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New estimates for the div-curl-grad operators and elliptic problems with L^1-data in the whole space and in the half-space

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Abstract

In this paper, we study the div-curl-grad operators and some elliptic problems in the whole space $\mathbb{R}^n$ and in the half-space $\mathbb{R}^n_+$, with $n \geq 2$. We consider data in weighted Sobolev spaces and in L^1.

Keywords: div-curl-grad operators, elliptic, half-space, weighted Sobolev spaces
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1. Introduction

The purpose of this paper is to present new results concerning the div-curl-grad operators and some elliptic problems in the whole space and in the half-space with data and solutions which live in L^1 or in weighted Sobolev spaces, expressing at the same time their regularity and their behavior at infinity. Recently, new estimates for L^1-vector field have been discovered by Bourgain, Brézis and Van Schaftingen (see [23], [10], [11], [12], [13], [15]) which yield in particular improved estimates for the solutions of elliptic systems in $\mathbb{R}^n$ or in a bounded domain $\Omega \subset \mathbb{R}^n$. Our work presented in this paper is naturally based on these very interesting results and our approach rests on the use of weighted Sobolev spaces.
This paper is organised as follows. In this section, we introduce some notations and the functional framework. Some results concerning the weighted Sobolev spaces and the spaces of traces are recalled. In Section 2 and Section 3, our work is focused on the div-grad-curl operators and elliptic problems in the whole space. After the case of the whole space, we then pass to the one of the half-space. Results in the half-space are presented in Section 4 (The div-grad opera-

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tors), Section 5 (Vector potentials) and in the last section of this paper (Elliptic problems).

In this paper, we use bold type characters to denote vector distributions or spaces of vector distributions with n components and $C > 0$ usually denotes a generic positive constant that may depends on the dimension n, the exponent p and possibly other parameters, but never on the functions under consideration. For any real number $1 < p < \infty$, we take p' to be the Hölder conjugate of p. Let Ω be an open subset in the n-dimensional real euclidean space. A typical point $\boldsymbol{x} \in \mathbb{R}^n$ is denoted by $\boldsymbol{x} = (\boldsymbol{x}', x_n)$, where $\boldsymbol{x}' = (x_1, x_2, ..., x_{n-1}) \in \mathbb{R}^{n-1}$ and $x_n \in \mathbb{R}$. Its distance to the origine is denoted by $r = |\boldsymbol{x}| = (x_1^2 + ... + x_n^2)^{1/2}$. Let $\overline{\mathbb{R}^n_+}$ denote the closure of the upper half-space $\mathbb{R}^n_+ = \{\boldsymbol{x} \in \mathbb{R}^n;\ x_n > 0\}$. In the half-space, its boundary is defined by $\Gamma \;=\; \{\, \boldsymbol{x} \in \mathbb{R}^n;\ x_n = 0 \,\} \;\equiv\; \mathbb{R}^{n-1}$. In order to control the behavior at infinity of our functions and distributions, we use for basic weight the quantity $\rho = \rho(r) = 1 + r$, which is equivalent to r at infinity. We define $\mathcal{D}(\Omega)$ to be the linear space of infinite differentiable functions with compact support on Ω. Now, let $\mathcal{D}'(\Omega)$ denote the dual space of $\mathcal{D}(\Omega)$, the space of distributions on Ω. For any $q \in \mathbb{N}$, $\mathscr{P}_q$ stands for the space of polynomials of degree $\leq q$. If q is strictly negative integer, we set by convention $\mathscr{P}_q = \{0\}$. Given a Banach space B, with dual space B' and a closed subspace X of B, we denote by $B' \perp X$ (or more simply $X^\perp$, if there is no ambiguity as to the duality product) the subspace of B' orthogonal to X, *i.e.*

$$B' \perp X = X^\perp = \{f \in B' |\ \forall\ v \in X,\ < f, v > = 0\} = (B/X)'.$$

The space $X^\perp$ is called the polar space of X in B' and is also denoted by X°. We also introduce the space

$$\boldsymbol{\mathcal{V}}(\Omega) \;=\; \{\, \boldsymbol{\varphi} \in \boldsymbol{\mathcal{D}}(\Omega),\ \operatorname{div} \boldsymbol{\varphi} = 0 \,\}.$$

In this paper, we want to consider some particular weighted Sobolev spaces (see [4], [5]). The open set Ω will be denote the whole space or the half-space. We begin by defining the space

$$W_0^{1,p}(\Omega) = \{u \in \mathcal{D}'(\Omega), \frac{u}{w_1} \in L^p(\Omega), \nabla u \in \mathbf{L}^p(\Omega)\},$$

where

$$w_1 = 1 + r \quad \text{if } p \neq n \quad \text{and} \quad w_1 = (1 + r)\ \ln(2 + r) \quad \text{if } p = n.$$

This space is a reflexive Banach space when endowed with the norm:

$$||\, u||_{W_0^{1,p}(\Omega)} = (||\frac{u}{w_1}||^p_{L^p(\Omega)} + ||\nabla u\, ||^p_{\mathbf{L}^p(\Omega)})^{1/p}.$$

We also introduce the space

$$W_0^{2,p}(\Omega) = \{u \in \mathcal{D}'(\Omega),\ \frac{u}{w_2} \in L^p(\Omega),\ \frac{\nabla u}{w_1} \in \mathbf{L}^p(\Omega),\ D^2 u \in \mathbf{L}^p(\Omega)\},$$

where

$$w_2 = (1+r)^2 \ \text{ if } p \notin \{\frac{n}{2}, n\} \quad \text{and} \quad w_2 = (1+r)^2 \ \ln(2+r), \quad \text{otherwise},$$

which is a Banach space endowed with its natural norm given by

$$||\, u||_{W_0^{2,p}(\Omega)} = (||\frac{u}{w_2}||^p_{L^p(\Omega)} + ||\frac{\nabla u}{w_1}||^p_{\mathbf{L}^p(\Omega)} + ||D^2 u||^p_{\mathbf{L}^p(\Omega)})^{1/p}.$$

We need to give also the definition of the following space

$$W^{1,p}_{-1}(\Omega) = \{u \in \mathcal{D}'(\Omega),\ \frac{u}{w_3} \in L^p(\Omega),\ \frac{\nabla u}{1+r} \in \mathbf{L}^p(\Omega)\},$$

where

$$w_3 = (1+r)^2 \ \text{ if } p \neq n/2 \quad \text{and} \quad w_3 = (1+r)^2 \ \ln(2+r), \quad \text{otherwise}.$$

This space is also a reflexive Banach space and we can show that $W_0^{2,p}(\Omega) \hookrightarrow W^{1,p}_{-1}(\Omega)$. Note that the logarithmic weight only appears if $p = n$ or $p = \frac{n}{2}$. From now on, when we write $W^{m,p}_\alpha(\Omega)$, it means that m, p and α are taken as in these above definitions of the weighted Sobolev spaces. It is also true for the generalized case of the weighted Sobolev spaces. The weights in these above definitions are chosen so that the corresponding space satisfies two properties. On the one hand, the space $\mathcal{D}(\overline{\Omega})$ is dense in $W^{m,p}_\alpha(\Omega)$. On the other hand, the following Poincar-type inequality holds in $W^{m,p}_\alpha(\Omega)$ (see [4], [5] and [6]). The semi-norm

$$|\,.\,|_{W^{m,p}_\alpha(\Omega)} \ = \ (\sum_{|\boldsymbol{\lambda}|=m} ||(1+r)^\alpha \partial^{\boldsymbol{\lambda}} u||^p_{L^p(\Omega)})^{1/p}$$

defines on $W^{m,p}_\alpha(\Omega)/\mathscr{P}_{q^*}$ a norm which is equivalent to the quotient norm,

$$\forall u \in W^{m,p}_\alpha(\Omega), \quad ||u||_{W^{m,p}_\alpha(\Omega)/\mathscr{P}_{q^*}} \leq C\, |u|_{W^{m,p}_\alpha(\Omega)}, \tag{1}$$

with $q^* = \inf(q, m-1)$, where q is the highest degree of the polynomials contained in $W^{m,p}_\alpha(\Omega)$. We define the space

$$\overset{\circ}{W}{}^{m,p}_\alpha(\Omega) = \overline{\mathcal{D}(\Omega)}^{||\,\cdot\,||_{W^{m,p}_\alpha(\Omega)}},$$

and its dual space, $W^{-m,p'}_{-\alpha}(\Omega)$, is a space of distributions. In addition, the semi-norm $|\,.\,|_{W^{m,p}_\alpha(\Omega)}$ is a norm on $\overset{\circ}{W}{}^{m,p}_\alpha(\Omega)$ that is equivalent to the full norm $||\,.\,||_{W^{m,p}_\alpha(\Omega)}$:

$$\forall u \in \overset{\circ}{W}{}^{m,p}_\alpha(\Omega), \quad ||u||_{W^{m,p}_\alpha(\Omega)} \leq C\, |u|_{W^{m,p}_\alpha(\Omega)}. \tag{2}$$

When $\Omega = \mathbb{R}^n$, we have $W^{m,p}_\alpha(\mathbb{R}^n) = \overset{\circ}{W}{}^{m,p}_\alpha(\mathbb{R}^n)$. We will now recall some properties of the weighted Sobolev spaces $W^{m,p}_\alpha(\Omega)$. All the local properties of $W^{m,p}_\alpha(\Omega)$ coincide with those of the classical Sobolev space $W^{m,p}(\Omega)$. A quick computation shows that for $m \geq 0$ and if $\dfrac{n}{p} + \alpha$ does not belong to

$\{i \in \mathbb{Z};\ i \leq m\}$, then $\mathscr{P}_{[m-n/p-\alpha]}$ is the space of all polynomials included in $W^{m,p}_{\alpha}(\mathbb{R}^n)$ (for $s \in \mathbb{R}$, $[s]$ stands for the integer part of s). For all $\boldsymbol{\lambda} \in \mathbb{N}^n$ where $0 \leq |\lambda| \leq 2m$ with $m = 1$ or $m = 2$, the mapping

$$u \in W^{m,p}_{\alpha}(\Omega) \to \partial^{\boldsymbol{\lambda}} u \in W^{m-|\boldsymbol{\lambda}|,p}_{\alpha}(\Omega)$$

is continuous. Recall the following Sobolev embeddings (see [1]):

$$W^{1,p}_0(\Omega) \hookrightarrow L^{p^*}(\Omega) \ \text{ where } \ p^* = \frac{np}{n-p} \ \text{ and } \ 1 < p < n.$$

Also recall

$$W^{1,n}_0(\mathbb{R}^n) \hookrightarrow BMO(\mathbb{R}^n).$$

The space BMO is defined as follows: A locally integrable function f belongs to BMO if

$$||\, f\, ||_{BMO} \ =: \sup_Q \frac{1}{|Q|} \int_Q |f(\boldsymbol{x}) - f_Q| d\boldsymbol{x} < \infty,$$

where the supremum is taken on all the cubes and $f_Q = \frac{1}{|Q|} \int_Q f(\boldsymbol{x})\, d\boldsymbol{x}$ is the average of $\boldsymbol{f}$ on Q. In the literature, we also find studies using the following spaces:

$$\widehat{W}^{1,p}(\Omega) \,=\, \overline{\mathcal{D}(\Omega)}^{\,||\,\nabla \cdot\,||_{\mathbf{L}^p}} \ \text{ and } \ \widehat{W}^{2,p}(\Omega) \,=\, \overline{\mathcal{D}(\Omega)}^{\,||\,\nabla^2 \cdot\,||_{\mathbf{L}^p}}.$$

In fact, when $\Omega = \mathbb{R}^n$ we can prove that $\widehat{W}^{1,p}(\mathbb{R}^n) = W^{1,p}_0(\mathbb{R}^n)$ if $1 < p < n$ and $\widehat{W}^{2,p}(\mathbb{R}^n) = W^{2,p}_0(\mathbb{R}^n)$ if $1 < p < n/2$. When $n/p - m \leq 0$ with $m = 1$ or $m = 2$, $\widehat{W}^{m,p}(\mathbb{R}^n)$ is not a space of distributions. For instance, in J. Deny, J. L. Lions [16], they show that $\widehat{W}^{1,2}(\mathbb{R}^2)$ is not a space of distributions. Without going into details, let (φ_ν) be a sequence of functions of $\mathcal{D}(\mathbb{R}^2)$ such that $||\nabla \varphi_\nu||_{\mathbf{L}^2(\mathbb{R}^2)}$ is a Cauchy sequence. Applying Proposition 9.3 [4], there exists a constant c_ν such that

$$\inf_{c \in \mathbb{R}} ||\, \varphi_\nu + c\, ||_{W^0_{-1,-1}(\mathbb{R}^2)} \ = \ ||\, \varphi_\nu + c_\nu\, ||_{W^0_{-1,-1}(\mathbb{R}^2)}$$

is also a Cauchy sequence and therefore converges. But this does not mean that φ_ν alone converges and from [4], $||\nabla \varphi_\nu||_{\mathbf{L}^2(\mathbb{R}^2)}$ tends to zero while $< \varphi_\nu, \psi >$ tends to infinity for many ψ of $\mathcal{D}(\mathbb{R}^2)$ instead of converging to a constant times the mean value of ψ. These considerations suggest that the space $\widehat{W}^{1,2}(\mathbb{R}^2)$ lacks the constant functions and is not a space of distributions.

In order to define the traces of functions of $W^{m,p}_{\alpha}(\mathbb{R}^n_+)$, we introduce for any $\sigma \in (0, 1)$ the space

$$W^{\sigma,p}_0(\mathbb{R}^n) \ = \ \{\, u \in \mathcal{D}'(\mathbb{R}^n); \quad w^{-\sigma} u \in L^p(\mathbb{R}^n) \ \text{ and } \int_{\mathbb{R}^n \times \mathbb{R}^n} \frac{|u(\boldsymbol{x}) - u(\boldsymbol{y})|^p}{|\boldsymbol{x} - \boldsymbol{y}|^{n+\sigma p}} d\boldsymbol{x}\, d\boldsymbol{y} < \infty\, \},$$

where

$$w = \rho \quad \text{if} \quad \frac{n}{p} \neq \sigma \quad \text{and} \quad w = \rho (\ln{(1+\rho)})^{1/\sigma} \quad \text{if} \quad \frac{n}{p} = \sigma.$$

It is a reflexive Banach space equipped with its natural norm

$$||u||_{W_0^{\sigma,p}(\mathbb{R}^n)} \; = \; \left(\left|\left| \frac{u}{w^\sigma} \right|\right|^p_{L^p(\mathbb{R}^n)} + \int_{\mathbb{R}^n \times \mathbb{R}^n} \frac{|u(\boldsymbol{x}) - u(\boldsymbol{y})|^p}{|\boldsymbol{x} - \boldsymbol{y}|^{n+\sigma p}} d\boldsymbol{x}\, d\boldsymbol{y} \right)^{1/p}.$$

Similarly, for any real number $\alpha \in \mathbb{R}$, we define the space

$$\begin{aligned} W_\alpha^{\sigma,p}(\mathbb{R}^n) \; = \; \{\, u \in \mathcal{D}'(\mathbb{R}^n); \quad & w^{\alpha-\sigma} u \in L^p(\mathbb{R}^n) \text{ and} \\ & \int_{\mathbb{R}^n \times \mathbb{R}^n} \frac{|\rho^\alpha(\boldsymbol{x}) u(\boldsymbol{x}) - \rho^\alpha(\boldsymbol{y}) u(\boldsymbol{y})|^p}{|\boldsymbol{x} - \boldsymbol{y}|^{n+\sigma p}} d\boldsymbol{x}\, d\boldsymbol{y} < \infty \,\}, \end{aligned}$$

where

$$w = \rho \quad \text{if} \quad \frac{n}{p} + \alpha \neq \sigma \quad \text{and} \quad w = \rho(\ln(1+\rho))^{1/(\sigma-\alpha)} \quad \text{if} \quad \frac{n}{p} + \alpha = \sigma.$$

For any $s \in \mathbb{R}^+$, we set

$$\begin{aligned} W_\alpha^{s,p}(\mathbb{R}^n) \; = \; \{\, & u \in \mathcal{D}'(\mathbb{R}^n);\; 0 \leq |\boldsymbol{\lambda}| \leq k,\; \rho^{\alpha-s+|\boldsymbol{\lambda}|}(\ln(1+\rho))^{-1} \partial^{\boldsymbol{\lambda}} u \in L^p(\mathbb{R}^n); \\ & k+1 \leq |\boldsymbol{\lambda}| \leq [s]-1,\; \rho^{\alpha-s+|\boldsymbol{\lambda}|} \partial^{\boldsymbol{\lambda}} u \in L^p(\mathbb{R}^n);\; |\boldsymbol{\lambda}| = [s], \partial^{\boldsymbol{\lambda}} u \in W_\alpha^{\sigma,p}(\mathbb{R}^n) \,\}, \end{aligned}$$

where

$$k = k(s,n,p,\alpha) = \begin{cases} s - \dfrac{n}{p} - \alpha & \text{if } \dfrac{n}{p} + \alpha \in \{\, \sigma, ..., \sigma + [s] \,\}, \\ -1, & \text{otherwise}, \end{cases}$$

with $\sigma = s - [s]$. It is a reflexive Banach space equipped with the norm

$$\begin{aligned} ||u||_{W_\alpha^{s,p}(\mathbb{R}^n)} \; = \; (& \sum_{0 \leq |\boldsymbol{\lambda}| \leq k} ||\rho^{\alpha-s+|\boldsymbol{\lambda}|}(\ln(1+\rho))^{-1} \partial^{\boldsymbol{\lambda}} u||^p_{L^p(\mathbb{R}^n)} \\ + & \sum_{k+1 \leq |\boldsymbol{\lambda}| \leq [s]-1} ||\rho^{\alpha-s+|\boldsymbol{\lambda}|} \partial^{\boldsymbol{\lambda}} u||^p_{L^p(\mathbb{R}^n)})^{1/p} + \sum_{|\boldsymbol{\lambda}|=[s]} ||\partial^{\boldsymbol{\lambda}} u||_{W_\alpha^{\sigma,p}(\mathbb{R}^n)}. \end{aligned}$$

We notice that this definition coincides with the previous definition of the weighted Sobolev spaces when $s = m$ is a nonnegative integer. If u is a function on $\mathbb{R}^n_+$, we denote its trace of order j on the hyperplane Γ by

$$\forall j \in \mathbb{N}, \quad \gamma_j u : \; \boldsymbol{x}' \longmapsto \frac{\partial^i u}{\partial x_n^j}(\boldsymbol{x}', 0).$$

Finally, we recall the following traces lemma due to Hanouzet [19] and extended by Amrouche-Nečasová [6] to this class of weighted Sobolev spaces.

Lemma 1.1. *The mapping*

$$\gamma = (\gamma_0, \gamma_1, ..., \gamma_{m-1}) : \; \mathcal{D}(\overline{\mathbb{R}^n_+}) \to \prod_{j=0}^{m-1} \mathcal{D}(\mathbb{R}^{n-1}),$$

can be extended to a linear continuous mapping, still denoted by γ*,*

$$\gamma : \; W_\alpha^{m,p}(\mathbb{R}^n_+) \to \prod_{j=0}^{m-1} W_\alpha^{m-j-1/p,p}(\mathbb{R}^{n-1}).$$

Moreover, γ *is surjective and* $\operatorname{Ker}\gamma = \overset{\circ}{W}{}_\alpha^{m,p}(\mathbb{R}^n_+)$.

In order to finish this section, we recall the definition of curl operator. When $n = 2$, we define the curl operator for distributions $\varphi \in \mathcal{D}'(\Omega)$ and $\boldsymbol{v} \in \boldsymbol{\mathcal{D}'}(\Omega)$ by

$$\mathbf{curl}\,\varphi = (\frac{\partial \varphi}{\partial x_2}, -\frac{\partial \varphi}{\partial x_1}) \quad \text{and} \quad \operatorname{curl} \boldsymbol{v} = \frac{\partial v_2}{\partial x_1} - \frac{\partial v_1}{\partial x_2}.$$

When $n = 3$, we define the curl operator of a distribution $\boldsymbol{v} \in \boldsymbol{\mathcal{D}'}(\Omega)$ as follows

$$\mathbf{curl}\,\boldsymbol{v} = (\frac{\partial v_3}{\partial x_2} - \frac{\partial v_2}{\partial x_3}, \frac{\partial v_1}{\partial x_3} - \frac{\partial v_3}{\partial x_1}, \frac{\partial v_2}{\partial x_1} - \frac{\partial v_1}{\partial x_2}).$$

The following properties are easily obtained.

$$\operatorname{curl}(\mathbf{curl}\ \varphi) = -\Delta\varphi \quad \text{with } n = 2,$$

$$-\Delta \boldsymbol{v} + \nabla(\operatorname{div} \boldsymbol{v}) = \begin{cases} \mathbf{curl}\,(\mathbf{curl}\,\boldsymbol{v}) & \text{when} \;\; n = 3, \\ \mathbf{curl}(\operatorname{curl} \boldsymbol{v}) & \text{when} \;\; n = 2. \end{cases}$$

2. The div-grad operators in the whole space

The following proposition was given by J. Bourgain and H. Brézis [10]. We give here a detailed proof.

Proposition 2.1. *Let* $f \in L^n(\mathbb{R}^n)$*. Then there exists* $\boldsymbol{u} \in \mathbf{L}^\infty(\mathbb{R}^n)$ *such that*

$$\operatorname{div} \boldsymbol{u} = f \quad \text{with} \quad \|\, \boldsymbol{u} \|_{\mathbf{L}^\infty(\mathbb{R}^n)} \leq C_n \|\, f \,\|_{L^n(\mathbb{R}^n)}. \tag{3}$$

Proof. We consider the following unbounded operator

$$A = \nabla : L^{n/(n-1)}(\mathbb{R}^n) \to \mathbf{L}^1(\mathbb{R}^n),$$

with

$$D(A) = W_0^{1,1}(\mathbb{R}^n) = \{v \in L^{n/(n-1)}(\mathbb{R}^n), \nabla v \in \mathbf{L}^1(\mathbb{R}^n)\}.$$

It is easy to see that A is closed: if $v_n \to v$ in $L^{n/(n-1)}$ and $\nabla v_n \to \boldsymbol{z}$ in $\mathbf{L}^1(\mathbb{R}^n)$, then we have $\boldsymbol{z} = \nabla v$. On the other hand, $D(A)$ is dense in $L^{n/(n-1)}(\mathbb{R}^n)$. We know that for all $u \in D(A)$,

$$\|\, u \,\|_{L^{n/(n-1)}(\mathbb{R}^n)} \leq C \,\|\, \nabla u \,\|_{\mathbf{L}^1(\mathbb{R}^n)}. \tag{4}$$

Thanks to Theorem II.20 [14], the adjoint operator

$$A^* \; = \; -\operatorname{div} \; : \mathbf{L}^\infty(\mathbb{R}^n) \to L^n(\mathbb{R}^n)$$

is surjective, *i.e.*, for all $f \in L^n(\mathbb{R}^n)$, there exists $\boldsymbol{u} \in \mathbf{L}^\infty(\mathbb{R}^n)$ such that $\operatorname{div} \boldsymbol{u} = f$. Then from Theorem II.5 [14], there exists $c > 0$ such that

$$\overline{\operatorname{div} B_{\mathbf{L}^\infty}(0,1)} \supset B_{L^n}(0, c), \tag{5}$$

where $B_E(0, \alpha)$ is the open ball in E of radius $\alpha > 0$ centered at origin. Let now $f \in L^n(\mathbb{R}^n)$ satisfying $f \neq 0$ and set

$$h \,=\, c \, \frac{f}{||\, f \,||_{L^n(\mathbb{R}^n)}}.$$

Therefore, we can deduce from (5) the existence of $\boldsymbol{v}_n$ in $\mathbf{L}^\infty(\mathbb{R}^n)$ satisfying $||\boldsymbol{v}_n||_{\mathbf{L}^\infty(\mathbb{R}^n)} \leq 1$ such that

$$\operatorname{div} \boldsymbol{v}_n \to h \;\text{ in } L^n(\mathbb{R}^n).$$

We can then extract a subsequence $(\boldsymbol{v}_{n_k})$ such that $\boldsymbol{v}_{n_k} \overset{*}{\rightharpoonup} \boldsymbol{v}$ in $\mathbf{L}^\infty(\mathbb{R}^n)$ with $||\, \boldsymbol{v} \,||_{\mathbf{L}^\infty(\mathbb{R}^n)} \leq 1$ and $h = \operatorname{div} \boldsymbol{v}$. Hence, we obtain the property (3) with $\boldsymbol{u} = \dfrac{1}{c} ||\, f \,||_{L^n(\mathbb{R}^n)} \boldsymbol{v}$. □

Remark 1. Proposition 2.1 can be improved by showing that $\boldsymbol{u}$ actually belongs to $\mathbf{W}^{1,n}_0(\mathbb{R}^n) \cap \mathbf{L}^\infty(\mathbb{R}^n)$ (see Theorem 2.3). Note that $W^{1,n}_0(\mathbb{R}^n) \hookrightarrow BMO(\mathbb{R}^n)$, but the corresponding embedding in $L^\infty(\mathbb{R}^n)$ does not take place.

Recall now De Rham's Theorem: let Ω be any open subset of $\mathbb{R}^n$ and let $\boldsymbol{f}$ be a distribution of $\boldsymbol{\mathcal{D}}'(\Omega)$ that satisfies:

$$\forall \boldsymbol{v} \in \boldsymbol{\mathcal{V}}(\Omega), \quad < \boldsymbol{f}, \boldsymbol{v} >_{\boldsymbol{\mathcal{D}}'(\Omega) \times \boldsymbol{\mathcal{D}}(\Omega)} \,=\, 0. \tag{6}$$

Then there exists π in $\mathcal{D}'(\Omega)$ such that $\boldsymbol{f} = \nabla \pi$. In particular, if $\boldsymbol{f} \in \mathbf{W}^{-1,p}_0(\mathbb{R}^n)$ with $1 < p < \infty$ and satisfies

$$\forall \boldsymbol{v} \in \boldsymbol{\mathcal{V}}(\mathbb{R}^n), \quad < \boldsymbol{f}, \boldsymbol{v} >_{\mathbf{W}^{-1,p}_0(\mathbb{R}^n) \times \mathbf{W}^{1,p'}_0(\mathbb{R}^n)} \,=\, 0, \tag{7}$$

then there exists a unique $\pi \in L^p(\mathbb{R}^n)$ such that $\boldsymbol{f} = \nabla \pi$ and the following estimate holds

$$||\, \pi \,||_{L^p(\mathbb{R}^n)} \;\leq\; C \,||\, \boldsymbol{f} \,||_{\mathbf{W}^{-1,p}_0(\mathbb{R}^n)}.$$

Similarly, if $\boldsymbol{f} \in \mathbf{L}^p(\mathbb{R}^n)$, with $1 < p < \infty$, and satisfies

$$\forall \boldsymbol{v} \in \boldsymbol{\mathcal{V}}(\mathbb{R}^n), \quad \int_\Omega \boldsymbol{f} \cdot \boldsymbol{v} \;=\; 0, \tag{8}$$

then $\boldsymbol{f} = \nabla \pi$ with $\pi \in W^{1,p}_0(\mathbb{R}^n)$. Note that $\boldsymbol{\mathcal{V}}(\mathbb{R}^n)$ is dense in $\mathbf{H}_p(\mathbb{R}^n)$ for all $1 \leq p < \infty$ (see Alliot-Amrouche [2] for $p > 1$ and Miyakawa [20] for $p = 1$), but is not dense in $\mathbf{H}_\infty(\mathbb{R}^n)$, where for any $1 \leq p \leq \infty$,

$$\mathbf{H}_p(\mathbb{R}^n) = \{ \boldsymbol{v} \in \mathbf{L}^p(\mathbb{R}^n);\; \operatorname{div} \boldsymbol{v} = 0 \}.$$

As $\boldsymbol{\mathcal{V}}(\mathbb{R}^n)$ is dense in

$$\mathbf{V}_0^{1,p'}(\mathbb{R}^n) = \left\{ \boldsymbol{v} \in \mathbf{W}_0^{1,p'}(\mathbb{R}^n);\ \text{div}\ \boldsymbol{v} = 0 \right\},$$

then (7) is equivalent to

$$\forall \boldsymbol{v} \ \in\ \mathbf{V}_0^{1,p'}(\mathbb{R}^n), \quad < \boldsymbol{f}, \boldsymbol{v} >_{\mathbf{W}_0^{-1,p}(\mathbb{R}^n) \times \mathbf{W}_0^{1,p'}(\mathbb{R}^n)} \ =\ 0. \tag{9}$$

The same property holds for the relation (8) with $\boldsymbol{\mathcal{V}}(\mathbb{R}^n)$ replaced by $\mathbf{H}_{p'}(\mathbb{R}^n)$.

The following corollary gives an answer when $\boldsymbol{f} \in \mathbf{L}^1(\mathbb{R}^n)$.

Corollary 2.2. *Assume that* $\boldsymbol{f} \in \mathbf{L}^1(\mathbb{R}^n)$ *satisfying*

$$\forall \boldsymbol{v} \in \mathbf{H}_\infty(\mathbb{R}^n), \quad \int_{\mathbb{R}^n} \boldsymbol{f} \cdot \boldsymbol{v}\, d\boldsymbol{x} = 0.$$

Then there exists a unique $\pi \in L^{n/(n-1)}(\mathbb{R}^n)$ *such that* $\boldsymbol{f} = \nabla\pi$ *with the estimate*

$$||\, \pi\, ||_{L^{n/(n-1)}(\mathbb{R}^n)} \leq C ||\, \boldsymbol{f}\, ||_{\mathbf{L}^1(\mathbb{R}^n)}. \tag{10}$$

Proof. This corollary can be proved because from (4), we have $\operatorname{Im}\nabla$ is closed subspace of $\mathbf{L}^1(\mathbb{R}^n)$, so that $[\mathbf{H}_\infty(\mathbb{R}^n)]^\circ = (\text{Ker div})^\circ = \operatorname{Im}\nabla$, where

$$[\mathbf{H}_\infty(\mathbb{R}^n)]^\circ \ =\ \left\{ \boldsymbol{f} \in \mathbf{L}^1(\mathbb{R}^n),\ \int_{\mathbb{R}^n} \boldsymbol{f} \cdot \boldsymbol{v}\, d\boldsymbol{x} = 0,\ \forall \boldsymbol{v} \in \mathbf{H}_\infty(\mathbb{R}^n) \right\}.$$

Recall that if E is a Banach space and M a subspace of the dual E', then the polar (or the orthogonal) of M is defined as follows

$$M^\circ \ =\ \{\, f \in E;\ < f, v > = 0,\ \forall v \in M\, \}.$$

$\square$

Remark 2. Observe first that the hypothesis of Corollary 2.2 implies that $\boldsymbol{f} \in \mathbf{L}_0^1(\mathbb{R}^n)$, *i.e.*, $\boldsymbol{f} \in \mathbf{L}^1(\mathbb{R}^n)$ and $\int_{\mathbb{R}^n} \boldsymbol{f} = 0$. Next, note that the conclusion of the above corollary shows that $\boldsymbol{f} \in \mathbf{W}_0^{-1,n/(n-1)}(\mathbb{R}^n)$ and then $\operatorname{div} \boldsymbol{f} \in W_0^{-2,n/(n-1)}(\mathbb{R}^n)$. Moreover, we have

$$\forall \lambda \in \mathscr{P}_1, \quad < \operatorname{div} \boldsymbol{f}, \lambda >_{W_0^{-2,n/(n-1)}(\mathbb{R}^n) \times W_0^{2,n}(\mathbb{R}^n)} \ =\ 0. \tag{11}$$

Now recall a result in J. Bourgain - H. Brézis (cf. [11] or [12]).

