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# ON $\left(K_{q}, k\right)$ STABLE GRAPHS WITH SMALL $k$ 

J.L. FOUQUET, H. THUILLIER, J.M. VANHERPE AND A.P. WOJDA


#### Abstract

A graph $G$ is $\left(K_{q}, k\right)$ (vertex) stable if it contains a copy of $K_{q}$ after deleting any subset of $k$ vertices. We show that for $q \geq 6$ and $k \leq \frac{q}{2}+1$ the only $\left(K_{q}, k\right)$ stable graph with minimum size is isomorphic to $K_{q+k}$.


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## 1. Introduction

For terms not defined here we refer to [1]. As usually, the order of a graph $G$ is the number of its vertices (it is denoted by $|G|$ ) and the size of $G$ is the number of its edges (it is denoted by $e(G)$ ). The degree of a vertex $v$ in a graph $G$ is denoted by $d_{G}(v)$, or simply by $d(v)$ if no confusion is possible. For any set $S$ of vertices, we denote by $G-S$ the subgraph induced by $V(G)-S$. If $S=\{v\}$ we write $G-v$ for $G-\{v\}$. When $e$ is an edge of $G$ we denote by $G-e$ the spanning subgraph $(V(G), E-\{e\})$. The disjoint union of two graphs $G_{1}$ and $G_{2}$ is denoted by $G_{1}+G_{2}$. The union of $p$ mutually disjoint copies of a graph $G$ is denoted by $p G$. A complete subgraph of order $q$ of $G$ is called a $q$-clique of $G$. The complete graph of order $q$ is denoted by $K_{q}$. When a graph $G$ contains a $q$-clique as subgraph, we say " $G$ contains a $K_{q}$ ".
The following notion was introduced by Dudek et al. in [2].
Definition 1.1. Let $H$ be a graph and $k$ be a natural number. A graph $G$ of order at least $k$ is said to be a $(H, k)$ stable graph if for any set $S$ of $k$ vertices the subgraph $G-S$ contains a graph isomorphic to $H$.

By $Q(H, k)$ we denote the size of a minimum $(H, k)$ stable graph. It is clear that if $G$ is an $(H, k)$ stable graph with minimum size then the graph obtained from $G$ by addition or deletion of some isolated vertices is also minimum $(H, k)$ stable. Hence we shall asume that all the graphs considered in the paper have no isolated vertices. A $(H, k)$ stable graph with minimum size shall be called a minimum $(H, k)$ stable graph.

Lemma 1.2. [2] Let $q$ and $k$ be integers, $q \geq 2, k \geq 1$. If $G$ is $(H, k)$ stable then, for every vertex $v$ of $G$, the graph $G-v$ is $(H, k-1)$ stable.
Proposition 1.3. [2] If $G$ is a $(H, k)$ stable graph with minimum size then every vertex as well as every edge is contained in a subgraph isomorphic to $H$.

[^0]Proof. Let $e$ be an edge of $G$ which is not contained in any subgraph of $G$ isomorphic to $H$, then $G-e$ would be a $(H, k)$ stable graph with less edges than $G$, a contradiction. Let $x$ be a vertex of $G$ and $e$ be an edge of $G$ incident with $x$, since $e$ is an edge of some subgraph isomorphic to $H$, say $H_{0}$, the vertex $x$ is a vertex of $H_{0}$.

## 2. Preliminary results

We are interested by minimum $\left(K_{q}, k\right)$ stable graphs (where $q$ and $k$ are integers such that $q \geq 2$ and $k \geq 0$ ). As a corollary to Proposition 1.3, every edge and every vertex of a minimum $\left(K_{q}, k\right)$ stable graph is contained in a $K_{q}$ (thus the minimum degree is at least $q-1$ ). Note that, for $q \geq 2$ and $k \geq 0$, the graph $K_{q+k}$ is $\left(K_{q}, k\right)$ stable, hence $Q\left(K_{q}, k\right) \leq\binom{ q+k}{2}$.
Definition 2.1. Let $H$ be a non complete graph on $q+t$ vertices $(t \geq 1)$. We shall say that $H$ is a near complete graph when it has a vertex $v$ such that

- $H-v$ is complete.
- $d_{H}(v)=q+r$ with $-1 \leq r \leq t-2$.

