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ANALYSIS OF THE UPWIND FINITE VOLUME METHOD FOR GENERAL INITIAL AND BOUNDARY VALUE TRANSPORT PROBLEMS

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Abstract. This paper is devoted to the convergence analysis of the upwind finite volume scheme for the initial and boundary value problem associated to the linear transport equation in any dimension, on general unstructured meshes. We are particularly interested in the case where the initial and boundary data are in L^∞ and the advection vector field v has low regularity properties, namely $v \in L^1(]0,T[,(W^{1,1}(\Omega))^d)$, with suitable assumptions on its divergence. In this general framework, we prove uniform in time strong convergence in $L^p(\Omega)$ with $p < +\infty$, of the approximate solution towards the unique weak solution of the problem as well as the strong convergence of its trace. The proof relies, in particular, on the Friedrichs' commutator argument, which is classical in the renormalized solutions theory.

Keywords. Finite volume methods - Transport equation - Renormalized solutions

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1. Introduction. This paper is devoted to the analysis of the upwind finite volume scheme for solving a general linear transport-reaction problem in any dimension. We are interested here in a low regularity framework for the data, still leading to existence and uniqueness of weak solutions, namely the one of renormalized solutions first introduced and studied in [DL89]. More precisely, we consider here the case where the transport vector-field may not be characteristic at the boundary of the domain. It is thus needed to use the trace theorems and the well-posedness results for the associated initial and boundary value problems given in [Boy05].

Our main result in the present paper is the proof of the uniform in time strong convergence in $L^p(\Omega), p < +\infty$, of the approximate solution given by the finite volume scheme towards the unique weak solution of the continuous problem with minimal assumptions on the data, and the meshes.

General notation. We shall adopt the following notation.

- $\mathcal{L}(f)$ will denote the Lipschitz constant of any Lipschitz continuous function f.
- For any real number x we define its positive and negative parts by $x^+ = (x+|x|)/2$, $x^- = (|x|-x)/2$, and we will often use that $x = x^+ - x^-$ and $|x| = x^+ + x^-$. Moreover, we denote by $\mathrm{sgn}(x)$ its sign, with the convention that $\mathrm{sgn}(0) = 0$.
- For any $a, b \in \mathbb{R}$, we define $[\![a,b]\!] = [a,b] \cap \mathbb{N}$.
- The characteristic function of a set A will be denoted by $\mathbb{1}_A$.

The continuous problem. Let $d \geq 1$, $\Omega \subset \mathbb{R}^d$ a bounded polygonal (or polyhedral) domain, and $T > 0$ given. We are interested here in the following initial and boundary value problem

$$\begin{cases} \partial_t \rho + \mathrm{div}\,(\rho v) + c\rho = 0, \quad \text{in }]0,T[\times\Omega, \\ \rho(0,\cdot) = \rho_0, \quad \text{in } \Omega, \\ \rho = \rho^{in}, \quad \text{on }]0,T[\times\Gamma, \text{ where } (v\cdot\boldsymbol{\nu}) < 0. \end{cases} \tag{1.1}$$

The general existence and uniqueness theory given in [Boy05, BF11] relies on the following assumptions

$$c \in L^1(]0,T[\times\Omega), \tag{1.2}$$

$$\begin{cases} v \in L^1(]0,T[,(W^{1,1}(\Omega))^d), & \text{(1.3a)} \\ (v\cdot\boldsymbol{\nu}) \in L^\alpha(]0,T[\times\Gamma), \text{ for some } \alpha > 1, & \text{(1.3b)} \end{cases}$$

$$\begin{cases} (c + \mathrm{div}\, v)^- \in L^1(]0,T[,L^\infty(\Omega)), & \text{(1.4a)} \\ (\mathrm{div}\, v)^+ \in L^1(]0,T[,L^\infty(\Omega)). & \text{(1.4b)} \end{cases}$$

The case where $c = \mathrm{div}\, v = 0$ and where Ω is a smooth domain is treated in [Boy05] and the extension to general data c, v and piecewise smooth domains is given in [BF11]. Associated to the vector field v, we introduce the measure $d\mu_v = (v\cdot\boldsymbol{\nu})\,dx\,dt$ on $]0,T[\times\Gamma$ and we denote by $d\mu_v^+$ (resp. $d\mu_v^-$) its positive (resp. negative) part in such a way that $|d\mu_v| = d\mu_v^+ + d\mu_v^-$. The support of $d\mu_v^+$ (resp. $d\mu_v^-$) is the outflow (resp. inflow) part of the boundary.

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This problem is the conservative form of the linear transport-reaction equation. As an example, for $c = -\operatorname{div} v$, we recover the usual non-conservative transport equation $\partial_t \rho + v \cdot \nabla \rho = 0$.

