



**HAL**  
open science

# Time compactness tools for discretized evolution equations and applications to degenerate parabolic PDEs

Boris Andreianov

► **To cite this version:**

Boris Andreianov. Time compactness tools for discretized evolution equations and applications to degenerate parabolic PDEs. 2011. hal-00561344v1

**HAL Id: hal-00561344**

**<https://hal.science/hal-00561344v1>**

Preprint submitted on 1 Feb 2011 (v1), last revised 24 Feb 2011 (v2)

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# Time compactness tools for discretized evolution equations and applications to degenerate parabolic PDEs

Boris Andreianov

**Abstract** We discuss several techniques for proving compactness of sequences of approximate solutions to discretized evolution PDEs, with applications to convergence of finite volume discretizations of degenerate parabolic equations. While the well-known Aubin-Simon kind functional-analytic techniques were recently generalized to the discrete setting by Gallouët and Latché [12], here we discuss direct techniques for estimating the time translates of approximate solutions in the space  $L^1$ . One important result is the Kruzhkov time compactness lemma. Further, we describe a specific technique that benefits from the order-preservation for the underlying PDE, and recall the well-known methods based on nonlinear weak-\* convergence and on the subsequent reduction of Young measures.

**Key words:** compactness, time translates, Kruzhkov lemma, order-preservation

## 1 Introduction

Let us think of evolution equations set on a cylindrical domain  $Q := (0, T) \times \Omega \subset \mathbb{R}^+ \times \mathbb{R}^N$ . Proving convergence of space-time discretizations of such equations often includes the three following steps : constructing discrete solutions and getting uniform (in appropriate discrete spaces) estimates; extracting a convergent subsequence; writing down a discrete weak formulation (e.g., with discretized test functions) and passing to the limit in the equation in order to infer convergence.

For the first step, obtention of estimates is greatly simplified by preservation, at the discrete level, of the key structure properties of the PDE (such as symmetry, coercivity, monotonicity of the diffusion operators involved; entropy dissipation, for the nonlinear convection operators in the degenerate parabolic case; etc.). For getting discrete *a priori* estimates, as in the continuous case test functions are often used; therefore, some analogues of integration-by-parts formulas and chain rules

---

Boris Andreianov  
CNRS UMR 6623, Besançon, France, e-mail: boris.andreianov@univ-fcomte.fr

are instrumental for the first step. For the examples we give in this paper, “discrete duality” type schemes (mimetic, co-volume, DDFV; see, e.g., [2] and references therein) can be used to guarantee an exact integration-by-parts feature. In contrast, chain rules for derivation in time or in space must be replaced by approximate analogues, often taking the form of convexity inequalities (see, e.g., [3], [2, Sect. 4]).

In this note, we give insight into convergence proofs for (different cases of) degenerate elliptic-parabolic-hyperbolic PDEs under the general form<sup>1</sup>

$$u = b(v), \quad u_t - \operatorname{div} [\mathbf{G}(v) - \mathbf{a}_0(\nabla\varphi(v))] + \psi(v) = f \quad \text{in } Q = (0, T) \times \Omega \quad (1)$$

with  $b(\cdot), \varphi(\cdot), \psi(\cdot)$  continuous<sup>2</sup> non-decreasing on  $\mathbb{R}$ , normalized by zero at zero, with a continuous convection flux  $\mathbf{G}(\cdot)$  and with  $\mathbf{a}_0 : \mathbb{R}^N \rightarrow \mathbb{R}^N$  of Leray-Lions type (see e.g. [1, 3];  $p$ -laplacian, with  $\mathbf{a}_0(\xi) = |\xi|^{p-2}\xi$  is a typical example). For the sake of simplicity, homogeneous Dirichlet boundary condition on  $(0, T) \times \partial\Omega$  is taken.