The previous definition generalizes Definition 1.5 in [3] initially given for $r \in$ $\{-1,0,1\}$ and the following lemma generalizes Proposition 2.1 in [3].
Lemma 2.2. Every minimum $\left(K_{q}, k\right)$ stable graph $G$, where $q \geq 3$ and $k \geq 1$, has no component $H$ isomorphic to a near complete graph.

Proof. Suppose, contrary to our claim, that $G$ has such a component $H$ and let $v$ be the vertex of $H$ such that $H-v$ is a clique of $G$. Then, $H$ has $q+t$ vertices with $q-1 \leq d(v) \leq q+t-2$ and $d(v)=q+r$. Since $G$ is minimum $\left(K_{q}, k\right)$ stable, $G-v$ is $\left(K_{q}, k-1\right)$ stable and is not $\left(K_{q}, k\right)$ stable. Then, $G-v$ contains a set $S$ with at most $k$ vertices intersecting every subgraph of $G-v$ isomorphic to a $K_{q}$. The graph $G-S$ contains some $K_{q}$ (at least one) and clearly every subgraph of $G-S$ isomorphic to a $K_{q}$ contains $v$. Since $N(v)$ is a $K_{q+r}$ and $N(v)-S$ contains no $K_{q},|N(v)-S| \leq q-1$. Since there exists a $K_{q}$ containing $v$ in $H-S$, $|N(v)-S|=q-1$ (and hence $|S \cap N(v)|=r+1$ ). Since $H-v-S$ contains no $K_{q}, H-v-S=N(v)-S$. Let $a$ be a vertex of $H-v$ not adjacent to $v$ and let $b$ be a vertex in $N(v)-S$, and consider $S^{\prime}=S-\{a\}+\{b\}$. We have $\left|S^{\prime}\right| \leq k$ and $G-S^{\prime}$ contains no $K_{q}$, a contradiction.

It is clear that $Q\left(K_{q}, 0\right)=\binom{q}{2}$ and the only minimum $\left(K_{q}, 0\right)$ stable graph is $K_{q}$. It is an easy exercise to see that $Q\left(K_{2}, k\right)=k+1$ and that the matching $(k+1) K_{2}$ is the unique minimum $\left(K_{2}, k\right)$ stable graph.
Theorem 2.3. [3] Let $G$ be a minimum $\left(K_{3}, k\right)$ stable graph, with $k \geq 0$. Then $G$ is isomorphic to $s K_{4}+t K_{3}$, for any choice of $s$ and $t$ such that $2 s+t=k+1$.
In [3] it was proved that if $q \geq 4$ and $k \in\{1,2\}$ then $Q\left(K_{q}, k\right)=\binom{q+k}{2}$ and the only minimum $\left(K_{q}, k\right)$ stable graph is $K_{q+k}$. We have proved also that if $q \geq 5$ then $Q\left(K_{q}, 3\right)=\binom{q+3}{2}$ and the only minimum $\left(K_{q}, 3\right)$ stable graph is $K_{q+3}$. Dudek, Szymański and Zwonek proved the following result.

Theorem 2.4. [2] For every $q \geq 4$, there exists an integer $k(q)$ such that $Q\left(K_{q}, k\right) \leq$ $(2 q-3)(k+1)$ for $k \geq k(q)$.

As a consequence of this last result, they have deduced that for every $k \geq \mathrm{k}(\mathrm{q})$ $K_{q+k}$ is not minimum ( $K_{q}, k$ ) stable.

Remark 2.5. From now on, throughout this section we assume that $q$ and $k$ are integers such that $q \geq 4, k \geq 1$ and for every $r$ such that $0 \leq r<k$ we have $Q\left(K_{q}, r\right)=\binom{q+r}{2}$ and the only minimum $\left(K_{q}, r\right)$ stable graph is $K_{q+r}$.
In view of Theorem 2.4, $k$ is bounded from above and we are interested to obtain the greatest possible value of $k$.

Lemma 2.6. Let $G$ be $a\left(K_{q}, k\right)$ stable graph such that $e(G) \leq\binom{ q+k}{2}$. Then either for every vertex $v$ we have $d(v) \leq q+k-2$ or $G$ is isomorphic to $K_{q+k}$.
Proof. Suppose that some vertex $v$ has degree at least $q+k-1$. By Lemma 1.2 the graph $G-v$ is $\left(K_{q}, k-1\right)$ stable, hence $Q\left(K_{q}, k-1\right) \leq e(G-v)=e(G)-d(v)$. Since $Q\left(K_{q}, k-1\right)=\binom{q+k-1}{2}$, we have
$\binom{q+k-1}{2} \leq e(G)-d(v) \leq\binom{ q+k}{2}-(q+k-1)=\binom{q+k-1}{2}$.
It follows that $e(G-v)=\binom{q+k-1}{K_{2}}, G-v$ is isomorphic to $K_{q+k-1}$ and $d(v)=q+k-1$. Hence, $G$ is isomorphic to $K_{q+k}$.

