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▶ **To cite this version:**

Cyril de Runz, Éric Desjardin, Michel Herbin, Frederic Piantoni. A new Method for the Comparison of two fuzzy numbers extending Fuzzy Max Order. Information Processing and Managment of Uncertainty in Knowledge-Based Systems, 2006, Paris, France. pp.127–133. hal-00560708

HAL Id: hal-00560708
https://hal.science/hal-00560708

Submitted on 29 Jan 2011

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A New Method for the Comparison of two Fuzzy Numbers extending Fuzzy Max Order

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Abstract

To obtain archaeological simulated maps, we need to compare dates of excavation data, represented by fuzzy numbers. Fuzzy Max Order (FMO) is a partial order relation on the set of the fuzzy numbers. But FMO is not able to compare two fuzzy numbers in some situations. In this paper, we propose a new method, Possibilistic Variation Order, extending FMO. We build new indices to order two fuzzy numbers which give us a weighted indication of the order obtained.

Keywords: Fuzzy order relation, fuzzy number, Fuzzy Max Order.

1 Introduction

Comparison of fuzzy quantities is a classical decision problem. In this context, many fuzzy ordering methods are proposed in literature [2, 16, 17]. Wang and Kerre [15] propose three classes of fuzzy ordering methods. In the first class, each method transforms fuzzy numbers by valuation [8]. Then they are compared according to their corresponding values as [1, 5, 9]. In the second class, reference sets are set up and fuzzy numbers are ranked comparing reference sets [4, 10]. In the last class, a fuzzy relation is build to make pairwise comparisons between the fuzzy quantities involved. In this class of ordering methods, the relation can be modelled using probabilities [18] or combination of indices [6, 7, 14].

In this paper, we propose such a combination to compare two fuzzy numbers.

Fuzzy Max Order (FMO), introduced in [13], is a partial order. But it is implicitly the basis of most of ordering methods which consist in extending FMO to a pseudo order [11]. In this paper, we propose to build new indices to compare fuzzy numbers when FMO can not be applied. The membership functions of a fuzzy number is considered in the framework of possibility theory. Our new method is based on the differences between the memberships functions of two fuzzy numbers (i.e. a variation of possibility). These indices permit us to give a degree of comparison.

First of all, we define fuzzy numbers and we explain FMO. Secondly, we quantify the difference of two fuzzy numbers when these ones are not comparable with FMO. A degree of comparison is stated. Thirdly, the transitivity of this new ordering method is studied. The next section is devoted to some literature examples (see in [2]) and we give an application on Geographic Information System(GIS) and archaeology. Finally, we present conclusion of this work.

2 Fuzzy Max Order: a partial order for fuzzy numbers

Let F be a fuzzy subset. The membership function f of the subset F is defined by $f : \mathbb{R} \longrightarrow [0, 1]$ where f is normal ($sup_{x \in \mathbb{R}} f(x) = 1$). The α-cuts of fuzzy subsets are defined by $F_\alpha = \{x \in \mathbb{R} \mid f(x) \geq \alpha\}$ with $\alpha > 0$. A fuzzy subset A of $\mathbb{R}$ is convex if and only if for

each α-cutes A_α, A_α is convex (A_α is a closed interval), α in [0,1]. We define a fuzzy number as a convex and normalized fuzzy subset.

Ramik and Raminek [13] introduced Fuzzy Max Order (FMO) as follow :

$F \preccurlyeq G$ iff sup $f_\alpha \leqslant$ sup g_α and inf $f_\alpha \leqslant$ inf g_α for each α in [0,1], where f_α and g_α are α-cuts of f and g respectively.

Considering the set of fuzzy numbers, FMO consists in applying the extension principle to the operators maximum and minimum on the interval [0,1].

Let F and G be two fuzzy numbers. The maximum of F and G is a fuzzy subset defined by the membership function $\widetilde{max}(f,g)$ where:

$$\widetilde{max}(f,g)(z) = \sup_{\substack{x,\,y \in \mathbb{R},\\ z = max(x,\,y)}} (min(f(x), g(y)))$$

The minimum of F and G is a fuzzy set defined by the membership function $\widetilde{min}(f,g)$ where:

$$\widetilde{min}(f,g)(z) = \sup_{\substack{x,\,y \in \mathbb{R},\\ z = min(x,\,y)}} (min(f(x), g(y)))$$

The images of $\widetilde{min}(f,g)$ and $\widetilde{max}(f,g)$ are illustrated by Figure 1 to 3.

