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# Dynamically ordered energy function for Morse-Smale diffeomorphisms on 3-manifolds

V. Grines\*      F. Laudenbach†      O. Pochinka‡

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## Abstract

This note deals with arbitrary Morse-Smale diffeomorphisms in dimension 3 and extends ideas from [3], [4], where gradient-like case was considered. We introduce a kind of Morse-Lyapunov function, called dynamically ordered, which fits well dynamics of diffeomorphism. The paper is devoted to finding conditions to the existence of such an energy function, that is, a function whose set of critical points coincides with the non-wandering set of the considered diffeomorphism. We show that the necessary and sufficient conditions to the existence of a dynamically ordered energy function reduces to the type of embedding of one-dimensional attractors and repellers of a given Morse-Smale diffeomorphism on a closed 3-manifold.

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*Keywords:* Morse-Smale diffeomorphism, Morse-Lyapunov function, energy function.

## 1 Introduction and formulation of the results

We limit ourselves to dimension 3, even if some concepts do not depend on dimension. Let  $M$  be a closed 3-manifold and  $f : M \rightarrow M$  be a Morse-Smale diffeomorphism, that is: its nonwandering set  $\Omega_f$  is finite, hence consists of periodic points;  $f$  is hyperbolic along  $\Omega_f$  and the stable and unstable manifolds have transverse intersections. For the first important results on Morse-Smale diffeomorphisms we refer to [11], [8].

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**Definition 1** A Morse function  $\varphi : M \rightarrow \mathbb{R}$  is said to be a *Lyapunov function* for  $f$  if:

- 1)  $\varphi(f(x)) < \varphi(x)$  for every  $x \notin \Omega_f$ ;
- 2)  $\varphi(f(x)) = \varphi(x)$  for every  $x \in \Omega_f$ .

Let us recall that a function  $\varphi : M \rightarrow \mathbb{R}$  is called a *Morse function* if all its critical points are non-degenerate. It is easy to construct Lyapunov functions for  $f$ . For instance, one considers the suspension of  $f$ , a 4-dimensional manifold which is fibered over the circle and is endowed with a Morse-Smale vector field  $X$  transverse to the fibration. Thus the method introduced by Smale in [12] applies and yields a Lyapunov function  $\Phi$  for  $X$ . The restriction of  $\Phi$  to the base fibre, identified with  $M$ , is a Lyapunov function for  $f$ .

The periodic points of  $f$  are easily seen to be critical points of its Lyapunov function  $\varphi$  and the index of  $\varphi$  at  $p \in \Omega_f$  equals the dimension of  $W_p^u$ . Moreover any periodic point  $p$  is a maximum of the restriction of  $\varphi$  to the unstable manifold  $W_p^u$  and a minimum of its restriction to the stable manifold  $W_p^s$  (see statement 4 below). If these extremums are non-degenerate then the invariant manifolds of  $p$  are transversal to all regular level sets of  $\varphi$  in some neighborhood  $U_p$  of  $p$ . This local property is useful for the construction of a (global) Lyapunov function. Next definition was introduced in [4].

**Definition 2** A Lyapunov function  $\varphi : M \rightarrow \mathbb{R}$  for the Morse-Smale diffeomorphism  $f : M \rightarrow M$  is called a *Morse-Lyapunov function* if every periodic point  $p$  is a non-degenerate maximum (resp. minimum) of the restriction of  $\varphi$  to the unstable (resp. stable) manifold  $W_p^u$  (resp.  $W_p^s$ ).

Among the Lyapunov functions of  $f$  those which are Morse-Lyapunov form a generic set in the  $C^\infty$ -topology (see [4], theorem 1). In general, a Morse-Lyapunov function may have critical points which are not periodic points of  $f$ .

**Definition 3** A Morse-Lyapunov function  $\varphi$  is called an *energy function* for a Morse-Smale diffeomorphism  $f$  if the set  $\text{Crit}_\varphi$  of critical points of  $\varphi$  coincides with  $\Omega_f$ .

D. Pixton [9] was the first who proved that there is a gradient-like diffeomorphism without heteroclinic curves on  $\mathbb{S}^3$  which has no an energy function. Let us recall that a Morse-Smale diffeomorphism  $f : M \rightarrow M$  is called *gradient-like* if for any pair of periodic points  $x, y$  ( $x \neq y$ ) the condition  $W_x^u \cap W_y^s \neq \emptyset$  implies  $\dim W_x^s < \dim W_y^s$ . It follows from the definition that a Morse-Smale diffeomorphism is gradient-like if and only if there are no *heteroclinic points* that is, intersection points of two-dimensional and one-dimensional invariant manifolds of different saddle points. Notice that two-dimensional invariant

manifolds of different saddle points of a gradient-like diffeomorphism may have a non-empty intersection along the so-called *heteroclinic curves* (see figure 1).

In [4] (Theorem 4) we gave necessary and sufficient conditions to the existence of a *self-indexing* energy function for a Morse-Smale diffeomorphism  $f : M \rightarrow M$  and showed that a non gradient-like diffeomorphism does not possess a self-indexing energy function. Here self-indexing means  $\varphi(p) = \dim W_p^u$  for every point  $p \in \Omega_f$ .

In the present paper we introduce the notion of *dynamically ordered* Morse-Lyapunov function for an arbitrary Morse-Smale diffeomorphism on 3-manifold. By the argument above-mentioned such a function always exists. We find conditions to the existence of such an energy function. We show that the necessary and sufficient conditions to the existence of dynamically ordered energy function is reduced to the type of embedding of one-dimensional attractors and repellers of a given Morse-Smale diffeomorphism on a closed 3-manifold. More details are given below.

Let  $f : M \rightarrow M$  be a Morse-Smale diffeomorphism. Following to S. Smale we introduce a partial order relation  $\prec$  on the set of periodic orbits of  $f$  in the following way:

$$\mathcal{O}_p \prec \mathcal{O}_r \iff W_{\mathcal{O}_p}^s \cap W_{\mathcal{O}_r}^u \neq \emptyset.$$

This definition means intuitively that all wandering points flow down along unstable manifolds to less elements. A sequence of different periodic orbits  $\mathcal{O}_p = \mathcal{O}_{p_0}, \mathcal{O}_{p_1}, \dots, \mathcal{O}_{p_k} = \mathcal{O}_r$  ( $k \geq 1$ ) such that  $\mathcal{O}_{p_0} \prec \mathcal{O}_{p_1} \prec \dots \prec \mathcal{O}_{p_k}$  is called a *chain of length  $k$  connecting  $\mathcal{O}_r$  to  $\mathcal{O}_p$* . The maximum length of such chains is called, by J. Palis in [8], the *behaviour* of  $\mathcal{O}_r$  relative to  $\mathcal{O}_p$  and denoted by  $beh(\mathcal{O}_r | \mathcal{O}_p)$ . For completeness it is assumed  $beh(\mathcal{O}_r | \mathcal{O}_p) = 0$  if  $W_{\mathcal{O}_r}^u \cap W_{\mathcal{O}_p}^s = \emptyset$ .

For each  $q \in \{0, 1, 2, 3\}$ , denote  $\Omega_q$  the subset of periodic points  $r$  such that  $\dim W_r^u = q$  and denote  $k_q$  the number of periodic orbits in the set  $\Omega_q$ . Set  $k_f = k_0 + k_1 + k_2 + k_3$  the number of all periodic orbits. For each periodic orbit  $\mathcal{O}_r$  set  $q_{\mathcal{O}_r} = \dim W_{\mathcal{O}_r}^u$  and  $b_{\mathcal{O}_r} = \max_{p \in \Omega_0} \{beh(\mathcal{O}_r | \mathcal{O}_p)\}$ .

It is clear that the condition  $\mathcal{O}_p \prec \mathcal{O}_r$  implies  $b_{\mathcal{O}_p} < b_{\mathcal{O}_r}$ . Notice that if  $\mathcal{O}_p \prec \mathcal{O}_r$  then  $q_{\mathcal{O}_p} \leq q_{\mathcal{O}_r}$ . Indeed, as the intersection  $W_{\mathcal{O}_p}^s \cap W_{\mathcal{O}_r}^u$  is transverse, the condition  $\mathcal{O}_p \prec \mathcal{O}_r$  implies the inequality  $\dim W_{\mathcal{O}_p}^s + \dim W_{\mathcal{O}_r}^u - 3 \geq 0$ . Then  $3 - q_{\mathcal{O}_p} + q_{\mathcal{O}_r} - 3 \geq 0$  and, hence,  $q_{\mathcal{O}_p} \leq q_{\mathcal{O}_r}$ .

