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# Polytopic control invariant sets for differential inclusion systems: a viability theory approach

Mirko Fiacchini, Sophie Tarbouriech and Christophe Prieur

**Abstract**— This paper presents a criterion to characterize control invariant polytopes for differential inclusion systems. The practice-oriented method, based on viability theory and convex analysis, can be applied to determine computational procedures to obtain families of control invariant polytopes. The criterion is based on a necessary and sufficient condition for viability to hold at any point on the boundary of a polytope.

## I. INTRODUCTION

The importance of invariance in control and systems analysis has been increasing since the first results on the topic, see the pioneering work [5]. Currently, well established results are available, mainly for linear systems, [15], [18], but also for nonlinear systems [1], [10], [14], see the monograph on invariance [7]. The problem of characterization and computation of invariant sets for particular classes of continuous-time nonlinear systems has been tackled in [12], [16], [17], mostly using LMI related approaches. The attention devoted to invariance is strongly due to its tight relation with many basic topics in control and systems analysis, such as stability, domain of attraction estimation and hard constraints satisfaction, among others. On the other hand, few computation-oriented results are available for generic nonlinear systems.

Viability theory, considered from the control point of view, concerns systems whose dynamics are given by a differential inclusion. Roughly speaking, they are systems for which the variation (or the successor, for discrete-time systems) of a state is a set rather than a point in the state space. Viability theory provides mathematical tools to characterize conditions for a set to be robust or control invariant for the system. Viability has been mainly developed by Aubin and co-authors, see [2]–[4], and applies to very general families of sets. See also [9], which proposes methods to construct viability kernels for multi-input single-output nonlinear systems affine in the control.

The main objective of this paper is to provide a characterization of control invariance and contractiveness of polytopes for differential inclusion systems. As common modeling frameworks, like uncertain systems or constrained controlled ones, can be cast in terms of differential inclusion

systems, the proposed approach applies to a wide class of systems. Differential inclusion can also be used to approximate nonlinear systems. In this paper, we focus on linear systems with state-dependent bounds on the input, which are intrinsically nonlinear. It has to be stressed that, while ellipsoidal invariant sets are more common in the context of nonlinear continuous-time systems, results in literature involving polytopes concern mainly linear systems, see [11], [13]. The aim is to apply analytical tools proper of viability theory and convex analysis, see [6], [8], [19], [20], to determine a computation-oriented criterion for characterizing polytopic control invariant sets for constrained continuous-time linear systems. Restricting our attention to polytopes, rather than to the generic sets dealt with in viability theory, allows us to use properties of convex analysis, which lead to tractable problems and more practical solutions. The price to pay is the introduction of a certain conservativeness.

The paper is organized as follows: Section II presents the problem statement, Section III recalls some definitions and results on viability theory. In Section IV the main results on control invariance of a polytope are stated. In Section V the presented method is applied to numerical examples. The paper ends with a section of conclusions.

## Notation

Given  $n \in \mathbb{N}$ , define  $\mathbb{N}_n = \{x \in \mathbb{N} : 1 \leq x \leq n\}$ . Given  $A \in \mathbb{R}^{n \times m}$ ,  $A_i$  with  $i \in \mathbb{N}_n$  denotes its  $i$ -th row. Operators  $\leq, \geq, <$  and  $>$  are intended to apply element-wise to vectors  $a, b \in \mathbb{R}^n$ . Given two sets  $D$  and  $E$  and a nonnegative scalar  $\alpha \geq 0$ , denote the set  $\alpha D = \{\alpha x : x \in D\}$ , and the Minkowski set addition is  $D \oplus E = \{x = d + e : d \in D, e \in E\}$ . The interior of  $D$  is denoted as  $\text{int}(D)$ , its boundary is  $\partial D$ . With  $\mathcal{S}(D)$  we denote the set of subsets of  $D$ . Given a set valued map  $F : \mathbb{R}^n \rightarrow \mathcal{S}(\mathbb{R}^m)$ , its domain is  $\text{dom}(F) = \{x \in \mathbb{R}^n : F(x) \neq \emptyset\}$  and its graph is  $\text{graph}(F) = \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^m : y \in F(x)\}$ .