Theorem 2.3. *For all* $f \in L^n(\mathbb{R}^n)$*, there exists* $\boldsymbol{w} \in \mathbf{W}_0^{1,n}(\mathbb{R}^n) \cap \mathbf{L}^\infty(\mathbb{R}^n)$ *such that* div $\boldsymbol{w} = f$ *and*

$$||\, \boldsymbol{w}\, ||_{\mathbf{W}_0^{1,n}(\mathbb{R}^n)} + ||\, \boldsymbol{w}\, ||_{\mathbf{L}^\infty(\mathbb{R}^n)} \ \leq\ C\, ||\, f\, ||_{L^n(\mathbb{R}^n)}.$$

Remark 3. Note that J. Bourgain and H. Brézis use the space $\widehat{W}^{1,n}(\mathbb{R}^n)$ (respectively, $\widehat{W}^{2,n}(\mathbb{R}^n)$), which is defined by the adherence of $\mathcal{D}(\mathbb{R}^n)$ for the norm $||\,\nabla.\,||_{\mathbf{L}^n(\mathbb{R}^n)}$ (respectively, the adherence of $\mathcal{D}(\mathbb{R}^n)$ for the norm $||\,\nabla^2.\,||_{\mathbf{L}^n(\mathbb{R}^n)}$) as their functional framework. Our choice is the weighted Sobolev space and it seems to us more adaptive (see the introduction in Section 1 for the explanation).

The second point of the following theorem is an extension of Corollary 2.2 and (7) with $p = n/(n-1)$.

Corollary 2.4. *i) There exists $C > 0$ such that for all $u \in L^{n/(n-1)}(\mathbb{R}^n)$, we have the following inequality*

$$||\,u\,||_{L^{n/(n-1)}(\mathbb{R}^n)} \;\leq\; C \inf_{\boldsymbol{g}+\boldsymbol{h}=\nabla u} (||\,\boldsymbol{g}\,||_{\mathbf{L}^1(\mathbb{R}^n)} + ||\,\boldsymbol{h}\,||_{\mathbf{W}_0^{-1,n/(n-1)}(\mathbb{R}^n)}) \tag{12}$$

with $\boldsymbol{g} \in \mathbf{L}^1(\mathbb{R}^n)$ and $\boldsymbol{h} \in \mathbf{W}_0^{-1,n/(n-1)}(\mathbb{R}^n)$.
ii) Let $\boldsymbol{f} \in \mathbf{L}^1(\mathbb{R}^n) + \mathbf{W}_0^{-1,n/(n-1)}(\mathbb{R}^n)$ satisfying the following compatibility condition

$$\forall \boldsymbol{v} \in \mathbf{V}_0^{1,n}(\mathbb{R}^n) \cap \mathbf{L}^\infty(\mathbb{R}^n), \quad < \boldsymbol{f}, \boldsymbol{v} > \,=\, 0. \tag{13}$$

Then there exists a unique $\pi \in L^{n/(n-1)}(\mathbb{R}^n)$ such that $\boldsymbol{f} = \nabla\pi$.

Proof. i) We consider two following operators

$$A = -\nabla : L^{n/(n-1)}(\mathbb{R}^n) \to \mathbf{L}^1(\mathbb{R}^n) + \mathbf{W}_0^{-1,n/(n-1)}(\mathbb{R}^n),$$

$$A^* = \mathrm{div} : \mathbf{W}_0^{1,n}(\mathbb{R}^n) \cap \mathbf{L}^\infty(\mathbb{R}^n) \to L^n(\mathbb{R}^n).$$

The rest of this proof is similar to the one of Proposition 2.1.
ii) The second point is a consequence of the first one. □

Remark 4. Remark that for all $u \in W_0^{1,1}(\mathbb{R}^n)$,

$$||\,u\,||_{L^{n/(n-1)}(\mathbb{R}^n)} \;\leq\; C\,||\,\nabla u\,||_{\mathbf{L}^1(\mathbb{R}^n)}, \tag{14}$$

and for all $u \in L^{n/(n-1)}(\mathbb{R}^n)$,

$$||\,u\,||_{L^{n/(n-1)}(\mathbb{R}^n)} \;\leq\; C\,||\,\nabla u\,||_{\mathbf{W}_0^{-1,n/(n-1)}(\mathbb{R}^n)}. \tag{15}$$

The inequality (14) is well-known. We now consider (15). It is shown in [4] that Δ is an isomorphism from $W_0^{2,n}(\mathbb{R}^n)/\mathscr{P}_1$ into $L^n(\mathbb{R}^n)$. By duality, we have

$$\Delta : \; L^{n/(n-1)}(\mathbb{R}^n) \to W_0^{-2,n/(n-1)}(\mathbb{R}^n) \perp \mathscr{P}_1$$

is also an isomorphism. Then for all $u \in L^{n/(n-1)}(\mathbb{R}^n)$, we can deduce

$$||\,u\,||_{L^{n/(n-1)}(\mathbb{R}^n)} \;\leq\; C\,||\,\Delta u\,||_{W_0^{-2,n/(n-1)}(\mathbb{R}^n)}. \tag{16}$$

Moreover, we can also see immediately that

$$\|\,\Delta u\,\|_{W_0^{-2,n/(n-1)}(\mathbb{R}^n)} \;=\; \|\,\mathrm{div}\,\nabla u\,\|_{W_0^{-2,n/(n-1)}(\mathbb{R}^n)}$$

and the following operator

$$\mathrm{div}:\; \mathbf{W}_0^{-1,n/(n-1)}(\mathbb{R}^n) \to W_0^{-2,n/(n-1)}(\mathbb{R}^n)$$

is continuous. Then

$$\|\,\Delta u\,\|_{W_0^{-2,n/(n-1)}(\mathbb{R}^n)} \;\leq\; C\,\|\,\nabla u\,\|_{\mathbf{W}_0^{-1,n/(n-1)}(\mathbb{R}^n)}$$

and we deduce easily (15). The inequality (12), stronger than (14) or (15) is especially interesting if $u \in L^{n/(n-1)}(\mathbb{R}^n)$ and $\nabla u \notin \mathbf{L}^1(\mathbb{R}^n)$.

Recall now the definition of Riesz transforms

$$R_j f \;=\; c_n\, p.v.\left(f * \frac{x_j}{|\boldsymbol{x}|^{n+1}}\right),\; j = 1, ..., n,$$

where $c_n \;=\; \Gamma\left(\frac{n+1}{2}\right)/\pi^{\frac{n+1}{2}}$. Also recall that their Fourier transforms satisfy $\widehat{R_j f} \;=\; i\dfrac{\xi_j}{|\xi|}\widehat{f}$.

The following corollary is proved in [11]. We give here a little different proof.

Corollary 2.5. *Assume that $F \in W_0^{1,n}(\mathbb{R}^n)$. Then there exists $\boldsymbol{Y} \in \mathbf{W}_0^{1,n}(\mathbb{R}^n) \cap \mathbf{L}^\infty(\mathbb{R}^n)$ such that*

$$F = \sum_{j=1}^n R_j Y_j.$$

Proof. Let $F \in W_0^{1,n}(\mathbb{R}^n)$ and define f by $\widehat{f}(\xi) = |\xi|\widehat{F}(\xi)$. Then we have $\widehat{R_j f}(\xi) = \widehat{\dfrac{\partial F}{\partial x_j}}(\xi)$. Therefore, $R_j f = \dfrac{\partial F}{\partial x_j} \in L^n(\mathbb{R}^n)$ for all $j = 1, ..., n$. Hence, $R_j R_j f \in L^n(\mathbb{R}^n)$ and we deduce $f = -\displaystyle\sum_{j=1}^n R_j R_j f \in L^n(\mathbb{R}^n)$. Thanks to Theorem 2.3, there exists $\boldsymbol{Y} \in \mathbf{W}_0^{1,n}(\mathbb{R}^n) \cap \mathbf{L}^\infty(\mathbb{R}^n)$ such that $f = \mathrm{div}\ \boldsymbol{Y}$, *i.e.*, $\widehat{f} = \displaystyle\sum_{j=1}^n i\xi_j Y_j$. Then, $\widehat{F} = \displaystyle\sum_{j=1}^n i\frac{\xi_j}{|\xi|}\widehat{Y_j}$, that means that $F = \displaystyle\sum_{j=1}^n R_j Y_j$. $\square$

An another result was established by J. Bourgain and H. Brézis [11] (see also H. Brézis and J. Van Schaftingen [15]).

Theorem 2.6. *Let Ω be a Lipschitz bounded open domain in $\mathbb{R}^n$.*
i) For all $\boldsymbol{\varphi} \in \mathbf{W}_0^{1,n}(\Omega)$, there exist $\boldsymbol{\psi} \in \mathbf{W}_0^{1,n}(\Omega) \cap \mathbf{L}^\infty(\Omega)$ and $\eta \in W_0^{2,n}(\Omega)$ such that

$$\boldsymbol{\varphi} = \boldsymbol{\psi} + \nabla\eta, \tag{17}$$

with the following estimate

$$\|\nabla \boldsymbol{\psi}\|_{\mathbf{L}^n(\Omega)} + \|\boldsymbol{\psi}\|_{\mathbf{L}^\infty(\Omega)} + \|D^2\eta\|_{\mathbf{L}^n(\Omega)} \leq C\,\|\nabla \boldsymbol{\varphi}\|_{\mathbf{L}^n(\Omega)}, \tag{18}$$

where C only depends on Ω.
ii) For all $\boldsymbol{\varphi} \in \mathbf{W}^{1,n}(\Omega)$, there exist $\boldsymbol{\psi} \in \mathbf{W}^{1,n}(\Omega) \cap \mathbf{L}^\infty(\Omega)$ and $\eta \in W^{2,n}(\Omega)$ such that (17) holds with $\boldsymbol{\psi} \cdot \boldsymbol{n} = 0$ on Γ and satisfying the following estimate

$$\|\boldsymbol{\psi}\|_{\mathbf{W}^{1,n}(\Omega)} + \|\boldsymbol{\psi}\|_{\mathbf{L}^\infty(\Omega)} + \|\eta\|_{\mathbf{W}^{2,n}(\Omega)} \leq C\,\|\boldsymbol{\varphi}\|_{\mathbf{W}^{1,n}(\Omega)}. \tag{19}$$

In the above theorem, $\mathbf{W}^{1,n}_0(\Omega)$ is the classical Sobolev space of functions in $\mathbf{W}^{1,n}(\Omega)$ vanishing on the boundary of Ω and $\mathbf{W}^{2,n}_0(\Omega)$ is the one of functions in $\mathbf{W}^{2,n}(\Omega)$ whose traces and the normal derivative are vanished on the boundary of Ω.

We now prove a similar result corresponding to weighted Sobolev spaces.

Corollary 2.7. *For all $\boldsymbol{\varphi} \in \mathbf{W}^{1,n}_0(\mathbb{R}^n)$, there exist $\boldsymbol{\psi} \in \mathbf{W}^{1,n}_0(\mathbb{R}^n) \cap \mathbf{L}^\infty(\mathbb{R}^n)$ and $\eta \in W^{2,n}_0(\mathbb{R}^n)$ such that*

$$\boldsymbol{\varphi} = \boldsymbol{\psi} + \nabla\eta,$$

with the following estimate

$$\|\boldsymbol{\psi}\|_{\mathbf{W}^{1,n}_0(\mathbb{R}^n)} + \|\boldsymbol{\psi}\|_{\mathbf{L}^\infty(\mathbb{R}^n)} + \|D^2\eta\|_{\mathbf{L}^n(\mathbb{R}^n)} \leq C\,\|\nabla \boldsymbol{\varphi}\|_{\mathbf{L}^n(\mathbb{R}^n)}. \tag{20}$$

Proof. Thanks to the density of $\boldsymbol{\mathcal{D}}(\mathbb{R}^n)$ in $\mathbf{W}^{1,n}_0(\mathbb{R}^n)$, there exists a sequence $(\boldsymbol{\varphi}_k)_{k\in\mathbb{N}^*}$ in $\boldsymbol{\mathcal{D}}(\mathbb{R}^n)$ that converges toward $\boldsymbol{\varphi}$ in $\mathbf{W}^{1,n}_0(\mathbb{R}^n)$. Let B_{r_k} be a ball such that $\operatorname{supp} \boldsymbol{\varphi}_k \subset B_{r_k}$ and we set $\boldsymbol{\varphi}'_k(\boldsymbol{x}) = \boldsymbol{\varphi}_k(r_k\boldsymbol{x})$. Then we deduce $\boldsymbol{\varphi}'_k \in \mathbf{W}^{1,n}_0(B_1)$. Applying Theorem 2.6, there exist $\boldsymbol{\psi}'_k \in \mathbf{W}^{1,n}_0(B_1) \cap \mathbf{L}^\infty(B_1)$ and $\eta'_k \in W^{2,n}_0(B_1)$ such that

$$\boldsymbol{\varphi}'_k = \boldsymbol{\psi}'_k + \nabla\eta'_k,$$

with the following estimate

$$\|\nabla \boldsymbol{\psi}'_k\|_{\mathbf{L}^n(B_1)} + \|\boldsymbol{\psi}'_k\|_{\mathbf{L}^\infty(B_1)} + \|D^2\eta'_k\|_{\mathbf{L}^n(B_1)} \leq C\,\|\nabla \boldsymbol{\varphi}'_k\|_{\mathbf{L}^n(B_1)}.$$

We now set

$$\boldsymbol{\psi}_k(\boldsymbol{x}) = \boldsymbol{\psi}'_k(\frac{\boldsymbol{x}}{r_k}) \quad \text{and} \quad \eta_k(\boldsymbol{x}) = r_k\,\eta'_k(\frac{\boldsymbol{x}}{r_k}).$$

Then we have

$$\boldsymbol{\varphi}_k = \boldsymbol{\psi}_k + \nabla\eta_k. \tag{21}$$

Moreover, since

$$\begin{aligned}
\|\nabla \boldsymbol{\psi}_k\|_{\mathbf{L}^n(\mathbb{R}^n)} &= \|\nabla \boldsymbol{\psi}'_k\|_{\mathbf{L}^n(B_1)},\\
\|\boldsymbol{\psi}_k\|_{\mathbf{L}^\infty(\mathbb{R}^n)} &= \|\boldsymbol{\psi}'_k\|_{\mathbf{L}^\infty(B_1)},\\
\|\nabla \boldsymbol{\varphi}_k\|_{\mathbf{L}^n(\mathbb{R}^n)} &= \|\nabla \boldsymbol{\varphi}'_k\|_{\mathbf{L}^n(B_1)},
\end{aligned}$$

we then have that

$$\|\nabla \boldsymbol{\psi}_k\|_{\mathbf{L}^n(\mathbb{R}^n)} + \|\boldsymbol{\psi}_k\|_{\mathbf{L}^\infty(\mathbb{R}^n)} \leq C\,\|\nabla \boldsymbol{\varphi}_k\|_{\mathbf{L}^n(\mathbb{R}^n)} \leq C\,\|\nabla \boldsymbol{\varphi}\|_{\mathbf{L}^n(\mathbb{R}^n)}. \tag{22}$$

Then there exists a sequence $(\boldsymbol{a}_k)$ in $\mathbb{R}^n$ such that $\boldsymbol{\psi}_k + \boldsymbol{a}_k$ is bounded in $\mathbf{W}_0^{1,n}(\mathbb{R}^n)$ and

$$\|\, \boldsymbol{\psi}_k + \boldsymbol{a}_k \,\|_{\mathbf{W}_0^{1,n}(\mathbb{R}^n)} \leq C \,\|\, \nabla \boldsymbol{\varphi} \,\|_{\mathbf{L}^n(\mathbb{R}^n)}. \tag{23}$$

As $\|\, \boldsymbol{\psi}_k \,\|_{\mathbf{L}^\infty(\mathbb{R}^n)}$ is also bounded, then the sequence $(\boldsymbol{a}_k)$ is bounded in $\mathbb{R}^n$. Then we can extract a subsequence, again denoted by $(\boldsymbol{a}_k)$, such that $\lim\limits_{k\to\infty} \boldsymbol{a}_k = \boldsymbol{a}$. We know that there exists $\boldsymbol{\psi}_0$ in $\mathbf{W}_0^{1,n}(\mathbb{R}^n)$ such that $\boldsymbol{\psi}_k + \boldsymbol{a}_k \rightharpoonup \boldsymbol{\psi}_0$ in $\mathbf{W}_0^{1,n}(\mathbb{R}^n)$ and

$$\|\, \boldsymbol{\psi}_0 \,\|_{\mathbf{W}_0^{1,n}(\mathbb{R}^n)} \leq C \,\|\, \nabla \boldsymbol{\varphi} \,\|_{\mathbf{L}^n(\mathbb{R}^n)}.$$

Then, we have $\boldsymbol{\psi}_k \rightharpoonup \boldsymbol{\psi}_0 - \boldsymbol{a}$ in $\mathbf{W}_0^{1,n}(\mathbb{R}^n)$. Moreover, $\boldsymbol{\psi}_k \overset{*}{\rightharpoonup} \boldsymbol{\psi}$ in $\mathbf{L}^\infty(\mathbb{R}^n)$, then it implies that $\boldsymbol{\psi} = \boldsymbol{\psi}_0 - \boldsymbol{a}$. In addition, we have the following estimate

$$\|\, \nabla \boldsymbol{\psi} \,\|_{\mathbf{L}^n(\mathbb{R}^n)} + \|\, \boldsymbol{\psi} \,\|_{\mathbf{L}^\infty(\mathbb{R}^n)} \leq C \,\|\, \nabla \boldsymbol{\varphi} \,\|_{\mathbf{L}^n(\mathbb{R}^n)}.$$

From (21), we can deduce that

$$\|\, D^2 \eta_k \,\|_{\mathbf{L}^n(\mathbb{R}^n)} \leq C \,\|\, \nabla \boldsymbol{\varphi} \,\|_{\mathbf{L}^n(\mathbb{R}^n)},$$

i.e., there exists $\alpha_k \in \mathscr{P}_1$ such that

$$\eta_k + \alpha_k \quad \text{is bounded in} \quad W_0^{2,n}(\mathbb{R}^n), \tag{24}$$

and there exists η_0 in $W_0^{2,n}(\mathbb{R}^n)$ such that $\eta_k + \alpha_k \rightharpoonup \eta_0$ in $W_0^{2,n}(\mathbb{R}^n)$. As $\boldsymbol{\varphi}_k$ and $\boldsymbol{\psi}_k$ are bounded in $\mathbf{W}_0^{1,n}(\mathbb{R}^n)$, it is also true for $\nabla \eta_k$, then from (24), we deduce that $\nabla \alpha_k$ is bounded in $\mathbf{W}_0^{1,n}(\mathbb{R}^n)$. Therefore, there exists a real sequence b_k such that $\alpha_k + b_k$ is bounded in $W_0^{2,n}(\mathbb{R}^n)$. Consequently, the sequence $\eta_k - b_k$ is bounded in $W_0^{2,n}(\mathbb{R}^n)$ and we can extract a subsequence, denoted in the same way, such that $\eta_k - b_k \rightharpoonup \eta$ in $W_0^{2,n}(\mathbb{R}^n)$. Then we have

$$\boldsymbol{\varphi}_k = \boldsymbol{\psi}_k + \nabla(\eta_k - b_k)$$

with the estimate

$$\|\, D^2(\eta_k - b_k) \,\|_{\mathbf{L}^n(\mathbb{R}^n)} \leq C \,\|\, \nabla \boldsymbol{\varphi} \,\|_{\mathbf{L}^n(\mathbb{R}^n)}.$$

We pass to limite in the above decomposition, we shall obtain $\boldsymbol{\varphi} = \boldsymbol{\psi} + \nabla \eta$ with

$$\|\, \nabla \boldsymbol{\psi} \,\|_{\mathbf{L}^n(\mathbb{R}^n)} + \|\, \boldsymbol{\psi} \,\|_{\mathbf{L}^\infty(\mathbb{R}^n)} + \|\, D^2 \eta \,\|_{\mathbf{L}^n(\mathbb{R}^n)} \leq C \,\|\, \nabla \boldsymbol{\varphi} \,\|_{\mathbf{L}^n(\mathbb{R}^n)},$$

and then we deduce (20). □

We have another version of Corollary 2.7.

Theorem 2.8. *For all $\boldsymbol{\varphi} \in \mathbf{W}_0^{1,n}(\mathbb{R}^n)$, there exist $\boldsymbol{\psi} \in \mathbf{W}_0^{1,n}(\mathbb{R}^n) \cap \mathbf{L}^\infty(\mathbb{R}^n)$ and $\eta \in W_0^{2,n}(\mathbb{R}^n)$ such that*

$$\boldsymbol{\varphi} = \boldsymbol{\psi} + \nabla \eta,$$

with the following estimate

$$\|\, \boldsymbol{\psi} \,\|_{\mathbf{W}_0^{1,n}(\mathbb{R}^n)} + \|\, \boldsymbol{\psi} \,\|_{\mathbf{L}^\infty(\mathbb{R}^n)} + \|\, \eta \,\|_{W_0^{2,n}(\mathbb{R}^n)} \leq C \,\|\, \boldsymbol{\varphi} \,\|_{\mathbf{W}_0^{1,n}(\mathbb{R}^n)}.$$

Proof. Let $\boldsymbol{\varphi} \in \mathbf{W}_0^{1,n}(\mathbb{R}^n)$. We resume the obtained functions $\boldsymbol{a}_k$, $\boldsymbol{\psi}_k$, $\boldsymbol{a}$, $\boldsymbol{\psi}_0$, $\boldsymbol{\psi}$ and η in the proof of Corollary 2.7. We have

$$|\boldsymbol{a}_k| = \frac{1}{|B_1|}\int_{B_1} |\boldsymbol{a}_k| \;\leq\; \frac{1}{|B_1|}\left(\int_{B_1} |\boldsymbol{a}_k|^n\right)^{1/n} |B_1|^{(n-1)/n}.$$

Then we deduce from (22) and (23) that

$$\begin{array}{rcl} |\boldsymbol{a}_k| \;\leq\; C\,||\,\boldsymbol{a}_k\,||_{\mathbf{L}^n(B_1)} & \leq & C\,||\,\boldsymbol{a}_k + \boldsymbol{\psi}_k\,||_{\mathbf{L}^n(B_1)} + C\,||\boldsymbol{\psi}_k\,||_{\mathbf{L}^n(B_1)} \\ & \leq & C\,||\,\nabla\boldsymbol{\varphi}\,||_{\mathbf{L}^n(\mathbb{R}^n)}. \end{array}$$

and

$$||\,\boldsymbol{a}\,||_{\mathbf{L}^\infty(\mathbb{R}^n)} \;\leq\; C\,||\,\nabla\boldsymbol{\varphi}\,||_{\mathbf{L}^n(\mathbb{R}^n)}.$$

As $\boldsymbol{\psi} \;=\; \boldsymbol{\psi}_0 - \boldsymbol{a}$, then

$$||\,\boldsymbol{\psi}\,||_{\mathbf{W}_0^{1,n}(\mathbb{R}^n)} \;\leq\; C\,||\,\nabla\boldsymbol{\varphi}\,||_{\mathbf{L}^n(\mathbb{R}^n)} \;\leq\; C\,||\,\boldsymbol{\varphi}\,||_{\mathbf{W}_0^{1,n}(\mathbb{R}^n)}.$$

As $\boldsymbol{\varphi} = \boldsymbol{\psi} + \nabla\eta$, then we have

$$||\,\nabla\eta\,||_{\mathbf{W}_0^{1,n}(\mathbb{R}^n)} \;\leq C\,||\,\boldsymbol{\varphi}\,||_{\mathbf{W}_0^{1,n}(\mathbb{R}^n)}.$$

Therefore, there exists $b \in \mathbb{R}$ such that

$$||\,\eta + b\,||_{W_0^{2,n}(\mathbb{R}^n)} \leq C\,||\,\nabla\eta\,||_{\mathbf{W}_0^{1,n}(\mathbb{R}^n)} \leq C\,||\,\boldsymbol{\varphi}\,||_{\mathbf{W}_0^{1,n}(\mathbb{R}^n)},$$

and the proof is complete. □

Remark 5. This theorem suggests this open question: if $\boldsymbol{\varphi} \in \mathbf{W}_0^{2,n/2}(\mathbb{R}^n)$, are there a function $\boldsymbol{\psi} \in \mathbf{W}_0^{2,n/2}(\mathbb{R}^n) \cap \mathbf{L}^\infty(\mathbb{R}^n)$ and a function $\eta \in W_0^{3,n/2}(\mathbb{R}^n)$ such that

$$\boldsymbol{\varphi} = \boldsymbol{\psi} + \nabla\eta,$$

with the corresponding estimate ?

We define now the space

$$\mathbf{X}(\mathbb{R}^n) = \{\,\boldsymbol{f} \in \mathbf{L}^1(\mathbb{R}^n),\ \operatorname{div}\boldsymbol{f} \in W_0^{-2,n/(n-1)}(\mathbb{R}^n)\},$$

which is Banach space endowed with the following norm

$$||\,\boldsymbol{f}\,\,||_{\mathbf{X}(\mathbb{R}^n)} \;=\; ||\,\boldsymbol{f}\,\,||_{\mathbf{L}^1(\mathbb{R}^n)} + ||\,\operatorname{div}\boldsymbol{f}\,\,||_{W_0^{-2,n/(n-1)}(\mathbb{R}^n)}. \tag{25}$$

Theorem 2.9. *Let $\boldsymbol{f} \in \mathbf{X}(\mathbb{R}^n)$. Then $\boldsymbol{f} \in \mathbf{W}_0^{-1,n/(n-1)}(\mathbb{R}^n)$ and we have the following inequality*

$$\forall\boldsymbol{\varphi} \;\in\; \mathbf{W}_0^{1,n}(\mathbb{R}^n) \cap \mathbf{L}^\infty(\mathbb{R}^n), \quad |\int_{\mathbb{R}^n} \boldsymbol{f}\cdot\boldsymbol{\varphi}\,| \;\leq\; C\,||\,\boldsymbol{f}\,||_{\mathbf{X}(\mathbb{R}^n)}||\boldsymbol{\varphi}\,||_{\mathbf{W}_0^{1,n}(\mathbb{R}^n)}. \tag{26}$$

Proof. We consider the following linear operator $\boldsymbol{\varphi} \xrightarrow{\boldsymbol{F}} \int_{\mathbb{R}^n} \boldsymbol{f} \cdot \boldsymbol{\varphi}$ defined on $\boldsymbol{\mathcal{D}}(\mathbb{R}^n)$. Thanks to Theorem 2.8, we have

$$\begin{aligned} | < \boldsymbol{F}, \boldsymbol{\varphi} > | &= | \int_{\mathbb{R}^n} \boldsymbol{f} \cdot (\boldsymbol{\psi} + \nabla\eta) | \\ &= | \int_{\mathbb{R}^n} \boldsymbol{f} \cdot \boldsymbol{\psi} - < \operatorname{div} \boldsymbol{f}, \eta >_{W_0^{-2,n/(n-1)}(\mathbb{R}^n) \times W_0^{2,n}(\mathbb{R}^n)} | \\ &\leq \| \boldsymbol{f} \|_{\mathbf{L}^1(\mathbb{R}^n)} \| \boldsymbol{\psi} \|_{\mathbf{L}^\infty(\mathbb{R}^n)} + \| \operatorname{div} \boldsymbol{f} \|_{W_0^{-2,n/(n-1)}} \| \eta \|_{W_0^{2,n}} \\ &\leq C \| \boldsymbol{f} \|_{\mathbf{X}(\mathbb{R}^n)} \| \boldsymbol{\varphi} \|_{\mathbf{W}_0^{1,n}(\mathbb{R}^n)}. \end{aligned}$$

As $\boldsymbol{\mathcal{D}}(\mathbb{R}^n)$ is dense in $\mathbf{W}_0^{1,n}(\mathbb{R}^n)$ and by applying Hahn-Banach Theorem, we can uniquely extend $\boldsymbol{F}$ by an element $\widetilde{\boldsymbol{F}} \in \mathbf{W}_0^{-1,n/(n-1)}(\mathbb{R}^n)$ satisfying

$$\| \widetilde{\boldsymbol{F}} \|_{\mathbf{W}_0^{-1,n/(n-1)}(\mathbb{R}^n)} \leq C \| \boldsymbol{f} \|_{\mathbf{X}(\mathbb{R}^n)}.$$

Besides, the linear operator $\boldsymbol{f} \longrightarrow \widetilde{\boldsymbol{F}}$ from $\mathbf{X}(\mathbb{R}^n)$ into $\mathbf{W}_0^{-1,n/(n-1)}(\mathbb{R}^n)$ is continuous and injective. Therefore, $\mathbf{X}(\mathbb{R}^n)$ can be identified to a subspace of $\mathbf{W}_0^{-1,n/(n-1)}(\mathbb{R}^n)$ with continuous and dense embedding. □

Remark 6. i) Let $\boldsymbol{f} \in \mathbf{X}(\mathbb{R}^n)$. Then $\int_{\mathbb{R}^n} \boldsymbol{f} = \mathbf{0}$ if and only if

$$\forall \lambda \in \mathscr{P}_1, \ < \operatorname{div} \boldsymbol{f}, \lambda >_{W_0^{-2,n/(n-1)}(\mathbb{R}^n) \times W_0^{2,n}(\mathbb{R}^n)} = 0.$$

Note that $\int_{\mathbb{R}^n} f_i = < f_i, 1 >_{W_0^{-1,n/(n-1)}(\mathbb{R}^n) \times W_0^{1,n}(\mathbb{R}^n)}$.

ii) Let $\boldsymbol{f} \in \mathbf{X}(\mathbb{R}^n)$ satisfying $\int_{\mathbb{R}^n} \boldsymbol{f} = \mathbf{0}$. Then we have the following inequality: for every $\boldsymbol{\varphi} \in \mathbf{W}_0^{1,n}(\mathbb{R}^n)$,

$$| < \boldsymbol{f}, \boldsymbol{\varphi} >_{\mathbf{W}_0^{-1,n/(n-1)}(\mathbb{R}^n) \times \mathbf{W}_0^{1,n}(\mathbb{R}^n)} | \leq C \| \boldsymbol{f} \|_{\mathbf{X}(\mathbb{R}^n)} \| \nabla \boldsymbol{\varphi} \|_{\mathbf{L}^n(\mathbb{R}^n)}.$$

Actually, we observe that for any $\boldsymbol{a} \in \mathbb{R}^n$,

$$| < \boldsymbol{f}, \boldsymbol{\varphi} > | = | < \boldsymbol{f}, \boldsymbol{\varphi} + \boldsymbol{a} > | \leq \| \boldsymbol{f} \|_{\mathbf{X}(\mathbb{R}^n)} \| \boldsymbol{\varphi} + \boldsymbol{a} \|_{\mathbf{W}_0^{1,n}(\mathbb{R}^n)}.$$

Consequently, taking the infinum, we have for every $\boldsymbol{\varphi} \in \mathbf{W}_0^{1,n}(\mathbb{R}^n)$ (see [4]):

$$| < \boldsymbol{f}, \boldsymbol{\varphi} > | \leq C \| \boldsymbol{f} \|_{\mathbf{X}(\mathbb{R}^n)} \| \nabla \boldsymbol{\varphi} \|_{\mathbf{L}^n(\mathbb{R}^n)}. \tag{27}$$

It is then easy to deduce the following corollary.

