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From Hermite polynomials to multifractional processes

Renaud Marty*

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Abstract

We establish an invariance principle where the limit process is a Hermite-type process. We also prove that this limit process is multifractional. Our main result is a generalization of results from [6] and [11] to a multifractional setting. It also generalizes the main result of [3] to a non-Gaussian framework.

1 Introduction

Hermite processes have attracted a lot of attention for many years because they have nice properties as they generalize fractional Brownian motion [6, 11]. Let \( m \in \mathbb{N}^* \) and \( H \in (1/2, 1) \). The Hermite process \( W_{m,H} \) of order \( m \) and Hurst index \( H \) can be defined for instance in terms of Dobrushin-Wiener-Itô integrals [5] as, for every \( t \in [0, \infty) \),

\[
W_{m,H}(t) = \int_{\mathbb{R}^m} f_{m,H}(x_1, \ldots, x_m, t) d\hat{B}_{x_1} \cdots d\hat{B}_{x_m} \tag{1}
\]

with

\[
f_{m,H}(x_1, \ldots, x_m, t) = C(m, H) \frac{\exp(it(x_1 + \cdots + x_m)) - 1}{i(x_1 + \cdots + x_m) |x_1 \cdots x_m|^{(2H-2+m)/2m}}
\]

where \( C(m, H) \) is a normalizing constant and \( d\hat{B} \) is the complex random measure corresponding to a standard Brownian motion \( \hat{B} \). Notice that for \( m = 1 \), the Hermite process \( W_{1,H} \) is the fractional Brownian motion with Hurst index \( H \).

An important property of Hermite processes is the invariance principle [4, 6, 10, 11], which can be stated as follows. Let \( X = \{X_j\}_{j \in \mathbb{N}} \) be a Gaussian stationary sequence of centered random variables with \( \mathbb{E}[X_0^2] = 1 \) and satisfying the property

\[
\mathbb{E}[X_0X_j] \sim cj^{2H-1}/m \quad \text{as} \quad j \to \infty \tag{2}
\]

where \( c \) is a positive real number. Notice that (2) is a long range property. We consider a function \( \phi \in L^2(e^{-x^2/2}dx) \) with Hermite rank equal to \( m \), and define the partial sum \( S_{\phi,H}^N(t) \) for every \( N \in \mathbb{N} \) and \( t \in [0, \infty) \) as

\[
S_{\phi,H}^N(t) = \frac{1}{N^H} \sum_{j=1}^{[Nt]} \phi(X_j). \tag{3}
\]

The invariance principle establishes that the finite-dimensional distributions of \( S_{\phi,H}^N \) converge, as \( N \) goes to infinity, to the Hermite process \( W_{m,H} \) with a suitable constant \( C(m, H) \).

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As fractional Brownian motion, Hermite processes of index $H$ are $H$–self-similar and $H'–$Hölder-continuous if and only if $H' < H$. As other fractional processes, a drawback of Hermite processes lies in their strong homogeneity of their properties, which are governed by the Hurst index $H$. In order to generalize fractional processes to less homogeneous processes, multifractional processes have been introduced, as for instance the class of multifractional Brownian motions [2, 8]. Multifractional processes have locally, but not globally, the same properties as fractional processes. These properties are governed by a function $h$ that substitutes for the constant $H$.

As for fractional Brownian motion and other Hermite processes, some nontrivial multifractional Gaussian processes satisfy invariance principle. Indeed, it is proven in [3] the following result. Let a Gaussian field $\{X_j(H)\}_{(j,H)\in \mathbb{N} \times (1/2,1)}$ satisfying some long-range assumptions and a continuous function $h$ taking its values in $(1/2,1)$. Then, the finite-dimensional distributions of the process

$$t \to S_h^N(t) = \sum_{j=1}^{[Nt]} \frac{X_j(h(j/N))}{N^{h(j/N)}}$$

converge to those of a centered Gaussian process $S_h$ with covariance given for $t, s \geq 0$ by:

$$\mathbb{E}[S_h(t)S_h(s)] = \int_0^t \int_0^s d\theta \int_0^s d\sigma R(h(\theta), h(\sigma))|\theta - \sigma|^{h(\theta)+h(\sigma)-2}$$

where $R$ is a continuous function and derived from long-range assumptions of the field $\{X_j(H)\}_{(j,H)\in \mathbb{N} \times (1/2,1)}$. The process $S_h$ is multifractional. If the function $h$ is constant, then the process $S_h$ is a fractional Brownian motion. The result above is then a generalization of classical invariance principle [9].

In this work, we generalize invariance principles presented above. We study the asymptotic behavior of a sequence generalizing both (3) and (4). In particular, this sequence is defined from a Gaussian field $\{X_j(H)\}_{(j,H)\in \mathbb{N} \times (1/2,1)}$ satisfying long-range properties, a function $\phi \in L^2(e^{-x^2/2}dx)$ with Hermite rank equal to $m$ and a Hurst function $h$. We get as a limit a multifractional process $S_{m,h}$ that depends on the integer $m$ and the function $h$. If the function is a constant $H$, then the limit process is the Hermite process with Hurst index $H$ and Hermite order $m$. If the integer $m$ is equal to 1, then the limit process corresponds to a Gaussian multifractional process of the class obtained in [3].

The paper is organized as follows. In Section 2 we recall some definitions and preliminary results about Hermite polynomials and multiple stochastic integrals, which are used throughout the paper. In Section 3 we establish the main result of the paper. Section 4 is devoted to the proof of the main result.

## 2 Preliminaries

In this section we give some definitions and recall some results we use throughout this paper.