But our main goal is to discuss the second step of the proofs, the one of getting compact<sup>3</sup> sequences of discrete solutions (when the compactification methods strongly utilize a particular structure of the underlying PDE, this step is in fact combined with the third step of passing to the limit). For linear problems, the two latter steps are somewhat trivial; indeed, mere functional-analytic bounds would lead to compactness in a weak topology, which is enough to pass to the limit from the discrete to the continuous weak formulation of the PDE. Thinking of nonlinear problems and passage to the limit in nonlinear terms, bounds in functional spaces can be sufficient when combined with basic compact embeddings; but this requires rather strong bounds involving e.g. some estimates of the derivatives. Regarding evolution PDEs of, say, porous medium type,  $L^p$  bounds are available on the space derivatives but not on the time derivatives (those belong to some *negative* Sobolev spaces). In this situation, either compactness in an *ad hoc* strong topology is needed; or the weak compactness coming from uniform boundedness should be combined with some compactification arguments (compensated compactness, Young measures and their reduction, etc.) that exploit in a non-trivial way the particular structure of the PDE in hand (div-curl structure, pseudomonotonicity, entropy inequalities, etc.).

In the continuous setting, one celebrated result is the Aubin-Lions lemma and its generalization by Simon [14]. To give an example most relevant for the applications we have in mind, let us simply state here that a sequence  $(u^h)_h$  bounded in  $L^1(0, T; W^{1,1}(\Omega))$  and such that  $(u_t^h)_h$  is bounded in  $L^1(0, T; W^{-1,1}(\Omega))$  is relatively compact in  $L^1(Q)$ , cf. [12]. In this situation, compactness property comes from an *a priori bound* on  $u^h$  in some space  $L^p(0, T, X)$  with  $X$  compactly embedded in  $L^1(\Omega)$  (e.g.,  $X = W^{1,1}(\Omega)$ ), and from a very basic use of the evolution PDE in hand: the PDE brings information on boundedness of the time derivatives  $u_t^h$  in some space

---

<sup>1</sup> see [4] and references therein for well-posedness theory of such “triple nonlinear” equations. These are mathematical models for porous media, sedimentation, Stefan problem, etc..

<sup>2</sup> Actually, we assume that either these functions are uniformly continuous or  $v$  is bounded *a priori*.

<sup>3</sup> Throughout the note, “compact” actually signifies “relatively compact”.

$L^q(0, T; Y)$  where  $Y$  can be a subspace of distributions on  $\Omega$  equipped with a rather weak topology (e.g.,  $Y = W^{-1,1}(\Omega)$ ). A discrete version of the Aubin-Simon lemma was recently proposed by Gallouët and Latché in [12]. A related result also taken from Simon [14] uses a bound on fractional time derivatives of  $u^h$ ; as it was demonstrated by Emmrich and Thalhammer in [9], this version is quite appropriate in the time-discretized setting, because time fractional derivatives of order less than  $1/2$  exist even for piecewise constant functions.

These results going back to [14] offer a very wide spectrum of applications, yet they are difficult to apply on degenerate parabolic problems with non-Lipschitz nonlinearities. Both of the above results are rather of a functional-analytic character; the difficulty comes from the fact that nonlinear mappings make bad correspondence between linear functional spaces. Yet this difficulty is not a fundamental one; roughly speaking, it is settled by a careful use of translation arguments and of moduli of continuity. In this note, we present a collection of complementary techniques that either involve as few functional spaces as possible<sup>4</sup> (e.g., only  $L^1(Q)$  is used e.g. in the Kruzhkov lemma, see § 2); or they make non-trivial use of the PDE in hand.

## 2 Direct estimation of time translates

By the Fréchet-Kolmogorov compactness criterion in  $L^1(Q)$ , uniform bounds on space and time translates of  $u^h$  are needed; in the setting of the present note, the first ones are readily available, and the difficulty is to estimate the time translates as

$$\forall h \int_0^{T-\delta} \int_{\Omega} |u^h(t+\delta) - u^h(t)| \leq \omega(\delta) \quad \text{with} \quad \lim_{\delta \rightarrow 0} \omega(\delta) = 0, \quad (2)$$

$\omega(\cdot)$  being a modulus of continuity, uniform in  $h$ . Here are two ways to obtain (2).