THEOREM 1.1 (Existence and uniqueness, [Boy05, BF11]). *We assume that assumptions* (1.2), (1.3), (1.4) *hold. For any* $\rho^0 \in L^\infty(\Omega)$ *and* $\rho^{in} \in L^\infty(]0,T[\times\Gamma, d\mu_v^-)$, *there exists a unique weak solution* $(\rho, \gamma(\rho)) \in L^\infty(]0,T[\times\Omega) \times L^\infty(]0,T[\times\Gamma, |d\mu_v|)$ *of* (1.1) *in the sense that*

$$\int_0^T \int_\Omega \rho(\partial_t \phi + v \cdot \nabla \phi - c\phi)\, dx\, dt - \int_0^T \int_\Gamma \gamma(\rho)\phi(v \cdot \boldsymbol{\nu})\, dx\, dt + \int_\Omega \rho^0 \phi(0,.)\, dx = 0, \quad \forall \phi \in \mathcal{C}_c^1([0,T[\times\overline{\Omega}), \tag{1.5}$$

the boundary condition being satisfied in the following sense

$$\gamma(\rho) = \rho^{in}, \quad d\mu_v^- \text{-almost everywhere.}$$

Moreover, the following properties are also proven in the same references:

- L^∞ **bound:**

$$\|\rho\|_{L^\infty(]0,T[\times\Omega)} \le \max(\|\rho_0\|_{L^\infty}, \|\rho^{in}\|_{L^\infty}) e^{\int_0^T \|(c+\operatorname{div} v)^-\|_{L^\infty(\Omega)}\, dt}. \tag{1.6}$$

- **Time regularity:** ρ lies in $\mathcal{C}^0([0,T], L^p(\Omega))$ for any $p < +\infty$ and $\rho(0) = \rho_0$.
- **Renormalization property:** For any smooth function $\beta : \mathbb{R} \mapsto \mathbb{R}$, the function $\beta(\rho)$ satisfies in the weak sense the problem

$$\begin{cases} \partial_t \beta(\rho) + \operatorname{div}(\beta(\rho) v) + c\beta'(\rho)\rho + (\operatorname{div} v)(\beta'(\rho)\rho - \beta(\rho)) = 0, & \text{in }]0,T[\times\Omega, \quad (1.7a) \\ \beta(\rho)(0,.) = \beta(\rho^0), & (1.7b) \\ \gamma(\beta(\rho)) = \beta(\gamma(\rho)), & \text{on }]0,T[\times\Gamma. \quad (1.7c) \end{cases}$$

Note that this property still holds for any continuous piecewise smooth function β (see Lemma 6.2).

Assumption (1.4a) clearly plays a fundamental role to obtain the L^∞ bound above. However, assumption (1.4b) is only useful in order to deduce the uniqueness property from the renormalization property through a Gronwall-like argument. Note that this last assumption can be slightly relaxed (see [Des96]) allowing to use Osgood's Lemma instead of Gronwall's Lemma. For instance, all the above results still hold if we assume the weaker condition that $e^{C(\operatorname{div} v)^+} \in L^1(]0,T[\times\Omega)$, for some $C > 0$.

Previously known results. The upwind finite volume method is the most classical stable method for the numerical approximation of linear transport problems (see for instance [EGH00, LeV02]). The method is first order for smooth solutions but it is well-known that, for non smooth initial data (say in $\mathrm{BV}(\Omega)$ or in some Sobolev space), the optimal convergence rate falls down to $1/2$, see for instance [DL11, MV07, Mer08, Kuz76, VV03, Des04a, Des04b]. As shown in [BGP05], contrary to what can be thought at first sight, the irregularity of the mesh is not the main reason for this behavior. In fact, this loss of convergence rate is mainly due to the numerical dissipation of the scheme which implies that discontinuities in the solution are smoothed along time even on regular grids thus leading to suboptimal convergence rate.

In all the results cited above, the transport vector field v is assumed to be at least Lipschitz-continuous (some of them assume further that v is constant) in order for the associated characteristic flow to be well defined and smooth enough, which is often one of the main tools in these analysis. Moreover, to our knowledge, the analysis of finite volume schemes for boundary value problems for linear hyperbolic equations is only addressed in [BGP05, CVV00] in the case of a constant vector field v (see for instance [OV06] for the case of nonlinear conservation laws).

The present study extends those results by accounting for less regular general vector fields and for L^∞ initial/boundary data. The prize to pay is that, to the best of our knowledge, no convergence rate is known in this general framework. Nevertheless, since the renormalized solution theory allows to define a suitable weak notion of characteristic flows for vector fields satisfying (1.3) (the so-called *regular Lagrangian flow*, see [DL07]), it should be possible to extend some of the results cited above concerning the convergence rate of the scheme to the current framework. We finally mention [Fet11] where, in the case of the mass conservation equation (that is for $c = 0$) and without boundary conditions (that is when $v \cdot \boldsymbol{\nu} = 0$ on the boundary), the weak-$\star$ convergence of the solution of the upwind scheme is proven in the above framework for meshes made of simplexes of $\mathbb{R}^d$.