Corollary 2.10. *Let* $\boldsymbol{f} \in \mathbf{L}^1(\mathbb{R}^n)$ *and* $\operatorname{div} \boldsymbol{f} = 0$. *Then* $\int_{\mathbb{R}^n} \boldsymbol{f} = \mathbf{0}$ *and for every* $\boldsymbol{\varphi} \in \mathbf{W}_0^{1,n}(\mathbb{R}^n) \cap \mathbf{L}^\infty(\mathbb{R}^n)$, *we have*

$$| \int_{\mathbb{R}^n} \boldsymbol{f} \cdot \boldsymbol{\varphi} | \leq C \| \boldsymbol{f} \|_{\mathbf{L}^1(\mathbb{R}^n)} \| \nabla \boldsymbol{\varphi} \|_{\mathbf{L}^n(\mathbb{R}^n)}.$$

Corollary 2.11. *Let* $\boldsymbol{f} \in \mathbf{L}^1(\mathbb{R}^3)$ *and* $\mathbf{curl}\,\boldsymbol{f} \in \mathbf{W}_0^{-2,3/2}(\mathbb{R}^3)$. *Then we have* $\boldsymbol{f} \in \mathbf{W}_0^{-1,3/2}(\mathbb{R}^3)$ *and the following estimate: for every* $\boldsymbol{\varphi} \in \mathbf{W}_0^{1,3}(\mathbb{R}^3) \cap \mathbf{L}^\infty(\mathbb{R}^3)$,

$$|\int_{\mathbb{R}^3} \boldsymbol{f} \cdot \boldsymbol{\varphi}| \;\leq\; C\,(\,||\,\boldsymbol{f}\,||_{\mathbf{L}^1(\mathbb{R}^3)} + ||\,\mathbf{curl}\,\boldsymbol{f}\,||_{\mathbf{W}_0^{-2,3/2}(\mathbb{R}^3)}\,)||\,\boldsymbol{\varphi}\,||_{\mathbf{W}_0^{1,3}(\mathbb{R}^3)}.$$

If moreover $\displaystyle\int_{\mathbb{R}^3} \boldsymbol{f} \;=\; \mathbf{0}$, *then for every* $\boldsymbol{\varphi} \in \mathbf{W}_0^{1,3}(\mathbb{R}^3) \cap \mathbf{L}^\infty(\mathbb{R}^3)$, *we have the following estimate*

$$|\int_{\mathbb{R}^3} \boldsymbol{f} \cdot \boldsymbol{\varphi}| \;\leq\; C\,(\,||\,\boldsymbol{f}\,||_{\mathbf{L}^1(\mathbb{R}^3)} + ||\,\mathbf{curl}\,\boldsymbol{f}\,||_{\mathbf{W}_0^{-2,3/2}(\mathbb{R}^3)}\,)||\nabla\boldsymbol{\varphi}\,||_{\mathbf{L}^3(\mathbb{R}^3)}.$$

Proof. Using Proposition 2.13, the proof is similar as the one of Theorem 2.9 and Remark 6. □

When $\boldsymbol{f} \in \mathbf{X}(\mathbb{R}^n)$, we can improve Corollary 2.2 as follows (recall that $\boldsymbol{\mathcal{V}}(\mathbb{R}^n)$ is not dense in $\mathbf{H}_\infty(\mathbb{R}^n)$).

Proposition 2.12. *Assume that* $\boldsymbol{f} \in \mathbf{X}(\mathbb{R}^n)$ *satisfying*

$$\forall\, \boldsymbol{v} \in \boldsymbol{\mathcal{V}}(\mathbb{R}^n), \quad \int_{\mathbb{R}^n} \boldsymbol{f} \cdot \boldsymbol{v}\, d\boldsymbol{x} = 0. \tag{28}$$

Then there exists a unique $\pi \in L^{n/(n-1)}(\mathbb{R}^n)$ *such that* $\boldsymbol{f} = \nabla\pi$ *and the following estimate holds*

$$||\,\pi\,||_{L^{n/(n-1)}(\mathbb{R}^n)} \;\leq\; C\,||\,\boldsymbol{f}\,||_{W_0^{-1,n/(n-1)}(\mathbb{R}^n)}.$$

Proof. This proposition is an immediate consequence of the embedding $\mathbf{X}(\mathbb{R}^n) \hookrightarrow \mathbf{W}_0^{-1,n/(n-1)}(\mathbb{R}^n)$ and De Rham's Theorem (see Alliot - Amrouche [2]). □

Remark 7. i) First remark that the hypothesis $\operatorname{div}\boldsymbol{f} \in W_0^{-2,n/(n-1)}(\mathbb{R}^n)$ of Proposition 2.12 is necessary because if $\boldsymbol{f} = \nabla\pi$ with $\pi \in L^{n/(n-1)}(\mathbb{R}^n)$, then $\boldsymbol{f} \in \mathbf{W}_0^{-1,n/(n-1)}(\mathbb{R}^n)$ and $\operatorname{div}\boldsymbol{f} = \Delta\pi \in W_0^{-2,n/(n-1)}(\mathbb{R}^n)$.
ii) Also note that (28) is equivalent to

$$\forall \boldsymbol{v} \in \boldsymbol{\mathcal{V}}(\mathbb{R}^n), \quad <\boldsymbol{f}, \boldsymbol{v}>_{\mathbf{W}_0^{-1,n/(n-1)}(\mathbb{R}^n)\times\mathbf{W}_0^{1,n}(\mathbb{R}^n)} \;=\; 0. \tag{29}$$

As $\boldsymbol{\mathcal{V}}(\mathbb{R}^n)$ is dense in $\mathbf{V}_0^{1,n}(\mathbb{R}^n)$, (29) is also equivalent to

$$\forall \boldsymbol{v} \in \mathbf{V}_0^{1,n}(\mathbb{R}^n), \quad <\boldsymbol{f}, \boldsymbol{v}>_{\mathbf{W}_0^{-1,n/(n-1)}(\mathbb{R}^n)\times\mathbf{W}_0^{1,n}(\mathbb{R}^n)} \;=\; 0. \tag{30}$$

Since vectors of the canonical basis of $\mathbb{R}^n$ belong to $\mathbf{V}_0^{1,n}(\mathbb{R}^n)$, we deduce that if (28) holds, then $\displaystyle\int_{\mathbb{R}^n} \boldsymbol{f} \;=\; \mathbf{0}$.

Proposition 2.13. *Let* $\boldsymbol{\varphi} \in \mathbf{W}_0^{1,3}(\mathbb{R}^3)$. *Then there exist* $\boldsymbol{\psi} \in \mathbf{W}_0^{1,3}(\mathbb{R}^3) \cap \mathbf{L}^\infty(\mathbb{R}^3)$ *and* $\boldsymbol{\eta} \in \mathbf{W}_0^{2,3}(\mathbb{R}^3)$ *such that*

$$\boldsymbol{\varphi} = \boldsymbol{\psi} + \mathbf{curl}\, \boldsymbol{\eta}$$

and we have the following estimate

$$||\nabla \boldsymbol{\psi}||_{\mathbf{L}^3(\mathbb{R}^3)} + ||\boldsymbol{\psi}||_{\mathbf{L}^\infty(\mathbb{R}^3)} + ||D^2 \boldsymbol{\eta}||_{\mathbf{L}^3(\mathbb{R}^3)} \; \leq \; C\, ||\nabla \boldsymbol{\varphi}||_{\mathbf{L}^3(\mathbb{R}^3)}.$$

Moreover, $\boldsymbol{\psi}$ *and* $\boldsymbol{\eta}$ *can be chosen such that*

$$||\boldsymbol{\psi}||_{\mathbf{W}_0^{1,3}(\mathbb{R}^3)} + ||\boldsymbol{\psi}||_{\mathbf{L}^\infty(\mathbb{R}^3)} + ||\boldsymbol{\eta}||_{\mathbf{W}_0^{2,3}(\mathbb{R}^3)} \; \leq \; C\, ||\boldsymbol{\varphi}||_{\mathbf{W}_0^{1,3}(\mathbb{R}^3)}.$$

Proof. Let $\boldsymbol{\varphi} \in \mathbf{W}_0^{1,3}(\mathbb{R}^3)$, then $\operatorname{div} \boldsymbol{\varphi} \in L^3(\mathbb{R}^3)$. Thanks to Theorem 2.3, there exists $\boldsymbol{\psi} \in \mathbf{L}^\infty(\mathbb{R}^3) \cap \mathbf{W}_0^{1,3}(\mathbb{R}^3)$ such that $\operatorname{div} \boldsymbol{\psi} = \operatorname{div} \boldsymbol{\varphi}$ and

$$||\, \boldsymbol{\psi}\, ||_{\mathbf{L}^\infty(\mathbb{R}^3)} + ||\, \boldsymbol{\psi}\, ||_{\mathbf{W}_0^{1,3}(\mathbb{R}^3)} \; \leq \; C\, ||\operatorname{div} \boldsymbol{\varphi}\, ||_{\mathbf{L}^3(\mathbb{R}^3)}.$$

Setting $\boldsymbol{z} = \boldsymbol{\varphi} - \boldsymbol{\psi}$. We know that there exists $\boldsymbol{\eta}_0 \in \mathbf{W}_0^{2,3}(\mathbb{R}^3)$ such that $-\Delta \boldsymbol{\eta}_0 = \mathbf{curl}\, \boldsymbol{z}$ satisfying the following estimate

$$||\boldsymbol{\eta}_0||_{\mathbf{W}_0^{2,3}(\mathbb{R}^3)} \; \leq \; C\, ||\nabla \boldsymbol{\varphi}||_{\mathbf{L}^3(\mathbb{R}^3)}.$$

However, $\operatorname{div} \boldsymbol{\eta}_0 \in W_0^{1,3}(\mathbb{R}^3)$ is harmonic, then we deduce $\operatorname{div} \boldsymbol{\eta}_0 = a$ with $a \in \mathbb{R}$. Therefore, we have

$$\mathbf{curl}\, (\mathbf{curl}\, \boldsymbol{\eta}_0) = \mathbf{curl}\, \boldsymbol{z}.$$

We set $\boldsymbol{y} = \boldsymbol{z} - \mathbf{curl}\, \boldsymbol{\eta}_0$. Then $\boldsymbol{y} \in \mathbf{W}_0^{1,3}(\mathbb{R}^3)$, $\operatorname{div} \boldsymbol{y} = 0$ and $\mathbf{curl}\, \boldsymbol{y} = \mathbf{0}$, *i.e.*, $\Delta \boldsymbol{y} = \mathbf{0}$. Then we can deduce $\boldsymbol{y} = \boldsymbol{b} \in \mathbb{R}^3$. Let $\boldsymbol{q} \in \mathscr{P}_1$ such that $\boldsymbol{b} = \mathbf{curl}\, \boldsymbol{q}$. We now set $\boldsymbol{\eta} = \boldsymbol{\eta}_0 + \boldsymbol{q}$. Then

$$\boldsymbol{\varphi} = \boldsymbol{\psi} + \mathbf{curl}\, \boldsymbol{\eta}$$

and we have the following estimate

$$||\boldsymbol{\psi}||_{\mathbf{W}_0^{1,3}(\mathbb{R}^3)} + ||\boldsymbol{\psi}||_{\mathbf{L}^\infty(\mathbb{R}^3)} + ||D^2 \boldsymbol{\eta}||_{\mathbf{L}^3(\mathbb{R}^3)} \; \leq \; C\, ||\nabla \boldsymbol{\varphi}||_{\mathbf{L}^3(\mathbb{R}^3)}.$$

□

We now introduce the following proposition.

Proposition 2.14. *Assume that* $u \in L^1_{loc}(\mathbb{R}^n)$ *satisfying* $\nabla u \in \mathbf{L}^1(\mathbb{R}^n)$. *Then there exists a unique constant* $K \in \mathbb{R}$ *such that* $u + K \in L^{n/(n-1)}(\mathbb{R}^n)$. *Moreover, we have*

$$||u + K||_{L^{n/(n-1)}(\mathbb{R}^n)} \leq C ||\nabla u||_{\mathbf{L}^1(\mathbb{R}^n)} \tag{31}$$

and

$$K = - \lim_{|x| \to \infty} \frac{1}{\omega_n} \int_{S_{n-1}} u(\sigma |x|) d\sigma, \tag{32}$$

where S_{n-1} *is the unit sphere of* $\mathbb{R}^n$ *and* ω_n *its surface.*

Proof. From Proposition 2.7 [21], it is easy to prove that there exists a unique constant $K \in \mathbb{R}$ verifying $u + K \in L^{n/(n-1)}(\mathbb{R}^n)$ and we have (31). Thanks to Lemma 1.3 [2], we have

$$\lim_{|x|\to\infty} |x|^{n-1} \int_{S_{n-1}} |u(\sigma|x|) + K| d\sigma = 0. \tag{33}$$

We set

$$D_R(r) = \int_{\{x\in\mathbb{R}^n, r<|x|<R\}} |\nabla u| dx.$$

By proceeding similarly as in [22], we have

$$D_R(r) \geq \int_{S_{n-1}} (\int_r^R |\frac{\partial u}{\partial \rho}|\rho^2) d\rho + \int_r^R \rho \int_{S_{n-1}} |\nabla^* u| d\sigma d\rho,$$

where $\nabla^* u$ is the projection of gradient of u on the unit sphere S_{n-1}.

$$|\nabla^* u|^2 = r^2[\ |\nabla u|^2 - |\frac{\partial u}{\partial \rho}|^2].$$

By Hölder's and Wirtinger's inequalities

$$\begin{aligned} D_R(r) \quad &\geq \int_{S_{n-1}} \int_r^R |\frac{\partial u}{\partial \rho}| d\rho \int_r^R \rho^{-2} d\rho \\ &+C \int_r^R (\int_{S_{n-1}} |u(|x|\sigma) - \frac{1}{\omega_n} \int_{S_{n-1}} u(|x|\sigma) d\sigma| d\sigma) \rho d\rho. \end{aligned}$$

By consequence, we have

$$\begin{aligned} D_R(r) \quad &\geq Cr \int_{S_{n-1}} |u(R\sigma) - u(r\sigma)| d\sigma \\ &+ C \int_r^R (\int_{S_{n-1}} |u(|x|\sigma) - \frac{1}{\omega_n} \int_{S_{n-1}} u(|x|\sigma) d\sigma| d\sigma) \rho d\rho. \end{aligned} \tag{34}$$

Since the both integrals on the right are non-negative, each is separately bounded by $D_R(r)$. Then, there exists a function $u^* \in L^1(S_{n-1})$ such that

$$\lim_{|x|\to\infty} \int_{S_{n-1}} |u(|x|\sigma) - u^*(|x|^\sigma)| d\sigma = 0,$$
$$\lim_{|x|\to\infty} \int_{S_{n-1}} u(|x|\sigma) d\sigma = \int_{S_{n-1}} u^*(\sigma) d\sigma.$$

Thanks to (33), we deduce that $u^* = -K$ and from (34) we have (32). □

Remark 8. We have a similar result as the above proposition in the case $u \in \mathcal{D}'(\mathbb{R}^n)$ and $\nabla u \in \mathbf{L}^p(\mathbb{R}^n)$ for all $p > 1$ (see Payne and Weinberger [22], Amrouche and Razafison [9], for example).

3. Vector potentials and elliptic problems in the whole space

Proposition 3.1. *There exists* $C > 0$ *such that for any* $\boldsymbol{u} \in \mathbf{L}^{3/2}(\mathbb{R}^3)$ *satisfying* $\mathbf{curl}\,\boldsymbol{u} \in \mathbf{L}^1(\mathbb{R}^3)$ *and* $\operatorname{div}\boldsymbol{u} = 0$*, we have the following estimate*

$$\|\,\boldsymbol{u}\|_{\mathbf{L}^{3/2}(\mathbb{R}^3)} \;\leq\; C\,\|\,\mathbf{curl}\,\boldsymbol{u}\|_{\mathbf{L}^1(\mathbb{R}^3)}. \tag{35}$$

Proof. Setting $\boldsymbol{f} = \mathbf{curl}\,\boldsymbol{u}$. Then $\boldsymbol{f}$ belongs to $\mathbf{W}_0^{-1,3/2}(\mathbb{R}^3)$ and for all $i = 1, 2, 3,$

$$<\, f_i, 1\, >_{W_0^{-1,3/2}(\mathbb{R}^3)\times W_0^{1,3}(\mathbb{R}^3)} \;=\; 0.$$

Therefore, there exists a unique solution $\boldsymbol{z} \in \mathbf{W}_0^{1,3/2}(\mathbb{R}^3)$ of $-\Delta\,\boldsymbol{z} = \boldsymbol{f}$ in $\mathbb{R}^3$ and satisfying

$$\|\,\boldsymbol{z}\,\|_{\mathbf{W}_0^{1,3/2}(\mathbb{R}^3)} \;\leq\; C\,\|\,\mathbf{curl}\,\boldsymbol{u}\,\|_{\mathbf{W}_0^{-1,3/2}(\mathbb{R}^3)} \leq\; C\,\|\,\mathbf{curl}\,\boldsymbol{u}\,\|_{\mathbf{L}^1(\mathbb{R}^3)}.$$

The last inequality is consequence of the embedding $\mathbf{X}(\mathbb{R}^3) \hookrightarrow \mathbf{W}_0^{-1,3/2}(\mathbb{R}^3)$ and because $\operatorname{div}\boldsymbol{f} = 0$ and $\boldsymbol{f} = \mathbf{curl}\,\boldsymbol{u} \in \mathbf{X}(\mathbb{R}^3)$. Moreover, it is easy to see that $\operatorname{div}\boldsymbol{z} = 0$ in $\mathbb{R}^3$. By setting $\boldsymbol{w} = \boldsymbol{u} - \mathbf{curl}\,\boldsymbol{z}$, we can easily deduce that $\Delta\,\boldsymbol{w} = \mathbf{0}$ in $\mathbb{R}^3$. Then $\boldsymbol{w} = \mathbf{0}$, $\boldsymbol{u} = \mathbf{curl}\,\boldsymbol{z}$ and we obtain the estimate (35). □

Proceeding similarly as in the proof of Proposition 2.1, we can show the following corollary.

Corollary 3.2. *Let* $\boldsymbol{f} \in \mathbf{L}^3(\mathbb{R}^3)$ *such that* $\operatorname{div}\boldsymbol{f} = 0$*. Then there exists* $\boldsymbol{u} \in \mathbf{L}^\infty(\mathbb{R}^3)$*, with* $\operatorname{div}\boldsymbol{u} = 0$ *and such that*

$$\mathbf{curl}\,\boldsymbol{u} = \boldsymbol{f}, \quad \text{with} \quad \|\,\boldsymbol{u}\|_{\mathbf{L}^\infty(\mathbb{R}^3)} \leq C\,\|\,\mathbf{curl}\,\boldsymbol{u}\|_{\mathbf{L}^3(\mathbb{R}^3)}.$$

In three-dimensional space, thanks to Theorem 2.8, we can deduce the following proposition.

Proposition 3.3. *Let* $\boldsymbol{f} \in \mathbf{L}^3(\mathbb{R}^3)$ *such that* $\operatorname{div}\boldsymbol{f} = 0$*. Then there exist* $\boldsymbol{\varphi} \in \mathbf{W}_0^{1,3}(\mathbb{R}^3)$*, unique up to a constant vector, and* $\boldsymbol{\psi} \in \mathbf{W}_0^{1,3}(\mathbb{R}^3) \cap \mathbf{L}^\infty(\mathbb{R}^3)$ *such that*

$$\mathbf{curl}\;\boldsymbol{\varphi} = \mathbf{curl}\;\boldsymbol{\psi} = \boldsymbol{f} \quad \text{and} \quad \operatorname{div}\boldsymbol{\varphi} \;=\; 0,$$

satisfying the following estimate

$$\|\,\nabla\boldsymbol{\varphi}\,\|_{\mathbf{L}^3(\mathbb{R}^3)} + \|\,\boldsymbol{\psi}\,\|_{\mathbf{W}_0^{1,3}(\mathbb{R}^3)} + \|\,\boldsymbol{\psi}\,\|_{\mathbf{L}^\infty(\mathbb{R}^3)} \leq C\|\,\boldsymbol{f}\,\|_{\mathbf{L}^3(\mathbb{R}^3)}.$$

Proof. From the hypothesis, we deduce $\mathbf{curl}\,\boldsymbol{f} \in \mathbf{W}_0^{-1,3}(\mathbb{R}^3)$. Then , from [4], there exists $\boldsymbol{\varphi} \in \mathbf{W}_0^{1,3}(\mathbb{R}^3)$, unique up to a constant vector, such that $-\Delta\boldsymbol{\varphi} = \mathbf{curl}\,\boldsymbol{f}$ in Ω and satisfying the following estimate

$$\inf_{\mathbf{a}\in\mathbb{R}^3} \|\boldsymbol{\varphi} + \boldsymbol{a}\|_{\mathbf{W}_0^{1,3}(\mathbb{R}^3)} \;\leq\; C\,\|\,\boldsymbol{f}\,\|_{\mathbf{L}^3(\mathbb{R}^3)}.$$

As $\operatorname{div}\boldsymbol{\varphi} \in L^3(\mathbb{R}^3)$ is harmonic, then we deduce $\operatorname{div}\boldsymbol{\varphi} = 0$. Consequently, we have

$$-\Delta\boldsymbol{\varphi} = \mathbf{curl}\,\mathbf{curl}\;\boldsymbol{\varphi} - \nabla\operatorname{div}\boldsymbol{\varphi} = \mathbf{curl}\,\mathbf{curl}\;\boldsymbol{\varphi}.$$

Therefore, we obtain $\mathbf{curl}\,(\,\mathbf{curl}\,\boldsymbol{\varphi}-\boldsymbol{f}\,)\;=\;\mathbf{0}$ in Ω. Setting $\boldsymbol{z}=\mathbf{curl}\,\boldsymbol{\varphi}-\boldsymbol{f}$. Then $\boldsymbol{z}\in\mathbf{L}^3(\mathbb{R}^3)$, div $\boldsymbol{z}=0$ and $\mathbf{curl}\,\boldsymbol{z}=\mathbf{0}$. Hence, we deduce $\Delta\boldsymbol{z}=\mathbf{0}$ and $\boldsymbol{z}=\mathbf{0}$, *i.e.*, $\mathbf{curl}\,\boldsymbol{\varphi}=\boldsymbol{f}$. Applying Theorem 2.8, there exist $\boldsymbol{\psi}\in\mathbf{W}_0^{1,3}(\mathbb{R}^3)\cap\mathbf{L}^\infty(\mathbb{R}^3)$ and $\eta\in W_0^{2,3}(\mathbb{R}^3)$ such that $\boldsymbol{\varphi}=\boldsymbol{\psi}+\nabla\eta$ in $\mathbb{R}^3$, with the following estimate

$$||\,\boldsymbol{\psi}\,||_{\mathbf{W}_0^{1,3}(\mathbb{R}^3)}+||\,\boldsymbol{\psi}\,||_{\mathbf{L}^\infty(\mathbb{R}^3)}\;\leq\;C\,||\,\nabla\,\boldsymbol{\varphi}\,||_{\mathbf{L}^3(\mathbb{R}^3)}\;\leq\;C\,||\,\boldsymbol{f}\,||_{\mathbf{L}^3(\mathbb{R}^3)}.$$

The function $\boldsymbol{\psi}$ is the required function, then the proof is finished. □

Remark 9. From the previous proposition, we have the following Helmholtz decomposition: for all $\boldsymbol{f}\in\mathbf{L}^3(\mathbb{R}^3)$, we have

$$\boldsymbol{f}=\mathbf{curl}\,\boldsymbol{\psi}+\nabla p \tag{36}$$

with $\boldsymbol{\psi}\in\mathbf{W}_0^{1,3}(\mathbb{R}^3)\cap\mathbf{L}^\infty(\mathbb{R}^3)$ and $p\in W_0^{1,3}(\mathbb{R}^3)$ and the following estimate

$$||\,\boldsymbol{\psi}\,||_{\mathbf{W}_0^{1,3}(\mathbb{R}^3)}+||\,\boldsymbol{\psi}\,||_{\mathbf{L}^\infty(\mathbb{R}^3)}+||\nabla p||_{\mathbf{L}^3(\mathbb{R}^3)}\;\leq\;C\,||\,\boldsymbol{f}\,||_{\mathbf{L}^3(\mathbb{R}^3)}. \tag{37}$$

Indeed, we have div $\boldsymbol{f}\;\in\;W_0^{-1,3}(\mathbb{R}^3)\perp\mathbb{R}$ because of $\boldsymbol{f}\;\in\;\mathbf{L}^3(\mathbb{R}^3)$. Then there exists $p\;\in\;W_0^{1,3}(\mathbb{R}^3)$, unique up to a constant, such that $\Delta p\;=\;\mathrm{div}\,\boldsymbol{f}$ and satisfying the following estimate

$$||\nabla p||_{\mathbf{L}^3(\mathbb{R}^3)}\;\leq\;C\,||\,\boldsymbol{f}\,||_{\mathbf{L}^3(\mathbb{R}^3)}.$$

The function $\boldsymbol{f}-\nabla p$ satisfies the hypothesis of Proposition 3.3, then we can decompose $\boldsymbol{f}$ as in (36) and we have the estimate (37).

Corollary 3.4. *There exists $C>0$ such that for all $\boldsymbol{u}\in\mathbf{L}^{3/2}(\mathbb{R}^3)$ satisfying* div $\boldsymbol{u}=0$*, we have the following inequality*

$$||\,\boldsymbol{u}||_{\mathbf{L}^{3/2}(\mathbb{R}^n)}\;\leq\;C\inf_{\boldsymbol{f}+\boldsymbol{g}=\mathbf{curl}\;\boldsymbol{u}}(||\,\boldsymbol{f}\,||_{\mathbf{L}^1(\mathbb{R}^3)}+||\,\boldsymbol{g}\,||_{\mathbf{W}_0^{-1,3/2}(\mathbb{R}^3)}) \tag{38}$$

with $\boldsymbol{f}\in\mathbf{L}^1(\mathbb{R}^3)$ and $\boldsymbol{g}\in\mathbf{W}_0^{-1,3/2}(\mathbb{R}^3)$.

Proof. We consider two following operators

$$A=\mathbf{curl}:\mathbf{H}_{3/2}(\mathbb{R}^3)\to\mathbf{L}^1(\mathbb{R}^3)+\mathbf{W}_0^{-1,3/2}(\mathbb{R}^3),$$

$$A^*=\mathbf{curl}:\mathbf{W}_0^{1,3}(\mathbb{R}^3)\cap\mathbf{L}^\infty(\mathbb{R}^3)\to\mathbf{H}_3(\mathbb{R}^3).$$

The rest of this proof is similar to the one of Proposition 2.1. □

The following corollary improves Corollary 2.10.

Corollary 3.5. *Let $\boldsymbol{f}\in\mathbf{L}^1(\mathbb{R}^3)$ such that* div $\boldsymbol{f}=0$*. Then for all $\boldsymbol{\varphi}\in\mathbf{W}_0^{1,3}(\mathbb{R}^3)$, we have the following estimate*

$$|<\boldsymbol{f},\boldsymbol{\varphi}>_{\mathbf{W}_0^{-1,3/2}(\mathbb{R}^3)\times\mathbf{W}_0^{1,3}(\mathbb{R}^3)}|\;\leq\;C\,||\,\boldsymbol{f}\,||_{\mathbf{L}^1(\mathbb{R}^3)}||\mathbf{curl}\,\boldsymbol{\varphi}||_{\mathbf{L}^3(\mathbb{R}^3)}. \tag{39}$$

Proof. First remark that from the hypothesis, we deduce $\boldsymbol{f} \in \mathbf{W}_0^{-1,3/2}(\mathbb{R}^3)$. Let $\boldsymbol{\varphi} \in \mathbf{W}_0^{1,3}(\mathbb{R}^3)$. Then we have $\mathbf{curl}\,\boldsymbol{\varphi} \in \mathbf{L}^3(\mathbb{R}^3)$. Thanks to Proposition 3.3, there exists $\boldsymbol{\psi} \in \mathbf{W}_0^{1,3}(\mathbb{R}^3) \cap \mathbf{L}^\infty(\mathbb{R}^3)$ such that $\mathbf{curl}\,\boldsymbol{\psi} = \mathbf{curl}\,\boldsymbol{\varphi}$ with the following estimate

$$|| \boldsymbol{\psi} ||_{\mathbf{W}_0^{1,3}(\mathbb{R}^3)} + || \boldsymbol{\psi} ||_{\mathbf{L}^\infty(\mathbb{R}^3)} \ \leq \ C \, || \, \mathbf{curl}\,\boldsymbol{\varphi} \, ||_{\mathbf{L}^3(\mathbb{R}^3)}. \tag{40}$$

Besides, there exists $\eta \in W_0^{2,3}(\mathbb{R}^3)$ such that $\boldsymbol{\varphi} = \boldsymbol{\psi} + \nabla\eta$ in $\mathbb{R}^3$. Then we have

$$\begin{aligned} < \boldsymbol{f}, \boldsymbol{\varphi} >_{\mathbf{W}_0^{-1,3/2}(\mathbb{R}^3)\times\mathbf{W}_0^{1,3}(\mathbb{R}^3)} \quad &= \int_{\mathbb{R}^3} \boldsymbol{f}\cdot\boldsymbol{\psi} + < \boldsymbol{f}, \nabla\eta >_{\mathbf{W}_0^{-1,3/2}(\mathbb{R}^3)\times\mathbf{W}_0^{1,3}(\mathbb{R}^3)} \\ &= \int_{\mathbb{R}^3} \boldsymbol{f}\cdot\boldsymbol{\psi}. \end{aligned}$$

Therefore, the estimate (39) is deduced from the estimate (40). □

Remark 10. We have another proof for the above corollary as follows: We can write that $\boldsymbol{f} = -\Delta\boldsymbol{u} = \mathbf{curl}\,\mathbf{curl}\,\boldsymbol{u}$, with $\boldsymbol{u} \in \mathbf{W}_0^{1,3/2}(\mathbb{R}^3)$ satisfying the following estimate

$$|| \, \boldsymbol{u} \, ||_{\mathbf{W}_0^{1,3/2}(\mathbb{R}^3)} \ \leq \ C \, || \, \boldsymbol{f} \, ||_{\mathbf{W}_0^{-1,3/2}(\mathbb{R}^3)} \leq C || \, \boldsymbol{f} \, ||_{\mathbf{L}^1(\mathbb{R}^3)}.$$

Then we deduce

$$\begin{aligned} | < \boldsymbol{f}, \boldsymbol{\varphi} >_{\mathbf{W}_0^{-1,3/2}(\mathbb{R}^3)\times\mathbf{W}_0^{1,3}(\mathbb{R}^3)} | \quad &= | < \mathbf{curl}\,\boldsymbol{u}, \mathbf{curl}\,\boldsymbol{\varphi} >_{\mathbf{L}^{3/2}(\mathbb{R}^3)\times\mathbf{L}^3(\mathbb{R}^3)} | \\ &\leq ||\mathbf{curl}\,\boldsymbol{u}||_{\mathbf{L}^{3/2}(\mathbb{R}^3)} ||\mathbf{curl}\,\boldsymbol{\varphi}||_{\mathbf{L}^3(\mathbb{R}^3)} \\ &\leq C \, || \, \boldsymbol{f} \, ||_{\mathbf{L}^1(\mathbb{R}^3)} ||\mathbf{curl}\,\boldsymbol{\varphi}||_{\mathbf{L}^3(\mathbb{R}^3)}. \end{aligned}$$

We now prove the following proposition.