### A discrete Kruzhkov lemma

- *The continuous setting (see [13])*

**Lemma 1 (Kruzhkov [13]).** *Assume that the families of functions  $(u^h)_h, (F_{\alpha}^h)_{h,\alpha}$  are bounded in  $L^1(Q)$  and satisfy  $\frac{\partial}{\partial t} u^h = \sum_{|\alpha| \leq m} D^{\alpha} F_{\alpha}^h$  in  $\mathcal{D}'(Q)$ . Assume that  $u^h$  can be extended outside  $Q$ , and one has<sup>5</sup>*

$$\int \int_Q |u^h(t, x+\delta) - u^h(t, x)| dx dt \leq \omega(\delta), \quad \text{with} \quad \lim_{\delta \rightarrow 0} \omega(\delta) = 0, \quad (3)$$

where  $\omega(\cdot)$  does not depend on  $h$ . Then  $(u^h)_h$  is (relatively) compact in  $L^1(Q)$ .

<sup>4</sup> Working in an  $h$ -independent space is an advantage for producing discrete versions of such arguments; yet the approach of [12] exhibits a simple and efficient use of  $h$ -dependent spaces.

<sup>5</sup> In practice, space translation estimates of the kind (3) can be obtained via an estimate of some discrete gradients; notice that estimates of kind (3) are stable upon composing  $(u^h)_h$  by a function  $b(\cdot)$  which is uniformly continuous (as in (1), we mean that  $u^h = b(v^h)$ ).

Clearly, this is an  $L^1_{loc}$  compactness result (one can apply the lemma locally in  $Q$ ).

• *A discrete version of the Kruzhkov lemma (see [7, 2])*

For problem (1), the value  $m = 1$  is relevant; we limit to this case the discussion of discrete analogues of Lemma 1.

To give an idea of discrete versions of the Kruzhkov lemma<sup>6</sup>, assume we are given a family of meshes of  $\Omega$  indexed by the size  $h > 0$ , satisfying mild proportionality restrictions (e.g., for the case of two-point flux finite volume schemes as described in [10], one needs for all neighbour volumes  $K, L$ ,  $\text{diam}(K) + \text{diam}(L) \leq \text{const } d_{K,L}$  uniformly in  $h$ ). Assume that in relation with the meshes, spaces of discrete functions  $\mathbb{R}_h$  and discrete fields  $(\mathbb{R}^N)_h$  are defined (each element  $u_h \in \mathbb{R}_h$  or  $\mathbf{F} \in (\mathbb{R}^N)_h$  is a piecewise constant on  $\Omega$  function reconstructed from the degrees of freedom of the discretization method). Assume we are given discrete gradient and discrete divergence operators  $\nabla^h$  and  $\text{div}^h$  mapping between these spaces.

Let  $(\delta_h)_h$  be the associated time steps and  $N_h$  is the entire part of  $T/\delta_h$ . Assume that with this notation, we are given discrete evolution equations under the form

$$\text{for } n \in [0, N_h], \quad \frac{b(v_h^{n+1}) - b(v_h^n)}{\delta_h} = \text{div}^h [\mathbf{F}_h^{n+1}] + f_h^{n+1} \quad (4)$$

where families of discrete functions  $(u_h^{n+1})_{n \in [0, N_h]}$ ,  $(f_h^{n+1})_{n \in [0, N_h]}$  and of discrete fields  $(\mathbf{F}_h^{n+1})_{n \in [0, N_h]}$  are bounded in  $L^1(Q)$ . Assume in addition that the discrete gradients  $(\nabla_h v_h^{n+1})_{n \in [0, N_h]}$  are bounded in  $L^1(Q)$  (and that this bound implies a uniform translation bound in space of the family  $v_h$ ; this is the case e.g. when discrete Poincaré inequalities can be proved). Under these assumptions, reproducing at the discrete level the proof of Lemma 1 as it is done in [7, 2], one concludes that the family  $(b(v^h))_h$  is relatively compact in  $L^1(Q)$ . Let us stress that the proof only uses one functional space: e.g. in different finite volume methods, all discrete objects (functions, fields, gradient, divergence) are naturally lifted to  $L^1(Q)$ .