2. The implicit upwind finite volume scheme.

2.1. Notation. We introduce here the main notation we need to define and analyse the finite volume method. A finite volume mesh of the domain Ω is a set $\mathcal{T} = (\kappa)_{\kappa\in\mathcal{T}}$ of closed connected polygonal subsets of $\mathbb{R}^d$, with disjoint interiors and such that $\overline{\Omega} = \bigcup_{\kappa\in\mathcal{T}} \kappa$.

The boundary of each control volume $\kappa \in \mathcal{T}$ can be written as the union of a finite number of edges/faces (we will often use the word "edge" even if $d > 2$) which are closed connected sets of dimension $d-1$ contained into hyperplanes.

We denote by $\mathcal{E}_\kappa$ the set of the faces/edges of κ. We assume that for any $\kappa, \mathcal{L}$ such that $\kappa \neq \mathcal{L}$ and $\kappa \cap \mathcal{L}$ is of co-dimension 1, then $\kappa \cap \mathcal{L} \in \mathcal{E}_\kappa \cap \mathcal{E}_\mathcal{L}$, in that case the corresponding face is denoted by $\kappa|\mathcal{L}$.

The set of all the faces in the mesh is denoted by $\mathcal{E}$ and $\mathcal{E}_{\rm bd}$ denote the subset of the faces which are included in the boundary $\partial\Omega$, $\mathcal{E}_{\rm int} = \mathcal{E} \setminus \mathcal{E}_{\rm bd}$ the set of the interior faces.

- For each $\kappa \in \mathcal{T}$, and $\sigma \in \mathcal{E}_\kappa$, we denote by $\boldsymbol{\nu}_{\kappa\sigma}$ the unit outward normal vector to κ on σ. If $\sigma = \kappa|\mathcal{L} \in \mathcal{E}_{\rm int}$, we shall sometimes use the notation $\boldsymbol{\nu}_{\kappa\mathcal{L}} = \boldsymbol{\nu}_{\kappa\sigma} = -\boldsymbol{\nu}_{\mathcal{L}\sigma}$. If $\sigma \in \mathcal{E}_{\rm bd}$, there is a unique $\kappa \in \mathcal{T}$ such that $\sigma \in \mathcal{E}_\kappa$ and then $\boldsymbol{\nu}_{\kappa\sigma}$ is nothing but the unit outward normal to $\partial\Omega$ and we may also write $\boldsymbol{\nu}_\sigma$ or $\boldsymbol{\nu}$ if no confusion is possible.
- We will denote by $|\kappa|$ (resp. $|\sigma|$) the d-dimensional Lebesgue measure of the control volume κ (resp. the $d-1$ dimensional measure of the face σ).
- The diameter of a control volume κ (resp. of an edge σ) shall be denoted by d_κ (resp. d_σ) and the size of the mesh is defined by $h_\mathcal{T} = \max_{\kappa\in\mathcal{T}} d_\kappa$.

We will need to measure the regularity of the mesh. To this end, we denote by $\mathrm{reg}(\mathcal{T})$, the smallest positive number such that

$$\|f\|_{L^1(\partial\kappa)} \leq \frac{\mathrm{reg}(\mathcal{T})}{d_\kappa}\|f\|_{W^{1,1}(\kappa)}, \quad \forall \kappa \in \mathcal{T}, \forall f \in W^{1,1}(\kappa). \tag{2.1}$$

In the convergence results given below we shall assume that $\mathrm{reg}(\mathcal{T})$ remains bounded as $h_\mathcal{T} \to 0$, which amounts to assume that the control volumes are not allowed to degenerate. For instance, in the case of control volumes which are simplexes, then the above assumption is nothing but the usual regularity assumption used in the finite element framework. Note finally, that (2.1) implies in particular (take $f = 1$) that

$$\sum_{\sigma\in\mathcal{E}_\kappa} d_\kappa|\sigma| \leq \mathrm{reg}(\mathcal{T})|\kappa|, \quad \forall \kappa \in \mathcal{T}.$$

It will be useful to associate a point $x_\kappa \in \kappa$ to each control volume $\kappa \in \mathcal{T}$. We may for instance choose x_κ to be the mass center of κ, if κ is convex. These points actually do not enter the definition of the scheme, they are only used as a tool in the analysis.

2.2. Definition of the scheme. Let us first define the discretization of the data needed to define our finite volume method (see Section 6.2 for further comments on the data).