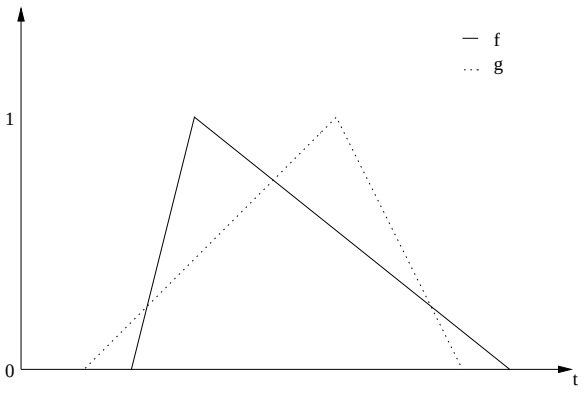


Figure 1: f and g membership functions of two fuzzy numbers F and G

$\widetilde{min}(F,G)$ (resp. $\widetilde{max}(F,G)$) is the fuzzy subset associated with $\widetilde{min}(f,g)$ (resp. $\widetilde{max}(f,g)$).

According to these definitions of $\widetilde{max}$ and $\widetilde{min}$ of two fuzzy numbers, the following three conditions (a) to (c) are equivalent [13]:

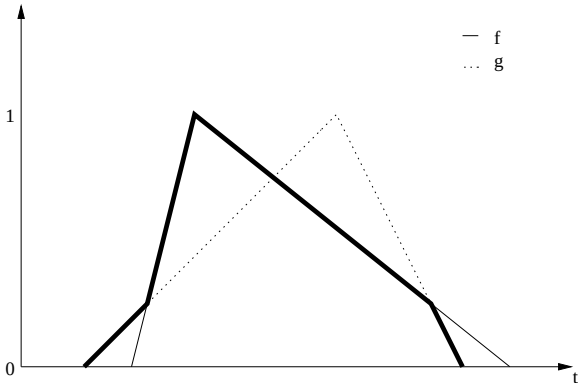


Figure 2: $\widetilde{min}(f,g)$ in bold

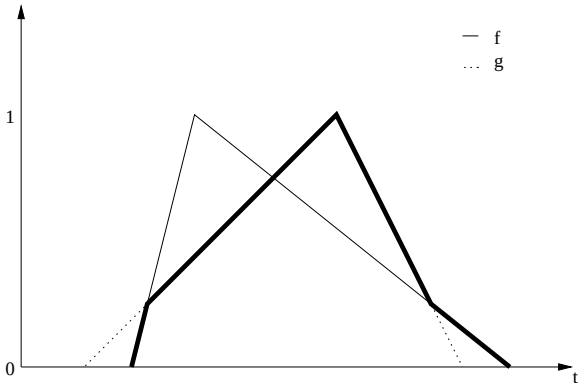


Figure 3: $\widetilde{max}(f,g)$ in bold

a) $F \preccurlyeq G$,

b) $\widetilde{max}(F,G) = G$,

c) $\widetilde{min}(F,G) = F$.

An other property of FMO is established in [13]: $\widetilde{max}(F,G)$ and $\widetilde{min}(F,G)$ of two fuzzy numbers F and G are fuzzy numbers.

FMO is a partial order on the set of fuzzy numbers. Figure 1 gives an example of two fuzzy numbers that FMO cannot compare. In the following, we propose an index to help us for the decision making in such a case.

3 Possibilistic Variation Order relation

3.1 Definition

We define the difference between two fuzzy numbers F and G in the framework of possibility distribution. The difference $f - g$ gives us informations to compare F and G. When

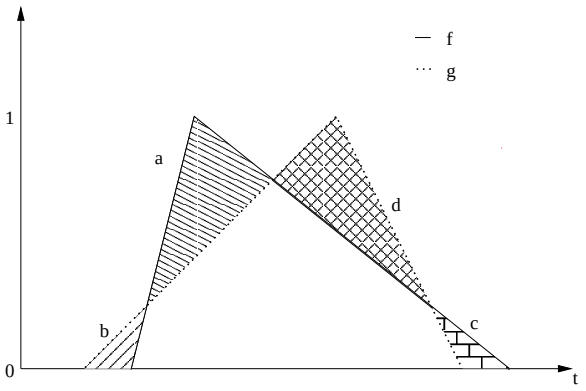


Figure 4: Four cases (a, b, c and d) in possibilistic variation

F and G are the same number, then $f - g$ is null. In other cases, this possibilistic variation is evaluated by integrating the absolute value of differences of membership functions (i.e. by using $\mathcal{A} = \int |f - g|$, illustrated by a,b,c and d on Figure 4).