Proposition 3.6. *Let $\boldsymbol{f} \in \mathbf{L}^1(\mathbb{R}^3)$ such that $\operatorname{div}\boldsymbol{f} = 0$. Then there exists a unique $\boldsymbol{\varphi} \in \mathbf{L}^{3/2}(\mathbb{R}^3)$ such that $\mathbf{curl}\ \boldsymbol{\varphi} = \boldsymbol{f}$ and $\operatorname{div}\boldsymbol{\varphi} = 0$ in $\mathbb{R}^3$ satisfying the following estimate*

$$|| \, \boldsymbol{\varphi} \, ||_{\mathbf{L}^{3/2}(\mathbb{R}^3)} \ \leq \ C \, || \, \boldsymbol{f} \, ||_{\mathbf{L}^1(\mathbb{R}^3)}.$$

Proof. From the definition of $\mathbf{X}(\mathbb{R}^3)$, we have $\boldsymbol{f} \in \mathbf{W}_0^{-1,3/2}(\mathbb{R}^3)$. As

$$\Delta : \mathbf{W}_0^{1,3/2}(\mathbb{R}^3) \longrightarrow \mathbf{W}_0^{-1,3/2}(\mathbb{R}^3) \perp \mathbb{R}$$

is an isomorphism (see [4]), then there exists a unique $\boldsymbol{h} \in \mathbf{W}_0^{1,3/2}(\mathbb{R}^3)$ such that $-\Delta\boldsymbol{h} = \boldsymbol{f}$ and we have the following estimate

$$|| \, \boldsymbol{h} \, ||_{\mathbf{W}_0^{1,3/2}(\mathbb{R}^3)} \leq C \, || \, \boldsymbol{f} \, ||_{\mathbf{L}^1(\mathbb{R}^3)}.$$

Moreover, we can see that $\operatorname{div}\boldsymbol{h} = 0$ and then $-\Delta\boldsymbol{h} = \mathbf{curl}\,\mathbf{curl}\ \boldsymbol{h}$. The proposition can be easily obtained by setting $\boldsymbol{\varphi} = \mathbf{curl}\ \boldsymbol{h}$. □

We have the following Helmholtz decomposition.

Corollary 3.7. *Let* $\boldsymbol{f} \in \mathbf{L}^1_0(\mathbb{R}^3)$ *such that* $\operatorname{div} \boldsymbol{f} \in W_0^{-2,3/2}(\mathbb{R}^3)$. *Then there exists a unique* $\boldsymbol{\varphi} \in \mathbf{L}^{3/2}(\mathbb{R}^3)$ *such that* $\operatorname{div} \boldsymbol{\varphi} = 0$ *and a unique* $\pi \in L^{3/2}(\mathbb{R}^3)$ *satisfying*

$$\boldsymbol{f} = \mathbf{curl}\, \boldsymbol{\varphi} + \nabla \pi$$

and the following estimate holds

$$\|\boldsymbol{\varphi}\|_{\mathbf{L}^{3/2}(\mathbb{R}^3)} + \|\pi\|_{L^{3/2}(\mathbb{R}^3)} \leq C \left(\|\boldsymbol{f}\|_{\mathbf{L}^1(\mathbb{R}^3)} + \|\operatorname{div} \boldsymbol{f}\|_{W_0^{-2,3/2}(\mathbb{R}^3)} \right).$$

Proof. It is clear that

$$\forall \lambda \in \mathscr{P}_1, \quad < \operatorname{div} \boldsymbol{f}, \lambda > = 0.$$

From the hypothesis and [4], there exists a unique $\pi \in L^{3/2}(\mathbb{R}^3)$ such that $\Delta \pi = \operatorname{div} \boldsymbol{f}$ and

$$\|\pi\|_{L^{3/2}(\mathbb{R}^3)} \leq C \|\operatorname{div} \boldsymbol{f}\|_{W_0^{-2,3/2}(\mathbb{R}^3)}.$$

Then, we deduce that $\boldsymbol{f} - \nabla \pi \in \mathbf{W}_0^{-1,3/2}(\mathbb{R}^3) \perp \mathbb{R}^3$. Therefore, there exists a unique $\boldsymbol{z} \in \mathbf{W}_0^{1,3/2}(\mathbb{R}^3)$ such that $-\Delta \boldsymbol{z} = \boldsymbol{f} - \nabla \pi$. Moreover, we see that $\operatorname{div} \boldsymbol{z} = 0$. Then, $\boldsymbol{f} - \nabla \pi = \mathbf{curl}\,\mathbf{curl} \boldsymbol{z}$. The proof is complete by setting $\boldsymbol{\varphi} = \mathbf{curl}\ \boldsymbol{z}$. $\square$

The following proposition is an extension of Proposition 3.6.

Proposition 3.8. *Let* $\boldsymbol{f} \in \mathbf{L}^1_0(\mathbb{R}^3) + \mathbf{W}_0^{-1,3/2}(\mathbb{R}^3)$ *such that* $\operatorname{div} \boldsymbol{f} = 0$ *and satisfying the following compatibility condition*

$$\forall i = 1, 2, 3, \quad < f_i,\ 1 > = 0.$$

Then there exists a unique $\boldsymbol{\varphi} \in \mathbf{L}^{3/2}(\mathbb{R}^3)$ *such that* $\mathbf{curl}\ \boldsymbol{\varphi} = \boldsymbol{f}$ *and* $\operatorname{div} \boldsymbol{\varphi} = 0$ *in* $\mathbb{R}^3$ *satisfying the following estimate*

$$\|\boldsymbol{\varphi}\|_{\mathbf{L}^{3/2}(\mathbb{R}^3)} \leq C \|\boldsymbol{f}\|_{\mathbf{L}^1(\mathbb{R}^3) + \mathbf{W}_0^{-1,3/2}(\mathbb{R}^3)}.$$

Proof. Let $\boldsymbol{f} = \boldsymbol{g} + \boldsymbol{h}$ with $\boldsymbol{g} \in \mathbf{L}^1_0(\mathbb{R}^3)$, $\boldsymbol{h} \in \mathbf{W}_0^{-1,3/2}(\mathbb{R}^3)$ and $\operatorname{div} \boldsymbol{f} = 0$. Then $\operatorname{div} \boldsymbol{g} = -\operatorname{div} \boldsymbol{h} \in W_0^{-2,3/2}(\mathbb{R}^3)$. Therefore we deduce $\boldsymbol{g} \in \mathbf{W}_0^{-1,3/2}(\mathbb{R}^3)$ and $\boldsymbol{g} \perp \mathbb{R}^3$. As $\boldsymbol{f} \in \mathbf{W}_0^{-1,3/2}(\mathbb{R}^3) \perp \mathbb{R}^3$, then there exists a unique $\boldsymbol{z} \in \mathbf{W}_0^{1,3/2}(\mathbb{R}^3)$ such that $-\Delta \boldsymbol{z} = \boldsymbol{f}$ and $\operatorname{div} \boldsymbol{z} = 0$. The proof is finished by choosing $\boldsymbol{\varphi} = \mathbf{curl}\, \boldsymbol{z}$. $\square$

In two-dimensional space, we have a similar result as Proposition 3.8.

Proposition 3.9. *Assume that* $\boldsymbol{f} \in \mathbf{L}^1_0(\mathbb{R}^2) + \mathbf{W}_0^{-1,2}(\mathbb{R}^2)$ *such that* $\operatorname{div} \boldsymbol{f} = 0$. *Then there exists* $\varphi \in L^2(\mathbb{R}^2)$ *such that* $\mathbf{curl}\ \varphi = \boldsymbol{f}$ *and satisfying the following estimate*

$$\|\varphi\|_{L^2(\mathbb{R}^2)} \leq C \|\boldsymbol{f}\|_{\mathbf{L}^1(\mathbb{R}^2) + \mathbf{W}_0^{-1,2}(\mathbb{R}^2)}.$$

Corollary 3.10. *Let* $\boldsymbol{f} \in \mathbf{X}(\mathbb{R}^n)$ *and* $\int_{\mathbb{R}^n} \boldsymbol{f} = \mathbf{0}$. *Then the following problem*

$$-\Delta \boldsymbol{u} = \boldsymbol{f} \quad \textit{in } \mathbb{R}^n$$

has a unique solution $\boldsymbol{u} \in \mathbf{W}_0^{1,n/(n-1)}(\mathbb{R}^n)$. *Moreover, we have* $\boldsymbol{u} = \mathcal{E}_n * \boldsymbol{f}$ *and* $\boldsymbol{u}$ *satisfies the following estimate*

$$\| \boldsymbol{u} \|_{\mathbf{W}_0^{1,n/(n-1)}(\mathbb{R}^n)} \leq C \| \boldsymbol{f} \|_{\mathbf{X}(\mathbb{R}^n)}.$$

Proof. This corollary is an immediate consequence of Theorem 2.9, Remark 6 and the fact that

$$\Delta : \; W_0^{1,n/(n-1)}(\mathbb{R}^n) \longrightarrow W_0^{-1,n/(n-1)}(\mathbb{R}^n) \perp \mathbb{R} \quad \text{if} \quad n \geq 3$$
$$\Delta : \; W_0^{1,2}(\mathbb{R}^2) \longrightarrow W_0^{-1,2}(\mathbb{R}^2) \perp \mathbb{R} \quad \text{if} \quad n = 2$$

are isomorphisms (cf. [4]). We recall that

$$W_0^{-1,n/(n-1)}(\mathbb{R}^n) \perp \mathbb{R} = \left\{ f \in W_0^{-1,n/(n-1)}(\mathbb{R}^n); \; < f, \; 1 > = 0 \right\}.$$

□

Remark 11. In particular, when $n = 2$, $\boldsymbol{f} \in \mathbf{L}^1(\mathbb{R}^2)$ and $\operatorname{div} \boldsymbol{f} = 0$, the solution given in Corollary 3.10 belongs to $\mathbf{L}^\infty(\mathbb{R}^n) \cap \mathscr{C}_0(\mathbb{R}^n)$. The reader can find this result in H. Brézis, J. Van Schaftingen [15] and J. Bourgain, H. Brézis [11].

Corollary 3.11. *Let* $f \in L^1(\mathbb{R}^n)$ *such that* $\partial_n f \in W_0^{-2,n/(n-1)}(\mathbb{R}^n)$.
i) Then we have $f \in W_0^{-1,n/(n-1)}(\mathbb{R}^n)$ *and the following estimate holds*

$$\| f \|_{W_0^{-1,n/(n-1)}(\mathbb{R}^n)} \leq C \left(\| f \|_{L^1(\mathbb{R}^n)} + \| \partial_n f \|_{W_0^{-2,n/(n-1)}(\mathbb{R}^n)} \right).$$

ii) Furthermore if $\int_{\mathbb{R}^n} f = 0$, *then there exists a unique* $u \in W_0^{1,n/(n-1)}(\mathbb{R}^n)$ *satisfying the following problem*

$$\Delta u = f \quad \text{in } \mathbb{R}^n,$$

and we have the following estimate

$$\| u \|_{W_0^{1,n/(n-1)}(\mathbb{R}^n)} \leq C \left(\| f \|_{L^1(\mathbb{R}^n)} + \| \partial_n f \|_{W_0^{-2,n/(n-1)}(\mathbb{R}^n)} \right).$$

Proof. This corollary can be obtained by applying Theorem 2.9 and Corollary 3.10 with $\boldsymbol{f} = (0, ..., 0, f)$. □

Remark 12. We know that if $f \in L^1(\mathbb{R}^n)$, then $\mathcal{E}_n * f \in L_w^{n/(n-2)}(\mathbb{R}^n)$ if $n \geq 3$ and $\nabla(\mathcal{E}_n * f) \in \mathbf{L}_w^{n/(n-1)}(\mathbb{R}^n)$. As $\partial_n f \in W_0^{-2,n/(n-1)}(\mathbb{R}^n)$ with $n \geq 2$, we deduce that $\partial_n(\mathcal{E}_n * f) \in L^{n/(n-1)}(\mathbb{R}^n)$, where

$$L_w^p(\mathbb{R}^n) = \left\{ f \text{ measurable on } \mathbb{R}^n;\ \sup_{t>0} t(|\{x \in \mathbb{R}^n; |f(x)| > t\}\ |)^{1/p} < \infty \right\}.$$

However, in Corollary 3.11, $\nabla(\mathcal{E}_n * f) \in \mathbf{L}^{n/(n-1)}(\mathbb{R}^n)$ and $\mathcal{E}_n * f \in L^{n/(n-2)}(\mathbb{R}^n)$. Then there is an anisotropic phenomenon.

Recall now a result in [2] concerning the Stokes problem in $\mathbb{R}^n$.

Theorem 3.12. *Let* $(\boldsymbol{f}, g) \in \mathbf{W}_0^{-1,p}(\mathbb{R}^n) \times L^p(\mathbb{R}^n)$ *satisfying the compatibility condition as follows*

$$\forall \boldsymbol{\lambda} \in \mathscr{P}_{[1-n/p']}, \quad < \boldsymbol{f}, \boldsymbol{\lambda} >_{\mathbf{W}_0^{-1,p}(\mathbb{R}^n) \times \mathbf{W}_0^{1,p'}(\mathbb{R}^n)} = 0. \tag{41}$$

Then the Stokes system $(\mathcal{S})$

$$-\Delta \boldsymbol{u} + \nabla \pi = \boldsymbol{f} \quad \text{and} \quad \operatorname{div} \boldsymbol{u} = g \quad \text{in } \mathbb{R}^n,$$

has a unique solution $(\boldsymbol{u}, \pi) \in \mathbf{W}_0^{1,p}(\mathbb{R}^n)/\mathscr{P}_{[1-n/p]} \times L^p(\mathbb{R}^n)$. *Moreover, we have the estimate*

$$\inf_{\boldsymbol{\lambda} \in \mathscr{P}_{[1-n/p]}} ||\, \boldsymbol{u} + \boldsymbol{\lambda}\, ||_{\mathbf{W}_0^{1,p}(\mathbb{R}^n)} + ||\, \pi\, ||_{L^p(\mathbb{R}^n)} \ \leq\ C \left(||\, \boldsymbol{f}\, ||_{\mathbf{W}_0^{-1,p}(\mathbb{R}^n)} + ||\, g\, ||_{L^p(\mathbb{R}^n)} \right).$$

Corollary 3.13. *Let* $(\boldsymbol{f}, g) \in \mathbf{X}(\mathbb{R}^n) \times L^{n/(n-1)}(\mathbb{R}^n)$ *satisfying the compatibility condition as follows*

$$< f_i\,,\, 1 > = 0 \quad \textit{for all } i = 1, 2, 3.$$

Then the Stokes system $(\mathcal{S})$ *has a unique solution* $(\boldsymbol{u}, \pi) \in \mathbf{W}_0^{1,n/(n-1)}(\mathbb{R}^n) \times L^{n/(n-1)}(\mathbb{R}^n)$ *and the following estimate holds*

$$||\, \boldsymbol{u}\, ||_{\mathbf{W}_0^{1,n/(n-1)}} + ||\, \pi\, ||_{L^{n/(n-1)}} \ \leq\ C \left(||\, \boldsymbol{f}\, ||_{\mathbf{W}_0^{-1,n/(n-1)}} + ||\, g\, ||_{L^{n/(n-1)}} \right).$$

Proof. This corollary is a consequence of $\mathbf{X}(\mathbb{R}^n) \hookrightarrow \mathbf{W}_0^{-1,n/(n-1)}(\mathbb{R}^n)$ and Theorem 3.12. □

4. The div-grad operators in the half-space

First of all, we introduce the following notations. If v is a function defined on $\mathbb{R}^n_+$, we set

$$v^*(\boldsymbol{x}', x_n) \ = \ \begin{cases} v(\boldsymbol{x}', x_n) & \text{if } x_n > 0, \\ v(\boldsymbol{x}', -x_n) & \text{if } x_n < 0, \end{cases} \tag{42}$$

and

$$v_*(\boldsymbol{x}', x_n) \;=\; \begin{cases} v(\boldsymbol{x}', x_n) & \text{if } x_n > 0, \\ -v(\boldsymbol{x}', -x_n) & \text{if } x_n < 0. \end{cases} \tag{43}$$

We also set

$$\mathbf{W}_1^{0,p}(\mathrm{div}; \mathbb{R}^n_+) \;=\; \{\, \boldsymbol{v} \,\in\, \mathbf{L}^p(\mathbb{R}^n_+);\;\; \mathrm{div}\, \boldsymbol{v} \,\in\, W_1^{0,p}(\mathbb{R}^n_+) \,\},$$

where $W_1^{0,p}(\mathbb{R}^n_+)$ is the subspace of functions u in $L^p(\mathbb{R}^n_+)$ which satisfy $|\boldsymbol{x}|u$ in $L^p(\mathbb{R}^n_+)$, and their normal traces are described in the following lemma (see C. Amrouche, S. Nečasová and Y. Raudin [7]):

Lemma 4.1. *The linear mapping*

$$\begin{aligned} \gamma_{e_n} :\; \boldsymbol{\mathcal{D}}(\overline{\mathbb{R}^n_+}) \;&\longrightarrow \mathcal{D}(\mathbb{R}^{n-1}), \\ \boldsymbol{v} \;&\longmapsto\; v_n|_\Gamma, \end{aligned}$$

can be extended to a linear continuous mapping

$$\gamma_{e_n} :\; \mathbf{W}_1^{0,p}(\mathrm{div}; \mathbb{R}^n_+) \;\longrightarrow\; W_0^{-1/p,p}(\mathbb{R}^{n-1}), \quad \text{for any } 1 < p < \infty.$$

Moreover, for all $\boldsymbol{v} \in \mathbf{W}_1^{0,p}(\mathrm{div}; \mathbb{R}^n_+)$ *and for all* $\varphi \in W_0^{1,p'}(\mathbb{R}^n_+)$*, we have the following Green formula*

$$\int_{\mathbb{R}^n_+} \boldsymbol{v} \cdot \nabla\varphi \, d\boldsymbol{x} + \int_{\mathbb{R}^n_+} \varphi \, \mathrm{div}\, \boldsymbol{v} \, d\boldsymbol{x} \;=\; - < v_n, \varphi >_{W_0^{-1/p,p}(\Gamma) \times W_0^{1/p,p'}(\Gamma)} .$$

Define now the following spaces

$$\mathbf{H}_p(\mathrm{div}; \mathbb{R}^n_+) \;=\; \{\, \boldsymbol{v} \,\in\, \mathbf{L}^p(\mathbb{R}^n_+);\; \mathrm{div}\, \boldsymbol{v} \,\in\, L^p(\mathbb{R}^n_+) \},$$

$$\overset{\circ}{\mathbf{H}}_p\,(\mathbb{R}^n_+) \;=\; \{\, \boldsymbol{v} \,\in\, \mathbf{L}^p(\mathbb{R}^n_+);\; \mathrm{div}\, \boldsymbol{v} \,=\, 0 \text{ in } \mathbb{R}^n_+;\; v_n \,=\, 0 \text{ on } \Gamma \,\},$$

$$\mathbf{V}_0^{1,p}(\mathbb{R}^n_+) \;=\; \{\, \boldsymbol{v} \,\in \overset{\circ}{\mathbf{W}}{}_0^{1,p}(\mathbb{R}^n_+);\; \mathrm{div}\, \boldsymbol{v} \,=\, 0 \text{ in } \mathbb{R}^n_+ \,\},$$

where $\overset{\circ}{\mathbf{W}}{}_0^{1,p}(\mathbb{R}^n_+)$ is the subspace of functions of $\mathbf{W}_0^{1,p}(\mathbb{R}^n_+)$ which are equal to zero on the boundary of $\mathbb{R}^n_+$.

Remark that from Lemma 4.1, if $\boldsymbol{v} \in \mathbf{L}^p(\mathbb{R}^n_+)$ and $\mathrm{div}\, \boldsymbol{v} = 0$ in $\mathbb{R}^n_+$, then $v_n \in W_0^{-1/p,p}(\mathbb{R}^{n-1})$.

We can show classically the following lemma.

Lemma 4.2. *For any* $1 \le p < \infty$*, we have that* $\boldsymbol{\mathcal{D}}(\overline{\mathbb{R}^n_+})$ *is dense in* $\mathbf{H}_p(\mathrm{div}; \mathbb{R}^n_+)$*. Moreover, for any* $1 < p < \infty$*, the following linear mapping*

$$\begin{aligned} \mathbf{H}_p(\mathrm{div}; \mathbb{R}^n_+) \;&\longrightarrow\; W_{-1}^{-1/p,p}(\Gamma) \\ \boldsymbol{v} \;&\longmapsto\; v_n \end{aligned}$$

is continuous and surjective.

Lemma 4.3. *For any $1 \le p < \infty$, we have*
1) $\boldsymbol{\mathcal{V}}(\mathbb{R}^n_+)$ *is dense in* $\overset{\circ}{\mathbf{H}}_p(\mathbb{R}^n_+)$,
2) $\boldsymbol{\mathcal{V}}(\mathbb{R}^n_+)$ *is dense in* $\mathbf{V}_0^{1,p}(\mathbb{R}^n_+)$.

Proof. **1)** We give the proof for the case $p = 1$. With similar arguments, it is then easy to consider the case $p > 1$. The idea consists in using Hahn-Banach Theorem and showing that $\boldsymbol{f} = \mathbf{0}$ if $\boldsymbol{f} \in [\overset{\circ}{\mathbf{H}}_1(\mathbb{R}^n_+)]'$ satisfying

$$\forall\, \boldsymbol{v} \in \boldsymbol{\mathcal{V}}(\mathbb{R}^n_+), \quad < \boldsymbol{f}, \boldsymbol{v} > = \mathbf{0}.$$

We know that there exists $\pi \in \mathcal{D}'(\mathbb{R}^n_+)$ such that $\boldsymbol{f} = \nabla\pi$. Thanks to [4], we have $\pi \in W^{1,p}_{loc}(\overline{\mathbb{R}^n_+})$ for any $p \ge 1$. On the other hand, $\pi \in \mathcal{C}^0(\overline{\mathbb{R}^n_+})$ and we can suppose $\pi(0) = 0$. So that

$$\forall\, \boldsymbol{x} \in \mathbb{R}^n_+, \quad |\pi(\boldsymbol{x})| \le |\boldsymbol{x}|.||\nabla\pi||_{\mathbf{L}^\infty(\mathbb{R}^n_+)}.$$

Let now $\psi \in \mathcal{C}^\infty(\mathbb{R}^n_+)$ with $0 \le \psi \le 1$ satisfying $\psi(\boldsymbol{x}) = 1$ if $|\boldsymbol{x}| \le 1$ and $\psi(\boldsymbol{x}) = 0$ if $|\boldsymbol{x}| \ge 2$. We set

$$\psi_k(\boldsymbol{x}) = \psi(\frac{\boldsymbol{x}}{k}) \quad \text{and} \quad \pi_k = \psi_k\pi.$$

Then $\pi_k \in L^\infty(\mathbb{R}^n_+)$ and $\operatorname{supp}\nabla\pi_k \subset B(0, 2k)$. Moreover, if $|\boldsymbol{x}| \le 2k$, then

$$|\nabla\pi_k(\boldsymbol{x})| \le \frac{C}{k}|\pi(\boldsymbol{x})| + C\,|\nabla\pi(\boldsymbol{x})| \le 3C||\nabla\pi||_{\mathbf{L}^\infty(\mathbb{R}^n_+)}.$$

Therefore, we have that $(\nabla\pi_k)_k$ is bounded in $\mathbf{L}^\infty(\mathbb{R}^n_+)$. In fact, we show that $\nabla\pi_k \overset{*}{\rightharpoonup} \nabla\pi$ in $\mathbf{L}^\infty(\mathbb{R}^n_+)$, *i.e.*,

$$\forall\varphi \in L^1(\mathbb{R}^n_+) \text{ and } \forall j = 1, ..., n, \quad \int_{\mathbb{R}^n_+} (\frac{\partial\pi_k}{\partial x_j} - \frac{\partial\pi}{\partial x_j})\varphi \longrightarrow 0$$

when $k \to \infty$. Let now $\boldsymbol{v} \in \overset{\circ}{\mathbf{H}}_1(\mathbb{R}^n_+)$. Thanks to Lemma 4.1, we have

$$\int_{\mathbb{R}^n_+} \nabla\pi_k \cdot \boldsymbol{v} = -\int_{\mathbb{R}^n_+} \pi_k \operatorname{div}\boldsymbol{v} = 0.$$

By passing to the limit in the above equation, we obtain

$$\int_{\mathbb{R}^n_+} \nabla\pi \cdot \boldsymbol{v} = 0 = < \boldsymbol{f}, \boldsymbol{v} > .$$

This ends the proof of the case $p = 1$.

2) We content ourselves here with the case $n = 3$ and $p > 1$. Proceeding similarly as in the proof of Lemma 5.6, we can show that if $\boldsymbol{f} \in \overset{\circ}{\mathbf{W}}{}^{1,p}_0(\mathbb{R}^3_+)$ such that $\operatorname{div}\boldsymbol{f} = 0$, then $\boldsymbol{f} = \mathbf{curl}\,\boldsymbol{\varphi}$ with $\boldsymbol{\varphi} \in \overset{\circ}{\mathbf{W}}{}^{2,p}_0(\mathbb{R}^3_+)$ and

$$||\,\boldsymbol{\varphi}\,||_{\mathbf{W}^{2,p}_0(\mathbb{R}^3_+)} \le C\,||\,\boldsymbol{f}\,||_{\mathbf{W}^{1,p}_0(\mathbb{R}^3_+)}.$$

As $\boldsymbol{\mathcal{D}}(\mathbb{R}^3_+)$ is dense in $\overset{\circ}{\mathbf{W}}{}^{2,p}_0(\mathbb{R}^3_+)$, there exists $\boldsymbol{\varphi}_k \in \boldsymbol{\mathcal{D}}(\mathbb{R}^3_+)$ such that $\boldsymbol{\varphi}_k \to \boldsymbol{\varphi}$ in $\mathbf{W}^{2,p}_0(\mathbb{R}^3_+)$. The sequence $\boldsymbol{f}_k = \mathbf{curl}\,\boldsymbol{\varphi}_k$ answers this question.

□

Lemma 4.4. *Let $1 < p < \infty$. The following properties are satisfied:*
i) The mapping

$$\nabla : \quad L^p(\mathbb{R}^n_+) \;\longrightarrow\; [\,\mathbf{V}^{1,p'}_0(\mathbb{R}^n_+)\,]^0$$

is an isomorphism.
ii) The mapping

$$\mathrm{div} : \; \overset{\circ}{\mathbf{W}}{}^{1,p'}_0(\mathbb{R}^n_+)/\mathbf{V}^{1,p'}_0(\mathbb{R}^n_+) \;\longrightarrow\; L^{p'}(\mathbb{R}^n_+)$$

is an isomorphism.

Proof. It suffices to show that the second operator is surjective. More generally, let $\varphi \in L^{p'}(\mathbb{R}^n_+)$ and $\boldsymbol{g} \in \mathbf{W}^{1-1/p',p'}_0(\mathbb{R}^{n-1})$ (instead of $\boldsymbol{g} = \mathbf{0}$). We know that there exists $\boldsymbol{u}_g \in \mathbf{W}^{1,p'}_0(\mathbb{R}^n_+)$ such that $\boldsymbol{u}_{\boldsymbol{g}} = \boldsymbol{g}$ on Γ. This shows that we can assume $\boldsymbol{g} = \mathbf{0}$. Then let $\pi \in W^{2,p'}_0(\mathbb{R}^n_+)$ be one solution of the following equation

$$\Delta\pi = \varphi \;\text{ in } \mathbb{R}^n_+ \quad\text{and}\quad \frac{\partial\pi}{\partial x_n} = 0 \;\text{ on } \Gamma.$$

Let $\psi_i \in W^{2,p'}_0(\mathbb{R}^n_+)$ with $i = 1, ..., n-1$ such that $\psi_i = 0$ and $\dfrac{\partial\psi_i}{\partial x_n} = \dfrac{\partial\pi}{\partial x_i}$ on Γ. We set

$$\boldsymbol{z} = \Big(\frac{\partial\psi_1}{\partial x_n}, ..., \frac{\partial\psi_{n-1}}{\partial x_n}, -\sum_{k=1}^{n-1}\frac{\partial\psi_k}{\partial x_k}\Big).$$

The function $\boldsymbol{u} = \nabla\pi - \boldsymbol{z}$ satisfies $\boldsymbol{u} \in \mathbf{W}^{1,p'}_0(\mathbb{R}^n_+)$, $\mathrm{div}\,\boldsymbol{u} = \varphi$ in $\mathbb{R}^n_+$ and $\boldsymbol{u} = \mathbf{0}$ on Γ. □

Remark 13. The property *i)* of Lemma 4.4 can be rewritten as follows: for any $\boldsymbol{f} \in \mathbf{W}^{-1,p}_0(\mathbb{R}^n_+)$ such that for any $\boldsymbol{v} \in \mathbf{V}^{1,p'}_0(\mathbb{R}^n_+)$ satisfying

$$< \boldsymbol{f}, \boldsymbol{v} > = \mathbf{0},$$

there exists a unique $\pi \in L^p(\mathbb{R}^n_+)$ such that $\boldsymbol{f} = \nabla\pi$ with the following estimate

$$\|\,\pi\,\|_{L^p(\mathbb{R}^n_+)} \;\le\; C\,\|\,\boldsymbol{f}\,\|_{\mathbf{W}^{-1,p}_0(\mathbb{R}^n_+)}.$$

We can improve this result as follows.

Theorem 4.5. *Assume $n \ge 3$ and $1 < p < \infty$. Let $\boldsymbol{f} \in \mathbf{W}^{-1,p}_0(\mathbb{R}^n_+)$ such that*

$$\forall\, \boldsymbol{v} \in \boldsymbol{\mathcal{V}}(\mathbb{R}^n_+), \quad < \boldsymbol{f}, \boldsymbol{v} > = \mathbf{0}.$$

Then $\boldsymbol{f} = \nabla\pi$, with $\pi \in L^p(\mathbb{R}^n_+)$.