Thus, it is possible to numerate all periodic orbits:  $\mathcal{O}_1, \dots, \mathcal{O}_{k_f}$  such way that this numeration preserves the partial order  $\prec$  and satisfies to following conditions:

$$q_{\mathcal{O}_i} < q_{\mathcal{O}_j} \Rightarrow i < j; \quad q_{\mathcal{O}_i} = q_{\mathcal{O}_j} \text{ and } b_{\mathcal{O}_i} < b_{\mathcal{O}_j} \Rightarrow i < j. \quad (1)$$

On figure 1 it is represented a phase portrait of a Morse-Smale diffeomorphism  $f : \mathbb{S}^3 \rightarrow \mathbb{S}^3$  with  $\Omega_f$  consisting of fixed points which are numerated according to conditions (1).

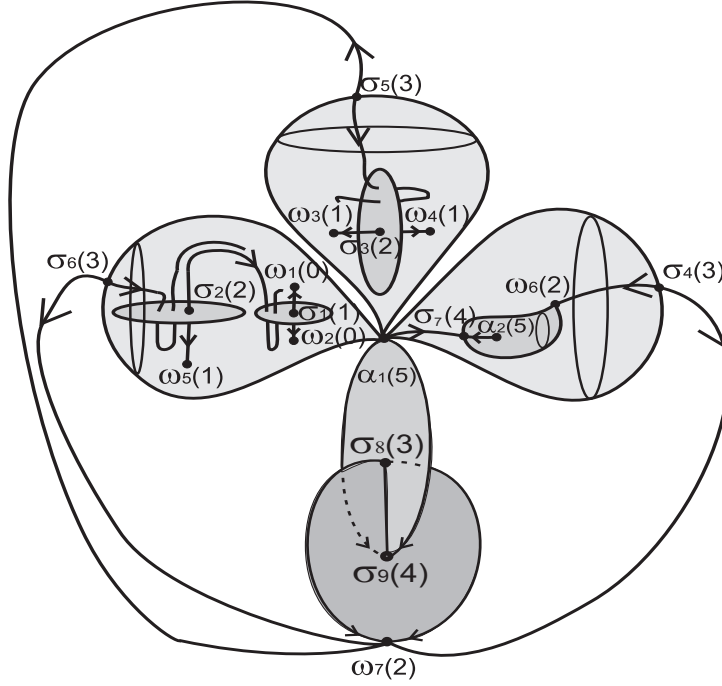


Figure 1: Phase portrait of a Morse-Smale diffeomorphism  $f : \mathbb{S}^3 \rightarrow \mathbb{S}^3$  with ordered periodic orbits according to conditions (1)

**Definition 4** A Morse-Lyapunov function  $\varphi$  is said to be dynamically ordered when  $\varphi(\mathcal{O}_i) = i$  for  $i \in \{0, \dots, k_f\}$ .

For each  $i = 1, \dots, k_f - 1$ , set  $A_i = \bigcup_{j=1}^i W_{\mathcal{O}_j}^u$ . It follows from [11] (theorem 2.3) that  $(cl W_{\mathcal{O}_i}^u) \setminus W_{\mathcal{O}_i}^u \subset A_{i-1}$ . Thus  $A_i$  is compact. It is well known (see, for example [10]) that the set  $A_i$  is an *attractor*, that is it has a *trapping neighborhood*  $M_i$  such that  $f(M_i) \subset int M_i$  ( $M_i$  is  $f$ -compressed) and  $\bigcap_{k \geq 0} f^k(M_i) = A_i$ .

We will say that  $A_i$  is  $q$ -dimensional attractor if  $q_{\mathcal{O}_i} = q$ .

For each  $i = 1, \dots, k_f - 1$ , denote by  $m_i$  the number of connected components of  $A_i$ . Notice that for each  $i = 1, \dots, k_0$ , the attractor  $A_i$  has dimension 0 and consists of  $m_i$  hyperbolic sinks.

**Proposition 1** Each zero-dimensional attractor  $A_i$  of the Morse-Smale diffeomorphism  $f$  on a 3-manifold  $M$  has a trapping neighborhood  $M_i$  consisting of  $m_i$  pairwise disjoint smooth 3-balls.

For each  $i = k_0 + 1, \dots, k_1$  the attractor  $A_i$  is one-dimensional and possesses a handle trapping neighborhood  $M_i$  in the sense of the next definition.

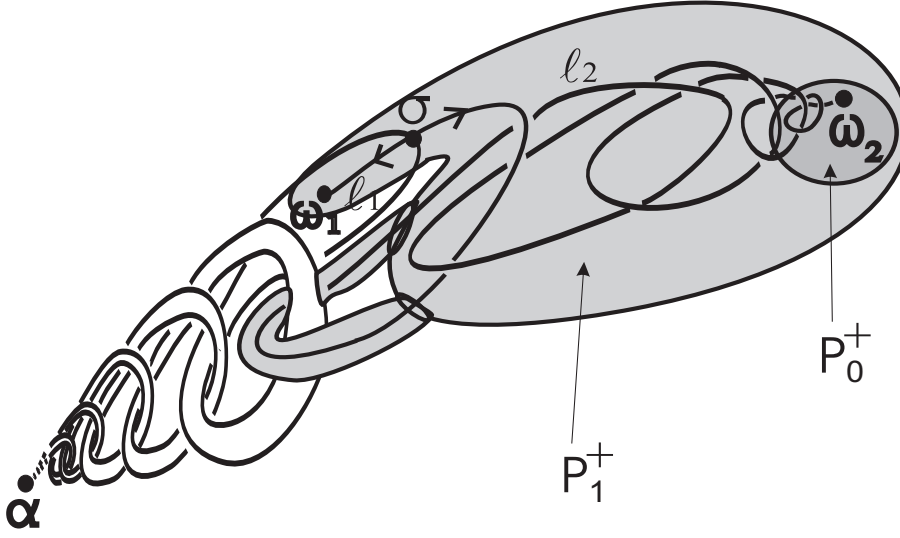


Figure 2: One-dimensional attractor of Pixton's example is not tightly embedded

**Definition 5** A trapping neighborhood  $M_i$  of a one-dimensional attractor  $A_i$  is called a handle if:

- 1)  $M_i$  has  $m_i$  connected components each of them being a smooth handle-body;
- 2)  $W_\sigma^s \cap M_i$  consists of exactly one two-dimensional closed disc for each saddle point  $\sigma \in \mathcal{O}_i$ .

**Proposition 2** Each one-dimensional attractor  $A_i$  of Morse-Smale diffeomorphism  $f$  on a 3-manifold  $M$  has a handle trapping neighborhood  $M_i$ .

Denote  $g_{M_i}$  the total genus of connected components of the handle trapping neighborhood  $M_i$ . For  $i = k_0 + 1, \dots, k_1$  denote by  $r_i$  the number of saddle points in  $A_i$ . Set  $g_i = m_i + r_i - m_{k_0}$ . It is easily shown that  $g_i \leq g_{M_i}$ .

**Definition 6** A handle trapping neighborhood  $M_i$  of some one-dimensional attractor  $A_i$  is said to be tight if  $g_{M_i} = g_i$ . One-dimensional attractor  $A_i$  possessing tight trapping neighborhood  $M_i$  is said to be tightly embedded.

By definition a repeller for  $f$  is an attractor for  $f^{-1}$ . Then a one-dimensional repeller for  $f$  is said to be *tightly embedded* if it is such an attractor for  $f^{-1}$ .

Notice that the property for a one-dimensional attractor (repeller) to be tightly embedded gives a topological information about the embedding of the unstable manifolds of its saddle periodic points. In the example which was constructed by D. Pixton in [9] the unique one-dimensional attractor  $cl W_\sigma^u$  has the following property: any 3-ball around  $cl W_\sigma^u$  intersects  $W_\sigma^s$  at more

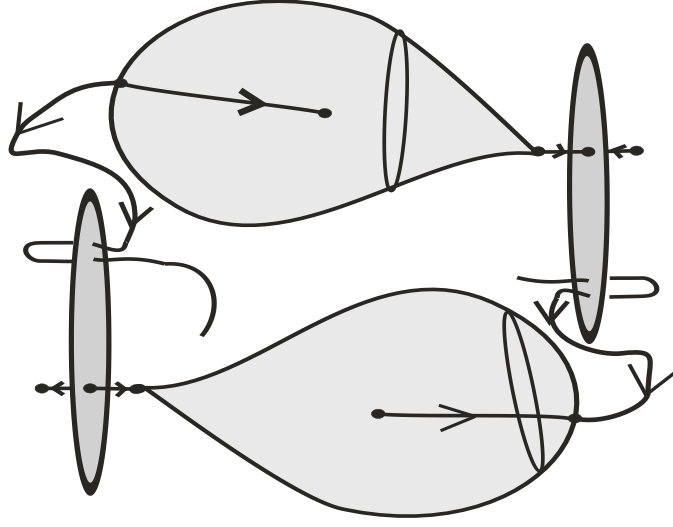


Figure 3: A Morse-Smale diffeomorphism without heteroclinic curves given on  $\mathbb{S}^3$

than one 2-disc (see figure 2, where are drawn the phase portrait of Pixton's diffeomorphism  $f$  and a 3-ball). Hence, this one-dimensional attractor is not tightly embedded.