### A classical technique for the “variational” setting

Following [1], by “variational” we mean a setting where the solution itself is an admissible test function in the weak formulation of the PDE; this situation occurs for (1). It typically comes along with *a priori* estimates that can be reproduced at the discrete level, provided the discretization is somewhat structure-preserving.<sup>7</sup>

<sup>6</sup> Here we give a rather heuristic presentation; see [7] and [2] for two precise formulations covering, e.g., standard two-point flux finite volume schemes, see [10], and DDFV schemes, see [2].

<sup>7</sup> Notice that for evolution PDEs governed by accretive in  $L^1(\Omega)$  operators, of which (1) is an example, time-implicit discretizations are better suited for structure preservation. Use of numerical schemes in space that possess a kind of discrete duality (mimetic, co-volume, DDFV schemes, etc.) enables getting discrete estimates analogous to the continuous ones. For notions of solution involving nonlinear test functions (e.g., entropy, renormalized solutions) orthogonality assumption on the meshes and isotropy assumption on the diffusion operator may be needed, see e.g. [3].

The technique of [1] used, in its finite volume version, e.g., in [10, 11, 3], is to integrate the equation in time from  $t$  to  $t + \delta$ , take  $u_h(t + \delta) - u_h(t)$  for test function, then integrate in  $(t, x)$ . On problem (1), this may lead to a uniform estimate

$$\forall h > 0 \quad \int_0^{T-\delta} \int_{\Omega} \left( b(v^h)(t + \delta) - b(v^h)(t) \right) \left( \varphi(v^h)(t + \delta) - \varphi(v^h)(t) \right) \leq \omega(\delta). \quad (5)$$

Then Lipschitz continuity of  $\varphi \circ b^{-1}$  (resp., of  $b \circ \varphi^{-1}$ ) can be used to infer uniform  $L^2$  time translates of  $\varphi(u^h)$  (resp., of  $b(u^h)$ ). Yet the corresponding  $L^1$  translates can be obtained in the case  $\varphi \circ b^{-1}$  (resp.,  $b \circ \varphi^{-1}$ ) is a merely continuous function.

• *A technique for  $L^1$  estimates involving non-Lipschitz nonlinearities (see [3])*

Consider the case where  $\tilde{\varphi} := \varphi \circ b^{-1}$  is a uniformly continuous function (moreover, it is non-decreasing). Let  $\pi$  be a concave modulus of continuity for  $\varphi \circ b^{-1}$ ,  $\Pi$  be its inverse, and set  $\tilde{\Pi}(r) = r\Pi(r)$ . Let  $\tilde{\pi}$  be the inverse of  $\tilde{\Pi}$ . Note that  $\tilde{\pi}$  is concave, continuous, and  $\tilde{\pi}(0) = 0$ . Set  $w = b(v^h)(t + \delta, x)$  and  $u = b(v^h)(t, x)$ . We have

$$\int_{\mathcal{Q}} |\tilde{\varphi}(w) - \tilde{\varphi}(u)| = \int_{\mathcal{Q}} \tilde{\pi}(\tilde{\Pi}(|\tilde{\varphi}(w) - \tilde{\varphi}(u)|)) \leq |\mathcal{Q}| \tilde{\pi} \left( \frac{1}{|\mathcal{Q}|} \int_{\mathcal{Q}} \tilde{\Pi}(|\tilde{\varphi}(w) - \tilde{\varphi}(u)|) \right).$$