Proof. Let $\boldsymbol{\varphi} \in \mathbf{W}_0^{1,p'}(\mathbb{R}^n)$. We define the operator

$$P\boldsymbol{\varphi}(\boldsymbol{x}', x_n) \;=\; \boldsymbol{\varphi}(\boldsymbol{x}', x_n) - \boldsymbol{\varphi}(\boldsymbol{x}', -x_n), \quad x_n > 0.$$

By duality, we also define the operator

$$< P^*\boldsymbol{f}, \boldsymbol{\varphi} > := < \boldsymbol{f}, P\boldsymbol{\varphi} >_{\mathbf{W}_0^{-1,p}(\mathbb{R}^n_+)\times \mathring{\mathbf{W}}_0^{1,p'}(\mathbb{R}^n_+)} .$$

It is clear to see that

$$P^* : \quad \mathbf{W}_0^{-1,p}(\mathbb{R}^n_+) \;\longrightarrow\; \mathbf{W}_0^{-1,p}(\mathbb{R}^n)$$

is continuous. Remark that, thanks to De Rham's Theorem, there exists $\theta \in \mathcal{D}'(\mathbb{R}^n_+)$ such that $\boldsymbol{f} = \nabla\theta$. On the other hand, as $\theta \in \mathcal{D}'(\mathbb{R}^n_+)$ and $\nabla\theta \in \mathbf{W}_0^{-1,p'}(\mathbb{R}^n_+)$, then $\theta \in L^{p'}_{loc}(\overline{\mathbb{R}^n_+})$ and $P^*\nabla\theta = \nabla\theta^*$. Then we have $\nabla\theta^* \in \mathbf{W}_0^{-1,p}(\mathbb{R}^n)$ and $\Delta\theta^* \in W_0^{-2,p}(\mathbb{R}^n)$. Moreover,

$$< \Delta\theta^*, 1 > = \; 0.$$

Thus there exists a unique $\lambda \in L^p(\mathbb{R}^n)$ such that $\Delta(\lambda - \theta^*) = 0$ in $\mathbb{R}^n$. However, $\nabla(\lambda - \theta^*)$ is harmonic and belongs to $\mathbf{W}_0^{-1,p}(\mathbb{R}^n)$. Consequently, $\nabla\theta^* = \nabla\lambda$ and there exists $C \in \mathbb{R}$ such that $\theta^* + C = \lambda$. The function $\pi = \theta + C \in L^p(\mathbb{R}^n_+)$ is the required solution. □

Remark 14. In Lemma 4.3, we have given a constructive proof for the density of the space $\boldsymbol{\mathcal{V}}(\mathbb{R}^n_+)$ in $\mathbf{V}_0^{1,p}(\mathbb{R}^n_+)$. Using Hahn-Banach Theorem and Theorem 4.5, a second proof of this result can be given.

We introduce the following proposition.

Proposition 4.6. *Let* $\boldsymbol{f} \in \mathbf{L}^1(\mathbb{R}^n_+)$ *such that*

$$\forall \boldsymbol{v} \in \mathbf{L}^\infty(\mathbb{R}^n_+) \quad \text{with div } \boldsymbol{v} = 0, \qquad \int_{\mathbb{R}^n_+} \boldsymbol{f} \cdot \boldsymbol{v} = 0.$$

Then there exists a unique $\pi \in L^{n/(n-1)}(\mathbb{R}^n_+)$ *satisfying* $\boldsymbol{f} = \nabla\pi$ *and the following estimate holds*

$$\|\,\pi\,\|_{L^{n/(n-1)}(\mathbb{R}^n_+)} \;\le\; C\,\|\,\boldsymbol{f}\,\|_{\mathbf{L}^1(\mathbb{R}^n_+)}.$$

Proof. Let $u \in L^{n/(n-1)}(\mathbb{R}^n_+)$ satisfying $\nabla u \in \mathbf{L}^1(\mathbb{R}^n_+)$. Then $u^* \in L^{n/(n-1)}(\mathbb{R}^n)$, $\nabla u^* \in \mathbf{L}^1(\mathbb{R}^n)$ and we have the following estimate

$$\|\,u\,\|_{L^{n/(n-1)}(\mathbb{R}^n_+)} \;\le\; \|\,u^*\,\|_{L^{n/(n-1)}(\mathbb{R}^n)} \;\le\; C\,\|\,\nabla u^*\,\|_{\mathbf{L}^1(\mathbb{R}^n)} \;\le\; 2C\,\|\,\nabla u\,\|_{\mathbf{L}^1(\mathbb{R}^n_+)}.$$

The remains of this proof is identical to the one of Corollary 2.2. □

We set $B_a^+ = \{\boldsymbol{x} \in \mathbb{R}^n_+, |\boldsymbol{x}\,| < a\}$ with $a \in \mathbb{R}$ and $a > 0$. We introduce the following lemma.

Lemma 4.7. *Let $f \in L^n_0(B^+_a)$. Then there exists $\boldsymbol{u} \in \mathbf{W}^{1,n}_0(B^+_a) \cap \mathbf{L}^\infty(B^+_a)$ such that* $\operatorname{div} \boldsymbol{u} = f$ *and we have the following estimate*

$$|| \boldsymbol{u} ||_{\mathbf{L}^\infty(B^+_a)} + || \nabla \boldsymbol{u} ||_{\mathbf{L}^n(B^+_a)} \leq C \, || f ||_{L^n(B^+_a)} \tag{44}$$

where C does not depend on a.

Proof. Let $f \in L^n_0(B^+_a)$ and we set $g(\boldsymbol{x}) = a \, f(a \, \boldsymbol{x})$ with $\boldsymbol{x} \in B^+_1$. Then we can deduce $g \in L^n_0(B^+_1)$. Thanks to Theorem 3 [10], there exists $\boldsymbol{v} \in \mathbf{W}^{1,n}_0(B^+_1) \cap \mathbf{L}^\infty(B^+_1)$ such that $\operatorname{div} \boldsymbol{v} = g$ and we have the following estimate

$$|| \boldsymbol{v} ||_{\mathbf{L}^\infty(B^+_1)} + || \nabla \boldsymbol{v} ||_{\mathbf{L}^n(B^+_1)} \leq C \, || g ||_{L^n(B^+_1)}. \tag{45}$$

We now set $\boldsymbol{u}(\boldsymbol{x}) = \boldsymbol{v}(\dfrac{\boldsymbol{x}}{a})$ with $\boldsymbol{x} \in B^+_a$. Then $\operatorname{div} \boldsymbol{u} = f$ and we have

$$\int_{B^+_a} |\nabla \boldsymbol{u}(\boldsymbol{x})|^n d\boldsymbol{x} = \int_{B^+_1} \frac{1}{a^n} |\nabla \boldsymbol{v}(\boldsymbol{t})|^n a^n d\boldsymbol{t} = \int_{B^+_1} |\nabla \boldsymbol{v}(\boldsymbol{t})|^n d\boldsymbol{t}. \tag{46}$$

Besides, we can similarly prove that

$$||\boldsymbol{u}||_{\mathbf{L}^\infty(B^+_a)} = ||\boldsymbol{v}||_{\mathbf{L}^\infty(B^+_1)} \text{ and } || f ||_{L^n(B^+_a)} = || g ||_{L^n(B^+_1)}. \tag{47}$$

The estimate (44) is deduced from (45), (46) and (47) and the proof is finished. □

We now give a similar result of Corollary 4.7 but the one is considered in the half-space.

Theorem 4.8. *Let $f \in L^n(\mathbb{R}^n_+)$. Then there exists $\boldsymbol{u} \in \overset{\circ}{\mathbf{W}}{}^{1,n}_0(\mathbb{R}^n_+) \cap \mathbf{L}^\infty(\mathbb{R}^n_+)$ such that* $\operatorname{div} \boldsymbol{u} = f$. *However, we have the following estimate*

$$|| \boldsymbol{u} ||_{\mathbf{L}^\infty(\mathbb{R}^n_+)} + || \boldsymbol{u} ||_{\mathbf{W}^{1,n}_0(\mathbb{R}^n_+)} \leq C \, || f ||_{L^n(\mathbb{R}^n_+)}. \tag{48}$$

Proof. It is easy to see that there exists a sequence $(f_k)_{k\in\mathbb{N}} \subset \mathcal{D}(\mathbb{R}^n_+)$ converging towards $f \in L^n(\mathbb{R}^n_+)$ because $\mathcal{D}(\mathbb{R}^n_+)$ is dense in $L^n(\mathbb{R}^n_+)$. Let $B^+_{r_k} \subset \mathbb{R}^n_+$ such that $\operatorname{supp} f_k \subset B^+_{r_k}$. Applying Lemma 4.7, there exists $u_k \in \mathbf{W}^{1,n}_0(B^+_{r_k}) \cap \mathbf{L}^\infty(B^+_{r_k})$ such that $\operatorname{div} \boldsymbol{u}_k = f_k$ and we have the following estimate

$$|| \boldsymbol{u}_k ||_{\mathbf{L}^\infty(B^+_{r_k})} + || \nabla \boldsymbol{u}_k ||_{\mathbf{L}^n(B^+_{r_k})} \leq C \, || f_k ||_{L^n(B^+_{r_k})} \leq C \, || f_k ||_{L^n(\mathbb{R}^n_+)}$$

where C only depends on n. By extending $\boldsymbol{u}_k$ in $\mathbb{R}^n_+$ by zero outside $B^+_{r_k}$ and denoting $\widetilde{\boldsymbol{u}_k}$ its extended function, we have

$$|| \widetilde{\boldsymbol{u}_k} ||_{\mathbf{L}^\infty(\mathbb{R}^n_+)} = || \boldsymbol{u}_k ||_{\mathbf{L}^\infty(B^+_{r_k})} \text{ and } || \widetilde{\boldsymbol{u}_k} ||_{\mathbf{W}^{1,n}_0(\mathbb{R}^n_+)} = || \boldsymbol{u}_k ||_{\mathbf{W}^{1,n}_0(B^+_{r_k})}.$$

Then $(\widetilde{\boldsymbol{u}_k})_k$ is bounded in $\overset{\circ}{\mathbf{W}}{}^{1,n}_0(\mathbb{R}^n_+) \cap \mathbf{L}^\infty(\mathbb{R}^n_+)$ and we can deduce that there exists a subsequence, again denoted by $(\widetilde{\boldsymbol{u}_k})_k$ such that $\widetilde{\boldsymbol{u}_k} \rightharpoonup \boldsymbol{u}$ in $\overset{\circ}{\mathbf{W}}{}^{1,n}_0(\mathbb{R}^n_+)$ and $\widetilde{\boldsymbol{u}_k} \overset{*}{\rightharpoonup} \boldsymbol{u}$ in $\mathbf{L}^\infty(\mathbb{R}^n_+)$. Hence, we have $\operatorname{div} \boldsymbol{u} = f$ and the estimate (48). □

Similarly to Corollary 2.4, we have the following result.

Corollary 4.9. *i) There exists $C > 0$ such that for all $u \in L^{n/(n-1)}(\mathbb{R}^n_+)$, we have the following estimate*

$$|| u ||_{L^{n/(n-1)}(\mathbb{R}^n_+)} \ \le \ C \inf_{\boldsymbol{f}+\boldsymbol{g}=\nabla u} (|| \boldsymbol{f} ||_{\mathbf{L}^1(\mathbb{R}^n_+)} + || \boldsymbol{g} ||_{\mathbf{W}_0^{-1,n/(n-1)}(\mathbb{R}^n_+)}) \tag{49}$$

with $\boldsymbol{f} \in \mathbf{L}^1(\mathbb{R}^n_+)$ and $\boldsymbol{g} \in \mathbf{W}_0^{-1,n/(n-1)}(\mathbb{R}^n_+)$.
ii) Let $\boldsymbol{f} \in \mathbf{L}^1(\mathbb{R}^n_+) + \mathbf{W}_0^{-1,n/(n-1)}(\mathbb{R}^n_+)$ satisfying the following compatibility condition

$$\forall \boldsymbol{v} \in \mathbf{V}_0^{1,n}(\mathbb{R}^n_+) \cap \mathbf{L}^\infty(\mathbb{R}^n_+), \quad < \boldsymbol{f}, \ \boldsymbol{v} > = \ 0. \tag{50}$$

Then there exists a unique $\pi \in L^{n/(n-1)}(\mathbb{R}^n_+)$ such that $\boldsymbol{f} = \nabla \pi$.

We define now the space

$$\mathbf{X}(\mathbb{R}^n_+) = \{ \boldsymbol{f} \in \mathbf{L}^1(\mathbb{R}^n_+), \ \mathrm{div}\, \boldsymbol{f} \in W_0^{-2,n/(n-1)}(\mathbb{R}^n_+) \}.$$

Theorem 4.10. *Let $\boldsymbol{f} \in \mathbf{X}(\mathbb{R}^n_+)$. Then $\boldsymbol{f} \in \mathbf{W}_0^{-1,n/(n-1)}(\mathbb{R}^n_+)$ and the following estimate holds*

$$|| \boldsymbol{f} ||_{\mathbf{W}_0^{-1,n/(n-1)}(\mathbb{R}^n_+)} \ \le \ C || \boldsymbol{f} ||_{\mathbf{X}(\mathbb{R}^n_+)}.$$

Proof. The proof is similar to the one of Theorem 2.9. □

Proposition 4.11. *Assume that $\boldsymbol{f} \in \mathbf{X}(\mathbb{R}^n_+)$ satisfying*

$$\forall \, \boldsymbol{v} \in \boldsymbol{\mathcal{V}}(\mathbb{R}^n_+), \quad \int_{\mathbb{R}^n_+} \boldsymbol{f} \cdot \boldsymbol{v} \, d\boldsymbol{x} = 0.$$

Then there exists a unique $\pi \in L^{n/(n-1)}(\mathbb{R}^n_+)$ such that $\boldsymbol{f} = \nabla \pi$ and the following estimate holds

$$|| \pi ||_{L^{n/(n-1)}(\mathbb{R}^n_+)} \ \le \ C || \boldsymbol{f} ||_{W_0^{-1,n/(n-1)}(\mathbb{R}^n_+)}.$$

Proof. This proposition is an immediate consequence of the embedding $\mathbf{X}(\mathbb{R}^n_+) \hookrightarrow \mathbf{W}_0^{-1,n/(n-1)}(\mathbb{R}^n_+)$ and Theorem 4.5. □

We now introduce the following theorem.

Theorem 4.12. *Let $\boldsymbol{\varphi} \in \overset{\circ}{\mathbf{W}}{}_0^{1,n}(\mathbb{R}^n_+)$. Then there exist $\boldsymbol{\psi} \in \overset{\circ}{\mathbf{W}}{}_0^{1,n}(\mathbb{R}^n_+) \cap \mathbf{L}^\infty(\mathbb{R}^n_+)$ and $\eta \in \overset{\circ}{W}{}_0^{2,n}(\mathbb{R}^n_+)$ such that*

$$\boldsymbol{\varphi} = \boldsymbol{\psi} + \nabla \eta.$$

Moreover, we have the following estimate

$$||\boldsymbol{\psi}||_{\mathbf{W}_0^{1,n}(\mathbb{R}^n_+)} + ||\boldsymbol{\psi}||_{\mathbf{L}^\infty(\mathbb{R}^n_+)} + ||\eta||_{W_0^{2,n}(\mathbb{R}^n_+)} \le C||\boldsymbol{\varphi}||_{\mathbf{W}_0^{1,n}(\mathbb{R}^n_+)}. \tag{51}$$

Proof. We know that $\boldsymbol{\mathcal{D}}(\mathbb{R}^n_+)$ is dense in $\overset{\circ}{\mathbf{W}}{}^{1,n}_0(\mathbb{R}^n_+)$, then there exists a sequence $(\boldsymbol{\varphi_k})_{k\in\mathbb{N}} \in \boldsymbol{\mathcal{D}}(\mathbb{R}^n_+)$ which converges towards $\boldsymbol{\varphi} \in \overset{\circ}{\mathbf{W}}{}^{1,n}_0(\mathbb{R}^n_+)$. Let now $B^+_{r_k}$ such that $\operatorname{supp}\boldsymbol{\varphi_k} \subset B^+_{r_k}$ and we set $\boldsymbol{\varphi}'_k(\boldsymbol{x}) = \boldsymbol{\varphi}_k(r_k\,\boldsymbol{x})$. Then we deduce $\boldsymbol{\varphi}'_k \in \mathbf{W}^{1,n}_0(B^+_1)$. Thanks to Theorem 2.6, there exist $\boldsymbol{\psi}'_k \in \mathbf{W}^{1,n}_0(B^+_1) \cap \mathbf{L}^\infty(B^+_1)$ and $\eta'_k \in W^{2,n}_0(B^+_1)$ such that $\boldsymbol{\varphi}'_k = \boldsymbol{\psi}'_k + \nabla\eta'_k$, with the following estimate

$$|| \nabla\boldsymbol{\psi}'_k ||_{\mathbf{L}^n(B^+_1)} + || \boldsymbol{\psi}'_k ||_{\mathbf{L}^\infty(B^+_1)} + || D^2\eta'_k ||_{\mathbf{L}^n(B^+_1)} \leq C\, || \nabla\boldsymbol{\varphi}'_k ||_{\mathbf{L}^n(B^+_1)}.$$

We now set

$$\boldsymbol{\psi}_k(\boldsymbol{x}) = \boldsymbol{\psi}'_k(\frac{\boldsymbol{x}}{r_k}) \quad \text{and} \quad \eta_k(\boldsymbol{x}) = r_k\,\eta'_k(\frac{\boldsymbol{x}}{r_k}).$$

Then we have $\boldsymbol{\varphi}_k = \boldsymbol{\psi}_k + \nabla\eta_k$. Proceeding similarly as in the proof of Corollary 2.7 and by passing to limit, we can show $\boldsymbol{\varphi} = \boldsymbol{\psi} + \nabla\eta$ with the following estimate

$$|| \nabla\boldsymbol{\psi} ||_{\mathbf{L}^n(\mathbb{R}^n_+)} + || \boldsymbol{\psi} ||_{\mathbf{L}^\infty(\mathbb{R}^n_+)} + || D^2\,\eta ||_{\mathbf{L}^n(\mathbb{R}^n_+)} \leq C|| \nabla\boldsymbol{\varphi} ||_{\mathbf{L}^n(\mathbb{R}^n_+)}.$$

The estimate (51) follows from the fact that the semi-norm $|| \nabla . ||_{\mathbf{L}^n(\mathbb{R}^n_+)}$ (respectively, $|| D^2 . ||_{\mathbf{L}^n(\mathbb{R}^n_+)}$) defines on $\overset{\circ}{\mathbf{W}}{}^{1,n}_0(\mathbb{R}^n_+)$ (respectively, $\overset{\circ}{W}{}^{2,n}_0(\mathbb{R}^n_+)$) a norm which is equivalent to the norm $|| . ||_{\mathbf{W}^{1,n}_0(\mathbb{R}^n_+)}$ (respectively, $|| . ||_{W^{2,n}_0(\mathbb{R}^n_+)}$). $\square$

Proposition 4.13. *Let $\boldsymbol{\varphi} \in \mathbf{W}^{1,n}_0(\mathbb{R}^n_+)$. Then there exists $\boldsymbol{\psi} \in \mathbf{W}^{1,n}_0(\mathbb{R}^n_+) \cap \mathbf{L}^\infty(\mathbb{R}^n_+)$ and $\eta \in W^{2,n}_0(\mathbb{R}^n_+)$ such that $\psi_n = 0$ on Γ satisfying*

$$\boldsymbol{\varphi} = \boldsymbol{\psi} + \nabla\eta$$

and the following estimate holds

$$|| \boldsymbol{\psi} ||_{\mathbf{L}^\infty(\mathbb{R}^n_+)} + || \boldsymbol{\psi} ||_{\mathbf{W}^{1,n}_0(\mathbb{R}^n_+)} + || \eta ||_{W^{2,n}_0(\mathbb{R}^n_+)} \leq C\, || \boldsymbol{\varphi} ||_{\mathbf{W}^{1,n}_0(\mathbb{R}^n_+)}. \tag{52}$$

Proof. As in Theorem 4.12, we can prove that $\boldsymbol{\varphi} = \boldsymbol{\psi}_0 + \nabla\eta_0$ such that $\boldsymbol{\psi}_0 \in \mathbf{W}^{1,n}_0(\mathbb{R}^n_+) \cap \mathbf{L}^\infty(\mathbb{R}^n_+)$ and $\eta_0 \in W^{2,n}_0(\mathbb{R}^n_+)$ with the following estimate

$$|| \boldsymbol{\psi}_0 ||_{\mathbf{L}^\infty(\mathbb{R}^n_+)} + || \boldsymbol{\psi}_0 ||_{\mathbf{W}^{1,n}_0(\mathbb{R}^n_+)} + || \eta_0 ||_{W^{2,n}_0(\mathbb{R}^n_+)} \leq C\, || \boldsymbol{\varphi} ||_{\mathbf{W}^{1,n}_0(\mathbb{R}^n_+)}.$$

Setting $\mu = \psi_{0n}$ on Γ, then $\mu \in L^\infty(\Gamma) \cap W^{1-1/n,n}_0(\Gamma)$. We can prove similarly as in Lemma 3.10 [15] that there exists $v \in W^{2,n}_0(\mathbb{R}^n_+) \cap W^{1,\infty}(\mathbb{R}^n_+)$ such that $v = 0$ and $\dfrac{\partial v}{\partial x_n} = \mu$ on Γ with the following estimates

$$|| v ||_{W^{2,n}_0(\mathbb{R}^n_+)} \leq C\, || \mu ||_{W^{1-1/n,n}_0(\Gamma)} \leq C\, || \boldsymbol{\psi}_0 ||_{\mathbf{W}^{1,n}_0(\mathbb{R}^n_+)}$$

and

$$|| \nabla v ||_{\mathbf{L}^\infty(\mathbb{R}^n_+)} \leq C\, || \mu ||_{L^\infty(\Gamma)} \leq C\, || \boldsymbol{\psi}_0 ||_{\mathbf{L}^\infty(\mathbb{R}^n_+)}.$$

The proof is complete by setting $\boldsymbol{\psi} = \boldsymbol{\psi}_0 - \nabla v$ and $\eta = \eta_0 + v$. $\square$

Remark 15. We can give another proof of the existence of $\boldsymbol{\varphi}$. We know that $\boldsymbol{f} \in \mathbf{W}_0^{-1,3/2}(\mathbb{R}^3_+) \perp \mathbb{R}^3$. Then, there exists a unique $\boldsymbol{z} \in \overset{\circ}{\mathbf{W}}{}_0^{1,3/2}(\mathbb{R}^3_+)$ satisfying $-\Delta \boldsymbol{z} = \boldsymbol{f}$, with div $\boldsymbol{z} = 0$ in $\mathbb{R}^3_+$. The function $\boldsymbol{\varphi} = \mathbf{curl}\, \boldsymbol{z}$ is the required function.

Proposition 4.14. *Let $\boldsymbol{\varphi} \in \mathbf{W}_0^{1,3}(\mathbb{R}^3_+)$. Then there exist $\boldsymbol{\psi} \in \mathbf{W}_0^{1,3}(\mathbb{R}^3_+) \cap \mathbf{L}^\infty(\mathbb{R}^3_+)$ and $\boldsymbol{\eta} \in \mathbf{W}_0^{2,3}(\mathbb{R}^3_+)$ such that*

$$\boldsymbol{\varphi} = \boldsymbol{\psi} + \mathbf{curl}\, \boldsymbol{\eta} \quad \text{with} \quad \boldsymbol{\psi}' = \mathbf{0} \text{ on } \Gamma$$

and we have the following estimate

$$||\, \boldsymbol{\psi}\, ||_{\mathbf{W}_0^{1,3}(\mathbb{R}^3_+)} + ||\, \boldsymbol{\psi}\, ||_{\mathbf{L}^\infty(\mathbb{R}^3_+)} + ||\, \boldsymbol{\eta}\, ||_{\mathbf{W}_0^{2,3}(\mathbb{R}^3_+)} \;\leq\; C\, ||\, \boldsymbol{\varphi}\, ||_{\mathbf{W}_0^{1,3}(\mathbb{R}^3_+)}.$$

Proof. Let $\boldsymbol{\varphi} \in \mathbf{W}_0^{1,3}(\mathbb{R}^3_+)$. Then $\boldsymbol{\varphi}^* \in \mathbf{W}_0^{1,3}(\mathbb{R}^3)$ (see at the beginning of this section for the notations), we can use Proposition 2.13: there exist $\boldsymbol{\psi}_0 \in \mathbf{W}_0^{1,3}(\mathbb{R}^3) \cap \mathbf{L}^\infty(\mathbb{R}^3)$ and $\boldsymbol{\eta}_0 \in \mathbf{W}_0^{2,3}(\mathbb{R}^3)$ such that $\boldsymbol{\varphi}^* = \boldsymbol{\psi_0} + \mathbf{curl}\, \boldsymbol{\eta_0}$ in $\mathbb{R}^3$, with the following estimate

$$||\boldsymbol{\psi}_0||_{\mathbf{W}_0^{1,3}(\mathbb{R}^3)} + ||\boldsymbol{\psi_0}||_{\mathbf{L}^\infty(\mathbb{R}^3)} + ||\boldsymbol{\eta}_0||_{\mathbf{W}_0^{2,3}(\mathbb{R}^3)} \;\leq\; C\, ||\boldsymbol{\varphi}||_{\mathbf{W}_0^{1,3}(\mathbb{R}^3_+)}.$$

As in the proof of Lemma 3.10 of [15], we can prove that there exist $\boldsymbol{\alpha} \in \mathbf{W}_0^{2,3}(\mathbb{R}^3_+)$ with $\nabla \boldsymbol{\alpha} \in \boldsymbol{L}^\infty(\mathbb{R}^3_+)$ satisfying $\boldsymbol{\alpha} = \mathbf{0}$ and $\frac{\partial \boldsymbol{\alpha}}{\partial x_3} = \boldsymbol{\psi_0}$ on Γ. Moreover, we have the estimate

$$||\boldsymbol{\alpha}||_{\mathbf{W}_0^{2,3}(\mathbb{R}^3_+)} + ||\nabla \boldsymbol{\alpha}||_{\mathbf{L}^\infty(\mathbb{R}^3_+)} \;\leq\; C\, (||\boldsymbol{\psi}_0||_{\mathbf{W}_0^{1,3}(\mathbb{R}^3)} + ||\boldsymbol{\psi_0}||_{\mathbf{L}^\infty(\mathbb{R}^3)}).$$

One has thus the conclusion with $\boldsymbol{\psi} = \boldsymbol{\psi}_0 - \mathbf{curl}\ \boldsymbol{\alpha}$ and $\boldsymbol{\eta} = \boldsymbol{\eta}_0 + \boldsymbol{\alpha}$. □

Remark 16. We have another result with the data $\boldsymbol{\varphi} \in \overset{\circ}{\mathbf{W}}{}_0^{1,3}(\mathbb{R}^3_+)$ (see Proposition 5.7).

Proposition 4.15. *i) If $\boldsymbol{f} \in \mathbf{L}^1(\mathbb{R}^3_+)$ with $\operatorname{div} \boldsymbol{f} = 0$, then $\displaystyle\int_{\mathbb{R}^3_+} f_3 = 0$.*

ii) If moreover $f_3 = 0$ on Γ, then $\displaystyle\int_{\mathbb{R}^3_+} \boldsymbol{f} = \mathbf{0}$.

Proof. i) Setting $\widetilde{\boldsymbol{f}} = (\boldsymbol{f}'_*,\ f_3^*)$. Then $\widetilde{\boldsymbol{f}} \in \mathbf{L}^1(\mathbb{R}^3)$, $\operatorname{div} \widetilde{\boldsymbol{f}} = 0$ in $\mathbb{R}^3$ and therefore $\displaystyle\int_{\mathbb{R}^3} \widetilde{\boldsymbol{f}} = 0$, *i.e.* $\displaystyle\int_{\mathbb{R}^3_+} f_3 = 0$.

ii) We have known $\displaystyle\int_{\mathbb{R}^3_+} f_3 = 0$. Setting $\widetilde{\boldsymbol{f}} = (\boldsymbol{f}'^*,\ f_{3*})$. Then $\widetilde{\boldsymbol{f}} \in \mathbf{L}^1(\mathbb{R}^3)$, $\operatorname{div} \widetilde{\boldsymbol{f}} = 0$ in $\mathbb{R}^3$ and therefore $\displaystyle\int_{\mathbb{R}^3} \widetilde{\boldsymbol{f}} = \mathbf{0}$ that implies $\displaystyle\int_{\mathbb{R}^3} \boldsymbol{f}' = \mathbf{0}$. □

Remark 17. i) Let $\boldsymbol{f} \in \mathbf{L}^1(\mathbb{R}^3_+)$ that satisfies the property: "For any $\eta \in W^{2,3}_0(\mathbb{R}^3_+)$ such that $\nabla\eta \in \mathbf{L}^\infty(\mathbb{R}^3_+)$ and $\eta = 0$ on Γ, we have $\displaystyle\int_{\mathbb{R}^3_+} \boldsymbol{f}\cdot\nabla\eta = 0$."
By taking $\eta = x_3$, then we obtain $\displaystyle\int_{\mathbb{R}^3_+} f_3 = 0$.
ii) Let $\boldsymbol{f} \in \mathbf{L}^1(\mathbb{R}^3_+)$ that satisfies the property:
"For any $\eta \in W^{2,3}_0(\mathbb{R}^3_+)$ such that $\nabla\eta \in \mathbf{L}^\infty(\mathbb{R}^3)$, we have $\displaystyle\int_{\mathbb{R}^3_+} \boldsymbol{f}\cdot\nabla\eta = 0$". By taking $\eta = x_i$ with $i = 1, 2$ and after taking $\eta = x_3$, we find $\displaystyle\int_{\mathbb{R}^3_+} \boldsymbol{f} = \mathbf{0}$.