Our main results are the following theorems.

**Theorem 1** *If a Morse-Smale diffeomorphism  $f : M \rightarrow M$  possesses a dynamically ordered energy function then all one-dimensional attractors and repellers of  $f$  are tightly embedded.*

**Definition 7** *A tight trapping neighborhood  $M_i$  of a one-dimensional attractor  $A_i$  is called strongly tight if  $M_i \setminus A_i$  is diffeomorphic to  $\partial M_i \times (0, 1]$ . A one-dimensional attractor  $A_i$  possessing a strongly tight trapping neighborhood  $M_i$  is said to be strongly tightly embedded.*

**Theorem 2** *Let  $f$  be a Morse-Smale diffeomorphism on a closed 3-manifold  $M$ . If all one-dimensional attractors and repellers of  $f$  are strongly tightly embedded then  $f$  possesses a dynamically ordered energy function.*

Notice that the condition in the last theorem is not necessary. For example in section 5 of paper [4] there was constructed a diffeomorphism on  $\mathbb{S}^2 \times \mathbb{S}^1$  possessing a dynamically ordered energy function, but whose one-dimensional attractor and repeller are not strongly tightly embedded.

The next theorem states a criterion for the existence of some dynamically ordered energy function for a Morse-Smale diffeomorphism without heteroclinic curves given on  $\mathbb{S}^3$ . It is based on paper [2], where it is specified interrelation

between topology of the ambient 3-manifold  $M$  and structure of the non-wandering set of a Morse-Smale diffeomorphism without heteroclinic curves given on  $M$ . In particular, the number  $k$  of saddles and the number  $l$  of sinks and sources satisfy the equality  $k = l - 2$  for any diffeomorphism of  $\mathbb{S}^3$  without heteroclinic curves (see statement 6 below). Methods from [1] for realizing Morse-Smale diffeomorphisms show that this class is not empty. Moreover, it contains diffeomorphisms with chains of intersections of saddle invariant manifolds of arbitrary length (see figure 3, where it is represented a phase portrait of a diffeomorphism from the class under consideration).

**Theorem 3** *A Morse-Smale diffeomorphism  $f : \mathbb{S}^3 \rightarrow \mathbb{S}^3$  without heteroclinic curves possesses a dynamically ordered energy function if and only if each one-dimensional attractor and repeller of  $f$  has a tight trapping neighborhood which is a union of 3-balls.*

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## 2 Auxiliary facts

In this section, we recall some statements that we need in the proof and give references.

**Statement 1** ( $\lambda$ -lemma, [8]). *Let  $x_0$  be a hyperbolic fixed point of a diffeomorphism  $f : M^n \rightarrow M^n$ ,  $\dim W_{x_0}^u = \ell$ ,  $0 < \ell < n$ , Let  $B^u \subset W_{x_0}^u$  and  $B^s \subset W_{x_0}^s$  be small  $\ell$ -disc and  $(n - \ell)$ -disc respectively centered at  $x_0$ . Let  $V := B^u \times B^s$  be their product in a chart about  $x_0$ . Let  $B$  be an  $\ell$ -disc transverse to  $W_{x_0}^s$  at  $q \neq x_0$ . Then, for any  $\varepsilon > 0$ , there exists a positive integer  $k_0$  such that the connected component of  $f^k(B) \cap V$  containing  $f^k(q)$  is  $\varepsilon$ - $C^1$ -close to  $B_u$  for each  $k \geq k_0$ .*

**Definition 8** *Let  $S$  be a compact surface properly embedded in a 3-manifold  $W$  (that is,  $\partial S \subset \partial W$ ). A disc  $D$  embedded in  $W$  with  $D \cap S = \partial D$  is called a compressing disc for  $S$  if the curve  $\partial D$  in  $S$  does not bound a disc in  $S$ .*

*The surface  $S$  is said to be compressible in  $W$  if there is a compressing disc for  $S$ . In the opposite case  $S$  is called incompressible<sup>1</sup>, if  $S$  is not the 2-sphere.*

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<sup>1</sup>It is well known to topologists that a bicollared surface is incompressible if and only if the inclusion  $S \hookrightarrow W$  induces an injection of fundamental groups.



**Statement 2** ([13], corollary 3.2) *Let  $S_g$  be an orientable surface of genus  $g \geq 1$  and let  $F$  be an incompressible orientable surface properly embedded in  $S_g \times [0, 1]$  such that  $\partial F \subset S_g \times \{1\}$ . Then there is a surface  $F_1 \subset S_g \times \{1\}$  which is homeomorphic to  $F$ , such that  $\partial F = \partial F_1$  and  $F \cup F_1$  bounds domain  $\Delta$  in  $S_g \times [0, 1]$  such that  $cl\Delta$  is homeomorphic to  $F \times [0, 1]$ , where  $cl(\cdot)$  stands for the closure.*

A particular case of statement 2 is the following fact.

**Corollary 1** ([5], theorem 3.3) *Let  $S_g$  be a closed orientable surface of genus  $g \geq 1$  and let surface  $F \subset int(S_g \times [0, 1])$  be a closed surface which has genus  $g$  and does not bound a domain in  $S_g \times [0, 1]$ . Then the closure of each connected component of  $S_g \times [0, 1] \setminus F$  is homeomorphic to  $S_g \times [0, 1]$ .*

**Proof:** According to the preceding statement, it is sufficient to check that  $F$  is incompressible. If  $F$  is compressible, there exists some incompressible surface  $F'$  whose genus  $g'$  is less than  $g$  and which still does not bound a domain in  $S_g \times [0, 1]$ . So  $F'$  is not a sphere and  $g' > 0$ . As  $F'$  is incompressible, the preceding statement tells us that  $F'$  is diffeomorphic to  $S_{g'}$ . Contradiction.  $\diamond$

**Statement 3** ([7], theorem 5.2) *Let  $M^n$  be a closed manifold,  $\varphi : M^n \rightarrow \mathbb{R}$  be a Morse function,  $C_q$  be the number of all its critical points with index  $q$ ,  $\beta_q(M^n)$  be the  $q$ -th Betti number and  $\chi(M^n)$  be the Euler characteristic. Then  $\beta_q(M^n) \leq C_q$  and  $\chi(M^n) = \sum_{q=0}^n (-1)^q C_q$ .*

**Statement 4** *Let  $\varphi : M^n \rightarrow \mathbb{R}$  be a Lyapunov function for a Morse-Smale diffeomorphism  $f : M^n \rightarrow M^n$ . Then*

- 1)  $-\varphi$  is Lyapunov function for  $f^{-1}$ ;
- 2) if  $p$  is a periodic point of  $f$  then  $\varphi(x) < \varphi(p)$  for every  $x \in W_p^u \setminus p$  and  $\varphi(x) > \varphi(p)$  for every  $x \in W_p^s \setminus p$ ;
- 3) if  $p$  is a periodic point of  $f$  then  $p$  is a critical point of  $\varphi$  whose index is  $\dim W_p^u$ .

**Statement 5** ([4], lemma 2.2) *Let  $f : M^n \rightarrow M^n$  be a Morse-Smale diffeomorphism on an  $n$ -dimensional manifold and let  $\mathcal{O}$  be a periodic orbit. For  $p \in \mathcal{O}$ , set  $q = \dim W_p^u$ . Then, there is some neighborhood  $U$  and an energy function  $\varphi : U \rightarrow \mathbb{R}$  for  $f$  such that  $(W_p^u \cap U) \subset Ox_1 \dots x_q$ ,  $(W_p^s \cap U) \subset Ox_{q+1} \dots x_n$  for Morse coordinates  $x_1, \dots, x_n$  of  $\varphi$  near  $p$  and  $\varphi(\mathcal{O}) = 0$ .*

**Statement 6** ([2], theorem) *Let  $M$  be a three-dimensional closed, connected, orientable manifold. Let  $f : M \rightarrow M$  be any Morse-Smale diffeomorphism without heteroclinic curves whose non-wandering set consists of  $k$  saddles and  $l$  sinks. The following holds true:*

- 1) if  $k = l - 2$ , then  $M$  is the 3-sphere;
  - 2) if  $k \neq l - 2$ , then  $M$  is the connected sum of  $\frac{k-l+2}{2}$  copies  $S^2 \times S^1$ .
- Conversely, for any non negative integers  $k, l, m$  such that  $\frac{k-l+2}{2} = m$ , there exists some Morse-Smale diffeomorphism  $f$  of  $M$  such that:
- a) the non-wandering set of  $f$  consists of  $k$  saddles and  $l$  sinks and sources;
  - b) the wandering set of  $f$  has no heteroclinic curves.