Since  $|\tilde{\varphi}(w) - \tilde{\varphi}(u)| \leq \pi(|w - u|)$ , we have  $\Pi(|\tilde{\varphi}(w) - \tilde{\varphi}(u)|) \leq |w - u|$  and

$$\tilde{\Pi}(|\tilde{\varphi}(w) - \tilde{\varphi}(u)|) = \Pi(|\tilde{\varphi}(w) - \tilde{\varphi}(u)|) |\tilde{\varphi}(w) - \tilde{\varphi}(u)| \leq |w - u| |\tilde{\varphi}(w) - \tilde{\varphi}(u)|.$$

Therefore, (5) implies an  $L^1$  estimate of the kind (2) on  $u^h = \varphi(v^h)$ :

$$\int_{\mathcal{Q}} |\varphi(v^h(t + \delta)) - \varphi(v^h(t))| \leq |\mathcal{Q}| \tilde{\pi} \left( \frac{1}{|\mathcal{Q}|} \int_{\mathcal{Q}} |w - u| |\tilde{\varphi}(w) - \tilde{\varphi}(u)| \right) = |\mathcal{Q}| \tilde{\pi} \left( \frac{1}{|\mathcal{Q}|} \omega(\delta) \right).$$

• *Use of contraction arguments and absorption terms (see [8])*

Let us mention one more possibility for getting estimates of kind (2) for (1), which takes advantage of the monotonicity of  $\psi(\cdot)$ . Assume  $\varphi = Id$  in (1); to shorten the arguments, assume  $f = 0$ . Then  $L^1$  translates in time of  $u^h = b(v^h)$  can be estimated with every of the two preceding methods, the Kruzhkov lemma and a direct estimation of translates with variational techniques. This makes  $(b(v^h))_h$  relatively compact; yet, when  $b^{-1}(\cdot)$  is discontinuous, no information on compactness of  $(v^h)_h$  is obtained this way. Now, let us use the translation (in time) invariance of the equation and the  $L^1$  contraction property natural for (1). This yields the estimate (see [8])

$$\int_{\Omega} |b(v_{\delta}^h) - b(v^h)|(T - \delta) + \int_s^{T-\delta} \int_{\Omega} |\psi(v_{\delta}^h) - \psi(v^h)| \leq \int_{\Omega} |b(v_{\delta}^h)(s) - b(v^h)(s)| \quad (6)$$

for all  $s \in (0, T - \delta)$ , where  $v_{\delta}^h(t) = v^h(t + \delta)$ . Integrating in  $s > \alpha > 0$ , using the time translation bound for  $(b(v^h))_h$  we get an  $L^1((\alpha, T) \times \Omega)$  estimate of time translates of  $\psi(v^h)$ . This is sufficient for  $L^1_{loc}$  compactness provided  $\psi(\cdot)$  is strictly increasing.

Discrete version of (6) (see [8]) assumes the  $L^1$  contraction property (linked to order preservation via the Crandall-Tartar lemma) is preserved at the discrete level.

### Applications to (1) and some other parabolic PDEs.

We treat different “doubly nonlinear” cases of (1), assuming that either  $b(\cdot)$  of  $\varphi(\cdot)$  is invertible and thus it can be taken to be identity  $Id(\cdot)$ , upon a change of unknown.

- *Application to a parabolic-hyperbolic PDE (see [3])*

In the case (1) with  $b = Id$ , provided  $L^p(Q)$  estimates of the discrete gradient of  $\varphi(v^h)$  are available, the space translates of  $\varphi(v^h)$  can be estimated uniformly (as well as the functions  $\varphi(v^h)$  themselves), and an estimate of the form (5) can be obtained. Then the above technique for exploiting (5) assesses the  $L^1(Q)$  compactness of  $(\varphi(v^h))_h$ , which is a first step of the convergence proof for this problem (see [3]).

- *Application to an elliptic-parabolic PDE with the structure condition (see [2])*

Assume that  $\varphi = Id$ . In this case, the Kruzhkov lemma can be used, see [2]; alternatively, estimate (5) allows to control the  $L^1$  time translates of  $b(v^h)$  similarly to what was described above. If the *structure condition*  $\mathbf{G}(v) = \mathbf{F}(b(v))$  is satisfied, compactness of  $(b(v^h))_h$  is enough to pass to the limit, see Alt and Luckhaus [1].