We separate $\mathcal{A}$ into four cases as following :

- if $\widetilde{min}(f,g) > \widetilde{max}(f,g)$ (i.e. the $\widetilde{min}$ is more possible than the $\widetilde{max}$)

  - if $f > g$ (F is more possible than G):

  $\mathcal{A}_{\widetilde{\mathbf{min}},\mathbf{f>g}} = \int_{\widetilde{min}(f,g)>\widetilde{max}(f,g),f>g} |f - g|$

  (part a on Figure 4),

  - if $g > f$ (G is more possible than F):

  $\mathcal{A}_{\widetilde{\mathbf{min}},\mathbf{g>f}} = \int_{\widetilde{min}(f,g)>\widetilde{max}(f,g),g>f} |f - g|$

  (part b on Figure 4),

- if $\widetilde{min}(f,g) < \widetilde{max}(f,g)$ (i.e. the $\widetilde{max}$ is more possible than the $\widetilde{min}$)

  - if $f > g$ (F is more possible than G):

  $\mathcal{A}_{\widetilde{\mathbf{max}},\mathbf{f>g}} = \int_{\widetilde{min}(f,g)<\widetilde{max}(f,g),f>g} |f - g|$

  (part c on Figure 4),

  - if $g > f$ (G is more possible than F):

  $\mathcal{A}_{\widetilde{\mathbf{max}},\mathbf{g>f}} = \int_{\widetilde{min}(f,g)<\widetilde{max}(f,g),g>f} |f - g|$

  (part d on Figure 4),

These cases determine the potentiality of each fuzzy number to be close to either $\widetilde{min}$ or $\widetilde{max}$.

We normalize the four cases defined on possibilistic variation by $\mathcal{A}$ value (except when F and G are the same number).

We could establish the following table :

Table 1: Table of comparison

$\mathcal{I}$	**f**	**g**
$\widetilde{min}$	$\mathcal{A}_{\widetilde{min},f>g}/\mathcal{A}$	$\mathcal{A}_{\widetilde{min},g>f}/\mathcal{A}$
$\widetilde{max}$	$\mathcal{A}_{\widetilde{max},f>g}/\mathcal{A}$	$\mathcal{A}_{\widetilde{max},g>f}/\mathcal{A}$

We defined now two indices using the previous table:

1. $\mathcal{I}_{f<g} = \mathcal{I}(f, min) + \mathcal{I}(g, \widetilde{max})$ gives us the index quantifying that F is close to $\widetilde{min}$ and G is close to $\widetilde{max}$.
2. $\mathcal{I}_{g<f} = \mathcal{I}(f, \widetilde{max}) + \mathcal{I}(g, \widetilde{min})$ gives us the index quantifying that G is close to $\widetilde{min}$ and F is close to $\widetilde{max}$.

We can verify that $\mathcal{I}_{f<g} + \mathcal{I}_{g<f} = 1$.

Then we define Possibilistic Variation Order (PVO) relation as follow :

- F is lower than G when $\mathcal{I}_{f<g} \geq \mathcal{I}_{g<f}$ and we have $F \preccurlyeq_{\mathcal{I}_{f<g}} G$,

If $F \preccurlyeq G$ using FMO then $F = \widetilde{min}(F, G)$ and $G = \widetilde{max}(F, G)$. Then we have $\mathcal{I}_{f<g} = 1$ and $\mathcal{I}_{g>f} = 0$ (except if $F = G$), so $F \preccurlyeq_1 G$ using PVO. Thus we consider that PVO is an extension of FMO (except for the identity of F and G).

Now we present PVO by an example.

3.2 Example

With figure 4 example, we build Table 2:

In this example, we obtain :

- $\mathcal{I}_{f<g} = 0.86$,
- $\mathcal{I}_{g<f} = 0.14$.

So we conclude that $F \preccurlyeq_{0.86} G$.

Table 2: Table of comparison

$\mathcal{I}$	**f**	**g**
$\widetilde{min}$	0.43	0.07
$\widetilde{max}$	0.07	0.43

4 Transitivity and PVO

PVO is not transitive:

Let F, G and H be three fuzzy numbers.