### 3 A trapping neighborhood for attractors

#### Proof of proposition 1

For each  $i = 1, \dots, k_0$ , let us prove the existence of a trapping neighborhood  $M_i$  consisting of  $m_i$  pairwise disjoint smooth 3-balls.

**Proof:** By construction the set  $A_i$  is the set of  $m_i$  hyperbolic sinks belonging to the periodic orbits numerated 1 to  $i$ . According to statement 5, there are some neighborhood  $U_{A_i} \subset W_{A_i}^s$  of  $A_i$  and an energy function  $\varphi_{A_i} : U_{A_i} \rightarrow \mathbb{R}$  for  $f$  with  $\varphi_{A_i}(A_i) = 0$ . For  $\varepsilon_i > 0$  small enough, the set  $M_i = \varphi_{A_i}^{-1}([0, \varepsilon_i])$  is a union of 3-balls. As  $\varphi_{A_i}$  is a Lyapunov function for  $f$  then  $f(M_i) \subset \text{int } M_i$ . As  $M_i \subset W_{A_i}^s$ , for every  $x \in M_i$ ,  $f^k(x)$  tends to  $A_i$  as  $k \rightarrow +\infty$ . As a consequence,  $A_i = \bigcap_{k \geq 0} f^k(M_i)$ .  $\diamond$

#### Proof of proposition 2

For each  $i = k_0 + 1, \dots, k_1$  let us prove the existence of a trapping neighborhood  $M_i$  of attractor  $A_i$  with the following properties:

- 1)  $M_i$  is a union of  $m_i$  pairwise disjoint handlebodies;
- 2)  $W_\sigma^s \cap M_i$  consists of exactly one two-dimensional closed disc for each saddle point  $\sigma \in \mathcal{O}_i$ .

**Proof:** According to proposition 1, there is a trapping neighborhood  $M_{k_0}$  consisting of  $m_{k_0}$  pairwise disjoint smooth 3-balls. Let us construct  $M_{i+1}$  for  $i = k_0$ .

Without loss of generality we can suppose that  $S_i = \partial M_i$  intersects  $W_{\mathcal{O}_{i+1}}^u$  transversely; let  $n_i$  be the number of intersection points. Set  $V_i = W_{\Omega_f \cap A_i}^s \setminus A_i$ . On this open set  $f$  acts freely and discontinuously. So, it is allowed to consider the quotient  $\hat{V}_i = V_i/f$ ; let  $p_i : V_i \rightarrow \hat{V}_i$  denote the natural projection. Notice that  $\hat{V}_i$  is made of the cobordism  $M_i \setminus \text{int } f(M_i)$  by gluing its boundaries by  $f$ . Then  $p_i(W_{\mathcal{O}_{i+1}}^u)$  is a union of knots which intersect  $p_i(S_i)$  transversely at  $n_i$  points. Thus, there is a tubular neighborhood  $\hat{T}_{i+1}^u \subset \hat{V}_i$  of  $p_i(W_{\mathcal{O}_{i+1}}^u)$  such that  $\hat{T}_{i+1}^u \cap p_i(S_i)$  consists of  $n_i$  2-discs. Set  $T_{i+1}^u = p^{-1}(\hat{T}_{i+1}^u)$ .

According to the  $\lambda$ -lemma (see statement 1),  $T_{i+1}^u \cup W_{\mathcal{O}_{i+1}}^s$  is a neighborhood of  $\mathcal{O}_{i+1}$ . According to statement 5, there are some neighborhood  $U_{\mathcal{O}_{i+1}} \subset (T_{i+1}^u \cup W_{\mathcal{O}_{i+1}}^s)$  of  $\mathcal{O}_{i+1}$  and an energy function  $\varphi_{\mathcal{O}_{i+1}} : U_{\mathcal{O}_{i+1}} \rightarrow \mathbb{R}$  for  $f$  with  $\varphi_{\mathcal{O}_{i+1}}(\mathcal{O}_{i+1}) = 0$ . When  $\varepsilon > 0$  is small enough, each connected component of  $H_{i+1} = \varphi_{\mathcal{O}_{i+1}}^{-1}((-\infty, \varepsilon])$  reads  $\{(x_1, x_2, x_3) \in U_{\mathcal{O}_{i+1}} \mid -x_1^2 + x_2^2 + x_3^2 \leq \varepsilon\}$  in local

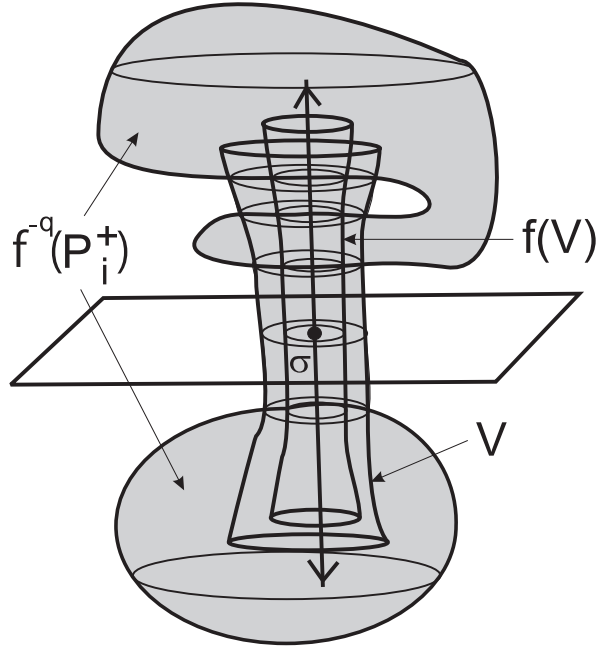


Figure 4: Construction of a handle trapping neighborhood for one-dimensional attractor

coordinates. According to the  $\lambda$ -lemma, when  $k \in \mathbb{N}$  is large enough,  $f^{-k}(S_i)$  intersects both  $H_{i+1}$  and  $f(H_{i+1})$ , and its intersection with these domains consists of  $n_i$  2-discs. Moreover  $f(H_{i+1}) \setminus \text{int } f^{-k}(M_i) \subset \text{int } H_{i+1}$  (see figure 4). Thus,  $M_{i+1} = f^{-k}(M_i) \cup H_{i+1}$  is a union of handlebodies, as it is obtained from the union of handlebodies  $f^{-k}(M_i)$  by gluing one-handles  $H_{i+1} \setminus \text{int } f^{-k}(M_i)$ . Moreover  $f(M_{i+1}) \subset \text{int } M_{i+1}$ ; indeed, it is true for a point  $x \in f^{-k}(M_i)$  as  $f^{-k}(M_i)$  is  $f$ -compressed and it is true for a point  $x \in (H_{i+1} \setminus f^{-k}(M_i))$  as  $f(H_{i+1}) \setminus \text{int } f^{-k}(M_i) \subset \text{int } H_{i+1}$ .

Let us prove the equality  $\bigcap_{k \geq 0} f^k(M_{i+1}) = A_{i+1}$ . As  $A_i \subset M_i$  and  $f^k(A_i) = A_i$  for  $k \in \mathbb{Z}$  then  $A_i \subset \bigcap_{k \geq 0} f^k(M_i)$ . Let us set  $\tilde{A}_i = \bigcap_{k \geq 0} f^k(M_i)$  and show that  $\tilde{A}_i = A_i$ . Assume contrary: there is a point  $x \in (\tilde{A}_i \setminus A_i)$ . Due to theorem 2.3 in [11] there is a point  $p \in (\Omega_f \setminus A_i)$  such that  $x \in W_p^u$ . As the set  $\tilde{A}_i$  is closed and invariant then  $cl(\mathcal{O}_x) \subset \tilde{A}_i$  and, hence,  $p \in \tilde{A}_i$ . This is a contradiction with the fact  $\tilde{A}_i \subset W_{A_i \cap \Omega_f}^s$ . A smoothing of the set  $M_{i+1}$  is the required handle trapping neighborhood.