- *Application to a cross-diffusion system (see [7])*

The following kind of models comes from population dynamics :

$$\begin{cases} u_t - D_1 \Delta u - \operatorname{div}((u+v)\nabla u + u\nabla v) = u(a_1 - b_1 u - c_1 v), \\ v_t - D_2 \Delta v - \operatorname{div}(v\nabla u + (u+v)\nabla v) = v(a_2 - b_2 u - c_2 v), \end{cases} \quad (7)$$

Natural estimates for approximate solutions of (7) are  $L^2$  bounds on  $\sqrt{1+u+v}\nabla u$ ,  $\sqrt{1+u+v}\nabla v$ ; this gives only an  $L^{4/3}$  bound on the diffusion fluxes in (7), thus we are not in a variational setting<sup>8</sup>. Therefore for existence obtained by convergence of finite volume approximations of the kind [10], the Kruzhkov lemma was used in [7]. Alternatively, the discrete Aubin-Simon lemma (see [12]) could be used here.

- *Application to convergence of some linearized implicit schemes (see [5])*

An elliptic-parabolic “bidomain” system is used for modelling heart electric activity. In [5], finite volume discretization of a simplified version of the bidomain model:

$$\begin{cases} v_t - \operatorname{div}[\mathbf{M}_i(\cdot)\nabla u_i] + H(v) = I_{ap}(\cdot), \\ v_t + \operatorname{div}[\mathbf{M}_e(\cdot)\nabla u_e] + H(v) = I_{ap}(\cdot), \end{cases} \quad v = u_i - u_e, \quad (8)$$

was considered; here, the “ionic current”  $H(\cdot)$  is a cubic polynomial. The nonlinear reaction term brings an estimate of  $v^h(v)$  which bounds  $v$  in  $L^4(Q)$ . Time-implicit DDFV discretization of (8) preserves this structure; then the problem falls into the “variational” framework<sup>9</sup> and time translates can be estimated like in [1, 10, 11]. From the practical point of view, it is important to accelerate computations, and to consider a linearized method where the discretization of the reaction term  $H(v)$  is not fully implicit. Unfortunately, for theoretical analysis  $L^4$  estimate for  $v^h$  is not available any more; only a weaker estimate can be obtained with interpolation arguments. In [5], we applied the Kruzhkov lemma to exploit this weaker estimate.

<sup>8</sup> from the practical point of view, e.g. the first equation cannot be tested with  $u(t+\delta)$ .

<sup>9</sup> Indeed, we have  $v^h$  bounded in  $L^4(Q)$  and  $H(v^h)$  is bounded in  $L^{4/3}(Q) = (L^4(Q))^*$ .

### 3 Advanced use of the underlying PDE features

#### Compactness from monotone penalization and order-preservation

For getting (6), we already used the order-preservation structure for (1). Its further use, in conjunction with penalization, may lead to an indirect compactness proof.

- *The structure needed for compactification*

Assume that one can prove *uniqueness* of a solution to the PDE under study, let us call it  $(Eq_0)$ . Assume that it can be embedded “continuously” into a family  $(Eq_\varepsilon)$  of perturbed PDEs having the property that  $v_{\varepsilon_1}^h \leq v_{\varepsilon_2}^h$  when  $\varepsilon_1 \leq \varepsilon_2$ , where  $v_{\varepsilon_1}^h, v_{\varepsilon_2}^h$  are the associated discrete solutions. Continuity in  $\varepsilon \in [-1, 1]$  means, we assume that limits as  $\varepsilon \rightarrow 0$  (if any) of exact solutions  $v_\varepsilon$  of  $(Eq_\varepsilon)$  solve the limit equation  $(Eq_0)$ .