## II. PROBLEM STATEMENT

Consider the continuous-time system given by:

$$\dot{x}(t) \in F(x(t)), \quad (1)$$

where  $x(t) \in \mathbb{R}^n$  is the state at time  $t$ , and with  $F : \mathbb{R}^n \rightarrow \mathcal{S}(\mathbb{R}^m)$ , set valued map. Notice that this modeling framework, which is referred to as differential inclusion, encloses common systems such as uncertain systems and controlled ones. In fact, the solutions of the uncertain system  $\dot{x}(t) = f(x(t), w(t))$ , where  $w(t) \in W(x(t))$  is the uncertainty (or the parameter) with  $W(x) \subseteq \mathbb{R}^n$ , are those of (1) with

$$F(x) = f(x, W(x)) = \{y \in \mathbb{R}^m : y = f(x, w), w \in W(x)\}.$$

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Analogously, the trajectories of the controlled system  $\dot{x}(t) = f(x(t), u(t))$ , with bounds on the input  $u(t) \in U(x(t)) \subseteq \mathbb{R}^m$ , are those of system (1) with

$$F(x) = f(x, U(x)) = \{y \in \mathbb{R}^n : y = f(x, u), u \in U(x)\}.$$

Moreover, differential inclusion can be used to approximate the evolutions of a nonlinear system  $\dot{x}(t) = f(x(t))$ , provided that  $f(x) \in F(x)$  for all  $x \in \mathbb{R}^n$ .

The objective of this work is to design a computation-oriented method for obtaining polytopic control invariant sets for a controlled system with state-dependent bounds on the input. This means, as formalized below, that our aim is the characterization and computation of a (family of) set  $K \subseteq \mathbb{R}^n$  such that, for all  $x(0) = x_0 \in K$ , there exists an admissible control signal  $u(t) \in U(x(t))$  which permits to maintain the state  $x(t)$ , solution of (1), in  $K$  for  $t \geq 0$ .

### III. VIABILITY THEORY

We recall here some general definitions and results on viability theory, which is strongly associated to the research of Aubin and co-authors, see [2]–[4]. Many of those results are developed in the cited works, and references therein, under assumptions which are more general than those required in this paper. Since we are interested in characterizing and computing polytopic control invariant sets, we give the definitions and the properties for the case under analysis.

*Definition 1 (Viability properties [3]):* Consider the set  $K \subseteq \text{dom}(F)$ . A function  $x(\cdot)$  from  $[0, T]$  to  $\mathbb{R}^n$ , solution of (1), is called *viable* if  $x(t) \in K$  for all  $[0, T]$ . We say that  $K$  enjoys the *local viability* property or control invariance (for the set valued map  $F$ ) if, for any initial state  $x_0$  in  $K$ , there exist  $T > 0$  and a viable solution on  $[0, T]$  to differential inclusion (1). It enjoys the *global viability* property (or, simply, the *viability* property) if we can take  $T = +\infty$ .

Many of the results provided in the context of viability theory analysis are valid for generic nonempty sets  $K$  in the state space. An important tool on which those results are based is the contingent cone of set  $K$  at  $x \in K$ , denoted as  $T_K(x)$ , see [3]. When  $K$  is closed and convex, the contingent cone is equal to the closure of the tangent cone.

*Theorem 1 (Tangent cones of closed, convex subsets [3]):* For  $K \subseteq \mathbb{R}^n$ , closed and convex, its contingent cone  $T_K(x)$  coincides with the closure of the tangent cone, given by

$$C_K(x) = \overline{\bigcup_{h>0} \frac{K-x}{h}},$$

that is the closed cone spanned by  $K - x$ .

Then, for convex, closed set  $K$ , we have that  $T_K(x) = C_K(x)$ , which are closed and convex cones. Notice that the tangent cone of a closed, convex set  $K$  is  $\mathbb{R}^n$  at any point  $x \in \text{int}(K)$ . Hence, the definition of viability domain, involving the contingent cone in the general case, can be given directly in terms of the tangent cone for closed convex sets.

*Definition 2 (Viability domain [3]):* Let  $F : \mathbb{R}^n \rightarrow \mathcal{S}(\mathbb{R}^n)$  be a nontrivial set valued map. A closed, convex set  $K \subseteq \text{dom}(F)$  is a *viability domain* of  $F$  if and only if

$$\forall x \in K, \quad F(x) \cap C_K(x) \neq \emptyset. \quad (2)$$

Also some assumptions on the set valued map  $F(\cdot)$  have to be imposed in order to apply the Viability Theorem. A set valued map  $F : \mathbb{R}^n \rightarrow \mathcal{S}(\mathbb{R}^n)$  fulfilling such assumptions is defined as Marchaud map, which amounts to say that its graph and its domain are closed, the values of  $F(\cdot)$  are convex, and the growth of  $F(\cdot)$  is linear, see [3]. Such preliminaries are useful since it has been proved that any closed, convex viability domain  $K$  for system (1) with  $F(\cdot)$  Marchaud map enjoys the viability property.