For each positive integer $m \in \mathbb{N}$, the $m$th Hermite polynomial $P_m$ of is defined as, for every $x \in \mathbb{R}$,

$$P_m(x) = (-1)^m e^{x^2/2} \frac{d^m}{dx^m} e^{-x^2/2}.$$  

The family of the Hermite polynomials $\{P_m, m \in \mathbb{N}\}$ is an orthogonal basis of the space $L^2(e^{-x^2/2}dx)$ defined by

$$L^2(e^{-x^2/2}dx) = \left\{ \phi : \mathbb{R} \to \mathbb{C}, \phi \text{ measurable and } \int_{\mathbb{R}} |\phi(x)|^2 e^{-x^2/2}dx < \infty \right\}$$

with the inner product $\langle \cdot, \cdot \rangle$ defined as, for every $\phi_1$ and $\phi_2$ in $L^2(e^{-x^2/2}dx)$,

$$\langle \phi_1, \phi_2 \rangle = \int_{\mathbb{R}} \phi_1(x) \overline{\phi_2(x)} e^{-x^2/2} \frac{dx}{\sqrt{2\pi}}$$

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and whose the corresponding norm will be denoted as \( \| \cdot \| \). For every function \( \phi \in L^2(e^{-x^2/2}dx) \), there exists an integer \( m_\phi \) such that \( \langle \phi, P_{m_\phi} \rangle \neq 0 \) and \( \langle \phi, P_m \rangle = 0 \) for every \( m = 0, \ldots, m_\phi - 1 \). The integer \( m_\phi \) is called the Hermite index of the function \( \phi \). Hence, for every \( \phi \in L^2(e^{-x^2/2}dx) \),

\[
\phi = \sum_{m=0}^\infty \frac{\langle \phi, P_m \rangle}{m!} P_m = \sum_{m=m_\phi}^\infty \frac{\langle \phi, P_m \rangle}{m!} P_m
\]  

(6)

where the convergence of the series holds for the norm \( \| \cdot \| \). If \( X \) and \( Y \) are two Gaussian random variables \( \mathcal{N}(0,1) \), then, for every \( j \) and \( k \) in \( \mathbb{N}^* \),

\[
\mathbb{E}[P_j(X)P_k(Y)] = \begin{cases} 
  k!\mathbb{E}[XY]^k & \text{if } k = j, \\
  0 & \text{if } k \neq j.
\end{cases}
\]  

(7)

As a consequence, for every \( \phi \in L^2(e^{-x^2/2}dx) \) and \( X \sim \mathcal{N}(0,1) \),

\[
\mathbb{E}[|\phi(X)|^2] = \sum_{m=0}^\infty \frac{\langle \phi, P_m \rangle^2}{m!} < \infty
\]  

(8)

Others objects we strongly use in this paper are multiple Wiener-Itô integrals \([5, 7]\). Many notions of multiple Wiener-Itô integrals with respect to Brownian motion exist and are used to define processes as Hermite processes \([6, 11]\). Here we have chosen to use the so-called Dobrushin-Wiener-Itô integrals introduced in \([5]\). Let \( d \in \mathbb{N}^* \), \( f : \mathbb{R}^d \to \mathbb{C} \) be a square-integrable function, and \( B = \{B_x\}_{x \in \mathbb{R}} \) be a standard Brownian motion in \( \mathbb{R} \). In this paper the Dobrushin-Wiener-Itô integral of \( f \) is denoted \( \int_{\mathbb{R}^d} f \, dB_{x_1} \cdots dB_{x_d} \).

It is well-defined if \( f \) is even and symmetric, that is, if \( f \) satisfies, for every \( (x_1, \ldots, x_d) \in \mathbb{R}^d \),

\[
f(x_1, \ldots, x_d) = f(-x_1, \ldots, -x_d),
\]

and for every permutation \( \varsigma \) on \( \{1, \ldots, d\} \),

\[
f(x_1, \ldots, x_d) = f(x_{\varsigma(1)}, \ldots, x_{\varsigma(d)}).
\]

We refer the reader to \([5]\) for the precise definition of \( \int_{\mathbb{R}^d} f \, dB_{x_1} \cdots dB_{x_d} \). Here we only recall some properties that we use in the proof of the main result. The integral \( \int_{\mathbb{R}^d} f \, dB_{x_1} \cdots dB_{x_d} \) is Gaussian if and only if \( d = 1 \). In any case, it is a centered random variable and we can express its variance as

\[
\mathbb{E} \left[ \left( \int_{\mathbb{R}^d} f \, dB_{x_1} \cdots dB_{x_d} \right)^2 \right] = d! \int_{\mathbb{R}^d} |f(x_1, \ldots, x_d)|^2 dx_1 \cdots dx_d.
\]

We have a substitution formula for multiple integrals, using the self-similarity of the Brownian motion. For every \( a > 0 \), we have the equality in distribution

\[
\int_{\mathbb{R}^d} f(x_1, \ldots, x_d) \, dB_{ax_1} \cdots dB_{ax_d} \overset{\text{dist}}{=} a^{d/2} \int_{\mathbb{R}^d} f(a x_1, \ldots, a x_d) \, dB_{x_1} \cdots dB_{x_d}.
\]  

(9)

Another formula for change of variables is applied in this paper and is a consequence of Proposition 4.2 of \([5]\). Let \( z : \mathbb{R} \to \mathbb{C} \) be a bounded and measurable function satisfying \( z(x) = z(-x) \) and \( |z(x)| = 1 \) for every \( x \in \mathbb{R} \). Then, we have the equality in distribution

\[
\int_{\mathbb{R}^d} f(x_1, \ldots, x_d) \, dB_{x_1} \cdots dB_{x_d} \overset{\text{dist}}{=} \int_{\mathbb{R}^d} f(x_1, \ldots, x_d) z(x_1) \cdots z(x_d) \, dB_{x_1} \cdots dB_{x_d}.
\]

(10)

By linearity of the integral and the bounded convergence theorem, we can prove the following convergence lemma.
Lemma 1. Let \( \{f_n\}_N \) be a sequence of even and symmetric functions in \( L^2(\mathbb{R}^d, \mathbb{C}) \). We assume that there exist two even and symmetric functions \( f \) and \( f^* \) in \( L^2(\mathbb{R}^d, \mathbb{C}) \) such that, for a.e. \( x \in \mathbb{R}^d \), \( \lim_{N \to \infty} f_n(x) = f(x) \) and \( \sup_N |f_n(x)| \leq f^*(x) \). Then,

\[
\lim_{N \to \infty} \mathbb{E} \left[ \left| \int_{\mathbb{R}^d} f_N d\hat{B}_d - \int_{\mathbb{R}^d} f d\hat{B}_d \right|^2 \right] = 0.
\]

To conclude this section, we mention a result that relate Hermite polynomials to multiple integrals. Let \( \psi \) be an even and symmetric function in \( L^2(\mathbb{R}, \mathbb{C}) \), \( m \) be a positive integer and \( P_m \) be the \( m \)-th Hermite polynomial defined as previously. The random integral \( \int_{\mathbb{R}} \psi(\xi) d\hat{B}_\xi \) defines a centered Gaussian variable. If \( \mathbb{E} \left[ \int_{\mathbb{R}} |\psi(\xi)|^2 d\xi \right] = 1 \) then we have almost surely

\[
P_m \left( \int_{\mathbb{R}} \psi(x) d\hat{B}_x \right) = \int_{\mathbb{R}^m} \psi(x_1) \cdots \psi(x_m) d\hat{B}_{x_1} \cdots d\hat{B}_{x_m}.
\]

