Selfstress states identification and localization in modular tensegrity grids

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ABSTRACT: The design of a modular tensegrity grid requires the determination of its selfstress states, before choosing an appropriate combination defining the system's initial stresses. However, the computation of the vectorial basis associated with selfstress states generally produces results that are difficult to exploit. We therefore propose two different strategies to identify and localize selfstress states in a modular tensegrity grid more pertinently. The first is based on a heuristic approach that exploits the system's structural composition of modularity and regularity. The second is numerical and aims at redefining the vectors of the basis in a more convenient and useful way. Two methods based on transformations of the vectorial basis of selfstress states have been developed for a minimal number of involved components. Finally, we suggest a selfstress state classification based on the number of components and their localization as well as on their mechanical behavior.

Keywords: Tensegrity Grid, Selfstress, Localization, Classification.

1. BACKGROUND AND OBJECTIVES

Tensegrity systems have been initially investigated by engineers or artists like R. B. Fuller [1] and K. Snelson [2]. Developments have lead to the identification of basic modules that could be assembled to create grids [3]. The realization of several physical systems has demonstrated the feasibility of such conceptual design. Examples of a plane and a curved grid, both realized at Montpellier University, are presented in Fig. 1.

Various structural compositions can be considered according to their basic module (triplex, quadruplex, or hexaplex, see Fig. 2). Such structure could be envisaged for building grids with specific boundary geometries that can meet different architectural requirements.

From a mechanical point of view, tensegrity grids are selfstressed equilibrate spatial structures composed of compressed struts connected to cables in tension. These initial stresses contribute to the system's rigidity and stability. The rigidity is increased by including the geometric stiffness in the tangent stiffness of each component, but also by the fact that all the cable elements are rigid provided they remain tensioned. The intensity of the selfstress should hence ensure their non slackening when external actions are considered. In the opposite case, a slackening of cables results in a rigidity loss and in possible mechanisms and instability phenomena. However, an excessive selfstress level is also inappropriate since it can lead to premature fatigue and disproportionate sizing of the components.

As a consequence, the determination of the suitable initial stresses distribution and intensity is a key point for a tensegrity grid design. Such a calculation requires characterizing the relationship between the grid form and the initial stresses in the components. This stage is called form-finding and can be achieved according to two approaches [4]: the form controlled ones, where possible stresses are determined from a given shape, and the force controlled ones, which consists in calculating the shapes associated to specified stresses.

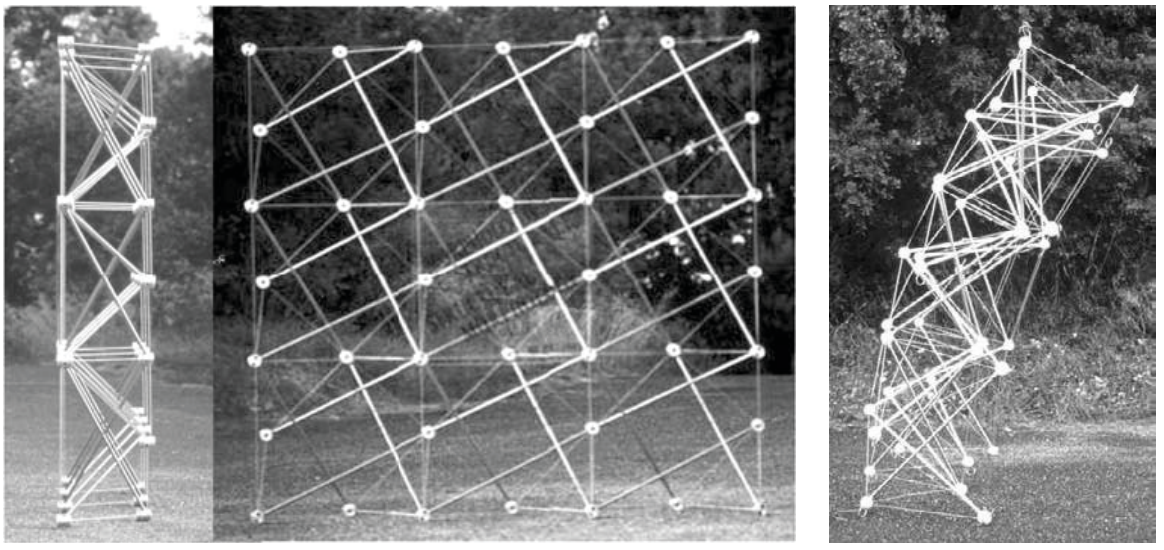

Figure 1. Plane and curved modular tensegrity grid.