Lemma 2.7. Let $G$ be a minimum $\left(K_{q}, k\right)$ stable graph. Then one of the following statements is true

- $G$ has no component isomorphic to $K_{q}$
- $Q\left(K_{q}, k-1\right)+\binom{q}{2} \leq Q\left(K_{q}, k\right)$

Proof. Suppose that some component $H$ of $G$ is isomorphic to a $K_{q}$. If $G-H$ is not $\left(K_{q}, k-1\right)$ stable, then there is a set $S$ with at most $k-1$ vertices intersecting each $K_{q}$ of $G-H$. Then, for any vertex $a$ of $H, S+a$ intersects each $K_{q}$ of $G$ while $S$ has at most $k-1$ vertices, a contradiction. Hence $G-H$ is $\left(K_{q}, k-1\right)$ stable and we have $Q\left(K_{q}, k-1\right) \leq e(G-H)=Q\left(K_{q}, k\right)-\binom{q}{2}$.

Lemma 2.8. [3] Let $G$ be a minimum $\left(K_{q}, k\right)$ stable graph and let $u$ be a vertex of degree $q-1$. Then one of the following statements is true

- $\forall v \in N(u) \quad d(v) \geq q+1$
- $Q\left(K_{q}, k-1\right)+3(q-2) \leq Q\left(K_{q}, k\right)$

Proof. By Proposition 1.3, since $d(u)=q-1,\{u\}+N(u)$ induces a complete graph on $q$ vertices. Assume that some vertex $w \in N(u)$ has degree $q+r$ where $r=-1$ or $r=0$, and let $v$ be a neighbour of $u$ distinct from $w$. Since the degree of $u$ in $G-v$ is $q-2$, no edge incident with $u$ can be contained in a $K_{q}$ of $G-v$. Since $G-v$ is $\left(K_{q}, k-1\right)$ stable, we can delete the $q-2$ edges incident with $u$ in $G-v$ and the resulting graph $G^{\prime}$ is still $\left(K_{q}, k-1\right)$ stable. By deleting $v$, we have $e(G-v) \leq e(G)-(q-1)$ and hence

$$
e\left(G^{\prime}\right) \leq e(G)-(q-1)-(q-2)
$$

In $G^{\prime}$, the degree of $w$ is now $q+r-2$. Hence, no edge incident with $w$ in $G^{\prime}$ can be contained in a $K_{q}$. Deleting these $q+r-2$ edges from $G^{\prime}$ leads to a graph $G^{\prime \prime}$ which remains to be ( $K_{q}, k-1$ ) stable. We get thus

$$
Q\left(K_{q}, k-1\right) \leq e\left(G^{\prime \prime}\right) \leq e(G)-(q-1)-(q-2)-(q+r-2)
$$

Since $e(G) \leq Q\left(K_{q}, k\right)$, the result follows.

Lemma 2.9. Let $G$ be a minimum $\left(K_{q}, k\right)$ stable graph, where $1 \leq k \leq 2 q-6$, and let $v$ be a vertex of degree $q-1$. Then for every vertex $w \in N(v)$ we have $d(w) \geq q+1$.

Proof. Suppose, contrary to the assertion of the lemma, that $d(w) \leq q$ for some vertex $w \in N(v)$. By Lemma 2.8, we have $Q\left(K_{q}, k-1\right)+3(q-2) \leq Q\left(K_{q}, k\right)$. Since $Q\left(K_{q}, k-1\right)=\binom{q+k-1}{2}$ and $Q\left(K_{q}, k\right) \leq\binom{ q+k}{2}$ we have $\binom{q+k-1}{2}+3 q-6 \leq\binom{ q+k}{2}$. Then we obtain $k \geq 2 q-5$, a contradiction.