Applying arguments above in series for each  $i = k_0 + 1, \dots, k_1 - 1$  we will get a handle trapping neighborhood  $M_{i+1}$  for  $A_{i+1}$ .  $\diamond$

**Proposition 3** *The one-dimensional attractor  $A_{k_1}$  is connected.*

**Proof:** Firstly, let us prove that any trapping neighborhood  $M_{k_1}$  of  $A_{k_1}$  is connected. Let us assume the contrary:  $M_{k_1}$  is a union of pairwise disjoint closed sets  $B_1$  and  $B_2$ . As  $M_{k_1}$  is  $f$ -compressed then without loss of generality we can suppose that  $f(B_i) \subset \text{int } B_i$ ,  $i = 1, 2$ . By construction,  $\tilde{B}_1 = \bigcup_{k \geq 0} f^{-k}(\text{int } B_1)$ ,  $\tilde{B}_2 = \bigcup_{k \geq 0} f^{-k}(\text{int } B_2)$  are pairwise disjoint open sets and  $\tilde{B}_1 \cup \tilde{B}_2 = W_{\Omega_0 \cup \Omega_1}^s$ .

On the other hand  $W_{\Omega_0 \cup \Omega_1}^s = M \setminus W_{\Omega_2 \cup \Omega_3}^s$  and, hence,  $W_{\Omega_0 \cup \Omega_1}^s$  is connected as  $\dim M = 3$  and  $\dim W_{\Omega_2 \cup \Omega_3}^s \leq 1$ . This is a contradiction.

Thus  $A_i$  is connected as intersection of nested connected compact sets  $M_i \supset f(M_i) \supset \dots \supset f^k(M_i) \supset \dots$   $\diamond$

Using notation introduced after proposition 2, due to proposition 3 we have  $m_{k_1} = 1$  and, hence  $g_{k_1} = 1 + |\Omega_1| - |\Omega_0|$ , where  $|\cdot|$  stands for the cardinality. We shall equip with a tilde all notations concerning repellers. Then  $\tilde{g}_{k_1} = 1 + |\Omega_2| - |\Omega_3|$  by the same argument applied to  $f^{-1}$ .

**Statement 7** ([4], lemma 3.3) *For any Morse-Smale diffeomorphism  $f : M \rightarrow M$  we have that  $g_{k_1} = \tilde{g}_{k_1}$ .*

## 4 Necessary condition for existence of dynamically ordered energy function

### Proof of theorem 1

Let us prove that if a Morse-Smale diffeomorphism  $f : M^3 \rightarrow M^3$  has a dynamically ordered energy function, then its one-dimensional attractors and repellers are tightly embedded.

**Proof:** Notice that  $f$  and  $f^{-1}$  possess dynamically ordered energy functions simultaneously. Indeed, if  $\varphi : M \rightarrow \mathbb{R}$  is such function for  $f$  then  $\tilde{\varphi} = k_f + 1 - \varphi : M \rightarrow \mathbb{R}$  is such function for  $f^{-1}$  (see statement 4). Therefore, it is enough to prove the fact for attractors.

Let  $\varphi : M \rightarrow \mathbb{R}$  be a dynamically ordered energy function for  $f : M \rightarrow M$ . By definition, the set  $A_{k_0}$  consists of the union of all sinks and  $m_{k_0} = |A_{k_0}|$ . Each sink is a critical point of index 0 for  $\varphi$  (see statement 4) and  $\varphi(\mathcal{O}_i) = i$ ,  $i = 1, \dots, k_0$ . According to the Morse theory, there is  $\varepsilon_{k_0} > 0$  such that  $M_{k_0} = \varphi^{-1}([1, k_0 + \varepsilon_{k_0}])$  is a union of  $m_{k_0}$  3-balls. One checks that  $M_{k_0}$  is a trapping neighborhood for  $A_{k_0}$  (see proposition 1).

For  $i \in \{k_0 + 1, \dots, k_1\}$ , let us prove that there is  $\varepsilon_i > 0$  such that  $M_i = \varphi^{-1}([1, i + \varepsilon_i])$  is a tight trapping neighborhood. First, let us prove the fact for  $M_{i+1}$ , when  $i = k_0$ . As  $\varphi$  is a Morse-Lyapunov function then there is  $\varepsilon_{i+1} > 0$  such that  $W_\sigma^s \cap M_{i+1}$  consists of exactly one closed 2-disc for each saddle point  $\sigma \in \mathcal{O}_{i+1}$ . With our notation after proposition 2, the number of points in

the orbit  $\mathcal{O}_{i+1}$  equals  $r_{i+1} - r_i$ . According to Morse theory,  $M_{i+1}$  has the homotopy type of  $M_i$  with gluing of  $(r_{i+1} - r_i)$  1-handles<sup>2</sup>. As  $M_i$  is a union of handlebodies (3-balls) then  $M_{i+1}$  is also such. Denote by  $g_{M_{i+1}}$  the sum of genus of handlebodies from  $M_{i+1}$ . According to statement 3,  $\chi(M_{i+1}) = m_{k_0} - r_{i+1}$ . Then  $-g_{M_{i+1}} + m_{i+1} = m_{k_0} - r_{i+1}$  or  $g_{M_{i+1}} = g_{i+1}$ . As  $\varphi$  is a Lyapunov function for  $f$  then  $M_{i+1}$  is  $f$ -compressed. Similar to proposition 1 equality  $\bigcap_{k \geq 0} f^k(M_{i+1}) = A_{i+1}$  is proved.

Applying this argument in series for each  $i = k_0, \dots, k_1 - 1$  we will prove that that  $M_i$  is a tight trapping neighborhood.  $\diamond$

## 5 Construction of a dynamically ordered energy function for $f$

Now  $f$  is a Morse-Smale diffeomorphism on a closed 3-manifold  $M$  and its one-dimensional attractors and repellers are strongly tightly embedded. For each  $i = k_0, \dots, k_1$  set  $S_i = \partial M_i$ ,  $N_i = W_{A_i \cap \Omega_f}^s$  and  $V_i = N_i \setminus A_i$ . According to corollary 1,  $V_i$  is diffeomorphic to  $S_i \times \mathbb{R}$ . We recall that notations related to repellers are equipped with a tilde.

Let us notice that we can suppose, without loss of generality, that  $M_{k_1} \subset (M \setminus \tilde{M}_{\tilde{k}_1})$  (in the opposite case we can consider trapping neighborhoods  $f^k(M_{k_0}), \dots, f^k(M_{k_1})$  having the required property when  $k \in \mathbb{N}$  is large enough). The open sets  $V_{k_1}$  and  $\tilde{V}_{\tilde{k}_1}$  coincide; both are obtained from  $M$  by removing  $A_{k_1}$  and  $\tilde{A}_{\tilde{k}_1}$ . Moreover,  $V_{k_1}$  is connected as  $\dim(A_{k_1} \cup \tilde{A}_{\tilde{k}_1}) \leq 1$ , it the same for  $M_{k_1}$ ,  $\tilde{M}_{\tilde{k}_1}$ ,  $N_{k_1}$ ,  $\tilde{N}_{\tilde{k}_1}$ ,  $A_{k_1}$ ,  $\tilde{A}_{\tilde{k}_1}$ . Due to statement 7 the handlebodies  $M_{k_1}$  and  $\tilde{M}_{\tilde{k}_1}$  have the same genus. Thus, according to corollary 1,  $K = M \setminus (M_{k_1} \cup \tilde{M}_{\tilde{k}_1})$  is a product.

We are going to describe shortly how to construct a dynamically ordered energy function  $\varphi : M \rightarrow \mathbb{R}$  for  $f$ . After that, we shall prove the needed lemmas.

1. We construct a dynamically ordered energy function  $\varphi_{k_0}$  on  $M_{k_0}$  using lemma 4.
2. We construct a dynamically ordered energy functions  $\varphi_{k_0+1}, \dots, \varphi_{k_1}$  successively on  $M_{k_0+1}, \dots, M_{k_1}$  using lemmas 2, 1 and 3.
3. By changing  $f$  to  $f^{-1}$  we can similarly construct a dynamically ordered energy function  $\psi$  for  $f^{-1}$  on  $\tilde{M}_{\tilde{k}_1}$ . Then  $\tilde{\varphi}_{\tilde{k}_1} = k_f + 1 - \psi$  is a dynamically ordered energy function for  $f$  on  $\tilde{M}_{\tilde{k}_1}$ .
4. Finally, we find a common extension to  $K$  of the functions  $\varphi_{k_1}$  and  $\tilde{\varphi}_{\tilde{k}_1}$  due to lemma 1.