Assume that for  $\varepsilon \neq 0$ , the corresponding sequence  $(v_\varepsilon^h)_h$  is well defined and it converges to an exact solution  $v_\varepsilon$  of  $(Eq_\varepsilon)$ . Then solutions  $(v_0^h)_h$  to the discretized equation  $(Eq_0)$  converge, as  $h \rightarrow 0$ , to the unique solution of  $(Eq_0)$ . For the proof, write

$$v_{-1}^h \leq v_{-1/2}^h \leq \dots \leq v_{-1/m}^h \leq \dots \leq v_0^h \leq \dots \leq v_{1/m}^h \leq \dots \leq v_{1/2}^h \leq v_1^h, \quad (9)$$

and pass to the limit as  $h \rightarrow 0$  to define  $v_{\pm 1/m} := \lim_{h \rightarrow 0} v_{\pm 1/m}^h$  (up to extraction of a subsequence) solution to  $(Eq_{\pm 1/m})$ ; then, (9) is inherited at the limit (except that  $(v_0^h)_h$  may not have a limit). By monotonicity, we can define  $\underline{v} := \lim_{m \rightarrow \infty} v_{-1/m}$ ,  $\bar{v} := \lim_{m \rightarrow \infty} v_{1/m}$ ; furthermore,  $\underline{v} \leq \liminf_{h \rightarrow 0} v_0^h \leq \limsup_{h \rightarrow 0} v_0^h \leq \bar{v}$ . Both  $\underline{v}, \bar{v}$  solve  $(Eq_0)$ ; by uniqueness,  $(v_0^h)_h$  converges to  $\underline{v} = \bar{v}$  the solution of  $(Eq_0)$ .

- *Application to an elliptic-parabolic PDE without the structure condition (see [8])*

We assume that  $\varphi = Id$ ,  $\psi = 0$  in (1). We have seen that compactness of  $(b(v^h))_h$  can be established, e.g., with the Kruzhkov lemma. Under the *structure condition*  $\mathbf{G}(v) = \mathbf{F}(b(u))$ , this is enough to pass to the limit in the equation. But in general, no control of time oscillations of  $\mathbf{G}(v^h)$  is available, and the method of [1] breaks down. Yet it is enough to add penalization term of the form  $\psi_\varepsilon(v) = \varepsilon(\arctan v \mp \frac{\pi}{2} \text{sign } \varepsilon)$  to get into the setting where (6) can be exploited to control discrete solutions  $(v_\varepsilon^h)_h$  and to pass to the limit, as  $h \rightarrow 0$ , for the  $\psi_\varepsilon$ -penalized equation  $(1_\varepsilon)$ . The order-preservation assumptions of the above method being fulfilled due to the choice of  $\psi_\varepsilon$ , we get convergence of  $(v^h)_h$  in the cases where uniqueness for (1) can be shown.

#### Use and reduction of Young measures

Let us recall well-known techniques (which are not specific to *time* compactness).

- *Nonlinear weak-\* convergence and its description (see references in [10, 11, 3, 6])*

Equi-integrable sequences  $(u^h)_h$  of  $L^1(Q)$  functions are relatively compact in the nonlinear weak-\* sense; namely, for all Carathéodory map  $F$  such that  $(F(\cdot, u^h(\cdot)))_h$  is equi-integrable,  $L^1$ -weak limit of  $F(\cdot, u^h(\cdot))$  equals to  $\int F(\cdot, \lambda) d\nu(\lambda; \cdot)$  for some family of probability measures  $(\nu(\cdot; t, x))_{t,x}$  on the set of values of  $(u^h)_h$ . For  $\mathbb{R}$ -valued sequences, equivalent description in terms of the distribution function of



$v(\cdot)$  is very convenient, see [10, 11]. Finally, if the Young measure  $v(\cdot; t, x)$  reduces to a Dirac measure  $\delta(\cdot - u(t, x))$ , then strong convergence of  $u^h$  to  $u$  is inferred.