*Theorem 2 (Viability theorem [3]):* Consider a Marchaud map  $F : \mathbb{R}^n \rightarrow \mathcal{S}(\mathbb{R}^n)$  and a closed, convex  $K \subseteq \text{dom}(F)$ . If  $K$  is a viability domain, then for any state  $x_0 \in K$ , there exists a viable solution on  $[0, +\infty)$  to the differential inclusion (1).

This means that, for every initial condition in  $K$ , closed and convex, there exists a trajectory of system (1) which remains in  $K$  at any time  $t \in [0, +\infty)$ , if there exists a "direction" belonging to the map  $F(x(t))$  and to the tangent cone of  $K$  at  $x(t)$ , see Figure 1. In this case, in fact, considering such direction at any time, the trajectory would always head towards the interior of set  $K$  (or on the boundary).

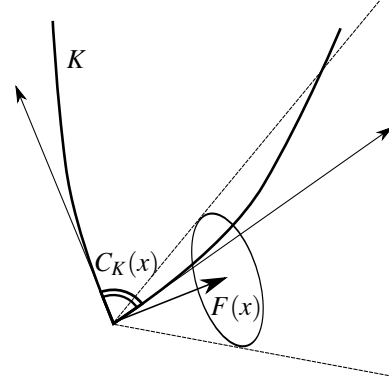


Fig. 1. Viability condition.

### IV. POLYTOPIC CONTROL INVARIANT SETS

The results presented in this section, representing the main contributions of that paper, provide a computation-oriented characterization of control invariance for polytopes.

Consider a polytope in the state space containing the origin in its interior,  $\Omega = \{x \in \mathbb{R}^n : Hx \leq 1\}$ , with  $H \in \mathbb{R}^{n_h \times n}$ , and the linear controlled system

$$\dot{x}(t) = Ax(t) + Bu(t), \quad (3)$$

with  $u(t) \in U(x(t))$  control input. The input bounding set  $U(x(t)) \subseteq \mathbb{R}^m$  is assumed to be the state-dependent polytope

$$U(x) = \{u \in \mathbb{R}^m : Lu \leq P(x)\}, \quad (4)$$

with  $L \in \mathbb{R}^{n_u \times m}$  and  $P : \mathbb{R}^n \rightarrow \mathbb{R}^{n_u}$ , and such that  $F(x) = Ax \oplus BU(x)$  is Marchaud. Notice that if  $U(x)$  is Marchaud then  $F(x) = Ax \oplus BU(x)$  is Marchaud too, see [3].

*Remark 1:* No loss of generality is induced by considering system (3)-(4) in spite of (1). In fact, given  $F(x)$  determining (1), Marchaud and with  $F(x)$  polytopic, for any  $A \in \mathbb{R}^{n \times n}$  and

defining  $B = I_n$  and  $U(x) = (-Ax) \oplus F(x)$ , the differential inclusion (1) can be written in terms of (3).

The Minkowski function is introduced here, see [7] and references therein for some properties.

*Definition 3:* Given a compact, convex set  $K \in \mathbb{R}^n$  with  $0 \in \text{int}(K)$ , the Minkowski function of  $K$  at  $x \in \mathbb{R}^n$  is

$$\Psi_K(x) = \min_{\alpha \geq 0} \{\alpha \in \mathbb{R} : x \in \alpha K\}.$$

In the case of a polytopic set  $\Omega$  containing the origin in its interior, the Minkowski function is given by

$$\Psi_\Omega(x) = \min_{\alpha \geq 0} \{\alpha \in \mathbb{R} : H_j x \leq \alpha, \forall j \in \mathbb{N}_{n_h}\} = \max_{j \in \mathbb{N}_{n_h}} \{H_j x\}.$$

The objective is to determine a condition for the level sets of the Minkowski function to be control invariant sets, within a region  $\Gamma \subseteq \mathbb{R}^n$ . To achieve the purpose, it is sufficient to prove that there exists  $u(x) \in U(x)$  such that  $\Psi_\Omega(x) \leq 0$ , for all  $x \in \Gamma$ , since it implies that  $\Psi_\Omega(x)$  is a nonincreasing function and since  $\alpha\Omega \subseteq \beta\Omega$  if and only if  $\alpha \leq \beta$ . In what follows we provide conditions, based on the concept of viability, to ensure  $\Psi_\Omega(x) \leq 0$ , which implies that the level sets of  $\Psi_\Omega(\cdot)$  are control invariant sets for system (3) with constrained input. Furthermore, we aim at determining the greatest region in the state space where such conditions are satisfied and then stability can be ensured by a proper selection of the control input.