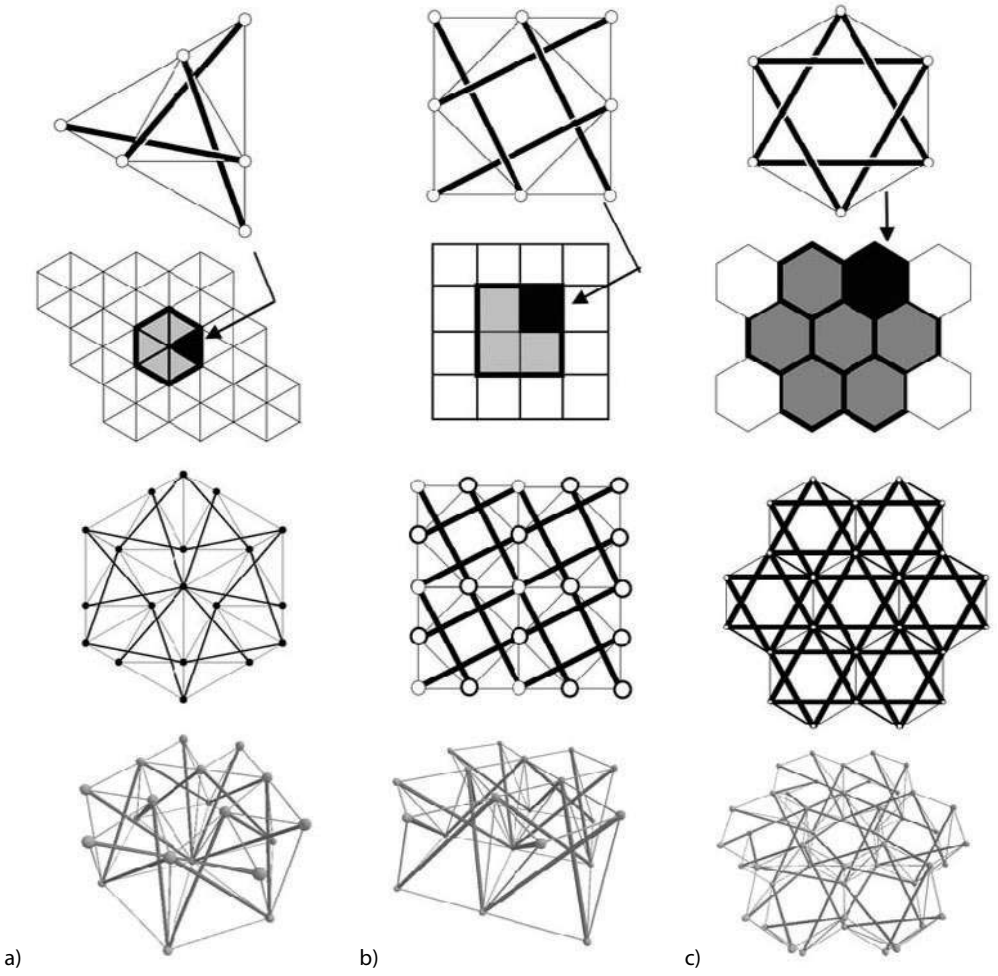

Figure 2. Grids based on triplex (a), quadruplex (b) or hexaplex (c) modules.

Once the distribution and intensity of the initial stresses have been chosen, the designer has secondly to analyze the mechanical behavior of the structure (stiffness and resistance). It can lead to other stresses specification until a suitable behavior is ensured. The last stage of the design is related to the realization of the grid. The main issue is the installation and assembling strategy to follow in order to obtain the

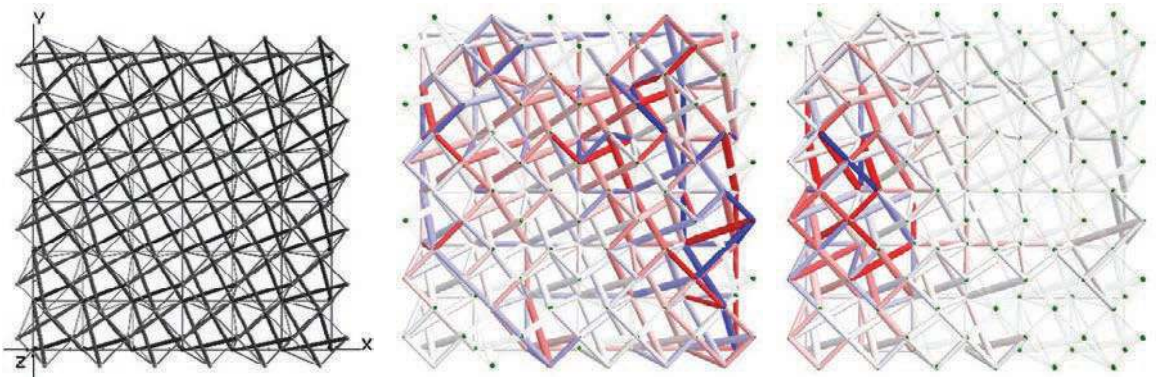

Figure 3. Quadruplex grid and two computed selfstress states.

required stresses in the components. Several approaches have been proposed for that purpose [5].

This study deals with the form controlled strategy and is devoted to the initial stresses identification for modular tensegrity grids. The objective is to propose adapted methods for the localization of the selfstress states in this type of structure. The calculations used until now to determine the initial stresses are most of the time based on the computation of the kernel of the system equilibrium matrix to provide a basis of the vectorial subspace of the selfstress states. However, the results obtained by this approach are generally difficult to exploit for engineers. For instance, the null space basis for a square grid composed of 6×6 quadruplex modules has 124 selfstress vectors. Two of them, directly stemming from the computation, are shown in Fig. 3 (the red and blue colors respectively correspond to the compression and tension intensities). The difficulty to interpret and to exploit such result is observable. Nevertheless, the designer has to determine a pertinent combination of these vectors in order to define the grid selfstress. The latter has indeed an incidence on the stiffness, strength and stability of the structure, and a precise and adapted knowledge of the different selfstress states is therefore necessary.

The objective of this study is to propose useful and user friendly methods to identify and to localize more pertinently the selfstress states in a modular tensegrity grid. The first approach is based on a heuristic construction using the system structural composition properties of modularity and regularity (symmetry and periodicity of the components). The second strategy is numerical and its objective is to rewrite the vectors of the basis in a more convenient and useful way. Two methods have been thus developed, based on several transformations of the selfstress states vectorial basis, to have a maximum of zero terms in the vectors.