Lemma 2.10. Let $G$ be a minimum $\left(K_{q}, k\right)$ stable graph, where $q \geq 5$ and $1 \leq k \leq$ $q-1$. Then the minimum degree of $G$ is at least $q$.
Proof. $\quad$ Suppose that there is a vertex $v$ of degree $q-1$ and let $w$ be a neighbour of $v$. Since $q-1 \leq 2 q-6$, by Lemma 2.9, $w$ has degree at least $q+1$. By Lemma 1.2 the graph $G-w$ is $\left(K_{q}, k-1\right)$ stable. In that graph $v$ is not contained in any $K_{q}$ since its degree is $q-2$. Hence $G-\{w, v\}$ is still $\left(K_{q}, k-1\right)$ stable. We have $e(G-\{w, v\})=e(G)-(d(v)+d(w)-1) \leq e(G)-2 q+1$. Since $Q\left(K_{q}, k-1\right)=\binom{q+k-1}{2}$ and $Q\left(K_{q}, k\right) \leq\binom{ q+k}{2}$ we have $\binom{q+k-1}{2} \leq e(G)-2 q+1 \leq\binom{ q+k}{2}-2 q+1$. It follows that $k \geq q$, a contradiction.

Lemma 2.11. Let $G$ be a minimum $\left(K_{q}, k\right)$ stable graph, where $q \geq 5$ and $1 \leq k \leq$ $q-1$, and let $v$ be a vertex of degree $q$. Then $N(v)$ is complete.

Proof. Assume, by contradiction, that $v$ is a vertex of degree $q$ and $N(v)$ contains two non adjacent vertices $a$ and $b$. Let $w \in N(v)$ distinct from $a$ and $b$ ( $w$ must exist since $q \geq 4$ ). By Lemma 1.2 the graph $G-w$ is $\left(K_{q}, k-1\right)$ stable. In that graph $v$ is not contained in a $K_{q}$ since its two neighbours $a$ and $b$ are not adjacent. Hence $G-\{w, v\}$ is still $\left(K_{q}, k-1\right)$ stable. By Lemma 2.10, $d(w) \geq q$ and hence $e(G-\{w, v\})=e(G)-(d(v)+d(w)-1) \leq e(G)-2 q+1$. We have, as in the proof of Lemma 2.10, $\binom{q+k-1}{2} \leq e(G)-2 q+1 \leq\binom{ q+k}{2}-2 q+1$, and we obtain $k \geq q$, a contradiction.

Lemma 2.12. Let $G$ be a minimum $\left(K_{q}, k\right)$ stable graph, where $q \geq 4$ and $2 \leq$ $k \leq \frac{q}{2}+1$, and let $v$ be a vertex of degree at least $q+1$. Then either $N(v)$ induces a complete graph or there exists an ordering $v_{1}, \ldots, v_{d(v)}$ of the vertices of $N(v)$ such that $\left\{v_{1}, \ldots, v_{q-1}\right\}$ induces a complete graph and $v_{d(v)-1} v_{d(v)}$ is not in $E(G)$. Moreover, there exists a vertex $w$ in $\left\{v_{1}, \ldots, v_{q-1}\right\}$ adjacent to $v_{d(v)-1}$ and $v_{d(v)}$.
Proof. Suppose that the subgraph induced by $N(v)$ is not complete and let $a$ and $b$ be two non adjacent neighbours of $v$. Assume that every complete graph on $q-1$ vertices contained in $N(v)$ intersects $\{a, b\}$. The graph $G-\{a, b\}$ is $\left(K_{q}, k-2\right)$ stable. In that graph, $v$ is not contained in a $K_{q}$, hence $G-\{a, b, v\}$ is still $\left(K_{q}, k-2\right)$ stable. We have $e(G-\{a, b, v\})=e(G)-(d(v)+d(a)+d(b)-2) \leq$ $e(G)-3 q-1$. Since $Q\left(K_{q}, k-2\right)=\binom{q+k-2}{2}$ and $Q\left(K_{q}, k\right) \leq\binom{ q+k}{2}$, we have $\binom{q+k-2}{2} \leq e(G-\{a, b, v\}) \leq e(G)-3 q-1 \leq\binom{ q+k}{2}-3 q-1$ and hence $\frac{q}{2}+2 \leq k$, a contradiction with $k \leq \frac{q}{2}+1$.