Condition  $\Psi_\Omega(x) \leq 0$  is equivalent to prove that, for every  $x \in \Gamma$ , there exists a  $u(x) \in U(x)$  such that  $(Ax + Bu(x))$  lies within the tangent cone of the level set of function  $\Psi_\Omega(\cdot)$ . Then, denoted  $\Omega(x) = \Psi_\Omega(x)\Omega$ , the viability condition (2) is

$$\forall x \in \Gamma, \quad (Ax \oplus BU(x)) \cap C_{\Omega(x)}(x) \neq \emptyset. \quad (5)$$

Notice that  $\Omega(x)$  is the smallest level set of  $\Psi_\Omega(\cdot)$  containing  $x$ . The tangent cone of  $\Omega(x)$  at  $x$  (which lies on the boundary of  $\Omega(x)$ , by construction) is given by

$$C_{\Omega(x)}(x) = \{v \in \mathbb{R}^n : H_k v \leq 0, \quad \forall k \in \arg \max_{j \in \mathbb{N}_{n_h}} \{H_j x\}\}.$$

Hence, suppose that the state  $x$  is known and denote with  $k = k(x) \in \mathbb{N}_{n_h}$  an index such that  $H_j x \leq H_k x$  for all  $j \in \mathbb{N}_{n_h}$ , that is, such that  $H_k x = \Psi_\Omega(x)$ . This is equivalent to  $k \in k_\Omega(x)$ , where  $k_\Omega(x) \subseteq \mathbb{N}_{n_h}$  is defined as

$$k_\Omega(x) = \arg \max_{j \in \mathbb{N}_{n_h}} \{H_j x\}. \quad (6)$$

Given  $k \in \mathbb{N}_{n_h}$  define

$$R_k = \{x \in \mathbb{R}^n : H_i x \leq H_k x, \quad \forall i \in \mathbb{N}_{n_h}\}, \quad (7)$$

that is, the region of points  $x \in \mathbb{R}^n$  such that  $k \in k_\Omega(x)$ . Notice that regions  $R_k$ , with  $k \in \mathbb{N}_{n_h}$ , can have nonempty intersections. The viability condition (5) is satisfied at  $x$  if there exists  $u = u(x) \in U(x)$  such that

$$H_k Ax + H_k Bu \leq 0, \quad (8)$$

for all  $k \in k_\Omega(x)$ . For every  $k \in \mathbb{N}_{n_h}$  and any  $x \in R_k$ , as in (7), we define an optimization problem as follows.

*Definition 4 (Primal problem):* Consider the system (3) with input bounds (4). Given  $k \in \mathbb{N}_{n_h}$  and  $x \in R_k$ , consider the following optimization problem:

$$\begin{aligned} \alpha_k^*(x) = \min_{\alpha, u} \quad & \alpha, \\ \text{s.t.} \quad & H_i x \leq \alpha, \quad \forall i \in \mathbb{N}_{n_h}, \\ & \tau(H_k Ax + H_k Bu) + (H_k x - \alpha) \leq 0, \\ & L_j u \leq P_j(x), \quad \forall j \in \mathbb{N}_{n_u}, \\ & \alpha \geq 0, \end{aligned} \quad (9)$$

with  $\tau > 0$ .

*Remark 2:* There is no direct connection between the second inequality in (9) and the Euler Approximating System (EAS), which is a discrete-time system often used in spite of the continuous-time one for computational purposes, see [7]. The inequality has a geometrical meaning, it is valid for every positive  $\tau$  and, most importantly, it does not introduce any approximation error. In fact, the condition for viability based on (9) is necessary and sufficient, as illustrated below.

A computation-oriented condition for the viability to be satisfied at a point  $x \in \mathbb{R}^n$  stems from the following property.

*Proposition 1:* Given  $k \in \mathbb{N}_{n_h}$  and  $x \in R_k$ , the optimal solution  $\alpha_k^*(x)$  of the primal problem (9), is such that  $\Psi_\Omega(x) = \alpha_k^*(x)$  if and only if there exists  $u \in U(x)$  such that condition (8) holds at  $x$ . Furthermore,  $\Psi_\Omega(x) < \alpha_k^*(x)$  if and only if condition (8) is not satisfied at  $x$  for any  $u \in U(x)$ .