### 3 Main result

We let \( m \in \mathbb{N}^* \) and define

\[
b_m = 1 - \frac{1}{2m}.
\]

We consider the Gaussian field \( X = \{X_n(H), n \in \mathbb{N}, H \in (b_m, 1)\} \) defined as, for every \( n \geq 0 \) and \( H \in (b_m, 1) \),

\[
X_n(H) = \int_{-\pi}^\pi \exp(\text{i}nx) g(H, x)|x|^{1/2-H} d\hat{B}_x
\]

where \( B \) is a standard Brownian motion and \( g : (b_m, 1) \times (-\pi, \pi) \to \mathbb{C} \) is a measurable function satisfying the following properties.

- For every \( (H, x) \in (b_m, 1) \times [-\pi, \pi] \), \( g(H, x) = g(H, -x) \). This property ensures that the field \( X \) is real.
- For every \( H \in (b_m, 1) \),

\[
\int_{-\pi}^\pi |g(H, x)|^2 |x|^{1-2H} dx = 1
\]

so that \( \mathbb{E}[X_n(H)^2] = 1 \).

- The function \( g \) is twice continuously differentiable on \( (b_m, 1) \times (-\pi, \pi) \). We then define, for every \( (H, x) \in (b_m, 1) \times (-\pi, \pi) \), \( g_0(H) = g(H, 0) \) and \( g_1(H, x) = \int_0^1 (\partial g / \partial \xi)(H, \xi) d\xi \) so that \( g = g_0 + g_1 \) and, for every compact set \( K \) of \( (b_m, 1) \),

\[
\lim_{x \to 0} \sup_{H \in K} \left( |g_1(H, x)| + \left| \frac{\partial g_1}{\partial H}(H, x) \right| \right) = 0.
\]

The assumptions above ensure that the covariance function satisfies the uniform long-range property of [3]. In particular, for every compact set \( K \subset (b_m, 1) \), we have

\[
\lim_{j \to -\infty} \sup_{(H_1, H_2) \in K^2} |(j-k)^2 - H_1 - H_2| \mathbb{E}[X_j(H_1)X_k(H_2)] - R(H_1, H_2)| = 0
\]

where

\[
R(H_1, H_2) = g_0(H_1)g_0(H_2) \int_{\mathbb{R}} \exp(\text{i}x|x|^{1-H_1-H_2} dx
\]

for every \( (H_1, H_2) \in (b_m, 1) \).
We consider a continuously differentiable function \( h : [0, \infty) \to (1/2, 1) \) and a function \( \phi \in L^2(e^{-x^2/2}dx) \) with Hermite rank equal to \( m \in \mathbb{N}^+ \). We let
\[
\tilde{h} := 1 + \frac{h - 1}{m} : [0, \infty) \to (b_m, 1).
\]
We define for every \( t \geq 0 \) and \( N > 0 \)
\[
S_{\phi,h}^N(t) := \sum_{j=1}^{\lfloor Nt \rfloor} \phi(X_j(h_j^N))) \frac{N}{Nh(j/N)},
\]
with
\[
h_j^N := 1 + \frac{h(j/N) - 1}{m} = \tilde{h}(j/N).
\]
Now we can state the main result of this paper.

**Theorem 1.** As \( N \to \infty \), the finite-dimensional distributions of \( S_{\phi,h}^N \) converge to those of \( S_{m,h} \) defined for every \( t \geq 0 \) as
\[
S_{m,h}(t) := \int_{\mathbb{R}^m} f_{m,h}(x_1, \ldots, x_m, t)d\tilde{B}_{x_1} \cdots d\tilde{B}_{x_m}
\]
with, for every \((x_1, \ldots, x_m, t) \in \mathbb{R}^m \times [0, \infty),\)
\[
f_{m,h}(x_1, \ldots, x_m, t) = \int_0^t \exp \left( i\theta \sum_{i=1}^m x_i \right) \tilde{g}(\theta) |x_1 \cdots x_m|^{1/2 - \tilde{h}(\theta)} d\theta
\]
where
\[
\tilde{g} = \frac{\langle \phi, P_m \rangle}{m!} \left( g_0 \circ \tilde{h} \right)^m
\]
The process \( S_{m,h} \) is continuous (up to a modification) and locally self-similar: for every \( t \geq 0 \),
\[
\lim\limits_{\varepsilon \to 0^+} \text{dist.} \left\{ \frac{S_{m,h}(t + \varepsilon u) - S_{m,h}(t)}{\varepsilon h(t)} \right\}_{u \geq 0} = \{T_{m,h,t}(u)\}_{u \geq 0}
\]
where \( \lim_{\varepsilon \to 0^+} \) stands for the limit in distribution in the space of continuous functions endowed with the uniform norm on every compact set and, for every \( u \geq 0 \),
\[
T_{m,h,t}(u) = \tilde{g}(t) \int_{\mathbb{R}^m} \frac{\exp(iu \sum_{i=1}^m x_i) - 1}{i \sum_{i=1}^m x_i |x_1 \cdots x_m|^{h(t) - 1/2}} d\tilde{B}_{x_1} \cdots d\tilde{B}_{x_m}
\]
Theorem 1 establishes that sequences of processes defined as (16), in particular from a Hurst function \( h \), converge to multifractional processes with Hurst function \( h \). This has been observed in [3] in the particular case \( \phi \equiv 1 \) where the limit process is \( S_{1,h} \), which is a centered Gaussian process of covariance
\[
(t, s) \mapsto \mathbb{E}[S_{1,h}(t)S_{1,h}(s)] = \int_0^t \int_0^s d\sigma R(h(\theta), h(\sigma))
\]
with \( R \) defined by (15). Theorem 1 is then an extension of the main result of [3], which assumes \( \phi \equiv 1 \), to any case where \( \phi \in L^2(e^{-x^2/2}dx) \).