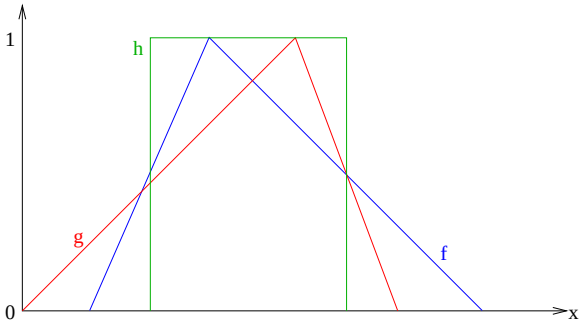


Figure 5: Illustration of the no transitivity of PVO

In Figure 5 we have $G \preccurlyeq_{0.53} F$, $F \preccurlyeq_{0.5} H$ and $H \preccurlyeq_{0.55} G$, so PVO is not a transitive relation.

PVO is not an order relation on fuzzy numbers set. It is only an order method helping decision making when FMO cannot be used. The main advantage of this method is that all fuzzy numbers are comparable pairwise when using FMO extended by PVO.

5 Classical literature examples

In this section, we take all the examples in [2]. In the first one, F and G can be ranked using FMO, unlike the four others.

5.1 First example : Figure 6

This example can be ranked by FMO and $G \preccurlyeq F$. Our indices values are :

- $\mathcal{I}_{f<g} = 0$,
- $\mathcal{I}_{g<f} = 1$.

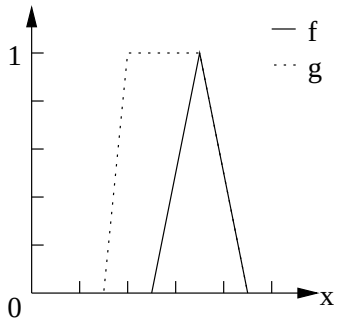


Figure 6: First classical example

So we have $G \preccurlyeq_1 F$ too.

5.2 Second example : Figure 7

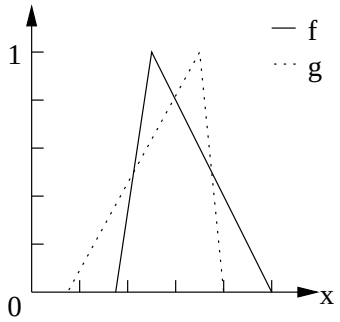


Figure 7: Second classical example

Our indices values are :

- $\mathcal{I}_{f<g} = 0.39$,
- $\mathcal{I}_{g<f} = 0.61$.

So we have $G \preccurlyeq_{0.61} F$. Note that this case is different of the case of Figure 4 where $F \preccurlyeq_{0.86} G$.

5.3 Third example : Figure 8

Our indices values are :

- $\mathcal{I}_{f<g} = 0.81$,
- $\mathcal{I}_{g<f} = 0.19$.

So we have $F \preccurlyeq_{0.81} G$.

5.4 Fourth example : Figure 9

Our indices values are :

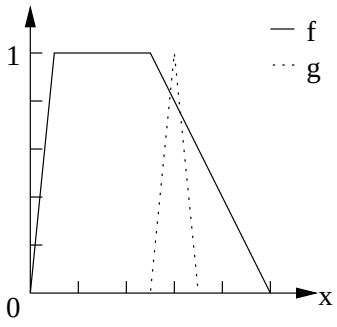


Figure 8: Third classical example

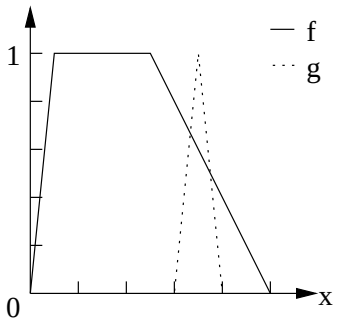


Figure 9: Fourth classical example

- $\mathcal{I}_{f<g} = 0.92$,
- $\mathcal{I}_{g<f} = 0.08$.

So we have $F \preccurlyeq_{0.92} G$.

5.5 Fifth example : Figure 10

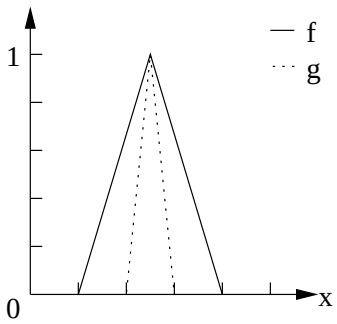


Figure 10: Fifth classical example

Our indices values are :

- $\mathcal{I}_{f<g} = 0.5$,
- $\mathcal{I}_{g<f} = 0.5$.