2. SELFSTRESS STATES IDENTIFICATION AND LOCALIZATION

2.1 Tensegrity Selfstress Basis

In a tensegrity system, the equilibrium of one node i connected to elements j may be expressed as [6]

$$\{f_i\}+\sum_j\{N_{ij}\}=0 \tag{1}$$

where f_i are the external actions acting upon the node N_{ij} and the internal normal forces in an element j (tension or compression). Hence, the equilibrium of the whole system, with b component and D degrees of freedom, may be written

$$[A]\{q\}=\{F\} \tag{2}$$

where $[A]$ is the equilibrium matrix (D rows and b columns), $\{F\}$ is the vector of external actions (D components) and $\{q\}$ is the vector of the force density coefficients (b components). The force density coefficients are the ratios of the normal forces divided by the component lengths; they allow to write in a more convenient way the equilibrium matrix (length calculations are unnecessary).

If no external action is considered on the structure, it comes

$$[A]\{q\}=\{0\}. \tag{3}$$

The selfstress states are hence the elements of a basis for the kernel of the equilibrium matrix, i.e. a basis for the vectorial subspace of the solutions of Eqn. (3)

$$\{S\}\in Ker[A]. \tag{4}$$

Moreover, the number of selfstress states is

$$s=b-r_A \tag{5}$$

(where r_A is the rank of the equilibrium matrix $[A]$).

The s vectors $\{S_i\}$ form the selfstress states basis [S]. They can be numerically computed by numerous software that calculates the kernel of a given matrix (Matlab®, Mathematica®, Maple® etc.). However, as

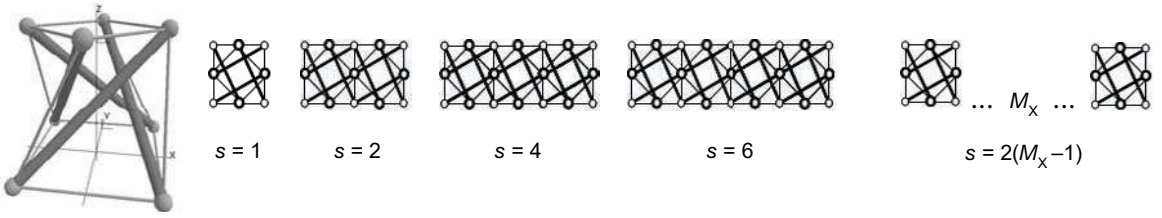

Figure 4. Number of selfstress states in a quadruplex beam.

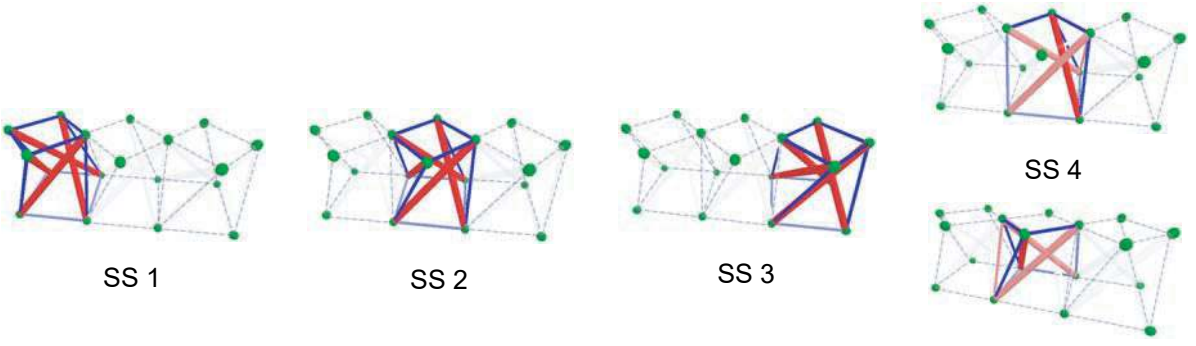

Figure 5. The four selfstress states in a 3 × 1 quadruplex beam.

shown in Fig. 3, the obtained selfstress states are generally difficult to understand and to use by the engineer.

2.2 Modular Tensegrity Beams

Beams composed by a linear assembly of tensegrity modules have been initially studied at Montpellier University [7]. One of the first steps was to determine the number of selfstress states according to the number of modules. A formula, depending on the nature of the module, has been put forward [8]. An example of a quadruplex beam is presented in Fig. 4.

If this beam has three quadruplex modules, the identification of the four selfstress states can be achieved according to the structural composition. Since every module keeps its individual selfstress state after assembling, the three corresponding selfstress states can be easily localized (SS 1 to 3, see Fig. 5). They are determined from the computed basis by using the simplex method where force density coefficients are specified on some elements [9]. For instance, the first selfstress state (SS 1) is obtained by prescribing a compression equal to −1 for one strut of the first module and equal to 0 for one strut of the other modules. The same procedure occurs for the second and third modules (SS 2 and 3). Nevertheless, one selfstress state remains to be identified (SS 4). Several representations are possible, depending on the linear combinations of the selfstress basis considered for writing the associated vector. We have chosen to consider the case of a maximum number of zeros in this vector, that is to say a minimum of involved components. This method will be presented in section 2.4.2. The result is not unique, but two interesting possibilities are illustrated in Fig. 5 (SS 4). We observe that, for these two vectors, the elements involved are located within the central module.

2.3 Selfstresses and Structural Composition

As previously shown for tensegrity beams, the repetitiveness and regularity of the modules may be used to identify the selfstress states in a modular tensegrity grid [10]. If we consider a simple case, based on assembling of 2D modules (Fig. 6), the evolution from one to several modules is characterized in the selfstress states basis [S] by the periodicity of non-zero terms of the vector $\{S_1\}$ associated to one single module.