Thus, $N(v)-\{a, b\}$ contains a $K_{q-1}$ and we can order the vertices of $N(v)$ in such a way that the $q-1$ first ones $v_{1}, \ldots, v_{q-1}$ induce a complete graph and the two last vertices $v_{d(v)-1}$ and $v_{d(v)}$ are not adjacent, as claimed.
Set $d(v)=q+r$ with $r \geq 1$. By Proposition 1.3, the edges $v v_{q+r-1}$ and $v v_{q+r}$ are contained in two distinct $q$-cliques, say $Q_{1}$ and $Q_{2}$. Since $v_{q+r-1}$ and $v_{q+r}$ are not adjacent, each $Q_{i}$ contains at most $r$ vertices in $N(v)-\left\{v_{1}, \ldots, v_{q-1}\right\}$. Hence, each $Q_{i}$ must have at least $q-r-1$ vertices in $\left\{v_{1} \ldots v_{q-1}\right\}$. Since $N(v)$ is not complete and $e(G) \leq\binom{ q+k}{2}$, by Lemma 2.6 we have $r \leq k-2$. Since $k \leq \frac{q}{2}+1, Q_{1}$ (as well as $Q_{2}$ ) has at least $q-r-1 \geq q-k+1 \geq \frac{q}{2}$ vertices in $\left\{v_{1} \ldots v_{q-1}\right\}$. Hence $Q_{1}$ and $Q_{2}$ have at least one common vertex $w$ in $N(v)$, and the lemma follows.

Lemma 2.13. Let $G$ be a minimum $\left(K_{q}, k\right)$ stable graph, where $q \geq 5$ and $2 \leq$ $k \leq \frac{q}{2}+1$, and let $H$ be a component of $G$. Assume that $v$ is a vertex of maximum degree in $H$. Then either $H$ is complete or the subgraph induced by $N(v)$ contains no complete subgraph on $d(v)-1$ vertices.

Proof. Assume that $H$ is not complete. First, we prove that the maximum degree in $H$ is at least $q+1$. If the minimum degree in $H$ is at least $q+1$, we are done. If there exists a vertex $u$ of degree $q-1$ in $H$ then, by Lemma 2.9, the degree of any vertex of $N(u)$ is at least $q+1$. If there exists a vertex $u$ of degree $q$ then, by Lemma 2.11, $N(u) \cup\{u\}$ induces a $K_{q+1}$, and hence, since $H$ is connected, there exists a vertex in $H-(N(u) \cup\{u\})$ having at least one neighbour $w$ in $N(u)$ and $d(w) \geq q+1$. Thus, the maximum degree in $H$ is $q+r$ with $r \geq 1$.
Since $H$ is not complete and $v$ is a vertex of maximum degree in $H$, the subgraph induced on $N(v)$ is not complete. By Lemma 2.12, there exists an ordering $\left\{v_{1} \ldots v_{q+r}\right\}$ of the vertices of $N(v)$ such that $\left\{v_{1} \ldots v_{q-1}\right\}$ induces a complete graph and $v_{q+r-1} v_{q+r}$ is not an edge of $G$. If the subgraph induced by $N(v)$ contains a complete subgraph on $q+r-1$ vertices then without loss of generality we may suppose that it contains the vertex $v_{q+r-1}$.
Assume that $\left\{v_{1}, \ldots, v_{q+r-2}, v_{q+r-1}\right\}$ induces a complete graph. Since $G$ is a minimum $\left(K_{q}, k\right)$ stable graph, by Proposition 1.3, the edge $v v_{q+r}$ must be contained in a $K_{q}$. Hence $v_{q+r}$ has at least $q-2$ neighbours in $\left\{v_{1}, \ldots, v_{q+r-2}\right\}$. Let us denote by $A$ this set of neighbours. Since the subgraph induced by $\left(N(v)-\left\{v_{q+r}\right\}\right)$ is complete, every vertex in $A$ has degree $q+r$ in $G$ (let us say that these vertices are saturated). Henceforth, every vertex in $A$ has no neighbour outside $N(v) \cup\{v\}$. By Lemma 2.2, the $(q+r)$-clique $\left(N(v)-\left\{v_{q+r}\right\}\right) \cup\{v\}$ is a proper subgraph of $H-\left\{v_{q+r}\right\}$. Since $H$ is connected, there exists a vertex $w$ outside $N(v) \cup\{v\}$ adjacent to a vertex $u$ in $N(v)$. Clearly, the vertex $u$ has also degree $q+r$ and it has no other neighbour outside $N(v) \cup\{v\}$ than $w$. The edge $u w$ being contained in a $K_{q}$ by Proposition 1.3, $w$ must have at least $q-2$ common neighbours with $u$ in $N(v)$. Let us denote by $B$ the set of neighbours of $w$ in $N(v)$. It is easy to see that every vertex in $B$ is saturated. Since $A$ and $B$ are disjoint, we have $2 q-3 \leq|A \cup B| \leq|N(v)|=q+r$ and hence $q \leq r+3$. Since $r \leq k-2$ by Lemma 2.6, we have $q \leq k+1$. Thus, we obtain $q \leq \frac{q}{2}+2$, which implies $q \leq 4$, a contradiction.
Hence the subgraph induced by the vertices $\left\{v_{1}, \ldots, v_{q+r-2}, v_{q+r-1}\right\}$ is not complete, and the lemma follows.