If we assume that \( h \equiv H \in (b_m, 1) \), then Theorem 1 is the main result of [6, 11]. In particular, the limit process \( S_{m,h} \) can be written as \( W_{m,H} \) in (1) with the constant
\[
C(m,H) = \frac{\langle \phi, P_m \rangle}{m!} \left( g_0 \left( \tilde{H} \right) \right)^m = \frac{\langle \phi, P_m \rangle}{m!} \left( \frac{R(\tilde{H}, \tilde{H})}{\int_{\mathbb{R}} e^{i\xi |\xi|^{1-2\tilde{h}}} d\xi} \right)^{m/2}
\]
where
\[
\tilde{H} := 1 + \frac{H - 1}{m} \in (b_m, 1).
\]
4 Proof of Theorem 1

The proof of Theorem 1 is organized as follows. In Subsection 4.1 we establish a technical lemma we then use throughout the proof of Theorem 1. We prove the convergence of $S_{\phi,h}^N$ in Subsection 4.2 and the regularity properties of $S_{m,h}$ in Subsection 4.3.

4.1 Technical lemma

In the following lemma, we prove for every $T > 0$ the existence of a function $\tilde{f}_T$ that is useful in the sequel of the proof to establish uniform bounds.

Lemma 2. For every $T > 0$, there exists a function $\tilde{f}_T \in L^2(\mathbb{R}^m, \mathbb{R})$ so that, for almost every $x \in \mathbb{R}^m$ and for every $t \in [0, T]$ and $H \in [\min h, \max h]$,

$$\left| \frac{e^{it \sum_{i=1}^m x_i - 1}}{x_1 \cdots x_m} \right|^{H-1/2} \sum_{i=1}^m x_i \right| \leq \tilde{f}_T(x),$$

Proof. For every $(x_1, \ldots, x_m) \in \mathbb{R}^m$ we define

$$L(x_1, \ldots, x_m) = (1 + |\ln |x_1 \cdots x_m||)^2$$

and

$$\tilde{f}_T(x_1, \ldots, x_m) = \max_{H \in [\min h, \max h]} \frac{L(x_1, \ldots, x_m)}{|x_1 \cdots x_m|^{2H-1}} \leq \tilde{f}_T(x_1, \ldots, x_m)^2.$$

It is then enough to prove that, for $H \in [\min \tilde{h}, \max \tilde{h}]$, the function

$$(x_1, \ldots, x_m) \mapsto \frac{T^2 \left| \sum_{i=1}^m x_i \right|^{H-1/2} \sum_{i=1}^m x_i \right|^{H-1}}{|x_1 \cdots x_m|^{2H-1}} L(x_1, \ldots, x_m)$$

is integrable. We successively make the substitutions $y_j = x_1 + \cdots + x_j$ for every $j \in \{1, \ldots, m\}$, $z_k = y_k/y_{k+1}$ for every $k \in \{1, \ldots, m-1\}$ and $z_m = y_m$ to get

$$\int_{\mathbb{R}^m} T^2 \left| \sum_{i=1}^m x_i \right|^{H-1/2} \sum_{i=1}^m x_i \right|^{H-1} L(x_1, \ldots, x_m) dx_1 \cdots dx_m$$

$$= \int_{\mathbb{R}^m} \frac{T^2 \left| y_1 \right| y_2 - y_1 \cdots y_m - y_{m-1} \right|^{H-1/2} \sum_{i=1}^m y_i \right|^{H-1} L(y_1, y_2, \ldots, y_m) dy_1 \cdots dy_m$$

$$= \int_{\mathbb{R}^m} \frac{T^2 \left| z_1 \right| z_2 - z_1 \cdots z_m - z_{m-1} \right|^{H-1/2} \sum_{i=1}^m z_i \right|^{H-1} \int_{\mathbb{R}} d\zeta_{m-1}$$

$$= \int_{\mathbb{R}^m} \frac{T^2 \left| z_1 \right| z_2 - z_1 \cdots z_m - z_{m-1} \right|^{H-1/2} \sum_{i=1}^m z_i \right|^{H-1} L \left( \prod_{k=1}^m z_k, \left( 1 - z_1 \right) \prod_{k=2}^m z_k, \ldots, \left( 1 - z_1 \right) \prod_{k}^m z_k \right).$$

The right-hand side above can be bounded by a finite sum of terms of the form

$$\int_{\mathbb{R}^m} \frac{T^2 \left| z_1 \right| z_2 - z_1 \cdots z_m - z_{m-1} \right|^{H-1/2} \sum_{i=1}^m z_i \right|^{H-1} \int_{\mathbb{R}} d\zeta_{m-1} \times \cdots$$

where $k$, $j$, $\mu$ and $\nu$ are integer. The terms of the form (19) are finite since $H \in (1-1/(2m), 1)$ and Bertrand’s test. This concludes the proof. □
4.2 Convergence of $S_{q,h}^N$

We first deal with the study of $S_{m,h}^N$ defined for every $t \geq 0$ by

$$S_{m,h}^N (t) = \sum_{j=1}^{[Nt]} P_m (X_j (h_j^N)) \frac{1}{N^{h_j^N/N}}.$$  

From now on, we denote $\prod_{l=1}^m d\tilde{B}_x$ by $d\tilde{B}_x^{\otimes m}$ when $x = (x_1, \cdots, x_d)$.

**Lemma 3.** The process $S_{m,h}^N$ is equal in distribution to the process $\tilde{S}_{m,h}^N$ defined for every $t \geq 0$ by

$$\tilde{S}_{m,h}^N (t) = \int_{(-\pi, \pi)^m} \frac{1}{N^{1/m}} \prod_{l=1}^m \exp (ijx_l/N) g(h_j^N, x_l/N) |x_l|^{1/2-h_j^N} d\tilde{B}_x^{\otimes m}.$$  

**Proof.** Using (11) we obtain, almost surely,

$$P_m (X_j (h_j^N)) = \int_{(-\pi, \pi)^m} \prod_{l=1}^m \exp (ijx_l) g(h_j^N, x_l) |x_l|^{1/2-h_j^N} d\tilde{B}_x^{\otimes m}.$$  

We then have

$$S_{m,h}^N (t) = \sum_{j=1}^{[Nt]} \frac{1}{N^{1-m/2}} \int_{(-\pi, \pi)^m} \prod_{l=1}^m \exp (ijx_l) g(h_j^N, x_l) |x_l|^{1/2-h_j^N} d\tilde{B}_x^{\otimes m}.$$  

Making the substitution $x \to x/N$ and using (9) we get

$$S_{m,h}^N \overset{\text{dist.}}{=} t \sum_{j=1}^{[Nt]} \frac{1}{N} \int_{(-\pi, \pi)^m} \prod_{l=1}^m \exp (ijx_l/N) g(h_j^N, x_l/N) |x_l|^{1/2-h_j^N}.$$  

This concludes the proof by linearity of the multiple integral. \qed

Now we aim to prove the convergence of $\tilde{S}_{m,h}^N (t)$ in $L^2(\Omega, \mathbb{R})$ for every $t$. To this goal, we introduce the functions

$$f^N : [0, \infty) \times \mathbb{R}^m \to \mathbb{C},$$

$$(t, x) \mapsto \frac{1}{N} \prod_{l=1}^m \exp (ijx_l/N) g(h_j^N, x_l/N) |x_l|^{1/2-h_j^N}$$

and we state the following lemma.