This approach may be extended to 3D modular tensegrity grids. An application is presented in Fig. 7 for a 2 × 3 quadruplex grid. The direct computation of the rank of equilibrium matrix indicates that this structure has nine selfstress states (s = 9 with Eqn. 5). However, the use of the grid structural composition allows localizing them without complicated calculations:

a. Since each module has one selfstress state, six repetitive selfstress states are generated in the system.
b. Since each line composed of three modules has one selfstress state (the “central” SS 4 in a 1 × 3

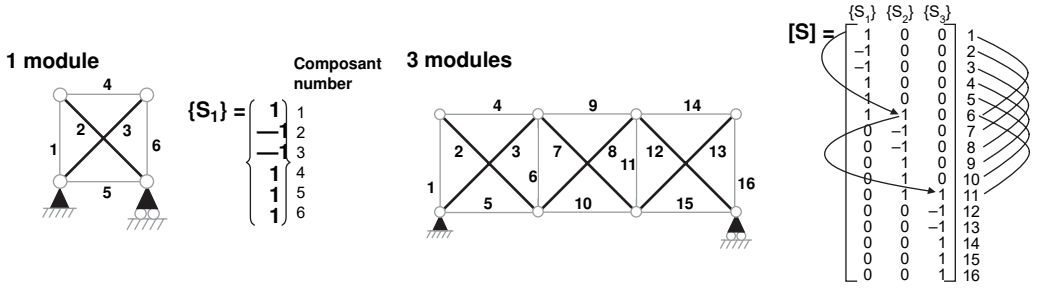

Figure 6. Modular periodicity in a 2D example.

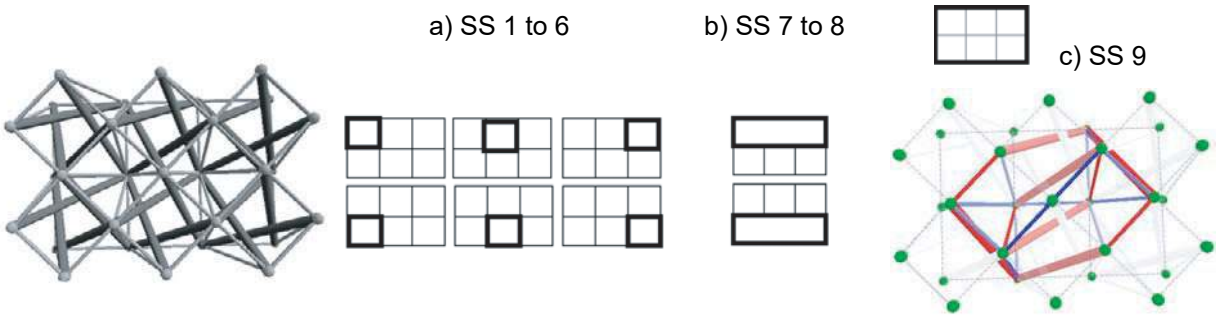

Figure 7. Selfstress localization in a 2 × 3 quadruplex grid (s = 9).

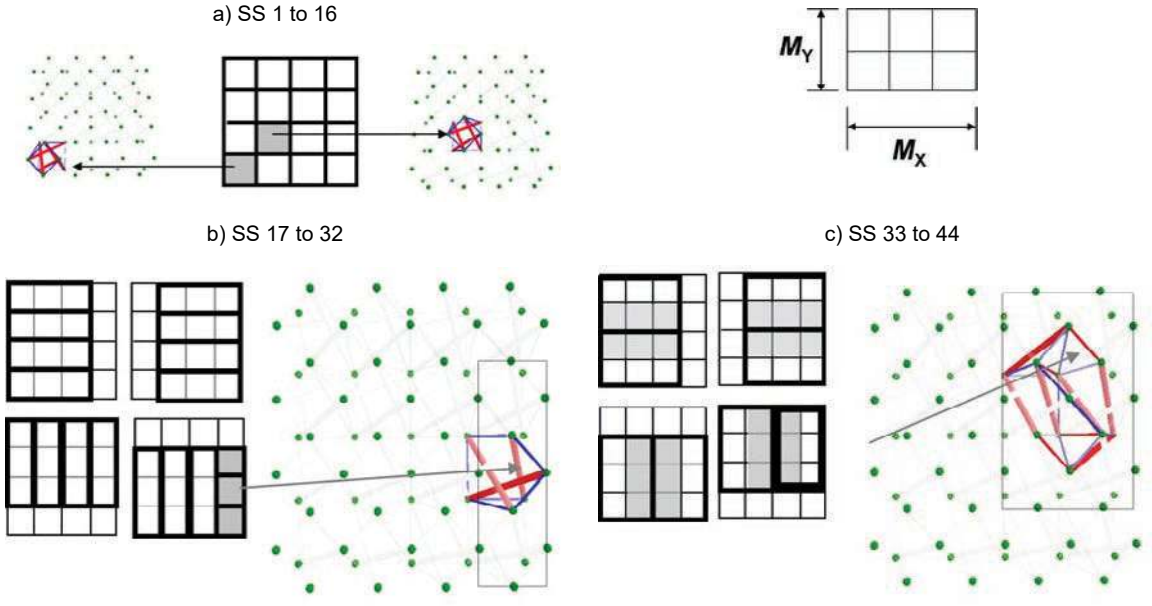

Figure 8. Selfstress localization in a 4 × 4 quadruplex grid (s = 44).

quadruplex beam shown in Fig. 5); we obtain two repetitive selfstress states.

c. One selfstress state remains located within the entire 2 × 3 grid. Such a state may be represented as in Fig. 7 (SS 9) with a maximum number of zeros in the corresponding vector.

These nine selfstress states may be this way easily localized in the system. Such a result could be helpful for the engineer who has to generate the grid selfstress by a linear combination of these vectors.