---

<sup>2</sup>A 3-dimensional 1-handle is the product of an interval with a 2-disc.

**Definition 9** Let  $D$  be a subset of  $M$  which is diffeomorphic to product  $S \times [0, 1]$  for some (possibly non connected) surface  $S$ . Then  $D$  is said to be an  $(f, S)$ -compressed product when there is a diffeomorphism  $g : D \rightarrow S \times [0, 1]$  such that  $g^{-1}(S \times \{t\})$  bounds an  $f$ -compressed domain in  $M$  for any  $t \in [0, 1]$ .

**Remark 4** Let  $D$  be an  $(f, S)$ -compressed product. Then the function  $\varphi_D : D \rightarrow \mathbb{R}$  defined by  $\varphi_D(x) = t$  for  $x \in g^{-1}(S \times \{t\})$  is an energy function for  $f|_D$  with  $g^{-1}(S \times \{t\})$  as level set for every  $t \in [0, 1]$ .

**Definition 10** Let  $i \in \{k_0, \dots, k_1\}$ . A set  $P_i$  is said to be parallel to  $M_i$  if  $P_i$  is  $f$ -compressed union of  $m_i$  pairwise disjoint handlebodies with sum of genres  $g_i$  such that  $A_i \subset P_i \subset N_i$ .

**Lemma 1** Let  $i \in \{k_0, \dots, k_1\}$ . Let  $P_i$  and  $Q_i$  be sets which are parallel to  $M_i$  with  $Q_i \subset P_i$ . If there is a dynamically ordered energy function  $\varphi_{Q_i} : Q_i \rightarrow \mathbb{R}$  for  $f$  with  $S_{Q_i} = \partial Q_i$  as a level set then there is a dynamically ordered energy function  $\varphi_{P_i} : P_i \rightarrow \mathbb{R}$  for  $f$  with  $S_{P_i} = \partial P_i$  as a level set.

**Proof:** The arguments for a proof of the lemma are similar to the arguments for the proof of lemma 4.2 from [3] with some modifications.

Firstly notice that as  $V_i$  is diffeomorphic to  $S_i \times \mathbb{R}$  then, according to corollary 1,  $G_i = P_i \setminus \text{int } Q_i$  is a product. Secondly notice that  $f^n(S_{P_i})$  and  $f^n(S_{Q_i})$  are incompressible in  $V_i$  for any  $n \in \mathbb{Z}$ . Now let us construct the function  $\varphi_{P_i}$ .

There are two cases: 1)  $S_{P_i} \cap (\bigcup_{n>0} f^{-n}(S_{Q_i})) = \emptyset$  and 2)  $S_{P_i} \cap (\bigcup_{n>0} f^{-n}(S_{Q_i})) \neq \emptyset$ .

In case 1), let  $m$  be the first positive integer such that  $f^m(P_i) \subset \text{int } Q_i$ . If  $m = 1$ , then  $G_i$  is  $(f, S_i)$ -compressed product and remark 4 yields the required function.

If  $m > 1$ , the surfaces  $f^{-1}(S_{Q_i}), f^{-2}(S_{Q_i}), \dots, f^{-m+1}(S_{Q_i})$  are mutually “parallel”, that is: two by two they bound a product cobordism (see corollary 1). Therefore they subdivide  $G_i$  in  $(f, S_i)$ -compressed products and, hence, the required function exists due to remark 4.

In case 2), without loss of generality we may assume that  $S_{P_i}$  is transverse to  $\bigcup_{n>0} f^{-n}(S_{Q_i})$ , which implies that there is a finite family  $\mathcal{C}$  of intersection curves.

We are going to describe a process of decreasing the number of intersection curves by an isotopy of  $Q_i$  among handlebodies which are contained in  $P_i$  and parallel to  $M_i$ ; they will be all equipped with a dynamically ordered energy function for  $f$  which is constant on the boundary.

Firstly we consider all intersection curves which are homotopic to zero in  $S_{P_i}$ . Let  $c$  be an innermost such curve. Then there is a disc  $\delta \subset S_{P_i}$  which is bounded by  $c$  and such that  $\text{int } \delta$  contains no curves from the family  $\mathcal{C}$ . As

$c \subset f^{-n}(S_{Q_i})$  for some  $n$  and  $f^{-n}(S_{Q_i})$  is incompressible in  $V_i$ , then  $c$  bounds a disc  $d \subset f^{-n}(S_{Q_i})$ . The 2-sphere  $\delta \cup d$  is embedded and bounds a 3-ball  $b$  in  $N_i$  (when the component of  $f^{-n}(S_{Q_i})$  containing  $d$  is a 2-sphere, replace  $d$  by the complementary disc if necessary).

There are two occurrences: (a)  $f^n(b) \subset Q_i$  and (b)  $f^n(b) \subset f^{-1}(Q_i)$ . We define  $Q'_i$  as  $cl(Q_i \setminus f^n(b))$  in the case (a) and  $Q_i \cup f^n(b)$  in the case (b). The fact that  $c$  is an innermost curve implies  $f(Q_i) \subset Q'_i \subset Q_i$  in case (a) and  $Q_i \subset Q'_i \subset f^{-1}(Q_i)$  in case (b). In both cases there is a smooth approximation  $\tilde{Q}_i$  of  $Q'_i$  such that  $f(Q_i) \subset \text{int } \tilde{Q}_i \subset Q_i$  if (a),  $Q_i \subset \text{int } \tilde{Q}_i \subset f^{-1}(Q_i)$  if (b), and the number of intersection curves in  $S_{P_i} \cap (\bigcup_{n>0} f^{-n}(\partial \tilde{Q}_i))$  is less than the cardinality of  $\mathcal{C}$ .

In case (a),  $\varphi_{f(Q_i)} = \varphi_{Q_i} f^{-1} : f(Q_i) \rightarrow \mathbb{R}$  is a dynamically ordered energy function which is constant on the boundary. Therefore  $\tilde{Q}_i \setminus \text{int } f(Q_i)$  is an  $(f, S_i)$ -compressed product and, hence, due to remark 4, there is a similar function on  $\tilde{Q}_i$ . Similarly in case (b),  $\tilde{Q}_i$  is equipped with a dynamically ordered energy function which is constant on the boundary as  $\tilde{Q}_i \setminus \text{int } Q_i$  is an  $(f, S_i)$ -compressed product.

We will repeat this process until getting a set  $\hat{Q}_i$  which are parallel to  $M_i$  and such that  $S_{P_i} \cap (\bigcup_{n>0} f^{-n}(\partial \hat{Q}_i))$  does not contain curves which are homotopic zero in  $S_{P_i}$ .

Thus we may assume that  $S_{P_i} \cap (\bigcup_{n>0} f^{-n}(S_i))$  does not contain intersection curves which are homotopic to zero in  $S_{P_i}$ . We denote by  $m$  the largest integer such that  $f^m(S_{P_i}) \cap S_i \neq \emptyset$ . Let  $F$  be a connected component of  $f^m(S_{P_i}) \cap G_i$ . We have  $\partial F \subset \partial S_{Q_i}$ . We claim that  $F$  is incompressible in  $G_i$ . Indeed, if  $\delta$  is a disc in  $G_i$  with boundary  $\gamma \subset F$  then  $\gamma$  is homotopic to zero in  $f^m(S_{P_i})$  as  $f^m(S_{P_i})$  is incompressible in  $V_i$ . As none component of  $\partial F$  is homotopic to zero in  $f^m(S_{P_i})$ ,  $\gamma$  is homotopic to zero in  $F$ .