**Lemma 4.** For every $t \geq 0$, there exists a function $f^*_N \in L^2(\mathbb{R}^m, \mathbb{R})$ so that, for every $x \in \mathbb{R}^m$ and $N \in \mathbb{N}$,

$$|f^N (t, x)| \leq f^*_N (x).$$

**Proof.** We have

$$f^N (t, x) = 1_{(-\pi, \pi)^m} (x) \left( \frac{i}{1-e^{-1}} \sum_{l=1}^m \exp (ijx_l/N) \prod_{l=1}^{[Nt]} g(h_j^N, x_l/N) |x_l|^{1/2-h_j^N} \right)$$

where

$$G_j^N (x) = \prod_{l=1}^m \frac{g(h_j^N, x_l/N)}{|x_l|^{h_j^N - 1/2}}.$$
We write

\[ f^N(t, x) = f^{N, 1}(t, x) - f^{N, 2}(t, x) \]

with

\[ f^{N, 1}(t, x) = \mathbb{I}_{(-N\pi, N\pi)}(x) \left( \frac{i \sum_{l=1}^{m} x_l / N}{1 - e^{-i \sum_{l=1}^{m} x_l / N}} \sum_{j=1}^{[Nt]} \frac{1}{i \sum_{l=1}^{m} x_l} \right) \]

and

\[ f^{N, 2}(t, x) = \mathbb{I}_{(-N\pi, N\pi)}(x) \left( \frac{i \sum_{l=1}^{m} x_l / N}{1 - e^{-i \sum_{l=1}^{m} x_l / N}} \sum_{j=1}^{[Nt]} \frac{1}{i \sum_{l=1}^{m} x_l} \right) \times \left( \sum_{j=1}^{[Nt]} \frac{e^{(j-1) \sum_{l=1}^{m} x_l / N} - 1}{i \sum_{l=1}^{m} x_l} G_j(x) \right). \]

We first deal with \( f^{N, 1} \). Because \( g \) is bounded, there exists \( M_1 > 0 \) such that for every \( x \) and \( N \)

\[ \left| f^{N, 1}(t, x) \right| \leq M_1 \left| \frac{e^{[Nt] \sum_{l=1}^{m} x_l / N} - 1}{x_1 \cdots x_m} \right|^{1/2} \sum_{l=1}^{m} x_l. \]

Then, by Lemma 2, there exists a function \( \tilde{f}_{1, 1} \in L^2(\mathbb{R}, \mathbb{R}) \) so that for every \( x \) and \( N \),

\[ \left| e^{[Nt] \sum_{l=1}^{m} x_l / N} - 1 \right|^{1/2} \sum_{l=1}^{m} x_l \leq \tilde{f}_{1, 1}(x), \]

so that we get

\[ \left| f^{N, 1}(t, x) \right| \leq M_1 \tilde{f}_{1, 1}(x). \tag{20} \]

Now we deal with \( f^{N, 2} \). By using Taylor formula we obtain, for almost every \( x \),

\[ \left| G_j(x) - G_{j-1}(x) \right| \leq \frac{\max |N|}{N} \max_{H \in [\min h, \max h]} \left| \frac{- \ln |x_1 \cdots x_m|}{x_1 \cdots x_m} \prod_{l=1}^{m} \left( \frac{g(H, x_l / N)}{N} \right) \right| \]

\[ + \frac{1}{|x_1 \cdots x_m|^{|H-1/2|} \sum_{k=1}^{m} \left( \frac{\partial g}{\partial H} \left( H, \frac{x_k}{N} \right) \right) \prod_{l=1, l \neq k}^{m} g \left( H, \frac{x_l}{N} \right). \]

Since \( g \) and \( \partial g/\partial H \) are bounded, there exists a constant \( M_2 > 0 \), which depends only on \( h \) and \( g \), such that for almost every \( x \) and every \( N \)

\[ \left| f^{N, 2}(t, x) \right| \leq \frac{M_2}{N} \sum_{j=1}^{[Nt]} \left| \frac{e^{(j-1) \sum_{l=1}^{m} x_l / N} - 1}{\sum_{l=1}^{m} x_l} \right| \max_{H \in [\min h, \max h]} \left| \frac{1 + \ln |x_1 \cdots x_m|}{|x_1 \cdots x_m|^{|H-1/2|}} \right|. \]

As for \( f^{N, 2} \), by Lemma 2, there exists a function \( \tilde{f}_{1, 2} \in L^2(\mathbb{R}, \mathbb{R}) \) so that for almost every \( x \) and every \( N \) and \( j \),

\[ \left| \frac{e^{(j-1) \sum_{l=1}^{m} x_l / N} - 1}{\sum_{l=1}^{m} x_l} \max_{H \in [\min h, \max h]} \left| \frac{1 + \ln |x_1 \cdots x_m|}{|x_1 \cdots x_m|^{|H-1/2|}} \right| \leq \tilde{f}_{1, 2}(x), \]
Because of Lemmas 1 and 4, it suffices to prove that the function

\[ G = \frac{\kappa}{\kappa + \theta} \]

where, for \( f \in \mathbb{R} \),

\[ \frac{\kappa}{\kappa + \theta} \]

so that we get

\[ |f^{N,2}(t, x)| \leq M_2 \tilde{f}_{1,2}(x). \tag{21} \]

Hence, taking \( \tilde{f}_1 = M_1 \tilde{f}_{1,1} + M_2 \tilde{f}_{1,2} \) and combining (20) and (21) we conclude the proof.

The convergence of \( \tilde{S}_{m,h}^N(t) \) can now be established.