*Proof:* First notice that  $k \in k_\Omega(x)$  since  $x \in R_k$ . Suppose that condition (8) holds for a proper  $u \in U(x)$ . Then the second constraint in (9) is satisfied by the value  $\alpha = \Psi_\Omega(x) = H_k x$  and the first set of constraints are fulfilled, by definition of Minkowski function. On the other hand if the optimal value of problem (9) is given by the Minkowski function at  $x$ , i.e.  $\alpha_k^*(x) = \Psi_\Omega(x)$ , then the first set of constraints are satisfied by definition, in fact  $H_i x \leq \max_{j \in \mathbb{N}_{n_h}} \{H_j x\} = \Psi_\Omega(x) = \alpha$ , for all  $i \in \mathbb{N}_{n_h}$ . Moreover, from (6), it follows that  $H_k x = \alpha$  and then the second constraint in (9) becomes the condition (8). Hence, we can conclude that the solution  $\alpha_k^*(x)$  is equal to the Minkowski function at  $x$  if and only if the condition (8) is satisfied at  $x$ .

Furthermore it is easy to see that  $\Psi_\Omega(x) \leq \alpha_k^*(x)$ . In fact, the value of the Minkowski function at  $x$  would be attained by removing the second constraint (if the set  $U(x)$  is nonempty, clearly), that means, it would be the optimal over a greater feasibility region, and then a smaller or equal value should be obtained. Hence it can be concluded that the optimal solution of the optimization problem is equal to the Minkowski function at  $x$  if and only if condition (8) is satisfied at  $x$ . This implies that the optimal value is greater than  $\Psi_\Omega(x)$  if and only if (8) is not fulfilled at  $x$ . ■

In what follows, we are going to use the Lagrange multipliers and the dual optimization problem to pose geometric conditions for a region of the state space to be a control invariant set. First, it is worth recalling that for the case under analysis strong duality holds if the primal is feasible, being (9) a linear problem in the optimization variables  $\alpha$  and  $u$ , see [6], [8].

Applying classical results from duality for convex opti-

mization problems and defining the function

$$L_k(\beta, \delta, \sigma; x) = \sum_{i=1}^{n_h} \beta_i H_i x + \delta \tau H_k A x + \delta H_k x - \sum_{j=1}^{n_u} \sigma_j P_j(x),$$

we obtain the dual problem:

$$\begin{aligned} L_k^*(x) &= \max_{\beta, \delta, \sigma} L_k(\beta, \delta, \sigma; x), \\ \text{s.t.} \quad &\sum_{i=1}^{n_h} \beta_i + \delta \leq 1, \\ &\delta \tau H_k B + \sum_{j=1}^{n_u} \sigma_j L_j = 0, \\ &\beta \geq 0, \quad \delta \geq 0, \quad \sigma \geq 0, \end{aligned} \quad (10)$$

whose optimal value is such that  $L_k(\beta, \delta, \sigma; x) \leq L_k^*(x)$  for all feasible  $(\beta, \delta, \sigma)$ , clearly. Hence  $L_k^*(x)$  is the maximal lower bound of  $\alpha_k^*(x)$  and, from strong duality,  $L_k^*(x) = \alpha_k^*(x)$ . Then,  $L_k(\beta, \delta, \sigma; x) \leq \alpha_k^*(x)$  for any feasible solution of (10).

*Proposition 2:* A necessary and sufficient condition for inequality (8) to be satisfied at  $x \in R_k$  is

$$L_k^*(x) \leq H_k x, \quad (11)$$

or, equivalently  $L_k(\beta, \delta, \sigma; x) \leq H_k x$  for every feasible solution  $(\beta, \delta, \sigma)$  of (10). Furthermore, condition (8) is violated at  $x \in R_k$  if and only if

$$L_k^*(x) > H_k x, \quad (12)$$

or, equivalently, if there exists a feasible solution  $(\beta, \delta, \sigma)$  of (10) such that  $L_k(\beta, \delta, \sigma; x) > H_k x$ .

*Proof:* Recall that  $\Psi_\Omega(x) \leq \alpha_k^*(x)$  and that  $\Psi_\Omega(x) < \alpha_k^*(x)$  if and only if (8) does not hold at  $x \in R_k$ . Then (8) holds at  $x$  if and only if  $H_k x = \Psi_\Omega(x) = L_k^*(x) = \alpha_k^*(x)$ , which is implied by (11). Analogously, it can be proved that (12) entails that  $\alpha_k^*(x) > \Psi_\Omega(x)$  and then, as previously shown, violation of condition (8). ■

Posing the condition for viability as in (11), rather than by means of an equality constraint, permits to obtain convex optimization problems under adequate assumptions on  $U(x)$ .