Proposition 2.14. Let $G$ be a minimum $\left(K_{q}, k\right)$ stable graph, where $q \geq 6$ and $2 \leq k \leq \frac{q}{2}+1$. Then every component of $G$ is a complete graph.
Proof. Let $H$ be a component of $G$ and $v$ be a vertex of maximum degree in $H$. If the subgraph induced on $N(v)$ is complete then $H$ is obviously complete. We can thus suppose that $N(v)$ is not a clique. By Lemmas 2.10 and 2.11, the minimum degree is at least $q+1$, and hence $d(v)=q+r$ with $r \geq 1$.
Claim 2.14.1. The graph $G-(N(v) \cup\{v\})$ is $\left(K_{q}, k-r\right)$ stable.
Proof By Lemma 2.12, we can consider an ordering $v_{1}, \ldots, v_{q+r}$ of $N(v)$ such that the set $\left\{v_{1}, \ldots, v_{q-1}\right\}$ induces a $K_{q-1}, v_{q+r-1} v_{q+r} \notin E(G)$ and there is a vertex $w \in\left\{v_{1}, \ldots, v_{q-1}\right\}$ adjacent to $v_{q+r-1}$ and $v_{q+r}$. By Lemma 2.13, we can find two non adjacent vertices $a$ and $b$ in $N(v)-\left\{v_{q+r}\right\}$ and two non adjacent vertices $c$ and $d$ in $N(v)-\left\{v_{q+r-1}\right\}$. Let us note that since the set $\left\{v_{1}, \ldots, v_{q-1}\right\}$ induces a complete graph, it contains at most one vertex of the set $\{a, b\}$ and at most one vertex of $\{c, d\}$. Then, $\left|\left\{v_{1}, \ldots, v_{q-1}\right\} \cap\{w, a, b, c, d\}\right| \leq 3$.
Since $H$ is not complete, the graph $G$ is not complete and by Lemma 2.6 we have $r \leq k-2$. Since $k \leq \frac{q}{2}+1$ and $q \geq 6$, there exists a subset $A \subseteq\left\{v_{1} \ldots v_{q-1}\right\}$ such that