**Lemma 5.** For every \( t \geq 0 \), as \( N \to \infty \), \( \tilde{S}_{m,h}^N(t) \) converges in \( L^2(\Omega, \mathbb{R}) \) to \( \tilde{S}_{m,h}^\infty(t) \) given by

\[ \tilde{S}_{m,h}^\infty(t) = \int_{\mathbb{R}^m} d\tilde{B}_x \int_0^t \exp \left( i\theta \sum_{l=1}^m x_l \right) g_0 \left( \tilde{h}(\theta) \right)^m |x_1 \cdots x_m|^{1/2 - \tilde{h}(\theta)} d\theta. \]

**Proof.** Because of Lemmas 1 and 4, it suffices to prove that the function \( f^N(t, x) \) converges for almost every \( x \) to \( f^\infty(t, x) \) defined by

\[ f^\infty(t, x) := \int_0^t \exp \left( i\theta \sum_{l=1}^m x_l \right) g_0 \left( \tilde{h}(\theta) \right)^m |x_1 \cdots x_m|^{1/2 - \tilde{h}(\theta)} d\theta. \]

We let

\[ G_{j,0}^N(x) = g_0(h_j^N)^m |x_1 \cdots x_m|^{1/2 - h_j^N} \quad \text{and} \quad G_{j,1}^N(x) = G_j^N(x) - G_{j,0}^N(x) \]

where \( G_j^N(x) \) is defined as in the proof of Lemma 4. We also consider the same decomposition \( f^N = f^{N_1} - f^{N_2} \) as in the proof of Lemma 4 and we let

\[ f^{N_1} = f^{N_1,0} - f^{N_1,1} \quad \text{and} \quad f^{N_2} = f^{N_2,0} - f^{N_2,1} \]

where, for \( \kappa \in \{0, 1\}, \)

\[ f^{N,\kappa}(t, x) = \mathbb{I}_{\{N \pi, N \pi\}}(x) \frac{e^{i \sum_{l=1}^m x_l / N}}{1 - e^{-i \sum_{l=1}^m x_l / N}} G_{\{N \pi, \kappa\}}^N(x) e^{i \sum_{l=1}^m x_l / N} \frac{1}{i \sum_{l=1}^m x_l} \]

and

\[ f^{N_2}(t, x) = \mathbb{I}_{\{N \pi, N \pi\}}(x) \frac{i \sum_{l=1}^m x_l / N}{1 - e^{-i \sum_{l=1}^m x_l / N}} \]

\[ \times \sum_{j=1}^{[N/4]} \frac{e^{i(j-1) \sum_{l=1}^m x_l / N} - 1}{i \sum_{l=1}^m x_l} (G_{j,1}(x) - G_{j,0}^N(x)). \]

Because \( h \) and \( g_0 \) are continuously differentiable we get, for almost every \( x \),

\[ \lim_{N \to \infty} f^{N,1,0}(t, x) = g_0(\tilde{h}(\theta))^m \frac{e^{i \sum_{l=1}^m x_l / N} - 1}{i \sum_{l=1}^m x_l} \]

and

\[ \lim_{N \to \infty} f^{N,2,0}(t, x) = \int_0^t \frac{e^{i \theta \sum_{l=1}^m x_l / N} - 1}{i \sum_{l=1}^m x_l} \tilde{h}'(\theta) \frac{\partial}{\partial H} \left( \frac{g_0(H)^m}{|x_1 \cdots x_m|^{1/2 - H}} \right) \bigg|_{H = \tilde{h}(\theta)} d\theta, \]

so that

\[ \lim_{N \to \infty} \left( f^{N,1,0}(t, x) - f^{N,2,0}(t, x) \right) = f^\infty(t, x). \]

Now we deal with \( f^{N,1,1} \) and \( f^{N,2,1} \). We remark that we can express \( G_{j,1}^N(x) \) as

\[ G_{j,1}^N(x) = |x_1 \cdots x_m|^{1/2 - h_j^N} \sum_{k=1}^m g_1 \left( h_j^N, \frac{x_k}{N} \right) g_0 \left( h_j^N \right) \prod_{l=k+1}^m g \left( h_j^N, \frac{x_l}{N} \right). \]
Then, because of Lemma 2 and the boundedness of $g_0$ and $g$, there exist a constant $M_3 > 0$ and a function $\tilde{f}_{t,3} \in L^2(\mathbb{R}^m, \mathbb{R})$ such that for almost every $x$,

$$|f^{N,1,1}(t, x)| \leq M_3 \tilde{f}_{t,3}(x) \sum_{k=1}^{m} \sup_{H \in [\min h, \max h]} |g_1(H, \frac{x_k}{N})|,$$

so that $\lim_{N \to \infty} f^{N,1,1}(t, x) = 0$. Similarly, using Lemma 2, there exist a constant $M_4 > 0$ and a function $\tilde{f}_{t,4} \in L^2(\mathbb{R}^m, \mathbb{R})$ such that for almost every $x$,

$$|f^{N,2,1}(t, x)| \leq M_4 \tilde{f}_{t,4}(x) \sum_{k=1}^{m} \sup_{H \in [\min h, \max h]} \left( |g_1(H, \frac{x_k}{N})| + |\frac{\partial g_1}{\partial H}(H, \frac{x_k}{N})| \right),$$

so that $\lim_{N \to \infty} f^{N,2,1}(t, x) = 0$ and then

$$\lim_{N \to \infty} (f^{N,2,1}(t, x) - f^{N,2,1}(t, x)) = 0,$$

which concludes the proof.

The following lemma establishes that the convergence of $S^N_{\phi,h}$ can be reduced to the one of $S^N_{m,h}$ and, as a consequence of Lemma 3, to the one of $S^N_{m,h}$.