5. Vector potentials in the half-space

Proposition 5.1. *Let* $\boldsymbol{f} \in \mathbf{L}^3(\mathbb{R}^3_+)$ *such that* $\operatorname{div}\boldsymbol{f} = 0$ *in* $\mathbb{R}^3_+$. *Then there exists* $\boldsymbol{\varphi} \in \mathbf{W}^{1,3}_0(\mathbb{R}^3_+)$ *such that* $\boldsymbol{f} = \mathbf{curl}\,\boldsymbol{\varphi}$ *with* $\operatorname{div}\boldsymbol{\varphi} = 0$ *in* $\mathbb{R}^3_+$, $\varphi_3 = 0$ *on* Γ *and we have the following estimate*

$$||\,\boldsymbol{\varphi}\,||_{\mathbf{W}^{1,3}_0(\mathbb{R}^3_+)} \;\leq\; C\,||\,\boldsymbol{f}||_{\mathbf{L}^3(\mathbb{R}^3_+)}.$$

Proof. *i)* Setting $\widetilde{\boldsymbol{f}} = (\boldsymbol{f}'_*,\ f^*_3)$. It is easy to show $\widetilde{\boldsymbol{f}} \in \mathbf{L}^3(\mathbb{R}^3)$ and $\operatorname{div}\widetilde{\boldsymbol{f}} = 0$ in $\mathbb{R}^3$. Let $\boldsymbol{\theta} \in \mathbf{W}^{1,3}_0(\mathbb{R}^3)$ such that $-\Delta\boldsymbol{\theta} = \mathbf{curl}\,\widetilde{\boldsymbol{f}}$. Then $\operatorname{div}\boldsymbol{\theta} = 0$ and $\mathbf{curl}\,(\mathbf{curl}\,\boldsymbol{\theta} - \widetilde{\boldsymbol{f}}) = \mathbf{0}$ in $\mathbb{R}^3$. We now set $\boldsymbol{z} = \mathbf{curl}\,\boldsymbol{\theta} - \widetilde{\boldsymbol{f}}$ in $\mathbb{R}^3$ and can deduce $\boldsymbol{z} \in \mathbf{L}^3(\mathbb{R}^3)$, $\operatorname{div}\boldsymbol{z} = 0$ and $\mathbf{curl}\,\boldsymbol{z} = \mathbf{0}$ in $\mathbb{R}^3$. As $\Delta\boldsymbol{z} = \mathbf{0}$, then $\boldsymbol{z} = \mathbf{0}$ in $\mathbb{R}^3$. It means that $\widetilde{f} = \mathbf{curl}\,\boldsymbol{\theta}$ in $\mathbb{R}^3$ and $\boldsymbol{f} = \mathbf{curl}\,\boldsymbol{\theta}|_{\mathbb{R}^3_+}$ with $\operatorname{div}\boldsymbol{\theta} = 0$ in $\mathbb{R}^3_+$.
ii) Let $h \in W^{2,3}_0(\mathbb{R}^3_+)$ be a solution of the following equation

$$\Delta\, h \;=\; 0 \;\text{ in } \mathbb{R}^3_+ \quad \text{ and } \quad \frac{\partial h}{\partial x_3} \;=\; \theta_3 \;\text{ on } \Gamma.$$

Setting $\boldsymbol{\varphi} = \boldsymbol{\theta} - \nabla\, h$. Then we have $\boldsymbol{\varphi} \in \mathbf{W}^{1,3}_0(\mathbb{R}^3_+)$, $\operatorname{div}\boldsymbol{\varphi} = 0$ in $\mathbb{R}^3_+$ and $\varphi_3 = 0$ on Γ. $\square$

Theorem 5.2. *Let* $\boldsymbol{f} \in \mathbf{L}^3(\mathbb{R}^3_+)$ *such that* $\operatorname{div}\boldsymbol{f} = 0$ *in* $\mathbb{R}^3_+$ *and* $f_3 = 0$ *on* Γ. *Then there exists a unique* $\boldsymbol{\varphi} \in \mathbf{W}^{1,3}_0(\mathbb{R}^3_+)$ *such that* $\boldsymbol{f} = \mathbf{curl}\,\boldsymbol{\varphi}$ *with* $\operatorname{div}\boldsymbol{\varphi} = 0$ *in* $\mathbb{R}^3_+$ *and* $\boldsymbol{\varphi}' = \mathbf{0}$ *on* Γ. *Moreover, we have the following estimate*

$$||\,\boldsymbol{\varphi}\,||_{\mathbf{W}^{1,3}_0(\mathbb{R}^3_+)} \;\leq\; C\,||\,\boldsymbol{f}\,||_{\mathbf{L}^3(\mathbb{R}^3_+)}.$$

Proof. *i)* We start this proof by showing the uniqueness of $\boldsymbol{\varphi}$. Indeed, if $\boldsymbol{\varphi} \in \mathbf{W}^{1,3}_0(\mathbb{R}^3_+)$ such that $\mathbf{curl}\,\boldsymbol{\varphi} = \mathbf{0}$, $\operatorname{div}\boldsymbol{\varphi} = 0$ in $\mathbb{R}^3_+$ and $\boldsymbol{\varphi}' = \mathbf{0}$ on Γ, then there exists $q \in W^{2,3}_0(\mathbb{R}^3_+)$ such that $\boldsymbol{\varphi} = \nabla\, q$ in $\mathbb{R}^3_+$ with $\Delta\, q = 0$ in $\mathbb{R}^3_+$ and q is a constant on Γ. Consequently, q is a constant in $\mathbb{R}^3_+$ and we then deduce $\boldsymbol{\varphi} = \mathbf{0}$ in $\mathbb{R}^3_+$.
ii) We now consider the existence of $\boldsymbol{\varphi}$. Setting $\widetilde{\boldsymbol{f}} = (\boldsymbol{f}'^*,\ f_{3*})$. Then we have

$\widetilde{\boldsymbol{f}} \in \mathbf{L}^3(\mathbb{R}^3)$ and $\operatorname{div} \widetilde{\boldsymbol{f}} = 0$ in $\mathbb{R}^3$. Therefore, by Proposition 3.3, there exists $\boldsymbol{z} \in \mathbf{W}_0^{1,3}(\mathbb{R}^3)$ such that $\mathbf{curl}\, \boldsymbol{z} = \widetilde{\boldsymbol{f}}$ and $\operatorname{div} \boldsymbol{z} = 0$. Let now $\boldsymbol{w}$ be the vector field defined on $\mathbb{R}^3_-$ by

$$\boldsymbol{w}(\boldsymbol{x}', x_3) = (-z_1(\boldsymbol{x}', -x_3), -z_2(\boldsymbol{x}', -x_3), z_3(\boldsymbol{x}', -x_3)), \quad x_3 < 0. \tag{53}$$

By some easy calculations, we can show $\mathbf{curl}\, \boldsymbol{w} = \mathbf{curl}\, \boldsymbol{z}$ in $\mathbb{R}^3_-$. Then we have

$$\boldsymbol{w} = \boldsymbol{z} + \nabla \mu \quad \text{in } \mathbb{R}^3_- \tag{54}$$

where $\mu \in \mathcal{D}'(\mathbb{R}^3_-)$. As $\nabla \mu$ belongs to $\mathbf{W}_0^{1,3}(\mathbb{R}^3_-)$, we can show $\mu \in W_0^{2,3}(\mathbb{R}^3_-)$. Let $\mu_0 \in W_0^{2,3}(\mathbb{R}^3_+)$ (cf. [6]) such that

$$\Delta \mu_0 = 0 \quad \text{in } \mathbb{R}^3_+ \quad \text{and} \quad \mu_0 = \mu \quad \text{on } \Gamma.$$

We set $\boldsymbol{\varphi} = \boldsymbol{z} + \frac{1}{2} \nabla \mu_0$. Then we have $\mathbf{curl}\, \boldsymbol{\varphi} = \boldsymbol{f}$ and $\operatorname{div} \boldsymbol{\varphi} = 0$ in $\mathbb{R}^3_+$. Applying now the trace operator to the relation (54), we deduce that

$$2\boldsymbol{z}' = -\nabla' \mu = -\nabla' \mu_0 \quad \text{on } \Gamma,$$

i.e., $\boldsymbol{\varphi}' = \mathbf{0}$ on Γ. $\square$

Corollary 5.3. *Let* $\boldsymbol{f} \in \mathbf{L}^3(\mathbb{R}^3_+)$ *such that* $\operatorname{div} \boldsymbol{f} = 0$ *in* $\mathbb{R}^3_+$ *and* $f_3 = 0$ *on* Γ.
i) There exists a unique $\boldsymbol{\varphi} \in \mathring{\mathbf{W}}_0^{1,3}(\mathbb{R}^3_+)$ *such that* $\operatorname{div} \Delta \boldsymbol{\varphi} = 0$ *satisfying* $\boldsymbol{f} = \mathbf{curl}\, \boldsymbol{\varphi}$ *in* $\mathbb{R}^3_+$*. Moreover, we have the following estimate*

$$\| \boldsymbol{\varphi} \|_{\mathbf{W}_0^{1,3}(\mathbb{R}^3_+)} \leq C \| \boldsymbol{f} \|_{\mathbf{L}^3(\mathbb{R}^3_+)}.$$

ii) There exists $\boldsymbol{\psi} \in \mathring{\mathbf{W}}_0^{1,3}(\mathbb{R}^3_+) \cap \mathbf{L}^\infty(\mathbb{R}^3_+)$ *such that* $\boldsymbol{f} = \mathbf{curl}\, \boldsymbol{\psi}$ *in* $\mathbb{R}^3_+$ *and we have the following estimate*

$$\| \boldsymbol{\psi} \|_{\mathbf{W}_0^{1,3}(\mathbb{R}^3_+)} + \| \boldsymbol{\psi} \|_{\mathbf{L}^\infty(\mathbb{R}^3_+)} \leq C \| \boldsymbol{f} \|_{\mathbf{L}^3(\mathbb{R}^3_+)}. \tag{55}$$

Proof. *i)* First of all, we start this proof by showing the uniqueness of $\boldsymbol{\varphi}$. Let $\boldsymbol{\varphi} \in \mathring{\mathbf{W}}_0^{1,3}(\mathbb{R}^3_+)$ such that $\mathbf{curl}\, \boldsymbol{\varphi} = \mathbf{0}$ and $\operatorname{div} \Delta \boldsymbol{\varphi} = 0$ in $\mathbb{R}^3_+$. Then there exists $q \in W_0^{2,3}(\mathbb{R}^3_+)$ such that $\boldsymbol{\varphi} = \nabla q$ with $\Delta^2 q = 0$ in $\mathbb{R}^3_+$, q is a constant on Γ and $\dfrac{\partial q}{\partial x_3} = 0$ on Γ. We can deduce that q is a constant in $\mathbb{R}^3_+$ and then $\boldsymbol{\varphi} = \mathbf{0}$ in $\mathbb{R}^3_+$.
We now consider the existence $\boldsymbol{\varphi}$. Let $\chi \in W_0^{2,3}(\mathbb{R}^3_+)$ be a solution of the following system

$$\Delta^2 \chi = 0 \text{ in } \mathbb{R}^3_+, \quad \chi = 0 \text{ on } \Gamma, \quad \frac{\partial \chi}{\partial x_3} = z_3 \text{ on } \Gamma,$$

where $\boldsymbol{z}$ is the vector potential given by Theorem 5.2. We set $\boldsymbol{\varphi} = \boldsymbol{z} - \nabla \chi$. Then $\boldsymbol{f} = \mathbf{curl}\, \boldsymbol{\varphi}$ with $\boldsymbol{\varphi} \in \mathring{\mathbf{W}}_0^{1,3}(\mathbb{R}^3_+)$.

ii) The proof is easily obtained by applying Theorem 4.12. $\square$

From Theorem 5.2 and Corollary 5.3, we have the following Helmholtz decompositions.

Corollary 5.4. *Let* $\boldsymbol{f} \in \mathbf{L}^3(\mathbb{R}^3_+)$.
i) There exist $\pi \in W_0^{1,3}(\mathbb{R}^3_+)$ *unique up to an additive constant and a unique* $\boldsymbol{\varphi} \in \mathbf{W}_0^{1,3}(\mathbb{R}^3_+)$ *such that* $\operatorname{div} \boldsymbol{\varphi} = 0$ *in* $\mathbb{R}^3_+$ *and* $\boldsymbol{\varphi}' = \mathbf{0}$ *on* Γ *satisfying*

$$\boldsymbol{f} = \mathbf{curl}\, \boldsymbol{\varphi} + \nabla \pi. \tag{56}$$

Moreover, we have the following estimate

$$|| \boldsymbol{\varphi} ||_{\mathbf{W}_0^{1,3}(\mathbb{R}^3_+)} + ||\nabla \pi||_{\mathbf{L}^3(\mathbb{R}^3_+)} \leq C \, || \boldsymbol{f} ||_{\mathbf{L}^3(\mathbb{R}^3_+)}. \tag{57}$$

ii) There exist a unique $\boldsymbol{\varphi} \in \overset{\circ}{\mathbf{W}}{}_0^{1,3}(\mathbb{R}^3_+)$ *with* $\operatorname{div} \Delta \boldsymbol{\varphi} = 0$ *in* $\mathbb{R}^3_+$ *and* $\pi \in W_0^{1,3}(\mathbb{R}^3_+)$ *unique up to an additive constant satisfying* (56) *with the corresponding estimate.*

iii) There exist $\boldsymbol{\varphi} \in \overset{\circ}{\mathbf{W}}{}_0^{1,3}(\mathbb{R}^3_+) \cap \mathbf{L}^\infty(\mathbb{R}^3_+)$ *and* $\pi \in W_0^{1,3}(\mathbb{R}^3_+)$ *unique up to an additive constant and satisfying* (56) *with the corresponding estimate.*

Proof. *i)* Let $\boldsymbol{f} \in \mathbf{L}^3(\mathbb{R}^3_+)$. We consider the following problem $(\mathcal{P})$: Find $\pi \in W_0^{1,3}(\mathbb{R}^3_+)$ such that

$$\forall \mu \in W_0^{1,3/2}(\mathbb{R}^3_+), \quad \int_{\mathbb{R}^3_+} \nabla \pi \cdot \nabla \mu = \int_{\mathbb{R}^3_+} \boldsymbol{f} \cdot \nabla \mu.$$

Thanks to C. Amrouche [3], we know that the problem $(\mathcal{P})$ has a solution π, unique up to a constant, satisfying the following estimate

$$|| \nabla \pi ||_{\mathbf{L}^3(\mathbb{R}^3_+)} \leq C \, || \boldsymbol{f} ||_{\mathbf{L}^3(\mathbb{R}^3_+)}.$$

The function π is also solution of the problem as follows

$$\Delta \pi = \operatorname{div} \boldsymbol{f} \quad \text{in } \mathbb{R}^3_+ \quad \text{and} \quad \frac{\partial \pi}{\partial x_3} - f_3 = 0 \quad \text{on } \Gamma.$$

We set $\boldsymbol{h} = \boldsymbol{f} - \nabla \pi$. It is easy to see $\boldsymbol{h} \in \mathbf{L}^3(\mathbb{R}^3_+)$, $\operatorname{div} \boldsymbol{h} = 0$ in $\mathbb{R}^3_+$ and $h_3 = 0$ on Γ. Applying Theorem 5.2, we can decompose $\boldsymbol{f}$ as in (56) and we obtain the estimate (57).

ii) Proceeding similarly as in the case *i)* of this corollary, but at the end of this proof, instead of apply Theorem 5.2, we use Corollary 5.3 part *i)* to obtain (56).

iii) This proof is complete by proceeding similarly as the precendent cases and by applying Corollary 5.3 part *ii)*. $\square$

Remark 18. Using the above proof of part *i)*, it is easy to see that any $\boldsymbol{f} \in \boldsymbol{\mathcal{D}}(\mathbb{R}^3_+)$ can be uniquely decomposed as the form (56) with $\boldsymbol{\varphi} \in \mathbf{W}_0^{1,q}(\mathbb{R}^3_+)$ and $\pi \in W_0^{1,q}(\mathbb{R}^3_+)$ for all $q > 1$ with the corresponding estimate similar to (57) where $C = C(q)$.

Corollary 5.5. *Let* $\boldsymbol{f} \in \mathbf{L}^1(\mathbb{R}^3_+)$ *such that* $\operatorname{div} \boldsymbol{f} = 0$*. Then, for every* $\boldsymbol{\varphi} \in \mathring{\mathbf{W}}{}_0^{1,3}(\mathbb{R}^3_+)$*, we have the following estimate*

$$| < \boldsymbol{f}, \boldsymbol{\varphi} >_{\mathbf{W}_0^{-1,3/2}(\mathbb{R}^3_+) \times \mathring{\mathbf{W}}_0^{1,3}(\mathbb{R}^3_+)} | \leq C \, || \, \boldsymbol{f} \, ||_{\mathbf{L}^1(\mathbb{R}^3_+)} ||\mathbf{curl} \, \boldsymbol{\varphi}||_{\mathbf{L}^3(\mathbb{R}^3_+)}.$$

Proof. This proof is similar to the one of Corollary 3.5. □

Lemma 5.6. *Assume* $\boldsymbol{f} \in \mathring{\mathbf{W}}{}_0^{1,3}(\mathbb{R}^3_+)$ *such that* $\operatorname{div} \boldsymbol{f} = 0$*. Then there exists a unique* $\boldsymbol{\varphi} \in \mathring{\mathbf{W}}{}_0^{2,3}(\mathbb{R}^3_+)$ *such that* $\boldsymbol{f} = \mathbf{curl} \, \boldsymbol{\varphi}$ *with* $\operatorname{div} \Delta^2 \boldsymbol{\varphi} = 0$ *in* $\mathbb{R}^3_+$*. Moreover, we have the following estimate*

$$|| \, \boldsymbol{\varphi} \, ||_{\mathbf{W}_0^{2,3}(\mathbb{R}^3_+)} \leq C \, || \, \boldsymbol{f} \, ||_{\mathbf{W}_0^{1,3}(\mathbb{R}^3_+)}.$$

Proof. *i) First step:* Let $\boldsymbol{f} \in \mathring{\mathbf{W}}{}_0^{1,3}(\mathbb{R}^3_+)$ and $\operatorname{div} \boldsymbol{f} = 0$. We extend $\boldsymbol{f}$ to $\mathbb{R}^3$ as in the proof of Theorem 5.2 and denote $\widetilde{\boldsymbol{f}}$ the extended function that belongs to $\mathbf{W}_0^{1,3}(\mathbb{R}^3)$ and is divergence free. Then, from [4], there exists $\boldsymbol{y} \in \mathbf{W}_0^{2,3}(\mathbb{R}^3)$ such that

$$-\Delta \, \boldsymbol{y} = \mathbf{curl} \, \widetilde{\boldsymbol{f}} \quad \text{in } \mathbb{R}^3.$$

As $\operatorname{div} \boldsymbol{y} \in W_0^{1,3}(\mathbb{R}^3)$ and is harmonic, then $\operatorname{div} \boldsymbol{y} = a$ where a is a constant. Thus, we have $\mathbf{curl} \, (\mathbf{curl} \, \boldsymbol{y} - \widetilde{\boldsymbol{f}}) = \mathbf{0}$. Therfore, we deduce

$$-\Delta(\mathbf{curl} \, \boldsymbol{y}) = \mathbf{curl} \, \mathbf{curl} \, \widetilde{\boldsymbol{f}} = -\Delta \, \widetilde{\boldsymbol{f}},$$

it means that $\widetilde{\boldsymbol{f}} - \mathbf{curl} \, \boldsymbol{y} \in \mathbf{W}_0^{1,3}(\mathbb{R}^3)$ and again, $\widetilde{\boldsymbol{f}} - \mathbf{curl} \, \boldsymbol{y} = \boldsymbol{b}$ where $\boldsymbol{b}$ is a constant vector in $\mathbb{R}^3$. Thanks to Lemma 3.1 [17], there exists a polynomial $\boldsymbol{s} \in \mathscr{P}_1$ such that $\mathbf{curl} \, \boldsymbol{s} = \boldsymbol{b}$ and $\operatorname{div} \boldsymbol{s} = -a$. The function $\boldsymbol{z} := \boldsymbol{y} + \boldsymbol{s}$ belongs to $\mathbf{W}_0^{2,3}(\mathbb{R}^3)$ and $\widetilde{\boldsymbol{f}} = \mathbf{curl} \, \boldsymbol{z}$ in $\mathbb{R}^3$ with $\operatorname{div} \boldsymbol{z} = 0$ in $\mathbb{R}^3$. Let $\boldsymbol{w}$ be the vector field defined on $\mathbb{R}^3_-$ by

$$\boldsymbol{w}(\boldsymbol{x}', x_3) = (-z_1(\boldsymbol{x}', -x_3), -z_2(\boldsymbol{x}', -x_3), z_3(\boldsymbol{x}', -x_3)), \quad x_3 < 0.$$

It is easy to show $\mathbf{curl} \, \boldsymbol{w} = \mathbf{curl} \, \boldsymbol{z}$ in $\mathbb{R}^3_-$. Then we have $\boldsymbol{w} = \boldsymbol{z} + \nabla \theta$, with $\theta \in \mathcal{D}'(\mathbb{R}^3_-)$. Proceeding as in the end of the proof of Theorem 5.2, we can deduce that there exists $\boldsymbol{\zeta} \in \mathbf{W}_0^{2,3}(\mathbb{R}^3_+)$ such that

$$\boldsymbol{f} = \mathbf{curl} \, \boldsymbol{\zeta} \quad \text{and} \quad \operatorname{div} \boldsymbol{\zeta} = 0 \ \text{ in } \mathbb{R}^3_+, \quad \boldsymbol{\zeta}' = \mathbf{0} \ \text{ on } \Gamma.$$

ii) Second step: Let $\chi \in W_0^{3,3}(\mathbb{R}^3_+)$ be a solution of the following system

$$\Delta^2 \chi = 0 \text{ in } \mathbb{R}^3_+, \quad \chi = 0 \ \text{ and } \ \frac{\partial \chi}{\partial x_3} = \zeta_3 \text{ on } \Gamma.$$

We set $\boldsymbol{h} = \boldsymbol{\zeta} - \nabla\chi$. Then $\boldsymbol{h} \in \mathbf{W}_0^{2,3}(\mathbb{R}^3_+)$ and

$$\boldsymbol{f} = \mathbf{curl}\,\boldsymbol{h} \quad \text{and} \quad \operatorname{div}\Delta\boldsymbol{h} = 0 \ \text{ in } \mathbb{R}^3_+, \qquad \boldsymbol{h} = \mathbf{0} \ \text{ on } \Gamma.$$

We know that there exists $\mu \in W_0^{3,3}(\mathbb{R}^3_+)$ satisfying

$$\Delta^3\mu = 0 \text{ in } \mathbb{R}^3_+, \quad \mu = \frac{\partial\mu}{\partial x_3} = 0 \ \text{ and } \ \frac{\partial^2\mu}{\partial x_3^2} = \frac{\partial h_3}{\partial x_3} \text{ on } \Gamma.$$

iii) Third step: We set $\boldsymbol{\varphi} = \boldsymbol{h} - \nabla\mu$. Then $\boldsymbol{\varphi} \in \mathbf{W}_0^{2,3}(\mathbb{R}^3_+)$, $\mathbf{curl}\,\boldsymbol{\varphi} = \boldsymbol{f}$ in $\mathbb{R}^3_+$ and $\boldsymbol{\varphi} = \mathbf{0}$ on Γ. We have

$$\frac{\partial\boldsymbol{\varphi}}{\partial x_3} = \frac{\partial\boldsymbol{h}}{\partial x_3} - \frac{\partial}{\partial x_3}\nabla\mu = \frac{\partial\boldsymbol{h}}{\partial x_3} - \nabla\frac{\partial\mu}{\partial x_3} \ \text{ in } \mathbb{R}^3_+.$$

Then,

$$\frac{\partial\varphi_3}{\partial x_3} = \frac{\partial h_3}{\partial x_3} - \frac{\partial^2\mu}{\partial x_3^2} = 0 \ \text{ on } \Gamma.$$

As $\boldsymbol{f} = \mathbf{curl}\,\boldsymbol{h}$ in $\mathbb{R}^3_+$ and $\boldsymbol{f} = \boldsymbol{h} = \mathbf{0}$ on Γ, then $\dfrac{\partial\boldsymbol{h}'}{\partial x_3} = \mathbf{0}$ on Γ. But $\nabla'\dfrac{\partial\mu}{\partial x_3} = 0$ on Γ, then $\dfrac{\partial\boldsymbol{\varphi}'}{\partial x_3} = 0$ on Γ and $\boldsymbol{\varphi} \in \mathring{\mathbf{W}}_0^{2,3}(\mathbb{R}^3_+)$.

iv) Last step: The uniqueness of $\boldsymbol{\varphi}$ follows from the fact that if $\boldsymbol{\varphi} \in \mathring{\mathbf{W}}_0^{2,3}(\mathbb{R}^3_+)$ satisfying $\mathbf{curl}\,\boldsymbol{\varphi} = \mathbf{0}$ and $\operatorname{div}\Delta^2\boldsymbol{\varphi} = 0$ in $\mathbb{R}^3_+$, then $\boldsymbol{\varphi} = \nabla q$ in $\mathbb{R}^3_+$ with $q \in W_0^{3,3}(\mathbb{R}^3_+)$, $\Delta^3 q = 0$ in $\mathbb{R}^3_+$; and $q = c$, $\dfrac{\partial q}{\partial x_3} = \dfrac{\partial^2 q}{\partial x_3^2} = 0$ on Γ where c is a constant, *i.e.*, $q = c$ in $\mathbb{R}^3_+$ and $\boldsymbol{\varphi} = \mathbf{0}$ in $\mathbb{R}^3_+$. This proof is finished. □

Proposition 5.7. *Let* $\boldsymbol{\varphi} \in \mathring{\mathbf{W}}_0^{1,3}(\mathbb{R}^3_+)$. *Then there exist* $\boldsymbol{\psi} \in \mathring{\mathbf{W}}_0^{1,3}(\mathbb{R}^3_+) \cap \mathbf{L}^\infty(\mathbb{R}^3_+)$ *and* $\boldsymbol{\eta} \in \mathring{\mathbf{W}}_0^{2,3}(\mathbb{R}^3_+)$ *such that*

$$\boldsymbol{\varphi} = \boldsymbol{\psi} + \mathbf{curl}\,\boldsymbol{\eta} \quad \text{with} \quad \operatorname{div}\Delta^2\boldsymbol{\eta} = 0 \text{ in } \mathbb{R}^3_+.$$

However, we have the following estimate

$$||\boldsymbol{\psi}||_{\mathbf{W}_0^{1,3}(\mathbb{R}^3_+)} + ||\boldsymbol{\psi}||_{\mathbf{L}^\infty(\mathbb{R}^3_+)} + ||\boldsymbol{\eta}||_{\mathbf{W}_0^{2,3}(\mathbb{R}^3_+)} \le C\,||\boldsymbol{\varphi}||_{\mathbf{W}_0^{1,3}(\mathbb{R}^3_+)}.$$

Proof. From the hypothesis, we have $\operatorname{div}\boldsymbol{\varphi} \in L^3(\mathbb{R}^3_+)$. Thanks to Theorem 4.8, there exists $\boldsymbol{\psi} \in \mathring{\mathbf{W}}_0^{1,3}(\mathbb{R}^3_+) \cap \mathbf{L}^\infty(\mathbb{R}^3_+)$ such that $\operatorname{div}\boldsymbol{\psi} = \operatorname{div}\boldsymbol{\varphi}$ and we have the following estimate

$$||\,\boldsymbol{\psi}\,||_{\mathbf{L}^\infty(\mathbb{R}^3_+)} + ||\,\boldsymbol{\psi}\,||_{\mathbf{W}_0^{1,3}(\mathbb{R}^3_+)} \le C\,||\operatorname{div}\boldsymbol{\varphi}\,||_{L^3(\mathbb{R}^3_+)}.$$

We set $\boldsymbol{f} = \boldsymbol{\varphi} - \boldsymbol{\psi}$. Then $\boldsymbol{f} \in \mathring{\mathbf{W}}_0^{1,3}(\mathbb{R}^3_+)$ and $\operatorname{div}\boldsymbol{f} = 0$. Applying Lemma 5.6, there exists $\boldsymbol{\eta} \in \mathring{\mathbf{W}}_0^{2,3}(\mathbb{R}^3_+)$ such that $\boldsymbol{f} = \mathbf{curl}\,\boldsymbol{\eta}$. □

Corollary 5.8. *Assume that* $\boldsymbol{f} \in \mathbf{L}^1(\mathbb{R}^3_+)$ *and* $\mathbf{curl}\,\boldsymbol{f} \in \mathbf{W}_0^{-2,3/2}(\mathbb{R}^3_+)$*. Then* $\boldsymbol{f} \in \mathbf{W}_0^{-1,3/2}(\mathbb{R}^3_+)$ *and we have the following estimate*

$$||\,\boldsymbol{f}\,||_{\mathbf{W}_0^{-1,3/2}(\mathbb{R}^3_+)} \;\leq\; C\,(\,||\,\boldsymbol{f}\,||_{\mathbf{L}^1(\mathbb{R}^3_+)} + ||\,\mathbf{curl}\,\boldsymbol{f}\,||_{\mathbf{W}_0^{-2,3/2}(\mathbb{R}^3_+)}\,).$$

Proof. The proof is similar to the one of Theorem 2.9. $\square$

Proposition 5.9. *Let* $\boldsymbol{f} \in \mathbf{L}^1(\mathbb{R}^3_+)$ *such that* $\operatorname{div}\boldsymbol{f} = 0$ *in* $\mathbb{R}^3_+$*. Then there exists a unique* $\boldsymbol{\varphi} \in \mathbf{L}^{3/2}(\mathbb{R}^3_+)$ *such that* $\mathbf{curl}\;\boldsymbol{\varphi} = \boldsymbol{f}$*,* $\operatorname{div}\boldsymbol{\varphi} = 0$ *in* $\mathbb{R}^3_+$ *and* $\varphi_3 = 0$ *on* Γ *satisfying the following estimate*

$$||\,\boldsymbol{\varphi}\,||_{\mathbf{L}^{3/2}(\mathbb{R}^3_+)} \;\leq\; C\,||\,\boldsymbol{f}\,||_{\mathbf{L}^1(\mathbb{R}^3_+)}.$$

Proof. *i) First step:* We first consider the uniqueness of $\boldsymbol{\varphi}$. Let $\boldsymbol{\varphi} \in \mathbf{L}^{3/2}(\mathbb{R}^3_+)$ such that $\mathbf{curl}\,\boldsymbol{\varphi} = \mathbf{0}$, $\operatorname{div}\boldsymbol{\varphi} = 0$ in $\mathbb{R}^3_+$ and $\varphi_3 = 0$ on Γ. Then $\boldsymbol{\varphi} = \nabla q$ with $q \in W_0^{1,3/2}(\mathbb{R}^3_+)$, $\Delta q = 0$ in $\mathbb{R}^3_+$ and $\dfrac{\partial q}{\partial x_3} = 0$ on Γ. Therefore, from C. Amrouche [3], we deduce $q = 0$ and $\boldsymbol{\varphi} = \mathbf{0}$ in $\mathbb{R}^3_+$.

ii) Second step: We set $\widetilde{\boldsymbol{f}} = (\boldsymbol{f}'_*, f_3^*)$, then we can deduce $\operatorname{div}\widetilde{\boldsymbol{f}} = 0$. Thanks to Proposition 3.6, there exists a unique $\boldsymbol{Z} \in \mathbf{L}^{3/2}(\mathbb{R}^3)$ such that $\mathbf{curl}\;\boldsymbol{Z} = \widetilde{\boldsymbol{f}}$ and $\operatorname{div}\boldsymbol{Z} = 0$ in $\mathbb{R}^3$ satisfying the following estimate

$$||\,\boldsymbol{Z}\,||_{\mathbf{L}^{3/2}(\mathbb{R}^3)} \;\leq\; C\,||\,\widetilde{\boldsymbol{f}}\,||_{\mathbf{L}^1(\mathbb{R}^3)} \;\leq\; C\,||\,\boldsymbol{f}\,||_{\mathbf{L}^1(\mathbb{R}^3_+)}.$$

We set $\boldsymbol{z} = \boldsymbol{Z}|_{\mathbb{R}^3_+}$. Thanks to Lemma 4.1, we can deduce that $z_3 \in W_0^{-2/3,3/2}(\Gamma)$ on Γ. We know that the following problem

$$-\Delta h \,=\, 0 \;\text{ in } \mathbb{R}^3_+ \quad\text{ and }\quad \frac{\partial h}{\partial x_3} \,=\, z_3 \;\text{ on } \Gamma,$$

has a unique solution $h \in W_0^{1,3/2}(\mathbb{R}^3_+)$ (see [3]). The proof is complete by setting $\boldsymbol{\varphi} = \boldsymbol{z} - \nabla h$. $\square$

Remark 19. We can give a second proof of the existence of the vector potential $\boldsymbol{\varphi}$. As $\boldsymbol{f} \in \mathbf{W}_0^{-1,3/2}(\mathbb{R}^3_+)$, there exists a unique $\boldsymbol{z} \in \overset{\circ}{\mathbf{W}}{}_0^{1,3/2}(\mathbb{R}^3_+)$ satisfying $-\Delta\boldsymbol{z} = \boldsymbol{f}$, with $\operatorname{div}\,\boldsymbol{z} = 0$ in $\mathbb{R}^3_+$. The function $\boldsymbol{\varphi} = \mathbf{curl}\,\boldsymbol{z}$ is the required function.