Therefore, according to statement 2 there is some surface  $F_1 \subset S_{Q_i}$  diffeomorphic to  $F$ , with  $\partial F = \partial F_1$ , and  $F \cup F_1$  bounds a domain  $\Delta$  in  $G_i$  which, up to smoothing of the boundary, is diffeomorphic to  $F \times [0, 1]$ . We then define  $\tilde{Q}_i$  as  $Q'_i = Q_i \cup \Delta$  up to smoothing. By the choice of  $m$ ,  $\tilde{Q}_i$  is  $f$ -compressed as  $f(\Delta) \subset Q_i$ . As  $\tilde{Q}_i$  is obtained by an isotopy supported in a neighborhood of  $\Delta$  from  $Q_i$  then  $\tilde{Q}_i \setminus \text{int } Q_i$  is an  $(f, S_i)$ -compressed product. Thus we get a dynamically ordered energy function on  $\tilde{Q}_i$  with  $\partial \tilde{Q}_i$  as a level set. Arguing recursively, we are reduced to case 1).  $\diamond$

**Lemma 2** *Let  $i \in \{k_0, \dots, k_1 - 1\}$  and  $D_{i+1} = M_{i+1} \cap W_{\mathcal{O}_{i+1}}^s$ . There is a tubular neighborhood  $N(D_{i+1}) \subset M_{i+1}$  of  $D_{i+1}$  such that the set  $P_i = M_{i+1} \setminus \text{int } N(D_{i+1})$  is parallel to  $M_i$ .*

**Proof:** Choose a tubular neighborhood  $N(D_{i+1}) \subset M_{i+1}$  of  $D_{i+1}$  such that  $N(D_{i+1}) \cap A_i = \emptyset$  and  $P_i$  is  $f$ -compressed. Recall that  $m_i$  is the number of



connected components of  $A_i$ ,  $r_i$  the number of saddle points in the set  $A_i$  and put  $g_i = m_i + r_i - m_{k_0}$ . Then the number of point in the orbit  $\mathcal{O}_{i+1}$ , as the number of discs in the set  $D_{i+1}$ , equals  $r_{i+1} - r_i$ . Since  $(cl(W_{\mathcal{O}_{i+1}}^u) \setminus W_{\mathcal{O}_{i+1}}^u) \subset A_i$  and  $A_{i+1} = A_i \cup W_{\mathcal{O}_{i+1}}^u$ , then  $m_{i+1} \leq m_i$ . Thus  $(m_i - m_{i+1})$  saddle points from  $\mathcal{O}_{i+1}$  divide the set  $A_{i+1}$  and, accordingly,  $(m_i - m_{i+1})$  2-discs from the set  $D_{i+1}$  divide the set  $M_{i+1}$ . Moreover, removing such discs does not decrease the genus, but removing any of the remaining discs decreases the genus by 1. Thus the number  $m_{P_i}$  of connected components of  $P_i$  equals  $m_{i+1} + (m_i - m_{i+1}) = m_i$  and sum  $g_{P_i}$  of their genera equals  $g_{i+1} - (r_{i+1} - r_i - (m_i - m_{i+1})) = m_{i+1} + r_{i+1} - m_{k_0} - r_{i+1} + r_i + m_i - m_{i+1} = m_i + r_i - m_{k_0} = g_i$ . Since  $P_i$  contains  $A_i$ , is contained in  $N_i$  and is  $f$ -compressed, then it is parallel to  $M_i$ .  $\diamond$

**Lemma 3** *Let  $i \in \{k_0, \dots, k_1 - 1\}$  and  $P_i$  is as in lemma 2. If there is a dynamically ordered energy function  $\varphi_{P_i} : P_i \rightarrow \mathbb{R}$  for  $f$  with  $S_{P_i}$  as a level set then there is a dynamically ordered energy function  $\varphi_{i+1} : M_{i+1} \rightarrow \mathbb{R}$  for  $f$  with  $S_{i+1}$  as a level set.*

**Proof:** The arguments for a proof of the lemma are similar to the arguments of section 4.3 from [3] with some modifications.

According to lemma 5, the orbit  $\mathcal{O}_{i+1}$  has a neighborhood  $U_{\mathcal{O}_{i+1}} \subset M$  endowed with a Morse-Lyapunov function  $\varphi_{\mathcal{O}_{i+1}} : U_{\mathcal{O}_{i+1}} \rightarrow \mathbb{R}$  of  $f$  with  $\varphi_{\mathcal{O}_{i+1}}(\mathcal{O}_{i+1}) = i + 1$ .

Each connected component  $U_\sigma$  of  $U_{\mathcal{O}_{i+1}}$ ,  $\sigma \in \mathcal{O}_{i+1}$ , is endowed with Morse coordinates  $(x_1, x_2, x_3)$  as in the conclusion of lemma 5:  $\varphi_{\mathcal{O}_{i+1}}(x_1, x_2, x_3) = i + 1 - x_1^2 + x_2^2 + x_3^2$ , the  $x_1$ -axis is contained in the unstable manifold and the  $(x_2, x_3)$ -plane is contained in the stable manifold.

Without lost of generality we can suppose that  $\varphi_{P_i}(S_{P_i}) = i + 1$ . Moreover,  $\varepsilon > 0$  and  $N(D_{i+1})$  are chosen such that:

- (1)  $S_{P_i}$  and  $S_{P_i^\varepsilon} = \partial P_i^\varepsilon$  are transverse to all level sets from  $\varphi_{\mathcal{O}_{i+1}}^{-1}(i + 1 - \varepsilon)$  to  $\varphi_{\mathcal{O}_{i+1}}^{-1}(i + 1 + \varepsilon)$  and  $f(P_i) \subset \text{int } P_i^\varepsilon$ , where  $P_i^\varepsilon = \varphi_{P_i}^{-1}([1, i + 1 - \varepsilon])$ ;
- (2)  $\varphi_{\mathcal{O}_{i+1}}(f^{-1}(E_\varepsilon)) > i + 1 + \varepsilon$ , where  $E_\varepsilon = (P_i \setminus \text{int } (P_i^\varepsilon)) \cap \{i + 1 - \varepsilon \leq \varphi_{\mathcal{O}_{i+1}} \leq i + 1 + \varepsilon\}$  (it is possible as  $\varphi_{\mathcal{O}_{i+1}}(f^{-1}(\varphi_{\mathcal{O}_{i+1}}^{-1}(i + 1) \setminus \mathcal{O}_{i+1})) > i + 1$ );
- (3)  $H_{i+1} \subset \text{int } M_{i+1}$ , where  $H_{i+1}$  the closure of  $\{x \in U_{\mathcal{O}_{i+1}} \mid x \notin P_i, \varphi_{\mathcal{O}_{i+1}}(x) \leq i + 1 + \varepsilon\}$  (see figure 5).

Due to lemma 2 there is a smoothing  $Q_{i+1}$  of  $P_i \cup H_{i+1}$  which are parallel to  $M_{i+1}$ . Then, using lemma 1, we can construct a dynamically ordered energy function  $\varphi_{i+1} : M_{i+1} \rightarrow \mathbb{R}$  with a level set  $S_{i+1}$  if we have similar function on  $Q_{i+1}$ . Thus, it is enough to construct such function on  $Q_{i+1}$ .

Denote  $d_\varepsilon$  the part of the level set  $\varphi_{\mathcal{O}_{i+1}}^{-1}(i + 1 - \varepsilon)$  belonging to  $U_{\mathcal{O}_{i+1}} \setminus \text{int } P_i^\varepsilon$ . By construction  $d_\varepsilon$  is the union of 2-discs. Denote  $D'_\varepsilon$  the union of 2-discs in