Consider now the dual problem (10). Given  $k \in \mathbb{N}_{n_h}$  and  $x \in R_k$ , the problem is the maximization of a linear function over a polyhedral set in the space of variables  $\beta$ ,  $\delta$  and  $\sigma$ . In general case, the maximum is attained at some extreme point or the problem is unbounded. Since the primal optimum exists and is bounded, the analysis can be reduced to the extreme points of the feasibility region of the dual problem.

*Property 1:* The optimal value of the dual problem (10) is attained at an extreme point of the feasibility region.

*Proof:* Since the origin is an extreme point of the feasibility region of the dual problem (10), which is bounded above by the primal optimal value, the result is implied by Fundamental Theorem of Linear Programming, see [6]. ■

It is important to stress the fact that the feasibility region of the dual problem does not depend on the value of  $x$  but only on  $H_k$  (and on the structure of the system, clearly). Then the dual problem feasibility set is valid for every  $x \in R_k$  and its extreme points can be precomputed knowing  $H_k$  only.

*Proposition 3:* Given  $k \in \mathbb{N}_{n_h}$ , denote with  $(\beta^p, \delta^p, \sigma^p)$  the  $p$ -th extreme point of the feasibility region of the dual

problem (10), with  $p \in \mathbb{N}_{n_v}$ . The subset of  $R_k$ , defined in (7), such that the condition (8) is satisfied at  $x \in R_k$  by a  $u(x) \in U(x)$  is given by

$$\mathcal{V}_k = \bigcap_{p \in \mathbb{N}_{n_v}} \{x \in \mathbb{R}^n : L_k(\beta^p, \delta^p, \sigma^p; x) \leq H_k x\}. \quad (13)$$

Furthermore, the region of points  $x \in R_k$  for which the condition (8) is violated for every  $u \in U(x)$  is

$$\bar{\mathcal{V}}_k = \bigcup_{p \in \mathbb{N}_{n_v}} \{x \in \mathbb{R}^n : L_k(\beta^p, \delta^p, \sigma^p; x) > H_k x\}. \quad (14)$$

*Proof:* From Property 1 we have that, for every  $x \in R_k$ , there exists a  $p^* = p^*(x) \in \mathbb{N}_{n_v}$  such that

$$L_k(\beta, \delta, \sigma; x) \leq L_k(\beta^{p^*}, \delta^{p^*}, \sigma^{p^*}; x) = L_k^*(x),$$

for any feasible  $(\beta, \delta, \sigma)$ . From this and Proposition 2 the first claim follows. Analogous considerations prove the second claim. ■

It is worth stressing that  $\mathcal{V}_k$ , given by the intersection of subsets of the state space, is the exact region of all  $x \in R_k$  where condition (8) is satisfied for an adequate  $u(x) \in U(x)$ . The only optimization problem to solve for characterizing  $\mathcal{V}_k$  concerns the computation of the extremes of the dual problem, neither the computation of  $u(x)$  is required.

*Remark 3:* Notice that  $\mathcal{V}_k$  is the set of points in  $R_k$  for which condition (8) is satisfied by a  $u = u(x) \in U(x)$ , for a particular  $k \in k_\Omega(x)$ . This is equivalent to the condition of viability for all  $x \in \text{int}(R_k)$ , that is if  $k_\Omega(x) = \{k\}$ . Viability, in fact, should concern a condition on  $u \in U(x)$  involving every  $k \in k_\Omega(x)$ . Intriguing phenomena (as Zeno solutions, for instance) could occur at  $x \in \partial R_k$ , with  $k \in \mathbb{N}_{n_h}$ , and deserve more accurate considerations. The analysis of such boundary phenomena is one the objective of our future research.

From the computational point of view, it is important to notice that if  $P_j(x)$  is concave in  $x$  for all  $j \in \mathbb{N}_{n_u}$ , the functions  $L_k(\beta^p, \delta^p, \sigma^p; x)$  are convex and then  $\mathcal{V}_k$  is a convex set. Analogously, if  $P_j(x)$  is convex in  $x$  for all  $j \in \mathbb{N}_{n_u}$ , then the  $L_k(\beta^p, \delta^p, \sigma^p; x)$  is a concave function and  $\bar{\mathcal{V}}_k$  is a union of convex sets. The analysis of the different  $n_h$  regions  $R_k$ , one for any  $H_k$  with  $k \in \mathbb{N}_{n_h}$ , permits to obtain a polytopic viable domain.