We introduce the following proposition.

Proposition 5.10. *Let* $\boldsymbol{f} \in \mathbf{W}_0^{-1,3/2}(\mathbb{R}^3_+)$ *such that* $\operatorname{div}\boldsymbol{f} = 0$ *in* $\mathbb{R}^3_+$*. Then there exists a unique* $\boldsymbol{\varphi} \in \mathbf{L}^{3/2}(\mathbb{R}^3_+)$ *such that* $\mathbf{curl}\;\boldsymbol{\varphi} = \boldsymbol{f}$*,* $\operatorname{div}\boldsymbol{\varphi} = 0$ *in* $\mathbb{R}^3_+$ *and* $\varphi_3 = 0$ *on* Γ *satisfying the following estimate*

$$||\,\boldsymbol{\varphi}\,||_{\mathbf{L}^{3/2}(\mathbb{R}^3_+)} \;\leq\; C\,||\,\boldsymbol{f}\,||_{\mathbf{W}_0^{-1,3/2}(\mathbb{R}^3_+)}.$$

Proof. For the uniqueness of $\boldsymbol{\varphi}$, the proof is similar to the one of Proposition 5.9. Thanks to Corollary 5.3, the following operator

$$B \;=\; \mathbf{curl} \,:\, \overset{\circ}{\mathbf{W}}{}_0^{1,3}(\mathbb{R}^3_+) \longrightarrow \overset{\circ}{\mathbf{H}}_3\,(\mathbb{R}^3_+)$$

is linear, continuous and surjective. Moreover, as $\boldsymbol{\mathcal{V}}(\mathbb{R}^3_+)$ is dense in $\overset{\circ}{\mathbf{H}}_{3/2}\,(\mathbb{R}^3_+)$, then for all $\boldsymbol{u} \in \overset{\circ}{\mathbf{W}}{}_0^{1,3}(\mathbb{R}^3_+)$ and $\boldsymbol{v} \in \overset{\circ}{\mathbf{H}}_{3/2}\,(\mathbb{R}^3_+)$, we have

$$\int_{\mathbb{R}^3_+} \boldsymbol{v} \cdot \mathbf{curl}\,\boldsymbol{u} \;=<\; \boldsymbol{u},\; \mathbf{curl}\,\boldsymbol{v} \;>_{\overset{\circ}{\mathbf{W}}{}_0^{1,3}(\mathbb{R}^3_+)\times \mathbf{W}_0^{-1,3/2}(\mathbb{R}^3_+)}\,.$$

As $\overset{\circ}{\mathbf{H}}_3\,(\mathbb{R}^3_+)' = \overset{\circ}{\mathbf{H}}_{3/2}\,(\mathbb{R}^3_+)$, then the adjoint operator of B

$$B^* \;=\; \mathbf{curl} \,:\, \overset{\circ}{\mathbf{H}}_{3/2}\,(\mathbb{R}^3_+) \longrightarrow \mathbf{W}_0^{-1,3/2}(\mathbb{R}^3_+)$$

is also linear and continuous. It is easy to see that the kernel of the operator B, namely the space $\{\, \boldsymbol{v} \;\in \overset{\circ}{\mathbf{W}}{}_0^{1,3}(\mathbb{R}^3_+); \quad \mathbf{curl}\,\boldsymbol{v} \;=\; \mathbf{0}\,\}$ is same as the space $\mathbf{G} \;=\; \{\, \nabla q; \;\; q \in \overset{\circ}{W}{}_0^{2,3}(\mathbb{R}^3_+)\,\}$. Then the following operators

$$\mathbf{curl} : \overset{\circ}{\mathbf{W}}{}_0^{1,3}(\mathbb{R}^3_+)\,/\,\mathbf{G} \longrightarrow \overset{\circ}{\mathbf{H}}_3\,(\mathbb{R}^3_+), \quad \mathbf{curl} : \overset{\circ}{\mathbf{H}}_{3/2}\,(\mathbb{R}^3_+) \longrightarrow \mathbf{W}_0^{-1,3/2}(\mathbb{R}^3_+) \perp \mathbf{G}$$

are isomorphisms. As $\mathcal{D}(\mathbb{R}^3_+)$ is dense in $W_0^{2,3}(\mathbb{R}^3_+)$, then we can easily verify

$$\mathbf{W}_0^{-1,3/2}(\mathbb{R}^3_+) \perp \mathbf{G} \;=\; \{\boldsymbol{f} \,\in\, \mathbf{W}_0^{-1,3/2}(\mathbb{R}^3_+) \;\text{ and }\; \operatorname{div}\boldsymbol{f} \;=\; 0\,\}.$$

The proof is finished. □

Remark 20. As in the previous remark, we can give a more direct proof.

The following corollary is the generalized case of Proposition 5.9 and Proposition 5.10.

Corollary 5.11. *Let* $\boldsymbol{f} \in \mathbf{L}^1(\mathbb{R}^3_+) + \mathbf{W}_0^{-1,3/2}(\mathbb{R}^3_+)$ *such that* $\operatorname{div}\boldsymbol{f} = 0$ *in* $\mathbb{R}^3_+$*. Then there exists a unique* $\boldsymbol{\varphi} \in \mathbf{L}^{3/2}(\mathbb{R}^3_+)$ *such that* $\mathbf{curl}\;\boldsymbol{\varphi} = \boldsymbol{f}$*,* $\operatorname{div}\boldsymbol{\varphi} = 0$ *in* $\mathbb{R}^3_+$ *and* $\varphi_3 = 0$ *on* Γ *satisfying the following estimate*

$$||\,\boldsymbol{\varphi}\,||_{\mathbf{L}^{3/2}(\mathbb{R}^3_+)} \leq C||\,\boldsymbol{f}\,||_{\mathbf{L}^1(\mathbb{R}^3_+)+\mathbf{W}_0^{-1,3/2}(\mathbb{R}^3_+)}.$$

Proof. Let now $\boldsymbol{f} \;=\; \boldsymbol{g}+\boldsymbol{h}$ with $\boldsymbol{g} \in \mathbf{L}^1(\mathbb{R}^3_+)$, $\boldsymbol{h} \in \mathbf{W}_0^{-1,3/2}(\mathbb{R}^3_+)$ and $\operatorname{div}\boldsymbol{f} = 0$ in $\mathbb{R}^3_+$. Then $\operatorname{div}\boldsymbol{g} \,\in\, W_0^{-2,3/2}(\mathbb{R}^3_+)$ and we can deduce $\boldsymbol{g} \,\in\, \mathbf{W}_0^{-1,3/2}(\mathbb{R}^3_+)$. This corollary is a consequence of Proposition 5.9 and Proposition 5.10. □

Lemma 5.12. *Let* $\boldsymbol{f} \in \mathbf{L}^1(\mathbb{R}^3_+)$ *such that* $\operatorname{div}\boldsymbol{f} = 0$ *in* $\mathbb{R}^3_+$ *and* $f_3 = 0$ *on* Γ. *Then the unique solution* $\boldsymbol{\varphi} \in \mathbf{L}^{3/2}(\mathbb{R}^3_+)$ *given by Proposition* 5.9 *satisfies* $\boldsymbol{\varphi}' \in \mathbf{W}_0^{-2/3,3/2}(\mathbb{R}^2)$ *and*

$$||\,\boldsymbol{\varphi}'\,||_{\mathbf{W}_0^{-2/3,3/2}(\mathbb{R}^2)} \;\leq\; C\,||\,\boldsymbol{f}\,||_{\mathbf{L}^1(\mathbb{R}^3_+)}.$$

Moreover for $i = 1$ *or* 2, *we have* $< \varphi_i,\ 1 >_{W_0^{-2/3,3/2}(\mathbb{R}^2)\times W_0^{2/3,3}(\mathbb{R}^2)} = 0.$

Proof. As $\boldsymbol{\mathcal{V}}(\mathbb{R}^3_+)$ is dense in $\overset{\circ}{\mathbf{H}}_1(\mathbb{R}^3_+)$, there exists $\boldsymbol{f}_k \in \boldsymbol{\mathcal{V}}(\mathbb{R}^3_+)$ such that $\boldsymbol{f}_k$ converges to $\boldsymbol{f}$ in $\mathbf{L}^1(\mathbb{R}^3_+)$. Let now $\boldsymbol{\varphi}_k \in \mathbf{L}^{3/2}(\mathbb{R}^3_+)$ such that $\mathbf{f}_k = \mathbf{curl}\ \boldsymbol{\varphi}_k$, $\operatorname{div}\boldsymbol{\varphi}_k = 0$ in $\mathbb{R}^3_+$ and $\boldsymbol{\varphi}_k \cdot \boldsymbol{n} = 0$ on Γ, with the estimate

$$||\,\boldsymbol{\varphi}_k\,||_{\mathbf{L}^{3/2}(\mathbb{R}^3_+)} \;\leq\; C\,||\,\boldsymbol{f}_k\,||_{\mathbf{L}^1(\mathbb{R}^3_+)}.$$

Let also $\boldsymbol{\mu}' \in W_0^{2/3,3}(\mathbb{R}^2)$ such that $\mu_3 = 0$ on Γ. Then, there exists $\boldsymbol{u} \in \mathbf{W}_0^{1,3}(\mathbb{R}^3_+)$ such that $\boldsymbol{u} = \boldsymbol{\mu}$ on Γ satisfying the estimate

$$||\,\boldsymbol{u}\,||_{\mathbf{W}_0^{1,3}(\mathbb{R}^3_+)} \;\leq\; C\,||\,\boldsymbol{\mu}\,||_{W_0^{2/3,3}(\mathbb{R}^2)}.$$

However, we known that there also exist $\boldsymbol{v} \in \mathbf{W}_0^{1,3}(\mathbb{R}^3_+) \cap \mathbf{L}^{\infty}(\mathbb{R}^3_+)$ and $\eta \in W_0^{2,3}(\mathbb{R}^3_+)$ such that $\boldsymbol{u} = \boldsymbol{v} + \nabla\eta$ with the corresponding estimate. Thus, we obtain

$$\begin{aligned}\int_{\mathbb{R}^3_+} \boldsymbol{u}\cdot\mathbf{curl}\ \boldsymbol{\varphi}_k &= \int_{\mathbb{R}^3_+} \boldsymbol{v}\cdot\mathbf{curl}\ \boldsymbol{\varphi}_k \\ &= \int_{\mathbb{R}^3_+} \boldsymbol{\varphi}_k\cdot\mathbf{curl}\ \boldsymbol{u} + < \boldsymbol{\varphi}_k \times \boldsymbol{n},\ \boldsymbol{u} >_{\Gamma}.\end{aligned}$$

Hence, we have

$$\begin{aligned}|< \boldsymbol{\varphi}_k \times \boldsymbol{n},\ \boldsymbol{\mu} >_{\Gamma}| &\leq \|\mathbf{curl}\ \boldsymbol{\varphi}_k\|_{\mathbf{L}^1(\mathbb{R}^3_+)}\|\boldsymbol{v}\|_{\mathbf{L}^{\infty}(\mathbb{R}^3_+)} + \|\boldsymbol{\varphi}_k\|_{\mathbf{L}^{3/2}(\mathbb{R}^3_+)}\|\mathbf{curl}\ \boldsymbol{u}\|_{\mathbf{L}^3} \\ &\leq C(\|\mathbf{curl}\ \boldsymbol{\varphi}_k\|_{\mathbf{L}^1(\mathbb{R}^3_+)} + \|\boldsymbol{\varphi}_k\|_{\mathbf{L}^{3/2}(\mathbb{R}^3_+)})\|\boldsymbol{u}\|_{\mathbf{W}_0^{1,3}(\mathbb{R}^3_+)} \\ &\leq C(\|\mathbf{curl}\ \boldsymbol{\varphi}_k\|_{\mathbf{L}^1(\mathbb{R}^3_+)} + \|\boldsymbol{\varphi}_k\|_{\mathbf{L}^{3/2}(\mathbb{R}^3_+)})||\boldsymbol{\mu}||_{W_0^{2/3,3}(\mathbb{R}^2)}.\end{aligned}$$

Then

$$\|\boldsymbol{\varphi}_k \times \boldsymbol{n}\|_{W_0^{-2/3,3}(\mathbb{R}^2)} \leq C\|\boldsymbol{f}_k\|_{\mathbf{L}^1(\mathbb{R}^3_+)}$$

and by passage to the limite, we have

$$\|\boldsymbol{\varphi} \times \boldsymbol{n}\|_{W_0^{-2/3,3}(\mathbb{R}^2)} \leq C\|\boldsymbol{f}\|_{\mathbf{L}^1(\mathbb{R}^3_+)}$$

and

$$< \boldsymbol{\varphi} \times \boldsymbol{n},\ \boldsymbol{u} >_{\mathbf{W}_0^{-2/3,3/2}(\mathbb{R}^2)\times\mathbf{W}_0^{2/3,3}(\mathbb{R}^2)} = \int_{\mathbb{R}^3_+} \boldsymbol{v}\cdot\ \mathbf{curl}\ \boldsymbol{\varphi} + \int_{\mathbb{R}^3_+} \boldsymbol{\varphi}\cdot\mathbf{curl}\ \boldsymbol{u}.$$

Finally, the orthogonality relations are simple consequence of this last relation, because if $\boldsymbol{u} = \boldsymbol{e}_i$, then $\boldsymbol{v} = \boldsymbol{u}$ and the integral on $\mathbb{R}^3_+$ of $\mathbf{curl}\ \boldsymbol{\varphi}$ is zero.

$\square$

Theorem 5.13. *Let* $\boldsymbol{f} \in \mathbf{L}^1(\mathbb{R}^3_+)$ *such that* $\operatorname{div} \boldsymbol{f} = 0$ *in* $\mathbb{R}^3_+$ *and* $f_3 = 0$ *on* Γ. *Then there exists a unique* $\boldsymbol{\varphi} \in \mathbf{L}^{3/2}(\mathbb{R}^3_+)$ *such that* $\boldsymbol{f} = \mathbf{curl}\, \boldsymbol{\varphi}$ *with* $\operatorname{div} \boldsymbol{\varphi} = 0$ *in* $\mathbb{R}^3_+$ *and* $\boldsymbol{\varphi}' = \mathbf{0}$ *on* Γ. *Moreover, we have the following estimate*

$$|| \boldsymbol{\varphi} ||_{\mathbf{L}^{3/2}(\mathbb{R}^3_+)} \;\leq\; C \, || \boldsymbol{f} ||_{\mathbf{L}^1(\mathbb{R}^3_+)}.$$

Proof. Let $\boldsymbol{\psi} \in \mathbf{L}^{3/2}(\mathbb{R}^3_+)$ such that $\boldsymbol{f} = \mathbf{curl}\, \boldsymbol{\psi}$ with $\operatorname{div} \boldsymbol{\psi} = 0$ in $\mathbb{R}^3_+$ and $\psi_3 = 0$ on Γ, with the estimate

$$|| \boldsymbol{\psi} ||_{\mathbf{L}^{3/2}(\mathbb{R}^3_+)} \;\leq\; C \, || \boldsymbol{f} \; ||_{\mathbf{L}^1(\mathbb{R}^3_+)}.$$

Using Theorem 3.5 of Amrouche-Raudin [8], there exists a unique pair $(\boldsymbol{w}, \pi) \in \mathbf{W}_0^{1,3/2}(\mathbb{R}^3_+) \times L^{3/2}(\mathbb{R}^3_+)$ solution to

$$\begin{cases} -\Delta \boldsymbol{w} + \nabla \pi = \mathbf{curl}\, \boldsymbol{\psi} \quad \text{and} \quad \operatorname{div}\, \boldsymbol{w} = 0 \quad \text{in } \mathbb{R}^3_+, \\ w_3 = 0, \quad \partial_3 w_1 = \psi_2 \quad \text{and} \quad \partial_3 w_2 = -\psi_1 \quad \text{on } \Gamma. \end{cases}$$

It is easy to see that $\Delta \pi = 0$ in $\mathbb{R}^3_+$. Moreover,

$$\frac{\partial \pi}{\partial x_3} \;=\; \Delta w_3 + \frac{\partial \psi_2}{\partial x_1} - \frac{\partial \psi_1}{\partial x_2} \quad \text{in } \mathbb{R}^3_+,$$

and on Γ, we have

$$\begin{aligned} \frac{\partial \pi}{\partial x_3} \quad &= \; \frac{\partial}{\partial x_3}\left(\frac{\partial w_3}{\partial x_3}\right) + \frac{\partial \psi_2}{\partial x_1} - \frac{\partial \psi_1}{\partial x_2} \\ &= \; -\frac{\partial}{\partial x_1}\left(\frac{\partial w_1}{\partial x_3}\right) - \frac{\partial}{\partial x_2}\left(\frac{\partial w_2}{\partial x_3}\right) + \frac{\partial \psi_2}{\partial x_1} - \frac{\partial \psi_1}{\partial x_2} \;=\; 0. \end{aligned}$$

Consequently, $\pi = 0$ in $\mathbb{R}^3_+$. The proof is finished by setting $\boldsymbol{\varphi} = \boldsymbol{\psi} - \mathbf{curl}\, \boldsymbol{w}$. □

In two-dimensional space, we have a similar results as Corollary 5.11.

Proposition 5.14. *Assume that* $\boldsymbol{f} \in \mathbf{L}^1(\mathbb{R}^2_+) + \mathbf{W}_0^{-1,2}(\mathbb{R}^2_+)$ *such that* $\operatorname{div} \boldsymbol{f} = 0$ *in* $\mathbb{R}^2_+$. *Then there exists* $\varphi \in L^2(\mathbb{R}^2_+)$ *such that* $\mathbf{curl}\; \varphi = \boldsymbol{f}$, $\operatorname{div} \varphi = 0$ *in* $\mathbb{R}^2_+$ *and* $\varphi_2 = 0$ *on* Γ *satisfying the following estimate*

$$|| \varphi ||_{L^2(\mathbb{R}^2_+)} \leq C || \boldsymbol{f} ||_{\mathbf{L}^1(\mathbb{R}^2_+) + \mathbf{W}_0^{-1,2}(\mathbb{R}^2_+)}.$$

6. Elliptic problems in the half-space

The following theorem was given by C. Amrouche and S. Nečasová [6].

Theorem 6.1. *Let* $1 < p < \infty$ *and* $(f, g) \in W_0^{-1,p}(\mathbb{R}^n_+) \times W^{1/p',p}(\Gamma)$. *Then the following problem*

$$(\mathcal{L}_+) \; \begin{cases} -\Delta u \;=\; f \;\; \text{in } \mathbb{R}^n_+, \\ u \;=\; g \;\; \text{on } \Gamma \;=\; \mathbb{R}^{n-1}, \end{cases}$$

has a unique solution $u \in W_0^{1,p}(\mathbb{R}^n_+)$ *and we have the following estimate*

$$\| u \|_{W_0^{1,p}(\mathbb{R}^n_+)} \ \leq \ C\,(\, \| f \|_{W_0^{-1,p}(\mathbb{R}^n_+)} + \| g \|_{W^{1/p',p}(\mathbb{R}^{n-1})}).$$

The following result is a consequence of Theorem 4.10 and Theorem 6.1.

Corollary 6.2. *Let* $\boldsymbol{f} \in \mathbf{X}(\mathbb{R}^n_+)$. *Then the following problem*

$$-\Delta \boldsymbol{u} \,=\, \boldsymbol{f} \ \ in\ \mathbb{R}^n_+ \ \ \text{and} \ \ \boldsymbol{u} \,=\, \mathbf{0} \ \ \text{on}\ \Gamma \,=\, \mathbb{R}^{n-1}, \tag{58}$$

has a unique solution $\boldsymbol{u} \in \mathbf{W}_0^{1,n/(n-1)}(\mathbb{R}^n_+)$ *and we have the following estimate*

$$\| \boldsymbol{u} \|_{\mathbf{W}_0^{1,n/(n-1)}(\mathbb{R}^n_+)} \ \leq \ C \,\| \boldsymbol{f} \|_{\mathbf{X}(\mathbb{R}^n_+)}.$$

Corollary 6.3. *Let* $f \in L^1(\mathbb{R}^n_+)$ *such that* $\partial_n f \in W_0^{-2,n/(n-1)}(\mathbb{R}^n_+)$. *Then we have* $f \in W_0^{-1,n/(n-1)}(\mathbb{R}^n_+)$ *and Problem* $(\mathcal{L}_+)$ *with* $g = 0$ *on* Γ *has a unique solution* $u \in W_0^{1,n/(n-1)}(\mathbb{R}^n_+)$ *satisfying the following estimate*

$$\| u \|_{W_0^{1,n/(n-1)}(\mathbb{R}^n_+)} \ \leq \ C \, \Big(\| f \|_{L^1(\mathbb{R}^n_+)} \,+\, \| \partial_n f \|_{W_0^{-2,n/(n-1)}(\mathbb{R}^n_+)} \Big) \,.$$

Proof. This corollary can be obtained by applying Theorem 4.10 and Corollary 6.2 with $\boldsymbol{f} \,=\, (0, ..., 0, f)$. □

Lemma 6.4. *Let* $\boldsymbol{g}' \in \mathbf{L}^1(\Gamma)$ *such that* $\mathrm{div}'\boldsymbol{g}' \in W_0^{-2+\frac{1}{n},\frac{n}{n-1}}(\Gamma)$. *Then* $\boldsymbol{g}' \in \mathbf{W}_0^{-1+\frac{1}{n},\frac{n}{n-1}}(\Gamma)$ *and we have the estimate*

$$\|\boldsymbol{g}'\|_{\mathbf{W}_0^{-1+\frac{1}{n},\frac{n}{n-1}}(\Gamma)} \ \leq \ C\,(\, \|\boldsymbol{g}'\|_{\mathbf{L}^1(\Gamma)} + \|\mathrm{div}'\,\boldsymbol{g}'\|_{W_0^{-2+\frac{1}{n},\frac{n}{n-1}}(\Gamma)}).$$

Proof. Let $\boldsymbol{\mu} \in \boldsymbol{\mathcal{D}}(\Gamma)$ and $\boldsymbol{\varphi} \in \mathbf{W}_0^{1,n}(\mathbb{R}^n_+)$ such that $\boldsymbol{\varphi} \,=\, \boldsymbol{\mu}$ on Γ with the estimate

$$\| \boldsymbol{\varphi} \|_{\mathbf{W}_0^{1,n}(\mathbb{R}^n_+)} \ \leq \ C\|\boldsymbol{\mu}\|_{\mathbf{W}^{1-\frac{1}{n},n}(\Gamma)}.$$

Thanks to Proposition 4.13, there exist $\boldsymbol{\psi} \in \mathbf{W}_0^{1,n}(\mathbb{R}^n_+) \cap \mathbf{L}^\infty(\mathbb{R}^n_+)$ and $\eta \in W_0^{2,n}(\mathbb{R}^n_+)$ such that $\psi_n = 0$ on Γ satisfying $\boldsymbol{\varphi} = \boldsymbol{\psi} + \nabla\eta$ and the estimate

$$\|\boldsymbol{\psi}\|_{\mathbf{L}^\infty(\mathbb{R}^n_+)} + \|\boldsymbol{\psi}\|_{\mathbf{W}_0^{1,n}(\mathbb{R}^n_+)} + \|\eta\|_{W_0^{2,n}(\mathbb{R}^n_+)} \ \leq \ C\|\boldsymbol{\varphi}\|_{\mathbf{W}_0^{1,n}(\mathbb{R}^n_+)}. \tag{59}$$

Then

$$\begin{aligned} < \boldsymbol{g}, \boldsymbol{\mu} >_{\boldsymbol{\mathcal{D}}'(\Gamma)\times\boldsymbol{\mathcal{D}}(\Gamma)} \ &= \int_\Gamma \boldsymbol{g}' \cdot \boldsymbol{\psi}' - < \mathrm{div}'\,\boldsymbol{g}', \eta >_{W_0^{-2+\frac{1}{n},\frac{n}{n-1}}(\Gamma)\times W_0^{2-\frac{1}{n},n}(\Gamma)} \\ &\quad + < g_n, \varphi_n >_{W_0^{-1+\frac{1}{n},\frac{n}{n-1}}(\Gamma)\times W_0^{1-\frac{1}{n},n}(\Gamma)} \end{aligned}$$

and

$$\begin{aligned} |< \boldsymbol{g}, \boldsymbol{\mu} >_{\boldsymbol{\mathcal{D}}'(\Gamma)\times\boldsymbol{\mathcal{D}}(\Gamma)}| \quad & \leq ||\boldsymbol{g}'||_{\mathbf{L}^1(\Gamma)} ||\boldsymbol{\psi}'||_{\mathbf{L}^\infty(\mathbb{R}^n_+)} + ||\mathrm{div}'\boldsymbol{g}'||_{W_0^{-2+\frac{1}{n},\frac{n}{n-1}}(\Gamma)} \times \\ & \times ||\eta||_{W_0^{2-\frac{1}{n},n}(\Gamma)} + ||g_n||_{W_0^{-1+\frac{1}{n},\frac{n}{n-1}}(\Gamma)} ||\varphi_n||_{W_0^{1-\frac{1}{n},n}(\Gamma)} \\ & \leq C||\boldsymbol{\varphi}||_{\mathbf{W}_0^{1,n}(\mathbb{R}^n_+)} \leq C||\boldsymbol{\mu}||_{\mathbf{W}^{1-\frac{1}{n},n}(\Gamma)}. \end{aligned} \tag{60}$$

Thanks to the density of $\boldsymbol{\mathcal{D}}(\Gamma)$ in $\mathbf{W}_0^{1-\frac{1}{n},n}(\Gamma)$ and (60), we can deduce $\boldsymbol{g}' \in \mathbf{W}_0^{-1+\frac{1}{n},\frac{n}{n-1}}(\Gamma)$ and

$$||\boldsymbol{g}'||_{\mathbf{W}_0^{-1+\frac{1}{n},\frac{n}{n-1}}(\Gamma)} \ \leq \ C\,(\,||\boldsymbol{g}'||_{\mathbf{L}^1(\Gamma)} + ||\mathrm{div}'\,\boldsymbol{g}'||_{W_0^{-2+\frac{1}{n},\frac{n}{n-1}}(\Gamma)}).$$

□

Theorem 6.5. *Let* $\boldsymbol{g}' \in \mathbf{L}^1(\Gamma)$ *and* $g_n \in W_0^{-1+\frac{1}{n},\frac{n}{n-1}}(\Gamma)$. *If*

$$\int_\Gamma \boldsymbol{g}' \,=\, \mathbf{0}, \quad < g_n, 1 >= 0 \quad \text{and} \quad \mathrm{div}'\boldsymbol{g}' \in W_0^{-2+\frac{1}{n},\frac{n}{n-1}}(\Gamma), \tag{61}$$

then the system

$$-\Delta\,\boldsymbol{u} \,=\, \mathbf{0} \quad in\ \mathbb{R}^n_+ \quad \text{and} \quad \boldsymbol{u} = \boldsymbol{g} \text{ on } \Gamma \tag{62}$$

has a unique very weak solution $\boldsymbol{u} \in \mathbf{L}^{n/(n-1)}(\mathbb{R}^n_+)$ *with the following estimate*

$$||\boldsymbol{u}||_{\mathbf{L}^{n/(n-1)}(\mathbb{R}^n_+)} \ \leq \ C\,(\,||\boldsymbol{g}'||_{\mathbf{L}^1(\Gamma)} + ||\mathrm{div}'\,\boldsymbol{g}'||_{W_0^{-2+\frac{1}{n},\frac{n}{n-1}}(\Gamma)} + ||g_n||_{W_0^{-1+\frac{1}{n},\frac{n}{n-1}}(\Gamma)}).$$

Proof. The system (62) is equivalent to the following one: Find $\boldsymbol{u}$ belonging $\mathbf{L}^{n/(n-1)}(\mathbb{R}^n_+)$ such that for all $\boldsymbol{v} \in \mathbf{W}_0^{2,n}(\mathbb{R}^n_+) \cap \overset{\circ}{\mathbf{W}}{}^{1,n}_{-1}(\mathbb{R}^n_+)$,

$$\int_{\mathbb{R}^n_+} \boldsymbol{u}\cdot\Delta\boldsymbol{v} = - < \boldsymbol{g}, \frac{\partial \boldsymbol{v}}{\partial x_n} >_{\mathbf{W}_0^{-1+\frac{1}{n},\frac{n}{n-1}}(\Gamma)\times\mathbf{W}_0^{1-\frac{1}{n},n}(\Gamma)} .$$

We know that, for all $\boldsymbol{F} \in \mathbf{L}^n(\mathbb{R}^n_+)$, there exists $\boldsymbol{v} \in \mathbf{W}_0^{2,n}(\mathbb{R}^n_+) \cap \overset{\circ}{\mathbf{W}}{}^{1,n}_{-1}(\mathbb{R}^n_+)$, unique up to an element of $x_n\mathbb{R}^n$, such that $-\Delta\boldsymbol{v} = \boldsymbol{F}$ in $\mathbb{R}^n_+$, $\boldsymbol{v} = \mathbf{0}$ on Γ and the following estimate holds

$$||\boldsymbol{v}||_{\mathbf{W}_0^{2,n}(\mathbb{R}^n_+)/x_n\mathbb{R}^n} \leq C||\boldsymbol{F}||_{\mathbf{L}^n(\mathbb{R}^n_+)}.$$

Then, from (61), we have for all $\boldsymbol{a} \in \mathbb{R}^n$,

$$\begin{aligned} & |< \boldsymbol{g}, \frac{\partial \boldsymbol{v}}{\partial x_n} >_{\mathbf{W}_0^{-1+\frac{1}{n},\frac{n}{n-1}}(\Gamma)\times\mathbf{W}_0^{1-\frac{1}{n},n}(\Gamma)}| \\ & = |< \boldsymbol{g}, \frac{\partial(\boldsymbol{v} + \boldsymbol{a}x_n)}{\partial x_n} >_{\mathbf{W}_0^{-1+\frac{1}{n},\frac{n}{n-1}}(\Gamma)\times\mathbf{W}_0^{1-\frac{1}{n},n}(\Gamma)}| \\ & \leq C||\boldsymbol{g}||_{\mathbf{W}_0^{-1+\frac{1}{n},\frac{n}{n-1}}(\Gamma)} ||\boldsymbol{v} + \boldsymbol{a}x_n||_{\mathbf{W}_0^{2,n}(\mathbb{R}^n_+)}. \end{aligned}$$

Consequently, taking the infinum, we have

$$| < \boldsymbol{g}, \frac{\partial \boldsymbol{v}}{\partial x_n} >_{\mathbf{W}_0^{-1+\frac{1}{n},\frac{n}{n-1}}(\Gamma)\times \mathbf{W}_0^{1-\frac{1}{n},n}(\Gamma)} | \leq C||\boldsymbol{g}||_{\mathbf{W}_0^{-1+\frac{1}{n},\frac{n}{n-1}}(\Gamma)} ||\boldsymbol{F}||_{\mathbf{L}^n(\mathbb{R}^n_+)}.$$

As $\int_\Gamma \boldsymbol{g}' = \mathbf{0}$ and $< g_n, 1 >= 0$, the linear operator

$$T: \ \boldsymbol{F} \ \longrightarrow < \boldsymbol{g}, \frac{\partial \boldsymbol{v}}{\partial x_n} >_{\mathbf{W}_0^{-1+\frac{1}{n},\frac{n}{n-1}}(\Gamma)\times \mathbf{W}_0^{1-\frac{1}{n},n}(\Gamma)}$$

is continuous on $\mathbf{L}^n(\mathbb{R}^n_+)$ and thanks to the Riesz reprensentation theorem, there exists a unique $\boldsymbol{u} \in \mathbf{L}^{n/(n-1)}(\mathbb{R}^n_+)$ such that $T(\boldsymbol{F}) = \displaystyle\int_{\mathbb{R}^n_+} \boldsymbol{u}\cdot \boldsymbol{F}$, *i.e.*, $\boldsymbol{u}$ is the solution of (62) with the desired estimate. □

In the following theorem, we consider the case of Neumann boundary conditions.