**Lemma 6.** For every $t \geq 0$, we have

$$\lim_{N \to \infty} \mathbb{E} \left( S^N_{\phi,h}(t) - \frac{\langle \phi, P_m \rangle}{m!} S^N_{m,h}(t) \right)^2 = 0.$$  

**Proof.** Since (6) we have

$$\mathbb{E} \left[ S^N_{\phi,h}(t) - \frac{\langle \phi, P_m \rangle}{m!} S^N_{m,h}(t) \right]^2 = \mathbb{E} \left[ \left( \sum_{j=1}^{[Nt]} \frac{1}{N^{h(j/N)}} \sum_{n=m+1}^{\infty} \frac{\langle \phi, P_n \rangle}{n!} P_n(X_j(h^N_j)) \right)^2 \right] \leq \mathbb{E} \left[ \left( \sum_{n=m+1}^{\infty} \frac{\langle \phi, P_n \rangle}{n!} P_n(X_k(h^N_k)) \right)^2 \right].$$

Because of (7) and (13) we get, for every $n \geq m + 1$,

$$\mathbb{E}[P_n(X_j(h^N_j))P_n(X_k(h^N_k))] = n! \mathbb{E}[X_j(h^N_j)X_k(h^N_k)]^n \leq n! \mathbb{E}[X_j(h^N_j)X_k(h^N_k)]^m,$$

so that

$$\mathbb{E} \left[ S^N_{\phi,h}(t) - \frac{\langle \phi, P_m \rangle}{m!} S^N_{m,h}(t) \right]^2 \leq \left( \sum_{n=m+1}^{\infty} \frac{\langle \phi, P_n \rangle^2}{n!} \right) \sum_{j=1}^{[Nt]} \sum_{k=1}^{[Nt]} \frac{\mathbb{E}[X_j(h^N_j)X_k(h^N_k)]^m}{N^{h(j/N)+h(k/N)}}.$$

Let $\eta > 0$. Using the representation of the field $X$, there exists $N_0 \in \mathbb{N}^*$ such that, for $|j - k| > N_0$ and $N \in \mathbb{N}^*$,

$$|\mathbb{E}[X_j(h^N_j)X_k(h^N_k)]| \leq \eta,$$

so that

$$\sum_{j=1}^{[Nt]} \sum_{k=1}^{[Nt]} \frac{\mathbb{E}[X_j(h^N_j)X_k(h^N_k)]^m}{N^{h(j/N)+h(k/N)}} \leq \sum_{j=1}^{[Nt]} \sum_{k=1}^{[Nt]} \frac{1_{|j-k| \leq N_0}}{N^{h(j/N)+h(k/N)}} + \eta \sum_{j=1}^{[Nt]} \sum_{k=1}^{[Nt]} 1_{|j-k| \geq 1} \frac{\mathbb{E}[X_j(h^N_j)X_k(h^N_k)]^m}{N^{h(j/N)+h(k/N)}}.$$
There exists $C_1(\eta) > 0$ such that
\[
\sum_{j=1}^{[Nt]} \sum_{k=1}^{[Nt]} \frac{1_{|j-k|\leq N_n}}{N^{k(j/N)+k(k/N)}} \leq \frac{C_\eta}{N^{2\min h-1}}.
\]
Moreover, because of the assumptions on $X_i$, there exists a constant $C_2 > 0$, which is independent on $\eta$, such that, for every $j, k$ and $N$,
\[
|\mathbb{E}[X_j(h_N^j)X_k(h_N^k)]| \leq C_2|j-k|^{h_N^j+h_N^k-2}.
\]
We then obtain
\[
\sum_{j=1}^{[Nt]} \sum_{k=1}^{[Nt]} \frac{|\mathbb{E}[X_j(h_N^j)X_k(h_N^k)]|^{m+1}}{N^{h(j/N)+h(k/N)}} \leq \frac{C_\eta}{N^{2\min h-1}} + \frac{\eta C_2}{N^2} \sum_{j=1}^{[Nt]} \sum_{k=1}^{[Nt]} 1_{|j-k|\geq 1} |j-k|^{m(h_N^j+h_N^k-2)}.
\]
Hence, for every $\eta > 0$
\[
\limsup_{N \to \infty} \mathbb{E} \left[ \left( S_{\phi,h}^N(t) - \frac{\langle \phi, P_m \rangle}{m!} S_{P_m,h}^N(t) \right)^2 \right] 
\leq \eta C_2 \left( \sum_{n=m+1}^{\infty} \frac{\langle \phi, P_n \rangle^2}{n!} \right) \int_0^t \int_0^t |\theta - \sigma|^{h(\theta)+h(\sigma)-2} d\theta d\sigma.
\]
The constants $\sum_{n=m+1}^{\infty} \frac{\langle \phi, P_n \rangle^2}{n!}$ and $\int_0^t \int_0^t |\theta - \sigma|^{h(\theta)+h(\sigma)-2} d\theta d\sigma$ are finite since (8) and $h > 1/2$ respectively. This concludes the proof. \hfill \square

Now we conclude this subsection by the following lemma.

**Lemma 7.** As $N \to \infty$, the finite-dimensional distributions of $S_{\phi,h}^N$ converge to those of $S_{m,h}$, which can be defined for every $t \geq 0$ as:
\[
S_{m,h}(t) := \frac{\langle \phi, P_m \rangle}{m!} \int_{\mathbb{R}^m} f^\infty(t, x_1, \ldots, x_m) d\widehat{B}_{x_1} \cdots d\widehat{B}_{x_m}.
\]

*Proof.* We fix $n \in \mathbb{N}$, $(t_1, \ldots, t_n) \in [0, \infty)^n$ and a Lipschitz bounded function $\Psi : \mathbb{R}^n \to \mathbb{R}$. We define $\phi_m = \langle \phi, P_m \rangle/m!$. We have
\[
\left| \mathbb{E}[\Psi(S_{\phi,h}^N(t_1), \ldots, S_{\phi,h}^N(t_n))] - \mathbb{E}[\Psi(S_{m,h}(t_1), \ldots, S_{m,h}(t_n))] \right| \leq E_1^N + E_2^N
\]
where
\[
E_1^N = \left| \mathbb{E} \left[ \Psi(S_{\phi,h}^N(t_1), \ldots, S_{\phi,h}^N(t_n)) - \Psi(\phi_m S_{P_m,h}^N(t_1), \ldots, \phi_m S_{P_m,h}^N(t_n)) \right] \right|
\]
and
\[
E_2^N = \left| \mathbb{E} \left[ \Psi(\phi_m S_{P_m,h}^N(t_1), \ldots, \phi_m S_{P_m,h}^N(t_n)) \right] - \mathbb{E}[\Psi(S_{m,h}(t_1), \ldots, S_{m,h}(t_n))] \right|
\]
Because $\Psi$ is Lipschitz and using Cauchy-Schwartz inequality, there exists $C_1 > 0$ so that, for every $N$,
\[
E_1^N \leq C_1 \sum_{j=1}^n \sqrt{\mathbb{E} \left[ \left( S_{\phi,h}^N(t_j) - \phi_m S_{P_m,h}^N(t_j) \right)^2 \right]}.
\]
Thus, as for $E$ or almost every $x$ with (10) and (9) we have
\[ E_N^2 = \sqrt{E\left[ \left( \phi_m \tilde{S}^N_{m,h}(t_1), \ldots, \phi_m \tilde{S}^N_{m,h}(t_n) \right) - \psi(S_{m,h}(t_1), \ldots, S_{m,h}(t_n)) \right]^2} \]

As a consequence, from Lemma 5,
\[ \lim_{N \to \infty} E_N^2 = 0. \]

We conclude the proof by combining (23), (24) and (25).