## V. ILLUSTRATIVE EXAMPLES

*Example 1:* Consider the linear system (3) with matrices

$$A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad (15)$$

and constraints on the input is  $U(x) = U = \{u \in \mathbb{R}^2 : \|u\|_\infty \leq 1\}$ . The trajectories in absence of control are given by the circumferences of the circles centered in the origin. It is, then, immediate to check that any circle in the state space is a viable set for the dynamic system. On the other hand, our aim here is to use this simple explanatory example to illustrate how the proposed results can be used for computing a family of control invariant sets and a region where viability condition holds for  $\Omega(x)$ .

Consider the set  $\Omega = \{x \in \mathbb{R}^2 : \|x\|_1 \leq 1\}$ . The objective is to compute the maximum  $\gamma \geq 0$  such that  $\mu\Omega$  are a

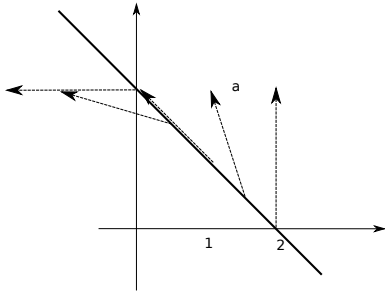


Fig. 2. Autonomous directions on the boundary of  $\gamma\Omega$  in  $R_1$ .

control invariant polytopes for every  $\mu$  such that  $0 \leq \mu \leq \gamma$ . Considering  $H_k = [1 \ 1]$ , the region under analysis is the first quadrant, i.e.  $R_k = \{x \in \mathbb{R}^2 : x \geq 0\}$ . By geometric inspection it can be noticed that the "critical" point in  $R_k$  for viability of  $\gamma\Omega$  is  $x = [\gamma \ 0]^T$ , see Figure 2. Actually, moving  $x$  along the facet of  $\gamma\Omega$  from  $[\gamma \ 0]^T$  to  $[0 \ \gamma]^T$ , the direction of the autonomous system, i.e.  $Ax$ , is such that  $H_k Ax$  decreases, becoming negative from  $x = [0.5\gamma \ 0.5\gamma]^T$ . Notice that if  $H_k Ax \leq 0$  then viability condition holds at  $x$  simply posing  $u = [0 \ 0]^T$ . By geometric inspection it can also be concluded that the maximal  $\gamma_k$  for which  $\gamma_k\Omega$  satisfies the viability condition in the region  $R_k$  is  $\gamma_k = 2$ , see Figure 3. We expect to recover the same value applying the presented results.

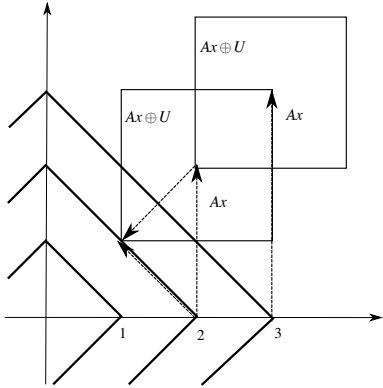


Fig. 3. Sets  $\gamma\Omega$  and system dynamics.

The dual problem feasibility region for the case under analysis is given by the following constraints:

$$\begin{cases} \delta\tau + \sigma_1 - \sigma_3 = 0, \\ \delta\tau + \sigma_2 - \sigma_4 = 0, \\ \sum_{i=1}^{n_h} \beta_i + \delta \leq 1, \\ \beta \geq 0, \quad \delta \geq 0, \quad \sigma \geq 0. \end{cases}$$

Notice that, for any possible value of  $\delta$ , there exist admissible values of  $\sigma$  such that the linear equality constraints hold. The first constraint, for instance, is satisfied by every pair of values  $\sigma_3 \geq 0$  and  $\sigma_1 \geq 0$  such that their difference is equal to  $\delta\tau$ . The extreme values of  $\beta$  and  $\delta$  are given by the extreme points of the region, in their subspace, contained between the simplex and the origin. Then the extreme values of  $\delta$  are 0 or 1. For  $\delta = 0$  we have that  $\sigma$  is such that  $\sigma_1 = \sigma_3$  and  $\sigma_2 = \sigma_4$ ,

and then the extremes are  $\sigma_1 = \sigma_3 = 0$  or  $\sigma_1 = \sigma_3 = +\infty$  and  $\sigma_2 = \sigma_4 = 0$  or  $\sigma_2 = \sigma_4 = +\infty$ . The infinite values can be discarded, since the related function  $L_k(\beta, \delta, \sigma; x)$  would be equal to  $-\infty$ , then leading to trivial inequalities in the definition of  $\mathcal{V}_k$  and  $\bar{\mathcal{V}}_k$ , (see (13) and (14)). Then the only interesting extremes are given by  $\sigma_i = 0$ , for  $i \in \mathbb{N}_4$ . The other possibility is  $\delta = 1$  and then the finite extreme value of  $\sigma$  is  $\sigma = [0 \ 0 \ \tau \ \tau]^T$ . Thus the finite extreme values of the dual problem feasibility region are