A second example is presented in Fig. 8 for a 4×4 quadruplex grid. All the selfstress states can be determined and localized by using an identical approach without any computation. Thus, 16 repetitive SS corresponding to the modules (a), 16 other repetitive SS associated with the 1 × 3 beams (b) and 12 repetitive SS for the 2 × 3 grids located within the structure (c) can be identified. This finally gives 44 selfstress states; the result is confirmed by a numerical computation of s = 44.

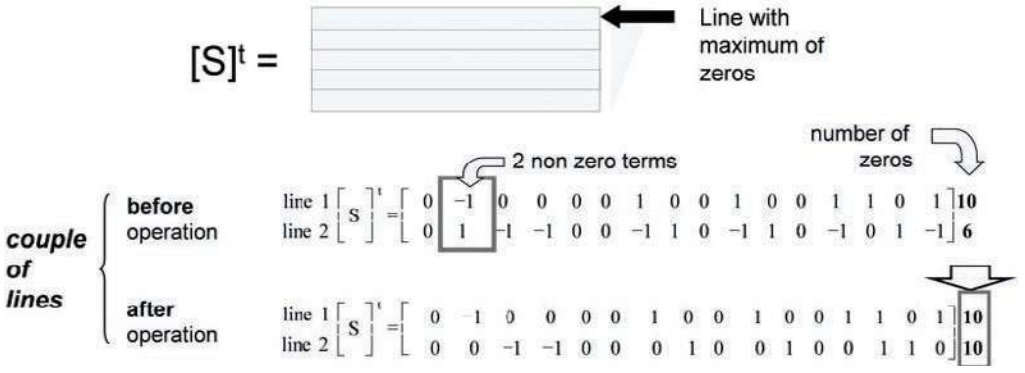

Figure 9. Gauss conditioned method.

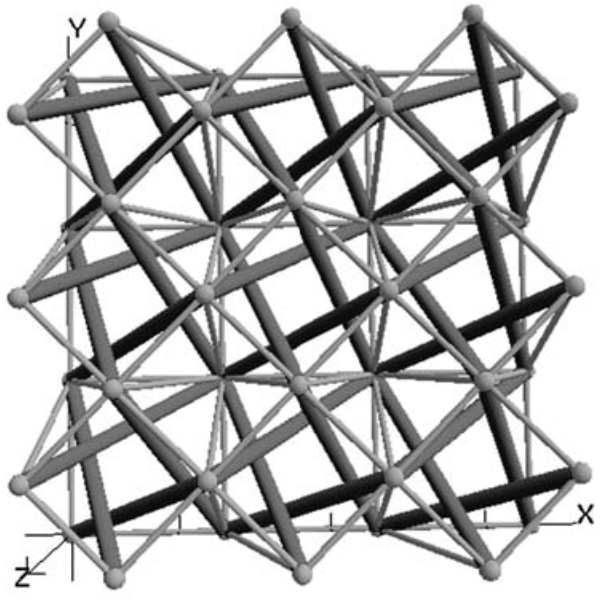

Figure 10. 3 × 3 quadruplex grid (s = 19).

An extension of this method is proposed in the case of a M_X by M_Y quadruplex grid. The number of the associated selfstress states is then (see Table 1)

- $s = M_X M_Y$ if $M_X \leq 2$ and $M_Y \leq 2$.
- $s = a + b + c$ if $M_X > 2$ and $M_Y \leq 2$, with $a = M_X M_Y$ (total number of modules), $b = M_Y (M_X - 2)$ (number of 1×3 horizontal beams) and $c = (M_Y - 1)(M_X - 2)$ (number of 2×3 horizontal grids).
- $s = a + b + c + d + e$ if $M_X > 2$ and $M_Y > 2$, with a, b and c as defined above, $d = M_X(M_Y - 2)$ (number of 1 × 3 vertical beams) and $e = (M_X - 1)(M_Y - 2)$ (number of 2 × 3 vertical grids).

2.4 Selfstress Reduction

Together with the grids' structural composition, we have developed numerical approaches to identify the selfstress states in modular tensegrity grids. The objective is to reduce the number of elements that are involved in each selfstress state. This requires maximizing the number of null terms in the corresponding vectors of the basis. This problem of finding a convenient sparse and localized basis for the kernel of a matrix is known as "the sparse null space basis problem" and has been already studied by different authors [11, 12]. It has been shown that, in general and for large matrices especially, it is not possible to find an algorithm which always converges to the optimal solution in a reasonable time.

Several numerical tests have led us to consider two different approaches: the Gauss conditioned method and a specific "reduction" method developed in our group. Both processes produce interesting and useful results in the case of modular grids, because of their specific matrices. Their application to tensegrity systems in general may, however, not always be optimally efficient.

2.4.1 Gauss Conditioned Method

The initial vectorial basis [^{i}S] coming from the software computation (Matlab®, Mathematica®, Maple® etc.) is first transposed and organized in accordance with the number of zeros in every line (Fig. 9, top). The two first lines are selected and their consecutive pairs of coefficients are considered until two non-zero values form a first couple. A linear

Table 1. Number of selfstress states in a M_x by M_Y quadruplex grid

M_X	1	2	2	3	3	3	4	4	4	5	5	6	7	8	9	10
M_Y	1	1	2	1	2	3	2	4	5	5	6	6	7	8	9	10
s	***1***	***2***	***4***	***4***	***9***	***19***	***14***	***44***	***59***	***79***	***99***	***124***	***179***	***244***	***319***	***404***