4.3 Continuity and local self-similarity of $S_{m,h}$

We first prove the local self-similarity in the sense of the finite-dimensional distributions. Then we prove the continuity of $S_{m,h}$. Finally, we establish a tightness property for the family $\{ (\varepsilon^{-h(t)}(S_{\phi,h}(t + \varepsilon u) - S_{\phi,h}(t)))_{u \geq 0} \}_{\varepsilon > 0}$ using Kolmogorov lemma [1] to conclude with the proof of the local self-similarity property in the space of continuous functions.

By making the substitution $\theta \to \varepsilon \theta + t$, we get
\[ S_{m,h}(t + \varepsilon u) - S_{m,h}(t) = \int_{\mathbb{R}^m} \exp \left( i \left( \sum_{l=1}^{m} x_l \right) \right) f_1(t, u, x, \varepsilon) d\tilde{B}_{x_1} \cdots d\tilde{B}_{x_m} \]

where
\[ f_1(t, u, x, \varepsilon) = \varepsilon^{1-h(t)} \int_0^u \exp \left( i \sum_{l=1}^{m} x_l \right) \tilde{g}(\varepsilon \theta + t) |x_1 \cdots x_m|^{1/2 - \tilde{h}(\varepsilon \theta + t)} d\theta. \]

Since (10) and (9) we have
\[ \left\{ S_{m,h}(t + \varepsilon u) - S_{m,h}(t) \right\}_{u \geq 0} \overset{\text{dist}}{=} \left\{ \int_{\mathbb{R}^m} f_2(t, u, x, \varepsilon) d\tilde{B}_{x_1} \cdots d\tilde{B}_{x_m} \right\}_{u \geq 0} \]

with
\[ f_2(t, u, x, \varepsilon) = \int_0^u \varepsilon^m(\tilde{h}(\varepsilon \theta + t) - \tilde{h}(t)) \exp \left( i \sum_{l=1}^{m} x_l \right) \tilde{g}(\varepsilon \theta + t) |x_1 \cdots x_m|^{1/2 - \tilde{h}(\varepsilon \theta + t)} d\theta. \]

For almost every $x \in \mathbb{R}^m$, every $u$ and $t$,
\[ \lim_{\varepsilon \to 0} f_2(t, u, x, \varepsilon) = \tilde{g}(t) \frac{\exp(iu \sum_{l=1}^{m} x_l) - 1}{i(u \sum_{l=1}^{m} x_l) |x_1 \cdots x_m|^{|h(t)| - 1/2}} \]

By an integration by part, we have
\[ f_2(t, u, x, \varepsilon) = \varepsilon^m(\tilde{h}(\varepsilon u + t) - \tilde{h}(t)) \tilde{g}(\varepsilon u + t) \frac{\exp(iu \sum_{l=1}^{m} x_l) - 1}{i(u \sum_{l=1}^{m} x_l) |x_1 \cdots x_m|^{|h(u + t)| - 1/2}} \]
\[ + \varepsilon \int_0^u \varepsilon^m(\tilde{h}(\varepsilon \theta + t) - \tilde{h}(t)) \exp(i \sum_{l=1}^{m} x_l) - 1 \frac{\exp(i \sum_{l=1}^{m} x_l) - 1}{i(\sum_{l=1}^{m} x_l) |x_1 \cdots x_m|^{|h(\varepsilon \theta + t)| - 1/2}} \]
\[ \times \left\{ \tilde{g}(\varepsilon \theta + t) + \tilde{h}(\varepsilon \theta + t) \tilde{g}(\varepsilon \theta + t) (\ln(e^m) - \ln |x_1 \cdots x_m|) \right\} d\theta \]
We fix $T > 0$ and $U > 0$. As a consequence of the identity above and because of the regularity of $\tilde{h}$ and $\tilde{g}$, there exists a constant $M_{T,U} > 0$ such that, for every $u \in [0,U]$, $t \in [0,T]$, $\varepsilon \in (0,1]$ and $x \in \mathbb{R}^m$,

$$|\tilde{f}_2(t,u,x,\varepsilon)| \leq M_{T,U} \tilde{f}_{T+U,2}(x)$$

(26)

where $\tilde{f}_{T+U,2}(x)$ is defined in Lemma 2. Since $\tilde{f}_{T+U,2}$ is square integrable and because of Lemma 1, this proves the local self-similarity of $S_{m,h}$ in the sense of the finite-dimensional distributions.

To prove the continuity of the $S_{m,h}$ we use Kolmogorov lemma. By making the same calculations as above we have, for every $t > s > 0$,

$$\mathbb{E}
\left[
(S_{\phi,h}(t) - S_{\phi,h}(s))^2
\right] = m!(t-s)^{2h(s)} \int_{\mathbb{R}^m} \tilde{f}_2(s,1,x,t-s)^2 \, dx_1 \cdots dx_m$$

If $t - s < 1$, because of (26) we then have

$$\mathbb{E}
\left[
(S_{\phi,h}(t) - S_{\phi,h}(s))^2
\right] \leq m! M_{1,1}(t-s)^{2h(s)} \int_{\mathbb{R}^m} \tilde{f}_{2,2}(x)^2 \, dx_1 \cdots dx_m,$$

which concludes the proof of the continuity of $S_{\phi,h}$.

Finally, in a similar way as just previously there exists a constant $C > 0$ such that, for every $u$ and $v$ satisfying $|u - v| < 1$,

$$\mathbb{E}
\left[
\frac{(S_{\phi,h}(t + \varepsilon u) - S_{\phi,h}(t + \varepsilon v))}{\varepsilon^{h(t)}}^2
\right] \leq C|u - v|^{2h(t)},$$

This prove the tightness of the family $\{\varepsilon^{-h(t)}(S_{\phi,h}(t + \varepsilon u) - S_{\phi,h}(t))\}_{u \geq 0}$ thanks to Kolmogorov lemma [1], and then the local self-similarity property of $S_{\phi,h}$.

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**References**