$$\begin{aligned} (\beta^1, \delta^1, \sigma^1) &= [1 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0]^T, \\ (\beta^2, \delta^2, \sigma^2) &= [0 \ 1 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0]^T, \\ (\beta^3, \delta^3, \sigma^3) &= [0 \ 0 \ 1 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0]^T, \\ (\beta^4, \delta^4, \sigma^4) &= [0 \ 0 \ 0 \ 1 \ 0 \ 0 \ 0 \ 0 \ 0]^T, \\ (\beta^5, \delta^5, \sigma^5) &= [0 \ 0 \ 0 \ 0 \ 1 \ 0 \ 0 \ \tau \ \tau]^T, \\ (\beta^6, \delta^6, \sigma^6) &= [0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0]^T. \end{aligned}$$

The resulting set of (nontrivial) constraints determining  $x \in R_k$  such that the viability condition holds for  $\Omega(x)$  are

$$\begin{cases} L_k(\beta^1, \delta^1, \sigma^1; x) = [1 \ 1]x \leq [1 \ 1]x, \\ L_k(\beta^2, \delta^2, \sigma^2; x) = [1 \ -1]x \leq [1 \ 1]x, \\ L_k(\beta^3, \delta^3, \sigma^3; x) = [-1 \ 1]x \leq [1 \ 1]x, \\ L_k(\beta^4, \delta^4, \sigma^4; x) = [-1 \ -1]x \leq [1 \ 1]x, \\ L_k(\beta^5, \delta^5, \sigma^5; x) = [1 \ 1]x + \tau[1 \ -1]x - 2\tau \leq [1 \ 1]x, \\ L_k(\beta^6, \delta^6, \sigma^6; x) = 0 \leq [1 \ 1]x, \end{cases}$$

and then

$$\begin{cases} x_2 \geq 0, \\ x_1 \geq 0, \\ [-1 \ -1]x \leq 0, \\ [1 \ -1]x \leq 2. \end{cases} \quad (16)$$

The region  $\mathcal{V}_k$ , and the half-spaces that determine it as in (16), are depicted in Figure 4. The shadowed regions are those points in the state space that violate constraints (16), and then the white portion of the space represents  $\mathcal{V}_k$ . Thus, the maximal  $\gamma_k$  such that  $\gamma_k\Omega$  satisfies the viability condition in  $R_k$  is 2, as expected. From symmetry, analogous results are obtained for every  $R_i$ , with  $i \in \mathbb{N}_4$ , and the resulting  $\gamma$ , obtained as the minimal  $\gamma_i$ , is 2. It is important to stress that no extreme point computation for  $\Omega$  is required, the half-space representation of the polytope is sufficient.

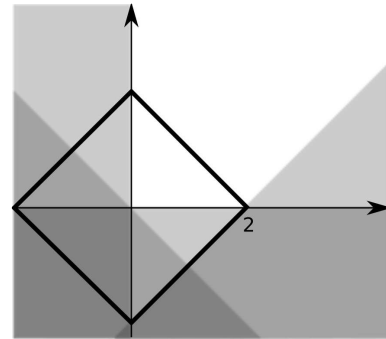


Fig. 4. Viability region in  $R_k$ .

*Example 2:* We consider now the same continuous-time dynamic system (15), with state-dependent bounds on the control input, that is  $U(x) = \{u \in \mathbb{R}^2 : Lu \leq P(x)\}$  with

Thus, the bounds on the input are boxes whose extreme values dependent on the state. Notice that the bounds are tighter as the state is further from the origin. We have to expect, then, that viability condition is satisfied in a region around the origin. Moreover, since  $P_j(\cdot)$  are concave in  $x$ , then functions  $L_k(\beta, \delta, \sigma; \cdot)$  are convex and then the viable region  $\mathcal{V}_k$  is also convex, being the intersection of convex sets. Functions  $L_k(\beta^p, \delta^p, \sigma^p; x)$  are the same as in the previous example for all  $p \neq 5$ . For  $p = 5$  the convex constraint results in:

Constraints and the region  $\mathcal{V}_k$  are depicted in Figure 5.

Fig. 5. Viability region in  $R_k$ .

## VI. CONCLUSIONS

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