- $|A|=r$
- $w \notin A$
- $A \cap\{a, b, c, d\}=\emptyset$

By repeated applications of Lemma 1.2, the graph $G_{1}$ obtained from $G$ by deleting $A$ is $\left(K_{q}, k-r\right)$ stable. In $G_{1}$, the degree of $v$ is equal to $q$.
Without loss of generality, suppose that $a$ is distinct from $v_{q+r-1}$ and $c$ is distinct from $v_{q+r}$. If there exists a $q$-clique in $G_{1}$ containing the edge $v v_{q+r-1}$ then $\left\{v_{1}, \ldots, v_{q+r-2}\right\}-A$ is a $(q-2)$-clique containing $a$. Since $a b$ is not an edge, we must have $b=v_{q+r-1}$, a contradiction to the fact that $a v_{q+r-1}$ is an edge. Thus, there is no $q$-clique in $G_{1}$ containing $v v_{q+r-1}$. Analogously, we prove that there is no $q$-clique in $G_{1}$ containing $v v_{q+r}$.
Hence, the graph $G_{2}$ obtained from $G_{1}$ by deletion of the edges $v v_{q+r-1}$ and $v v_{q+r}$ is still $\left(K_{q}, k-r\right)$ stable. In $G_{2}, v$ has degree $q-2$, so it is not contained in any $K_{q}$. We can thus delete $v$ and we get a $\left(K_{q}, k-r\right)$ stable graph $G_{3}$.
Since the maximum degree in $G$ is $q+r$, the degree of $w$ in $G_{3}$ is at most $q-1$. Recall that $w$ is adjacent to the two non adjacent vertices $v_{q+r-1}$ and $v_{q+r}$. Hence $w$ is not contained in any $K_{q}$ of $G_{3}$, which means that $G_{4}=G_{3}-w$ is still ( $K_{q}, k-r$ ) stable. Since the degree of each vertex in $\left\{v_{1}, \ldots, v_{q-1}\right\}-(A \cup\{w\})$ is at most $q-2$ in $G_{4}$, none of these vertices can be contained in any $K_{q}$ of $G_{4}$. Hence by deletion of these vertices we get again a $\left(K_{q}, k-r\right)$ stable graph $G_{5}$. We shall prove that none of the $r+1$ vertices $v_{q}, \ldots, v_{q+r}$ is contained in a $K_{q}$ of $G_{5}$.
Note that $G_{5}=G-\left\{v, v_{1}, \ldots, v_{q-1}\right\}$. For $q \leq j \leq q+r$, denote by $d_{j}$ the degree of the vertex $v_{j}$ in the subgraph induced by $\left\{v_{q}, \ldots, v_{q+r}\right\}$. Clearly we have $0 \leq d_{j} \leq r$. In $G$, by Proposition 1.3, the edge $v v_{j}$ is contained in a $K_{q}$. Hence $v_{j}$ is adjacent (in $G$ ) to at least $q-2-d_{j}$ vertices in $\left\{v_{1}, \ldots, v_{q-1}\right\}$. Since we have deleted the vertex $v$ and the vertices $v_{1}, \ldots, v_{q-1}$, we have thus $d_{G_{5}}\left(v_{j}\right) \leq q+r-\left(q-2-d_{j}\right)-1=r+1+d_{j}$. If $d_{j} \leq r-1$ then $d_{G_{5}}\left(v_{j}\right) \leq 2 r \leq 2(k-2) \leq q-2$ and there is no $K_{q}$ in $G_{5}$ containing $v_{j}$. The equality $d_{G_{5}}\left(v_{j}\right)=q-1$ can only be obtained when $d_{j}=r$, that is $v_{j}$ has $r$ neighbours in $v_{q} \ldots v_{q+r}$. Since $v_{q+r-1}$ and $v_{q+r}$ are not adjacent, $v_{j}$ is not contained in any $K_{q}$ of $G_{5}$.

Hence, the graph $G_{6}=G-(N(v) \cup\{v\})$ obtained from $G_{5}$ by deletion of all the vertices $v_{q}, v_{q+1} \ldots, v_{q+r}$ is still $\left(K_{q}, k-r\right)$ stable, and the claim follows.

## Claim 2.14.2.

$$
\begin{equation*}
\binom{q+k-r}{2}+q+r+\binom{q-1}{2}+\frac{1}{2}(r+1)(2 q-r-2)+1 \leq\binom{ q+k}{2} \tag{2.1}
\end{equation*}
$$

Proof To get back $G$ from $G-(N(v) \cup\{v\})$ we add, at least

- the $q+r$ edges incident with $v$,
- the $\binom{q-1}{2}$ edges of the $(q-1)$-clique induced by the set $\left\{v_{1} \ldots v_{q-1}\right\}$,
- the edges incident with $v_{q}, \ldots, v_{q+r}$ and not incident with $v$.

Let $l$ be the number of edges incident with $v_{q}, \ldots, v_{q+r}$ and not incident with $v$. We have

$$
\begin{equation*}
e(G-(N(v) \cup\{v\}))+q+r+\binom{q-1}{2}+l \leq e(G) \tag{2.2}
\end{equation*}
$$

In order to find a lower bound of the number of edges incident with the vertices $v_{q}, \ldots, v_{q+r}$, for each $i \in\{q, \ldots, q+r\}$ let us denote by $d_{i}$ the degree of the vertex $v_{i}$ in the subgraph induced by the set $\left\{v_{q}, \ldots, v_{q+r}\right\}$. Then,

$$
l=\frac{1}{2} \Sigma_{i=q}^{q+r} d_{i}+\Sigma_{i=q}^{q+r}\left(d_{G}\left(v_{i}\right)-1-d_{i}\right)=\Sigma_{i=q}^{q+r} d_{G}\left(v_{i}\right)-(r+1)-\frac{1}{2} \Sigma_{i=q}^{q+r} d_{i} .
$$