Theorem 6.6. *Let*

$$\boldsymbol{f}\in \mathbf{L}^1(\mathbb{R}^n_+), \ \boldsymbol{g}' \in \mathbf{L}^1(\Gamma) \quad \text{and} \quad g_n \in W_0^{-1+\frac{1}{n},\frac{n}{n-1}}(\Gamma) \tag{63}$$

satisfying the compatibility condition

$$\int_{\mathbb{R}^n_+} \boldsymbol{f}' + \int_\Gamma \boldsymbol{g}' = \mathbf{0} \quad \text{and} \quad \int_{\mathbb{R}^n_+} f_n + < g_n, 1 >= 0. \tag{64}$$

If

$$[\boldsymbol{f}, \boldsymbol{g}'] = \sup_{\xi\in W_0^{2,n}(\mathbb{R}^n_+), \ \xi\neq 0} \frac{| \int_{\mathbb{R}^n_+} \boldsymbol{f}\cdot \nabla\xi + \int_\Gamma \boldsymbol{g}'\cdot \nabla'\xi \,|}{||\,\xi\,||_{W_0^{2,n}(\mathbb{R}^n_+)}} < \infty \tag{65}$$

i.e div $\boldsymbol{f}\in [W_0^{2,n}(\mathbb{R}^n_+)]'$ *and div'* $\boldsymbol{g}' \in W_0^{-2+1/n,n/(n-1)}(\Gamma)$, *then the system*

$$-\Delta\, \boldsymbol{u} = \boldsymbol{f} \ \text{ in } \mathbb{R}^n_+ \quad \text{and} \quad \frac{\partial \boldsymbol{u}}{\partial x_n} = \boldsymbol{g} \text{ on } \Gamma \tag{66}$$

has a unique solution $\boldsymbol{u}\in \mathbf{W}_0^{1,n/(n-1)}(\mathbb{R}^n_+)$ *with the corresponding estimate.*

The proof is a direct consequence of the following lemma.

Lemma 6.7. *Let* $\boldsymbol{f}$, $\boldsymbol{g}'$ and g_n *such that* (63) - (65) *hold. For every* $\boldsymbol{\varphi} \in \mathbf{W}_0^{1,n}(\mathbb{R}^n_+) \cap \mathbf{L}^\infty(\mathbb{R}^n_+)$, *we have*

$$\begin{aligned} | \int_{\mathbb{R}^n_+} \boldsymbol{f}\cdot \boldsymbol{\varphi} + \int_\Gamma \boldsymbol{g}'\cdot \boldsymbol{\varphi}' + < g_n, \ \varphi_n > | \ &\leq \ C\,(||\boldsymbol{f}||_{\mathbf{L}^1(\mathbb{R}^n_+)} + ||\,\boldsymbol{g}'\,||_{\mathbf{L}^1(\Gamma)}|| + \\ &+ ||\,g_n\,||_{\mathbf{W}_0^{1-\frac{1}{n},n}(\Gamma)} + [\boldsymbol{f}, \boldsymbol{g}'\,])||\,\nabla\boldsymbol{\varphi}\,||_{\mathbf{L}^n(\mathbb{R}^n_+)} \end{aligned} \tag{67}$$

Proof. Write $\boldsymbol{\varphi} \in \mathbf{W}_0^{1,n}(\mathbb{R}^n_+) \cap \mathbf{L}^\infty(\mathbb{R}^n_+)$ as $\boldsymbol{\varphi} = \boldsymbol{\psi} + \nabla\eta$ according to Proposition 4.13. One has

$$\int_{\mathbb{R}^n_+} \boldsymbol{f}\cdot\boldsymbol{\varphi} + \int_\Gamma \boldsymbol{g}'\cdot\boldsymbol{\varphi}' + < g_n,\ \varphi_n >= \int_{\mathbb{R}^n_+} \boldsymbol{f}\cdot(\boldsymbol{\psi}+\nabla\eta) + \int_\Gamma \boldsymbol{g}'\cdot\boldsymbol{\psi}' + \\ + \int_\Gamma \boldsymbol{g}'\cdot\nabla'\eta + < g_n,\ \psi_n > + < g_n,\ \frac{\partial\eta}{\partial x_n} > .$$

We deduce then the estimate

$$\begin{aligned} | \int_{\mathbb{R}^n_+} \boldsymbol{f}\cdot\boldsymbol{\varphi} + \int_\Gamma \boldsymbol{g}'\cdot\boldsymbol{\varphi}' + < g_n,\ \varphi_n > | \ &\leq\ C\,(||\,\boldsymbol{f}\,||_{\mathbf{L}^1(\mathbb{R}^n_+)} + ||\,\boldsymbol{g}'\,||_{\mathbf{L}^1(\Gamma)}||+ \\ &+ ||\,g_n\,||_{\mathbf{W}_0^{1-\frac{1}{n},n}(\Gamma)} + [\boldsymbol{f},\boldsymbol{g}'])||\,\boldsymbol{\varphi}\,||_{\mathbf{W}_0^{1,n}(\mathbb{R}^n_+)}. \end{aligned} \tag{68}$$

Because of the compatibility conditions (64), this last relation also holds if we replace $\boldsymbol{\varphi}$ by $\boldsymbol{\varphi}+\boldsymbol{K}$, with $\boldsymbol{K}\in\mathbb{R}^n$. Finally the estimate (67) is consequence of the following Hardy inequality:

$$Inf_{\boldsymbol{K}\in\mathbb{R}^n}||\,\boldsymbol{\varphi}+\boldsymbol{K}\,||_{\mathbf{W}_0^{1,n}(\mathbb{R}^n_+)} \leq C||\,\nabla\boldsymbol{\varphi}\,||_{\mathbf{L}^n(\mathbb{R}^n_+)}.$$

□

Proof of Theorem 6.6. According to the estimate (67), we have

$$\int_{\mathbb{R}^n_+} \boldsymbol{f}\cdot\boldsymbol{\varphi} + \int_\Gamma \boldsymbol{g}'\cdot\boldsymbol{\varphi}' + < g_n,\ \varphi_n > = \int_{\mathbb{R}^n_+} \boldsymbol{\mathcal{F}} : \nabla\boldsymbol{\varphi}$$

with $\boldsymbol{\mathcal{F}} \in \mathbf{L}^{n/(n-1)}(\mathbb{R}^n_+)$ and

$$||\,\boldsymbol{\mathcal{F}}\,||_{\mathbf{L}^{n/(n-1)}(\mathbb{R}^n_+)} \ \leq\ C\,(||\,\boldsymbol{f}\,||_{\mathbf{L}^1(\mathbb{R}^n_+)} + ||\,\boldsymbol{g}'\,||_{\mathbf{L}^1(\Gamma)}|| + ||\,g_n\,||_{\mathbf{W}_0^{1-\frac{1}{n},n}(\Gamma)} + [\boldsymbol{f},\boldsymbol{g}']).$$

It means that $\boldsymbol{f} = -\mathrm{div}\,\boldsymbol{\mathcal{F}} \in \mathbf{W}_0^{-1,n/(n-1)}(\mathbb{R}^n_+)$ and $\mathcal{F}_{i1} = g_i$ on Γ for any $i = 1,\ldots,n$. It is easy to see that the problem (66) is equivalent to the following one:

Find $\boldsymbol{u} \in \mathbf{W}_0^{1,n/(n-1)}(\mathbb{R}^n_+)$ such that for any $\boldsymbol{\varphi} \in \mathbf{W}_0^{1,n)}(\mathbb{R}^n_+)$, we have

$$\int_{\mathbb{R}^n_+} \nabla\boldsymbol{u} : \nabla\boldsymbol{\varphi} \ = \ \int_{\mathbb{R}^n_+} \boldsymbol{\mathcal{F}} : \nabla\boldsymbol{\varphi}. \tag{69}$$

The regularity $\mathbf{L}^{n/(n-1)}(\mathbb{R}^n_+)$ of $\boldsymbol{\mathcal{F}}$ assures the existence of a unique solution $\boldsymbol{u} \in \mathbf{W}_0^{1,n/(n-1)}(\mathbb{R}^n_+)$ of the problem (66). □

Theorem 6.8. *Let* $\boldsymbol{f} \in \mathbf{L}^1(\mathbb{R}^n_+)$ *such that* $\mathrm{div}\,\boldsymbol{f} \in [W_0^{2,n}(\mathbb{R}^n_+) \cap \overset{\circ}{W}{}^{1,n}_{-1}(\mathbb{R}^n_+)]'$ *and* $\int_{\mathbb{R}^n_+} f_n = 0$. *Then* $\boldsymbol{f} \in \mathbf{W}_0^{-1,n/(n-1)}(\mathbb{R}^n_+)$ *and the system*

$$-\Delta\boldsymbol{u} = \boldsymbol{f} \ \text{ in } \mathbb{R}^n_+; \quad \boldsymbol{u}' = \boldsymbol{0} \quad \text{and} \quad \frac{\partial u_n}{\partial x_n} = 0 \ \text{ on } \Gamma \tag{70}$$

has a unique solution $\boldsymbol{u} \in \mathbf{W}_0^{1,n/(n-1)}(\mathbb{R}^n_+)$.

Proof. The problem (70) is equivalent to the following one: Find $\boldsymbol{u}$ belonging $\mathbf{W}_0^{1,n/(n-1)}(\mathbb{R}^n_+)$ such that $\boldsymbol{u}' = \mathbf{0}$ on Γ and for all $\boldsymbol{\varphi} = (\boldsymbol{\varphi}', \varphi_n) \in \mathring{\mathbf{W}}_0^{1,n}(\mathbb{R}^n_+) \times \mathcal{D}(\overline{\mathbb{R}^n_+})$,

$$\int_{\mathbb{R}^n_+} \nabla \boldsymbol{u} : \nabla \boldsymbol{\varphi} \ = <\boldsymbol{f}', \boldsymbol{\varphi}' >_{\mathbf{W}_0^{-1,n/(n-1)}(\mathbb{R}^n_+) \times \mathring{\mathbf{W}}_0^{1,n}(\mathbb{R}^n_+)} + \int_{\mathbb{R}^n_+} f_n \varphi_n. \tag{71}$$

Step 1: First of all, we shall show that for all $\boldsymbol{\varphi} \in \mathbf{W}_0^{1,n}(\mathbb{R}^n_+) \cap \mathbf{L}^\infty(\mathbb{R}^n_+)$ such that $\boldsymbol{\varphi}' = \mathbf{0}$ on Γ, we have

$$|\int_{\mathbb{R}^n_+} \boldsymbol{f} \cdot \boldsymbol{\varphi}| \ \leq \ C\,(||\,\boldsymbol{f}\,||_{\mathbf{L}^1(\mathbb{R}^n_+)} + ||\,\mathrm{div}\,\boldsymbol{f}\,||_{\mathcal{K}})||\,\boldsymbol{\varphi}\,||_{\mathbf{W}_0^{1,n}(\mathbb{R}^n_+)} \tag{72}$$

where $\mathcal{K} = [W_0^{2,n}(\mathbb{R}^n_+) \cap \mathring{W}{}_{-1}^{1,n}(\mathbb{R}^n_+)]'$ here. Indeed, for every $\boldsymbol{\varphi} \in \mathbf{W}_0^{1,n}(\mathbb{R}^n_+) \cap \mathbf{L}^\infty(\mathbb{R}^n_+)$ such that $\boldsymbol{\varphi}' = \mathbf{0}$ on Γ, let $\lambda \in W_0^{2,n}(\mathbb{R}^n_+)$ such that $\lambda = 0$ and $\dfrac{\partial \lambda}{\partial x_n} = \varphi_n$ on Γ with the following estimate

$$||\,\lambda\,||_{W_0^{2,n}(\mathbb{R}^n_+)} \ \leq \ C\,||\,\varphi_n\,||_{W_0^{1-1/n,n}(\Gamma)} \ \leq \ C\,||\,\boldsymbol{\varphi}\,||_{\mathbf{W}_0^{1,n}(\mathbb{R}^n_+)}.$$

By setting $\widetilde{\boldsymbol{\varphi}} = \boldsymbol{\varphi} - \nabla\lambda$, we can deduce $\widetilde{\boldsymbol{\varphi}} \in \mathring{\mathbf{W}}_0^{1,n}(\mathbb{R}^n_+)$. Thanks to Theorem 4.12, we have $\widetilde{\boldsymbol{\varphi}} = \boldsymbol{\psi} + \nabla\eta$ with $\boldsymbol{\psi} \in \mathring{\mathbf{W}}_0^{1,n}(\mathbb{R}^n_+) \cap \mathbf{L}^\infty(\mathbb{R}^n_+)$ and $\eta \in \mathring{\mathbf{W}}_0^{2,n}(\mathbb{R}^n_+)$. It implies $\boldsymbol{\varphi} = \boldsymbol{\psi} + \nabla\mu$ with $\mu = \eta + \lambda \in W_0^{2,n}(\mathbb{R}^n_+)$, $\mu = 0$ and $\dfrac{\partial\mu}{\partial x_n} = \varphi_n$ on Γ. Then, we have

$$\begin{aligned}
|\int_{\mathbb{R}^n_+} \boldsymbol{f} \cdot \boldsymbol{\varphi}| \quad &= \ |\textstyle\int_{\mathbb{R}^n_+} \boldsymbol{f} \cdot \boldsymbol{\psi} - <\,\mathrm{div}\,\boldsymbol{f}, \mu >| \\
&\leq \ ||\,\boldsymbol{f}\,||_{\mathbf{L}^1(\mathbb{R}^n_+)}||\,\boldsymbol{\psi}\,||_{\mathbf{L}^\infty(\mathbb{R}^n_+)} + ||\,\mathrm{div}\,\boldsymbol{f}\,||_{\mathcal{K}}||\,\mu\,||_{W_0^{2,n}(\mathbb{R}^n_+)} \\
&\leq C(\,||\,\boldsymbol{f}\,||_{\mathbf{L}^1(\mathbb{R}^n_+)} + ||\,\mathrm{div}\,\boldsymbol{f}\,||_{\mathcal{K}}\,)||\,\boldsymbol{\varphi}\,||_{\mathbf{W}_0^{1,n}(\mathbb{R}^n_+)},
\end{aligned}$$

with the estimate (72).

Step 2: Replacing now $\boldsymbol{\varphi}$ by $(\boldsymbol{\varphi}', \varphi_n + K)$ with $K \in \mathbb{R}$, then for all $\boldsymbol{\varphi} \in \mathbf{W}_0^{1,n}(\mathbb{R}^n_+) \cap \mathbf{L}^\infty(\mathbb{R}^n_+)$ such that $\boldsymbol{\varphi}' = \mathbf{0}$ on Γ and for all $K \in \mathbb{R}$, because $\int_{\mathbb{R}^n_+} f_n = 0$, we have by (72)

$$\begin{aligned}
|\int_{\mathbb{R}^n_+} (\boldsymbol{f}' \cdot \boldsymbol{\varphi}' + f_n\varphi_n)| \quad &\leq (\,||\,\boldsymbol{f}\,||_{\mathbf{L}^1(\mathbb{R}^n_+)} + ||\,\mathrm{div}\,\boldsymbol{f}\,||_{\mathcal{K}}\,)\times \\
&\times(||\,\boldsymbol{\varphi}'\,||_{\mathbf{W}_0^{1,n}(\mathbb{R}^n_+)} + ||\,\varphi_n + K\,||_{W_0^{1,n}(\mathbb{R}^n_+)}).
\end{aligned}$$

Taking the infinum, we obtain by Hardy inequality:

$$\begin{aligned}
|\int_{\mathbb{R}^n_+} (\boldsymbol{f}' \cdot \boldsymbol{\varphi}' + f_n\varphi_n)| \quad &\leq C(\,||\,\boldsymbol{f}\,||_{\mathbf{L}^1(\mathbb{R}^n_+)} + ||\,\mathrm{div}\,\boldsymbol{f}\,||_{\mathcal{K}}\,)\times \\
&\times(\,||\,\nabla\boldsymbol{\varphi}'\,||_{\mathbf{L}^n(\mathbb{R}^n_+)} + ||\,\nabla\varphi_n\,||_{L^n(\mathbb{R}^n_+)}).
\end{aligned}$$

Then,

$$\int_{\mathbb{R}^n_+} \boldsymbol{f}\cdot \boldsymbol{\varphi} \;=\; \int_{\mathbb{R}^n_+} \boldsymbol{\mathcal{F}} : \nabla \boldsymbol{\varphi}$$

with $\boldsymbol{\mathcal{F}} \in \mathbf{L}^{n/(n-1)}(\mathbb{R}^n_+)$ and

$$||\, \boldsymbol{\mathcal{F}}\, ||_{\mathbf{L}^{n/(n-1)}(\mathbb{R}^n_+)} \;\leq\; C\, (\, ||\, \boldsymbol{f}\, ||_{\mathbf{L}^1(\mathbb{R}^n_+)} + ||\, \mathrm{div}\, \boldsymbol{f}\, ||_{\mathcal{K}}\,).$$

It means that $\boldsymbol{f} = -\mathrm{div}\, \boldsymbol{\mathcal{F}} \in \mathbf{W}_0^{-1,n/(n-1)}(\mathbb{R}^n_+)$ and $\mathcal{F}_{nn} = 0$ on Γ. It is easy to see that the problem (71) is equivalent to the following one:
Find $\boldsymbol{u} \in \mathbf{W}_0^{1,n/(n-1)}(\mathbb{R}^n_+)$ such that $\boldsymbol{u}' = \mathbf{0}$ on Γ and for any $\boldsymbol{\varphi} = (\boldsymbol{\varphi}', \varphi_n) \in \mathring{\mathbf{W}}{}_0^{1,n}(\mathbb{R}^n_+) \times W_0^{1,n}(\mathbb{R}^n_+)$, we have

$$\int_{\mathbb{R}^n_+} \nabla \boldsymbol{u} : \nabla \boldsymbol{\varphi} \;=\; \int_{\mathbb{R}^n_+} \boldsymbol{\mathcal{F}} : \nabla \boldsymbol{\varphi}. \tag{73}$$

Taking $\varphi_n = 0$ and after that taking $\boldsymbol{\varphi}' = \mathbf{0}$ on the problem (73), we directs our attention to the resolution of two classic problems, one with Dirichlet boundary condition and another with Neumann boundary condition. The regularity $\mathbf{L}^{n/(n-1)}(\mathbb{R}^n_+)$ of $\boldsymbol{\mathcal{F}}$ assures the existence of a unique $\boldsymbol{u} \in \mathbf{W}_0^{1,n/(n-1)}(\mathbb{R}^n_+)$ of the problem (73).

□

Theorem 6.9. *Let* $\boldsymbol{f} \in \mathbf{L}^1(\mathbb{R}^n_+)$ *such that* $\displaystyle\int_{\mathbb{R}^n_+} \boldsymbol{f}' = \mathbf{0}$. *If*

$$[\, \boldsymbol{f}\,] \;=\; \sup_{\xi \in \mathcal{D}(\overline{\mathbb{R}^n_+}),\, \frac{\partial \xi}{\partial x_n} = 0 \;\; on\;\; \Gamma} \frac{|\, \int_{\mathbb{R}^n_+} \boldsymbol{f} \cdot \nabla \xi\, |}{||\, \xi\, ||_{W_0^{2,n}(\mathbb{R}^n_+)}} \;<\; \infty$$

holds, then the system

$$-\Delta \boldsymbol{u} \,=\, \boldsymbol{f} \;\text{ in } \mathbb{R}^n_+, \quad u_n = 0 \quad \text{and} \quad \frac{\partial \boldsymbol{u}'}{\partial x_n} = \mathbf{0} \quad \text{on } \Gamma$$

has a unique solution $\boldsymbol{u} \in \mathbf{W}_0^{1,n/(n-1)}(\mathbb{R}^n_+)$ *and*

$$||\, \boldsymbol{u}\, ||_{\mathbf{W}_0^{1,n/(n-1)}(\mathbb{R}^n_+)} \;\leq\; C\, (\, ||\boldsymbol{f}||_{\mathbf{L}^1(\mathbb{R}^n_+)} + [\, \boldsymbol{f}\,]\,).$$

Proof. Let $\boldsymbol{\varphi} \in \mathbf{W}_0^{1,n}(\mathbb{R}^n_+) \cap \mathbf{L}^\infty(\mathbb{R}^n_+)$ such that $\varphi_n = 0$ on Γ. Then there exist $\boldsymbol{\psi} \in \mathbf{W}_0^{1,n}(\mathbb{R}^n_+) \cap \mathbf{L}^\infty(\mathbb{R}^n_+)$ and $\eta \in W_0^{2,n}(\mathbb{R}^n_+)$ such that $\boldsymbol{\varphi} = \boldsymbol{\psi} + \nabla \eta$ with $\psi_n = \dfrac{\partial \eta}{\partial x_n} = 0$ on Γ. Moreover, we have

$$||\, \boldsymbol{\psi}_0\, ||_{\mathbf{L}^\infty(\mathbb{R}^n_+)} + ||\, \boldsymbol{\psi}_0\, ||_{\mathbf{W}_0^{1,n}(\mathbb{R}^n_+)} + ||\, \eta_0\, ||_{W_0^{2,n}(\mathbb{R}^n_+)} \;\leq\; C\, ||\, \boldsymbol{\varphi}\, ||_{\mathbf{W}_0^{1,n}(\mathbb{R}^n_+)}.$$

Then,

$$\int_{\mathbb{R}^n_+} \boldsymbol{f} \cdot \boldsymbol{\varphi} \;=\; \int_{\mathbb{R}^n_+} \boldsymbol{f} \cdot \boldsymbol{\psi} + \int_{\mathbb{R}^n_+} \boldsymbol{f} \cdot \nabla \eta$$

and

$$| \int_{\mathbb{R}^n_+} \boldsymbol{f} \cdot \boldsymbol{\varphi} | \;\leq\; C\,(\,\|\,\boldsymbol{f}\,\|_{\mathbf{L}^1(\mathbb{R}^n_+)} + [\,\boldsymbol{f}\,]\,)\|\,\boldsymbol{\varphi}\,\|_{\mathbf{W}^{1,n}_0(\mathbb{R}^n_+)}.$$

Let $\boldsymbol{K} = (\boldsymbol{K}', 0) \in \mathbb{R}^{n-1} \times \{0\}$. Then we have

$$\begin{aligned}
& | \int_{\mathbb{R}^n_+} \boldsymbol{f}' \cdot (\boldsymbol{\varphi}' + \boldsymbol{K}) + \int_{\mathbb{R}^n_+} f_n \varphi_n \,| \;=\; | \int_{\mathbb{R}^n_+} \boldsymbol{f} \cdot \boldsymbol{\varphi}\,| \\
& \leq\; C\,(\,\|\,\boldsymbol{f}\,\|_{\mathbf{L}^1(\mathbb{R}^n_+)} + [\,\boldsymbol{f}\,]\,)(\,\|\,\boldsymbol{\varphi}' + \boldsymbol{K}\,\|_{\mathbf{W}^{1,n}_{\mathbf{0}}(\mathbb{R}^n_+)} + \|\,\varphi_n\,\|_{W^{1,n}_0(\mathbb{R}^n_+)}\,) \\
& \leq\; C\,(\,\|\,\boldsymbol{f}\,\|_{\mathbf{L}^1(\mathbb{R}^n_+)} + [\,\boldsymbol{f}\,]\,)(\,\|\,\nabla \boldsymbol{\varphi}'\,\|_{\mathbf{L}^n(\mathbb{R}^n_+)} + \|\,\nabla \varphi_n\,\|_{\mathbf{L}^n(\mathbb{R}^n_+)}).
\end{aligned}$$

The remains of this proof is similar to Theorem 6.8. □

Recall now a result of C. Amrouche, S. Nečasová and Y. Raudin in [7] concerning the Stokes problem in $\mathbb{R}^n_+$.

Theorem 6.10. *For any* $\boldsymbol{f} \in \mathbf{W}^{-1,p}_0(\mathbb{R}^n_+)$, $h \in L^p(\mathbb{R}^n_+)$ *and* $\boldsymbol{g} \in \mathbf{W}^{1-1/p,p}_0(\Gamma)$, *then the Stokes system*

$$(\mathcal{S}^+)\;\begin{cases} -\Delta \boldsymbol{u} + \nabla \pi = \boldsymbol{f} & \text{in } \mathbb{R}^n_+, \\ \operatorname{div} \boldsymbol{u} = h & \text{in } \mathbb{R}^n_+, \\ \boldsymbol{u} = \boldsymbol{g} & \text{on } \Gamma = \mathbb{R}^{n-1}, \end{cases}$$

has a unique solution $(\boldsymbol{u}, \pi) \in \mathbf{W}^{1,p}_0(\mathbb{R}^n_+) \times L^p(\mathbb{R}^n_+)$. *Moreover, we have the estimate*

$$\|\,\boldsymbol{u}\,\|_{\mathbf{W}^{1,p}_0(\mathbb{R}^n_+)} + \|\,\pi\,\|_{L^p(\mathbb{R}^n_+)} \;\leq\; C(\,\|\,\boldsymbol{f}\,\|_{\mathbf{W}^{-1,p}_0(\mathbb{R}^n_+)} + \|\,h\,\|_{L^p(\mathbb{R}^n_+)} + \|\,\boldsymbol{g}\,\|_{\mathbf{W}^{1-1/p,p}_0(\Gamma)}).$$

As consequence, we obtain the following Helmholtz decomposition.

Corollary 6.11. *Let* $\boldsymbol{f} \in \mathbf{X}(\mathbb{R}^3_+)$. *Then there exists a unique* $\boldsymbol{\varphi} \in \mathbf{L}^{3/2}(\mathbb{R}^3_+)$ *such that* $\operatorname{div} \boldsymbol{\varphi} = 0$ *with* $\varphi_3 = 0$ *on* Γ *and a unique* $\pi \in L^{3/2}(\mathbb{R}^3_+)$ *satisfying*

$$\boldsymbol{f} = \mathbf{curl}\, \boldsymbol{\varphi} + \nabla \pi$$

and the following estimate holds

$$\|\,\boldsymbol{\varphi}\,\|_{\mathbf{L}^{3/2}(\mathbb{R}^3_+)} + \|\,\pi\,\|_{L^{3/2}(\mathbb{R}^3_+)} \;\leq\; C\,\|\,\boldsymbol{f}\,\|_{\mathbf{X}(\mathbb{R}^3_+)}.$$

Proof. Let $\boldsymbol{f} \in \mathbf{X}(\mathbb{R}^3_+)$. Then we have $\boldsymbol{f} \in \mathbf{W}^{-1,3/2}_0(\mathbb{R}^3_+)$. Thanks to Theorem 6.10, there exists a unique solution $(\boldsymbol{u}, \pi) \in \mathbf{W}^{1,3/2}_0(\mathbb{R}^3_+) \times L^{3/2}(\mathbb{R}^3_+)$ such that

$$-\Delta \boldsymbol{u} + \nabla \pi \;=\; \boldsymbol{f} \text{ in } \mathbb{R}^3_+, \quad \operatorname{div} \boldsymbol{u} \;=\; 0 \text{ in } \mathbb{R}^3_+, \quad \boldsymbol{u} \;=\; \mathbf{0} \text{ on } \Gamma.$$

This proof is finished by choosing $\boldsymbol{\varphi} \;=\; \mathbf{curl}\, \boldsymbol{u}$. □

Corollary 6.12. *Let* $(\boldsymbol{f}, h, \boldsymbol{g}) \in \mathbf{X}(\mathbb{R}^n) \times L^{n/(n-1)}(\mathbb{R}^n) \times \mathbf{W}_0^{1/n,n/(n-1)}(\Gamma)$. *Then the Stokes system* $(\mathcal{S}^+)$ *has a unique solution* $(\boldsymbol{u}, \pi) \in \mathbf{W}_0^{1,n/(n-1)}(\mathbb{R}^n_+) \times L^{n/(n-1)}(\mathbb{R}^n_+)$ *and the following estimate holds*

$$\begin{aligned} &\|\, \boldsymbol{u} \,\|_{\mathbf{W}_0^{1,n/(n-1)}(\mathbb{R}^n_+)} + \|\, \pi \,\|_{L^{n/(n-1)}(\mathbb{R}^n_+)} \\ &\leq \; C \left(\|\, \boldsymbol{f} \,\|_{\mathbf{W}_0^{-1,n/(n-1)}(\mathbb{R}^n_+)} + \|\, h \,\|_{L^{n/(n-1)}(\mathbb{R}^n_+)} + \|\, \boldsymbol{g} \,\|_{\mathbf{W}_0^{1/n,n/(n-1)}(\Gamma)} \right). \end{aligned}$$

Proof. This corollary is a consequence of $\mathbf{X}(\mathbb{R}^n_+) \hookrightarrow \mathbf{W}_0^{-1,n/(n-1)}(\mathbb{R}^n_+)$ and Theorem 6.10. □

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