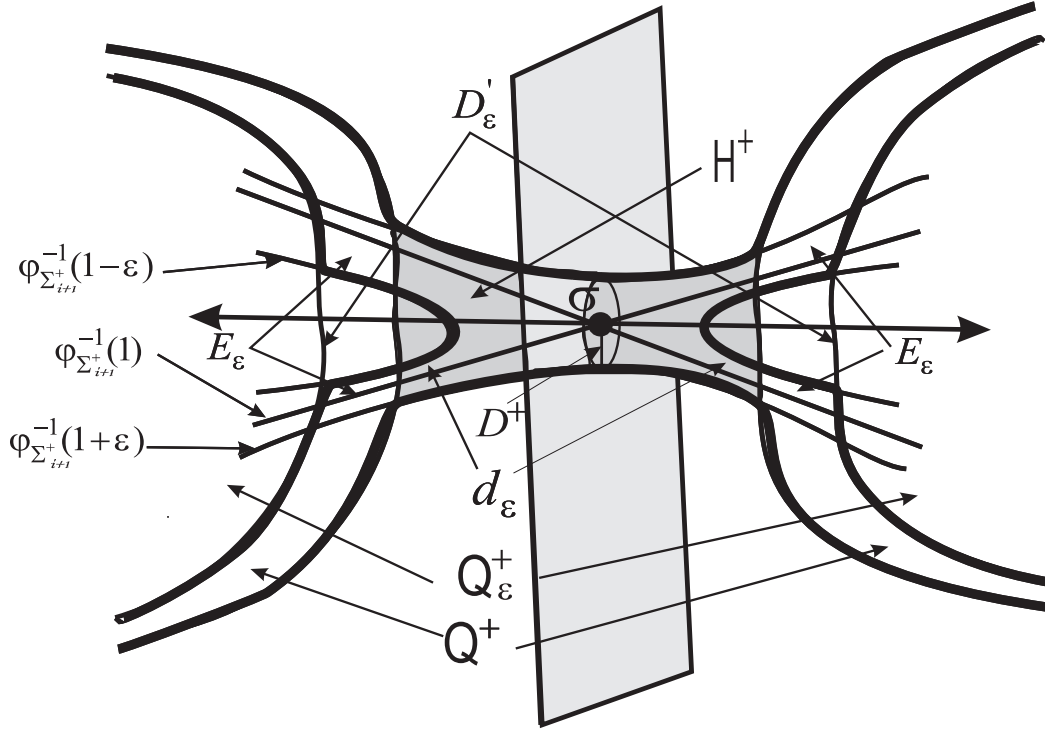


Figure 5: Illustration to proof of lemma 3

$\partial P_i^\varepsilon$  such that  $\partial d_\varepsilon = \partial D'_\varepsilon$ . Set  $S = P_i^\varepsilon \setminus D'_\varepsilon \cup d_\varepsilon$ . Denote  $Q(S)$  the union of handlebodies bounded by  $S$  and containing  $A_i$ . We check that  $Q(S)$  is  $f$ -compressed. Indeed, it is true for  $P_i^\varepsilon$ . Moreover,  $f(d_\varepsilon)$  does not intersect  $d_\varepsilon$  nor  $P_i^\varepsilon \setminus D'_\varepsilon$ . The first intersection is empty as  $\varphi_{\mathcal{O}_{i+1}}$  is a Lyapunov function and  $d_\varepsilon$  lies in a level set of it. The second one is empty as  $\varphi_{\mathcal{O}_{i+1}}(d_\varepsilon) = i + 1 - \varepsilon \leq \varphi_{\mathcal{O}_{i+1}}(x)$ , for any  $x \in U_{\mathcal{O}_{i+1}}$ ,  $x \in (P_i^\varepsilon \setminus D'_\varepsilon)$ . Thus  $Q(S)$  is parallel to  $M_{i+1}$  and, by lemma 1, there is a dynamically ordered energy function on  $Q(S)$  with  $S$  as a level set.

Let  $K$  be the domain between  $\partial Q_{i+1}$  and  $S$ . We define a function  $\varphi_K : K \rightarrow \mathbb{R}$  whose value is  $i + 1 + \varepsilon$  on  $\partial Q_{i+1}$ ,  $i + 1 - \varepsilon$  on  $S$ , coinciding with  $\varphi_{\mathcal{O}_{i+1}}$  on  $K \cap H_{i+1}$  and without critical points outside of  $H_{i+1}$ . This last condition is easy to satisfy as the domain in question is a product cobordism due to condition (1). With all the informations that we have on the image of  $f$ , it is easy to check that  $\varphi_K$  is a Lyapunov function. Indeed, it is obvious for the points of  $K$  which are not in  $H_{i+1}$ . Suppose that  $x \in (K \cap H_{i+1})$ . Then we have two possibilities: a)  $f(x) \in H_{i+1}$ ; b)  $f(x) \notin H_{i+1}$ . In the first case the conclusion follows from the Lyapunov property of  $\varphi_{\mathcal{O}_{i+1}}$ . In the second case we are going to show that  $f(x) \in \{\varphi_{\mathcal{O}_{i+1}} < i + 1 - \varepsilon\}$  and then the conclusion also holds. Suppose on the contrary that  $f(x) \notin \{\varphi_{\mathcal{O}_{i+1}} < i + 1 - \varepsilon\}$ . Then  $f(x)$  belongs to the domain  $E_\varepsilon$ . But it follows from the choice of  $E_\varepsilon$  (see condition

(2)) that  $f^{-1}(E_\varepsilon)$  does not intersect  $H_{i+1}$ . We get a contradiction.

Thus we get the desired function on  $Q_{i+1}$ .  $\diamond$

**Lemma 4** *There is a dynamically ordered energy function  $\varphi_{k_0}$  on  $M_{k_0}$ .*

**Proof:** By the construction the set  $A_{k_0}$  consists of  $m_{k_0}$  hyperbolic sinks and  $M_{k_0}$  consists of  $m_{k_0}$  pairwise disjoint 3-balls. According to statement 5, there is  $m_{k_0}$  pairwise disjoint 3-balls  $Q_{k_0} \subset M_{k_0}$  around  $A_{k_0}$  equipped a dynamically ordered energy function  $\varphi_{Q_{k_0}} : Q_{k_0} \rightarrow \mathbb{R}$  for  $f$  with  $\partial Q_{k_0}$  as a level set. According to lemma 1, there is similar function on  $M_{k_0}$ .  $\diamond$

## 6 Dynamically ordered energy function for diffeomorphisms on 3-sphere

In this section  $f : \mathbb{S}^3 \rightarrow \mathbb{S}^3$  is a Morse-Smale diffeomorphism without heteroclinic curve. We prove theorem 3: the one-dimensional attractors and repellers have tight trapping neighborhoods which are 3-balls if and only if there exists a dynamically ordered energy function for  $f$ .

**Proof:** Let  $i = k_0 + 1, \dots, k_1$ . Then  $A_i = \bigcup_{j=1}^i W_j^u$  is one-dimensional attractor with  $m_i$  connected components. Moreover, let  $r_i$  be a number of saddle points in  $A_i$  and  $g_i = m_i + r_i - |\Omega_0|$ , where  $|\Omega_0|$  is number of sinks of the diffeomorphism  $f$ . We first prove an auxiliary proposition.

**Proposition 5** *If  $f : \mathbb{S}^3 \rightarrow \mathbb{S}^3$  is a Morse-Smale diffeomorphism without heteroclinic curve then  $g_i = 0$  for each  $i = k_0 + 1, \dots, k_1$ .*

**Proof:** We start from  $g_{k_1}$ . According to proposition 3, the attractor  $A_{k_1}$  is connected, that is  $m_{k_1} = 1$  and, hence  $g_{k_1} = 1 + |\Omega_1| - |\Omega_0|$ . Due to statement 7,  $g_{k_1} = \tilde{g}_{k_1}$ , where  $\tilde{g}_{k_1} = 1 + |\Omega_2| - |\Omega_3|$ . By statement 6,  $2 + |\Omega_1 \cup \Omega_2| - |\Omega_0 \cup \Omega_3| = 0$  for any Morse-Smale diffeomorphism without heteroclinic curve on  $\mathbb{S}^3$ . Thus  $g_{k_1} + \tilde{g}_{k_1} = 0$  and, hence,  $g_{k_1} = 0$ .

Now let us show that  $g_{i+1} \geq g_i$ . Indeed, by the construction  $A_{i+1} = A_i \cup W_{i+1}^u$  and  $cl(A_{i+1}) \setminus A_{i+1} \subset A_i$ . In our denotation the orbit  $\mathcal{O}_{i+1}$  consists of  $r_{i+1} - r_i$  saddle points. Then  $m_{i+1} \geq m_i - (r_{i+1} - r_i)$  and, hence,  $g_{i+1} - g_i = m_{i+1} - m_i + r_{i+1} - r_i \geq 0$ .

Thus  $0 \leq g_{k_0+1} \leq g_{k_0+2} \leq \dots \leq g_{k_1} = 0$ . It means that  $g_i = 0$  for each  $i = k_0 + 1, \dots, k_1$ .  $\diamond$

**Necessity.** Let  $f : \mathbb{S}^3 \rightarrow \mathbb{S}^3$  be a Morse-Smale diffeomorphism without heteroclinic curve and possessing a dynamically ordered energy function  $\varphi$ . According to theorem 1, each one-dimensional attractor and repeller has a

tight trapping neighborhood. Due to proposition 5 this neighborhood is a union of 3-balls.

**Sufficiency.** Let  $f : \mathbb{S}^3 \rightarrow \mathbb{S}^3$  be a Morse-Smale diffeomorphism without heteroclinic curve, whose one-dimensional attractors and repellers have tight trapping neighborhoods which are 3-balls. Let us show that all one-dimensional attractors and repellers of  $f$  are strongly tightly embedded. We give a proof for attractors; for repellers the proof is similar by looking at  $f^{-1}$ .

Let  $i = k_0 + 1, \dots, k_1$  and  $M_i$  be a tight trapping neighborhood of a one-dimensional attractor  $A_i$ . By assumption  $M_i$  is a union of 3-balls. Since a pair of nested balls is unique up to diffeomorphism,  $K_i = M_i \setminus \text{int } f(M_i)$  is a disjoint union of copies of  $S^2 \times [0, 1]$ . As  $M_i \setminus A_i = \bigcup_{k \geq 0} f^k(M_i)$ , then  $M_i \setminus A_i$  is diffeomorphic to  $\partial M_i \times (0, 1]$ . Thus the attractor  $A_i$  is strongly tight and, according to theorem 2,  $f$  possesses a dynamically ordered energy function.

◇

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