\mathcal{G}} - |\mathbb{A}_0|^2_{\mathcal{G}_0}|$ can be controlled by those in $|\mathbb{A} - \mathbb{A}_0|_{\mathcal{G}_0}$ and $|\mathcal{G}_0 - \mathcal{G}|_{\mathcal{G}_0}$.

First, formally (recall $\mathbb{I} \sim g \sim g_0 \sim \mathbb{I}_0$) we have

$$|\mathbb{I}|^2 - |\mathbb{I}_0|_0^2 = g^{-2}\mathbb{I}^2 - g_0^{-2}\mathbb{I}_0^2 = (g^{-2} - g_0^{-2})\mathbb{I}^2 + g_0^{-2}(\mathbb{I}^2 - \mathbb{I}_0^2) \leq C(|g - g_0|_0 + |\mathbb{I} - \mathbb{I}_0|_0).$$

We also have (recall $V' \sim V \sim V_0 \sim V_0'$)

$$
\begin{aligned}
|(V^{-1}V')^2 - (V_0^{-1}V_0')^2| &\leq C_1|VV' - V_0V_0'| \\
&\leq C_1 V_0^{-2}|VV' - V_0V_0' + V^{-1}V'(V_0^2 - V^2)| \\
&\leq C(V_0^{-2}|VV' - V_0V_0'| + V_0^{-2}|V_0^2 - V^2|).
\end{aligned}
$$

Finally, we can write

$$V^2|\xi'|^2 - V_0^2|\xi_0'|_0^2 = a + b + c,$$

where

$$
\begin{aligned}
|a| &= V_0^{-2}||V^2\xi'|_0^2 - |V_0^2\xi_0'|_0^2| \leq V_0^{-2}|V^2\xi' + V_0^2\xi_0'|_0|V^2\xi' - V_0^2\xi_0'|_0 \\
&\leq CV_0^{-1}|V^2\xi' - V_0^2\xi_0'|_0, \\
|b| &= V_0^{-2}V^4||\xi'|^2 - |\xi'|_0^2| \leq C_1 V^2|\xi'|^2|g - g_0|_0 \\
&\leq C|g - g_0|_0, \\
|c| &= V_0^{-2}|V_0^2 - V^2|V^2|\xi'|^2 \leq CV_0^{-2}|V_0^2 - V^2|.
\end{aligned}
$$

$\square$

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