Since by Lemma 2.10 the minimum degree in $G$ is at least $q$, we have

$$
l \geq q(r+1)-(r+1)-\frac{1}{2} \Sigma_{i=q}^{q+r} d_{i}
$$

Since for every $i$ in $\{q, \ldots, q+r-2\} \quad d_{i} \leq r, d_{q+r-1} \leq r-1$ and $d_{q+r} \leq r-1$, we obtain

$$
l \geq q(r+1)-(r+1)-\frac{1}{2} r(r-1)-(r-1)
$$

and hence

$$
l \geq \frac{1}{2}(r+1)(2 q-r-2)+1
$$

By the assumption made at the beginning of the section (see Remark 2.5), a minimum $\left(K_{q}, k-r\right)$ stable graph has $\binom{q+k-r}{2}$ edges. Since $e(G) \leq\binom{ q+k}{2}$, the inequality (2.1) follows from Claim 2.14.1 and the inequality (2.2).

A simple calculation shows that the inequality

$$
q^{2}+q+2 \leq 2 k r
$$

can be obtained by starting from the inequality (2.1).
Since $r \leq k-2$ and $k \leq \frac{q}{2}+1$, we have $q^{2}+q+2 \leq 2 k(k-2) \leq(q+2)\left(\frac{q}{2}-1\right)$, hence $\frac{q^{2}}{2}+q+4 \leq 0$, a contradiction. Thus, $N(v)$ is a clique and the proposition follows.

## 3. Result

In [3], it is shown that if $G$ is minimum $\left(K_{q}, k\right)$ stable and the numbers $k$ and $q$ verify one of the following conditions:

- $k=1$ and $q \geq 4$
- $k=2$ and $q \geq 4$
- $k=3$ and $q \geq 5$
then $G$ is isomorphic to $K_{q+k}$.
Theorem 3.1. Let $G$ be a minimum $\left(K_{q}, k\right)$ stable graph, where $q \geq 6$ and $k \leq$ $\frac{q}{2}+1$. Then $G$ is isomorphic to $K_{q+k}$.
Proof. For $0 \leq k \leq 3$ the graph $G$ is isomorphic to $K_{q+k}$. Let $k$ be such that $4 \leq k \leq \frac{q}{2}+1$ and suppose that for every $r$ with $0 \leq r<k$ the only minimum ( $K_{q}, r$ ) stable graph is $K_{q+r}$. By Proposition 2.14, the graph $G$ is the disjoint union of $p$ complete graphs $H_{1} \equiv K_{q+k_{1}}, H_{2} \equiv K_{q+k_{2}}, \cdots, H_{p} \equiv K_{q+k_{p}}$. Suppose, without loss of generality, that $k_{1} \geq k_{2} \geq \cdots \geq k_{p} \geq 0$ and that there exist two components $H_{i}$ and $H_{j}$ with $i<j$ such that $k_{i}-k_{j} \geq 2$. By substituting $H_{i}^{\prime} \equiv K_{q+k_{i}-1}$ for $H_{i}$ and $H_{j}^{\prime} \equiv K_{q+k_{j}+1}$ for $H_{j}$, we obtain a new ( $K_{q}, k$ ) stable graph $G^{\prime}$ such that $e\left(G^{\prime}\right)=e(G)-\left(k_{i}-k_{j}-1\right)<e(G)$, which is a contradiction. Thus, for any $i$ and any $j, \quad 0 \leq\left|k_{i}-k_{j}\right| \leq 1$ (cf 2] Proposition 7).
To conclude that $G$ has a unique component, observe the following facts.
- The graphs $2 K_{q+l}$ and $K_{q+2 l+1}$ are both $\left(K_{q}, 2 l+1\right)$ stable, but if $2 l+1 \leq$ $\frac{q}{2}+1$ then $\binom{q+2 l+1}{2}<2\binom{q+l}{2}$
- The graphs $K_{q+l} \cup K_{q+l+1}$ and $K_{q+2 l+2}$ are both $\left(K_{q}, 2 l+2\right)$ stable but if $2 l+2 \leq \frac{q}{2}+1$ then $\binom{q+2 l+2}{2}<\binom{q+l+1}{2}+\binom{q+l}{2}$


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