• *Application to discretization of the elliptic  $p(x)$ -laplacian (cf. [6])*

The classical Minty argument (see e.g. [1, 2]) cannot be applied directly when the  $p(x)$ -laplacian discretized problem  $-\operatorname{div}^h[|\nabla^h u_h|^{p_h(\cdot)-2} \nabla^h u_h] = f^h$  is considered: discretizations  $p_h$  of  $p(\cdot)$  lead to estimates in “moving” spaces  $W^{1, p_h(\cdot)}$ . Using Young measures for  $(\nabla^h u_h)_h$  one can “pull everyone down” to the common space  $L^1$  and use the monotonicity of  $\xi \mapsto |\xi|^{p-2} \xi$  to reduce the Young measure (see [6]).

• *Application to a parabolic-hyperbolic PDE (see [3])*

For the case of (1) with  $b = Id$ , we have already justified the compactness of  $(\varphi(v^h))_h$  in  $L^1(Q)$ . The Minty argument or the above reduction of Young measures yield compactness of  $(\nabla^h \varphi(v^h))_h$ . Then, following the ideas of Tartar and DiPerna, a weak- $*$  limit of  $(v^h)_h$  is created and *entropy inequalities* are used to reduce the corresponding Young measure to a Dirac measure, in the way of [10, 11].

**Acknowledgements** The author thanks E. Emmrich for discussions on the above techniques.

## References

1. H.W. Alt and S. Luckhaus. Quasilinear elliptic-parabolic differential equations. *Mat. Z.* **183**: 311–341 (1983)
2. B. Andreianov, M. Bendahmane and F. Hubert. On 3D DDFV discretization of gradient and divergence operators. II. Discrete functional analysis tools and applications to degenerate parabolic problems. Preprint.
3. B. Andreianov, M. Bendahmane, and K.H. Karlsen. Discrete duality finite volume schemes for doubly nonlinear degenerate hyperbolic-parabolic equations. *J. Hyp. Diff. Eq.* **7**:1–67 (2010)
4. B. Andreianov, M. Bendahmane, K.H. Karlsen and S. Ouaro. Well-posedness results for triply nonlinear degenerate parabolic equations. *J. Diff. Eq.* **247**(1):277–302 (2009)
5. B. Andreianov, M. Bendahmane, K.H. Karlsen and Ch. Pierre. Convergence of Discrete Duality Finite Volume schemes for the macroscopic bidomain model of the heart electric activity. Preprint HAL, <http://hal.archives-ouvertes.fr/hal-00526047>.
6. B. Andreianov, M. Bendahmane and S. Ouaro. Structural stability for variable exponent elliptic problems, I: the  $p(x)$ -Laplacian kind problems. *Nonlinear Anal.* **73**(1):2–24 (2010)
7. B. Andreianov, M. Bendahmane and R. Ruiz Baier. Analysis of a finite volume method to solve a cross-diffusion population system. *Math. Models Meth. Appl. Sci.* 2011, to appear.
8. B. Andreianov and P. Wittbold. Convergence of approximate solutions to an elliptic-parabolic equation without the structure condition. Preprint 2011.
9. E. Emmrich and M. Thalhammer. Doubly nonlinear evolution equations of second order: Existence and fully discrete approximation. *J. Diff. Eq.* 2011, to appear.
10. R. Eymard, T. Gallouët, and R. Herbin. *Finite Volume Methods*. Handbook of Numerical Analysis, Vol. VII, P. Ciarlet, J.-L. Lions, eds., North-Holland, 2000.
11. R. Eymard, T. Gallouët, R. Herbin and A. Michel. Convergence of a finite volume scheme for nonlinear degenerate parabolic equations. *Numer. Math.* **92**(1):41–82 (2002)
12. T. Gallouët and J.-C. Latché. Compactness of discrete approximate solutions to parabolic PDEs - Application to a turbulence model. *Comm. on Pure and Appl. Anal.* 2011, to appear
13. S.N. Kruzhkov. Results on the nature of the continuity of solutions of parabolic equations and some of their applications. *Math. Notes* **6**(1):517-523 (1969)
14. J. Simon. Compact sets in the space  $L^p(0, T; B)$ . *Ann. Mat. Pura ed Appl.* **146**:65–96 (1987)