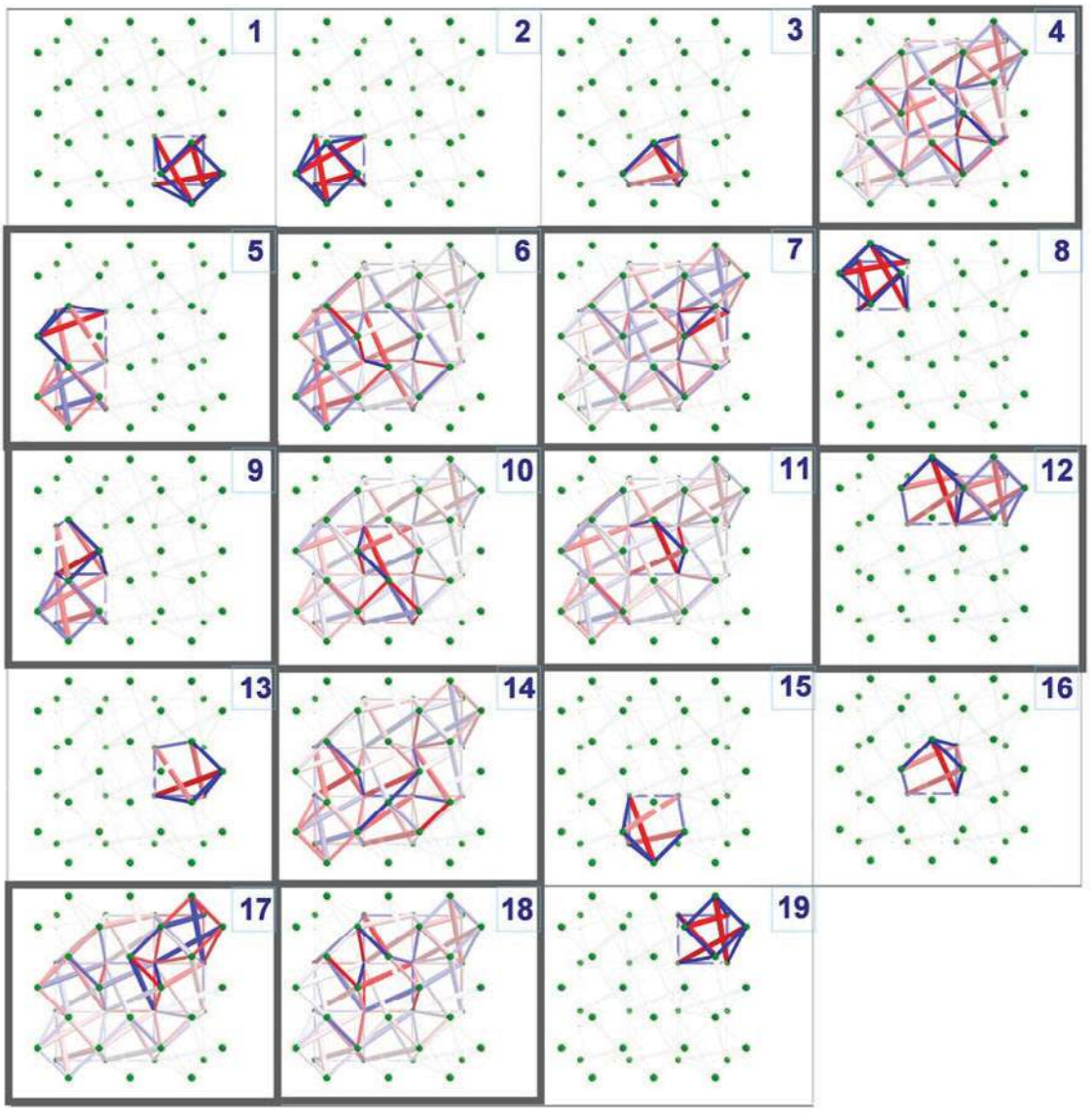

Figure 11. Initially computed basis [^{i}S] for a 3 × 3 quadruplex grid.

combination of these lines is then used to have null terms in the second line (Fig. 9). However, this operation is successful only if the number of zeros in the second line increases. In the opposite, another pair of coefficients is tested. The matrix is then reorganized to have the maximum number of zeros in the second line. The elimination process is then repeated between the second and the third lines. In the end of the process, a "reduced" matrix [^{r}S] is obtained. A "zero" value is defined on the basis of numerical tests that have shown that the best compromise is to consider a given number as null if its absolute value is smaller than 10^{-6}.

This approach is successful in the case of structures composed with a limited number of modules and, consequently, of selfstress states. The results show a practical limit close to $s < 15$, which is few (e.g. a 2 × 4 quadruplex grid). No mathematical evidence has been put forward yet to explain this limit. For larger structures, we have obtained satisfactory results by developing a specific "reduction method".

2.4.2 Reduction Method

This approach is based on the transformation of the initial matrix [^{i}S] (b rows and s columns, $b \times s$) to s "reduced" vectors $\{^{i}S_j\}$ (j = 1 to s), with a maximum of zero terms, by using appropriate vectors of coefficients $\{\alpha_j\}$:

$$\{^{r}S_j\} = [^{i}S]\{\alpha_j\} \quad (6)$$

Three steps are required and illustrated by the calculation of $\{\alpha_1\}$ and $[^{r}S_1]$:

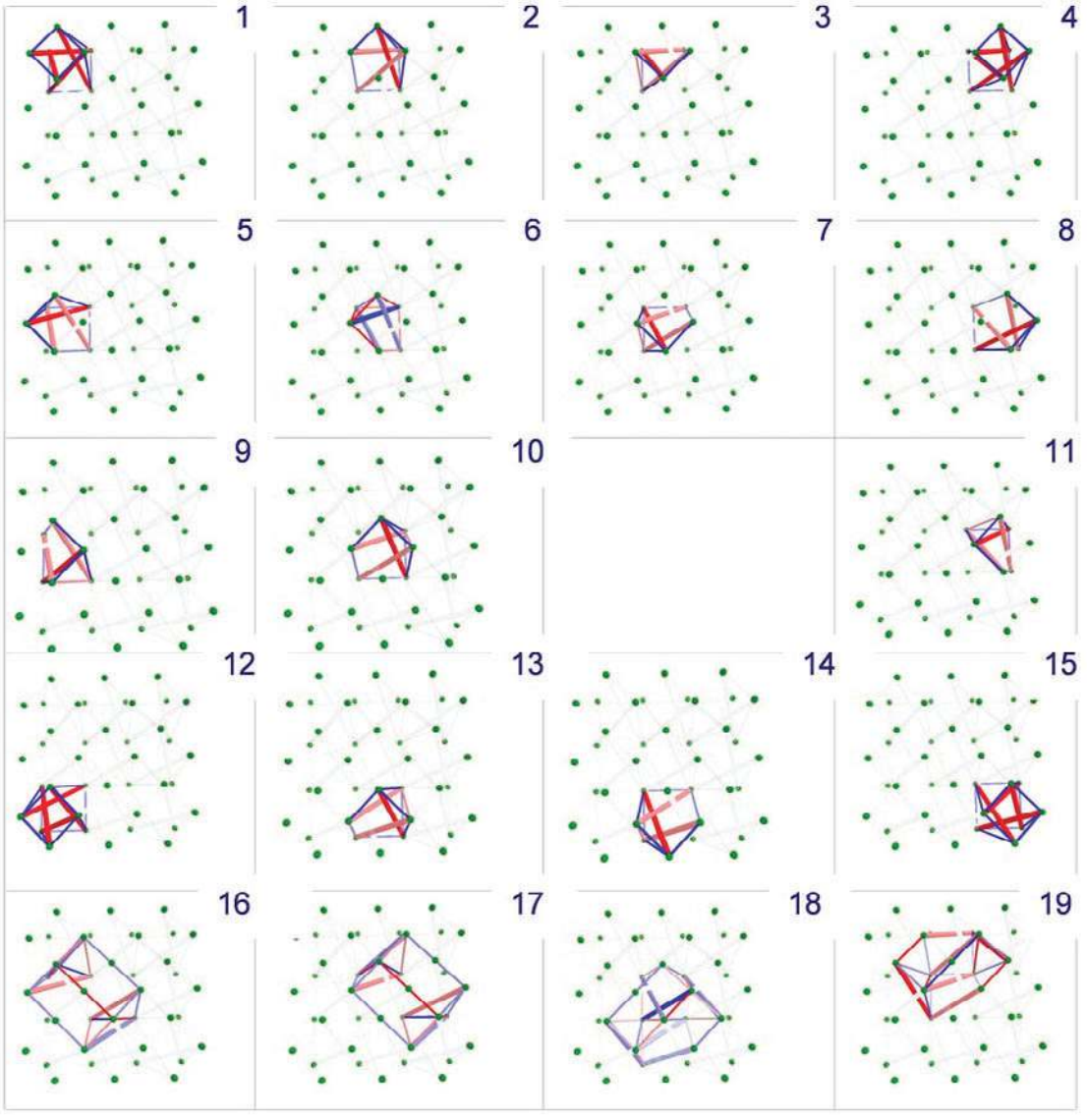

Figure 12. Reduced basis $[^{r}\mathrm{S}]$ for a 3×3 quadruplex grid.

- First, the matrix $[^{i}\mathrm{S}]$ is transformed into an auxiliary matrix $[^{a}\mathrm{S}](b' \times s)$ built by selecting the lines which are not multiple. For instance, the first line $^{i}\mathrm{S}_1$ is compared to another line k which is eliminated if $^{i}\mathrm{S}_k = \lambda\,{}^{i}\mathrm{S}_1$ (multiple lines, λ real number). The matrix $[^{a}\mathrm{S}]$ is then organized with the line with the fewest multiples at the top, and the line with the most multiples at the bottom.
- A square matrix $[^{s}\mathrm{S}](s \times s)$ is then extracted from $[^{a}\mathrm{S}]$ by selecting the first line (with the fewest multiples) and $(s - 1)$ independent ones among the other lines. It should, however, be ensured that the rank of $[^{s}\mathrm{S}]$ is equal to s.
- The vector $\{\alpha_1\}(1 \times s)$ is thus calculated according to

$$[^{s}\mathrm{S}]\{\alpha_1\} = \{\mathrm{I}_1\} \quad \text{with } \mathrm{I}_{11} = 1 \text{ and } \mathrm{I}_{1k} = 0 \text{ for } 1 > k \geq s \ (1^{\text{st}} \text{ column of the identity matrix}). \tag{7}$$

It comes

$$\{\alpha_1\} = [^{s}\mathrm{S}]^{-1}\ \{\mathrm{I}_1\} \tag{8}$$

and the searched first vector is

$$[^{r}\mathrm{S}_1] = [^{i}\mathrm{S}]\{\alpha_1\} = [^{i}\mathrm{S}][^{s}\mathrm{S}]^{-1}\{\mathrm{I}_1\}.$$

This process is repeated for $j = 2$ to s, by considering the vector $\{\mathrm{I}_j\}$ as the j^{th} column of the $s \times s$ identity matrix. The s obtained vectors generate therefore the reduced basis $[^{r}\mathrm{S}]$. It is hence is equal to

$$\left[{}^{r}\mathrm{S}\right] = \left[{}^{i}\mathrm{S}\right]\left[{}^{s}\mathrm{S}\right]^{-1}. \tag{9}$$

Figure 13. Selfstress classification.