So we have $F \preccurlyeq_{0.5} G$ and $G \preccurlyeq_{0.5} F$.

6 Application

Archaeological data can be represented by the possibility theory in fuzzy numbers. We have some dates as "near 1330 post J.C." or "middle age" for objects found in excavations. To give to the archaeologists some predictive maps, we want to use PVO results during spatial inferences. The degree, obtained with PVO, helps us to resolve some spatial and architectural contradictions of excavation maps. On the "Galerie Rémoise" site (Figure 11), we have two walls (WALL1 and WALL2) built in the first century, showing an architectural contradiction as one encroaches the other one. Our goal is to design a new map to simulate the archaeological hypothesis on dating. With FMO extended by PVO, we can propose a weighted indication on which wall had been built first.

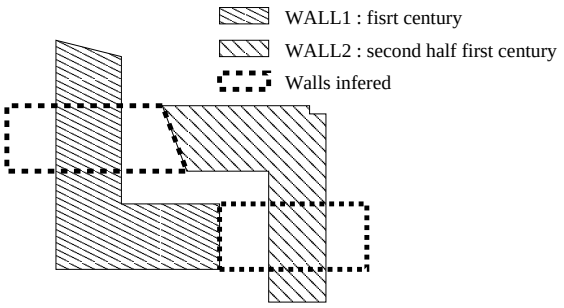


Figure 11: Extract of the "Galerie Rémoise" excavation

We represent these two building dates by fuzzy numbers F (date of WALL1) and G (date of WALL2) as Figure 12. The membership function f (resp g) of F (resp G) is defined as a trapezoidal where :

- Support(f)=[(t1-0,2(t2-t1)),(t2+0,2(t2-t1))],
- Kernel(f)=[(t1+0,2(t2-t1)),(t2-0,2(t2-t1))],
- t1 and t2 are respectively the beginning and the end of the periode.

We compare these two dates with well known fuzzy methods (Table 3) and PVO.

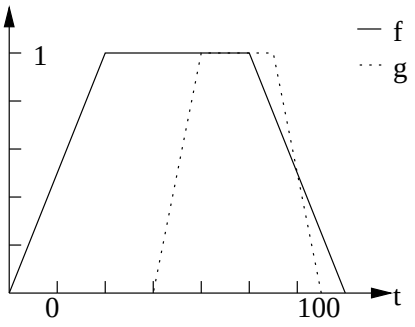


Figure 12: Fuzzy representation of the walls dates

Table 3: Order fuzzy methods applied to F and G

Methods	Result
Adamo [1] $\alpha = 1$	$F \prec G$
Adamo $\alpha = 0$	$G \prec F$
Fortemps and Rouben [9]	$F \prec G$
Dubois and Prade [7]	$F \prec G$
Kerre [10]	$F \prec G$
Chang [3]	$G \prec F$
PVO	$F \preccurlyeq_{0,95} G$

The classical methods give a boolean order. PVO moderate the order. This moderation permits to include a possibility degree. Using PVO, we obtain $F \preccurlyeq_{0,95} G$, so in our context, we suggest that WALL1 was built first with a possibility of 0,95.

7 Discussion and conclusion

This paper proposes a new method (PVO) extending Fuzzy Max Order [13]. It permits us to compare two fuzzy numbers when partial order FMO fails. When comparing two fuzzy numbers incomparable by FMO, PVO gives a possibilistic interpretation of an order decision. The decision is quantified by a ratio (see section 3), equals to one when the data are FMO comparable. FMO with PVO extension gives us a mean to systematize order decisions in the case of automated data processing. Note that PVO is not a transitive relation, then we could obtain inconsistencies in order decisions.

The goal of this work is to applied PVO for modelling data of archaeological excavations. Thousand of dates are stored on SIGRem [12] (GIS of Reims applied to historical data). We have to compare these ones to build maps modelling archaeological sites. Despite experts, thousand of order decisions could be inconsistent. But the maps give informations to help archaeologists to interpret data. In this framework of modelling, PVO is an help to simulate maps avoiding the lost of time due to thousand expert assessments.

In future works, we propose to develop models and simulations in SIGRem GIS using PVO approach for ordering data.

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