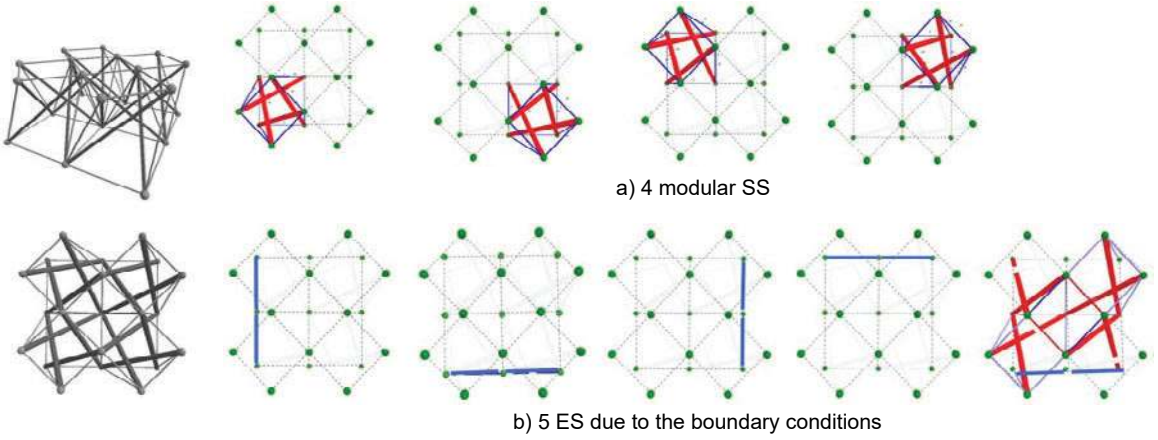

Figure 14. Selfstresses and exostress in a 2 × 2 quadruplex grid.

2.5 Application

An example is proposed for a 3 × 3 quadruplex grid with 19 selfstress states (Fig. 10). The computation of the initial basis with a software leads to the 19 vectors represented in Fig. 11. We observe that some of them (1, 2, 3, 8, 13, 15, 16, 19) are convenient since few elements are involved. However, the other selfstress states are more difficult to understand and to exploit. The basis [^{r}S] obtained after transformation of [^{i}S] with the reduction method is represented in Fig. 12. All the selfstress states are then optimally localized and accessible to further analysis:

- The 9 repetitive selfstress states associated to the modules (1, 2, 4, 5, 8, 10, 12, 14, 15);
- The 6 repetitive selfstress states of the 1 × 3 beams (3, 6, 7, 9, 11, 13);
- The 4 repetitive selfstress states associated to the 2 × 3 grids (16, 17, 18, 19).

The identification of repetitive selfstress states is, however, coincidental and not controlled. Their localization is not always ensured for larger grids. However, the results obtained with the method we suggest are more convenient and diagnostic compared with those given by the initial basis [^{i}S].

3. INITIAL STRESSES CLASSIFICATION

On the basis of the results shown above, we propose a classification of the selfstress states (Fig. 13):

- If a selfstress state matches with the mechanical behavior of the elements, that is to say tractions for cables and compressions for struts, it is called *compatible*, or *non-compatible* in the opposite case.
- If a selfstress state involves all the components of the structure, it is called *total*, or *partial* if at least one element is not concerned.
- For a partial selfstress state, three cases occur: 1) In the case of a sub-domain defined for generating a self-equilibrate and stable system, a partial selfstress state is called *modular.* 2) If the sub-domain is not self-equilibrated and stable, such a state is called *restricted*. 3) If the partial selfstress state involves several components diffusely distributed within the system, it is called *diffuse*.

These definitions only apply to selfstress states. However, initial forces in a tensegrity system can also be a consequence to its anchorage on supports. The boundary conditions may indeed generate additional initial stresses even if no external action is considered. In the case of a non isostatic anchoring, forces in some elements can occur in association with reactions at the supports. We prefer to distinguish these initial stresses and to call them *exostress states* (ES). They are not self-equilibrated and require non isostatic boundary conditions.

For instance, a 2 × 2 quadruplex grid has 4 four modular selfstress states when it is not fixed (Fig. 14, a).

However, if the nodes at the corners of the lower plane are pinned on the supports, then five additional exostresses can appear. Four of them are in this plane and involve pairs of aligned cables connecting two supports; the fifth one is more complex and concerns numerous components spatially distributed in the grid (Fig. 14, b).

The distinction between selfstresses and exostresses is important for further developments. The writing of the vectorial basis [S] could be indeed perturbed by the additional exostresses if non isostatic anchoring conditions are considered. In this case, the engineer should be able to distinguish them for an accurate analysis. In the presented example, such splitting is not difficult. Nevertheless, further developments are required to identify efficiently selfstresses and exostresses in general.

4. CONCLUSIONS

The computation of the vectorial basis of selfstress states in a tensegrity structure generally does not provide satisfactory results. Hence, two strategies for the identification and localization of selfstress states in a modular tensegrity grid are proposed. The first approach uses the regularity and repetitiveness of the components; several results are presented for a quadruplex grid. The second strategy is based on a maximum of zero terms in the writing of the selfstress vectors. Two methods to reduce the number of the components involved are presented with illustrative examples. Consequently, a classification of selfstress states is proposed with reference to the number and localization of the involved components and compatibility with their mechanical behavior. In case of a partial selfstress state, localization may be labeled *modular*, *restricted* or *diffuse*. The difference between selfstresses and exostresses due to boundary conditions is brought forward.

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