- For any $\kappa \in \mathcal{T}$, $n \in [\![0, N-1]\!]$, we define

$$c_\kappa^n = \frac{1}{\delta t|\kappa|}\int_{t^n}^{t^{n+1}}\int_\kappa c\,dx\,dt, \quad \text{and} \quad v_{\kappa\sigma}^n = \frac{1}{\delta t|\sigma|}\int_{t^n}^{t^{n+1}}\int_\sigma (v\cdot\boldsymbol{\nu}_{\kappa\sigma})\,dx\,dt, \ \forall \sigma \in \mathcal{E}_\kappa.$$

  Furthermore, if $\sigma \in \mathcal{E}_{\rm int}$, with $\sigma = \kappa|\mathcal{L}$ we shall use the notation $v_{\kappa\mathcal{L}}^n = v_{\kappa\sigma}^n = -v_{\mathcal{L}\sigma}^n$, and if $\sigma \in \mathcal{E}_\kappa \cap \mathcal{E}_{\rm bd}$ we will note $v_\sigma^n = v_{\kappa\sigma}^n$. We will often use the fact that, by Stokes' formula, we have

$$\sum_{\sigma\in\mathcal{E}_\kappa} |\sigma|v_{\kappa\sigma}^n = |\kappa|(\mathrm{div}\, v)_\kappa^n = \frac{1}{\delta t}\int_{t^n}^{t^{n+1}}\int_\kappa (\mathrm{div}\, v)\,dx\,dt. \tag{2.2}$$

- For any boundary edge $\sigma \in \mathcal{E}_{\rm bd}$ and any $n \in [\![0, N-1]\!]$, we define

$$\rho_\sigma^{in,n+1} = \frac{1}{\delta t|\sigma|}\int_{t^n}^{t^{n+1}}\int_\sigma \rho^{in}\,dx\,dt. \tag{2.3}$$

  Notice that ρ^{in} is *a priori* only given $d\mu_v^-$-almost everywhere so that in this formula we need, in fact, to consider an extension of ρ^{in} in $L^\infty(]0,T[\times\Gamma)$.

To simplify a little the notation, let us introduce $v_{\kappa\sigma}^{n+} = (v_{\kappa\sigma}^n)^+$ and $v_{\kappa\sigma}^{n-} = (v_{\kappa\sigma}^n)^-$. The implicit finite volume scheme we consider is the following: Find $(\rho_\kappa^n)_{\substack{n\in[\![0,N]\!]\\ \kappa\in\mathcal{T}}}$ such that

$$\begin{cases} |\kappa|\dfrac{\rho_\kappa^{n+1}-\rho_\kappa^n}{\delta t} + \displaystyle\sum_{\sigma\in\mathcal{E}_\kappa\cap\mathcal{E}_{\rm int}} |\sigma|\big(v_{\kappa\sigma}^{n+}\rho_\kappa^{n+1} - v_{\kappa\sigma}^{n-}\rho_\mathcal{L}^{n+1}\big) + \sum_{\sigma\in\mathcal{E}_\kappa\cap\mathcal{E}_{\rm bd}} |\sigma|v_{\kappa\sigma}^n\rho_\sigma^{n+1} \\ \qquad\qquad + |\kappa|c_\kappa^n\rho_\kappa^{n+1} = 0, \quad \forall n \in [\![0,N-1]\!], \forall \kappa \in \mathcal{T}, \\ \rho_\kappa^0 = \dfrac{1}{|\kappa|}\displaystyle\int_\kappa \rho^0\,dx, \quad \forall \kappa \in \mathcal{T}, \\ \rho_\sigma^{n+1} = \rho_\sigma^{in,n+1}, \quad \forall n \in [\![0,N-1]\!], \forall \sigma \in \mathcal{E}_{\rm bd}, \ \text{ s.t. } \ v_{\kappa\sigma}^n \leq 0, \\ \rho_\sigma^{n+1} = \rho_\kappa^{n+1}, \quad \forall n \in [\![0,N-1]\!], \forall \sigma \in \mathcal{E}_{\rm bd}, \ \text{ s.t. } \ v_{\kappa\sigma}^n > 0. \end{cases} \tag{2.4}$$

Remark 2.1.

- *For the pure advection equation, that is when $c = -\operatorname{div} v$, with (2.2), the scheme reads*

$$|\kappa|\frac{\rho_\kappa^{n+1}-\rho_\kappa^n}{\delta t} + \sum_{\sigma\in\mathcal{E}_\kappa\cap\mathcal{E}_{\rm int}} |\sigma| v_{\kappa\sigma}^{n-}\left(\rho_\kappa^{n+1}-\rho_\mathcal{L}^{n+1}\right) + \sum_{\sigma\in\mathcal{E}_\kappa\cap\mathcal{E}_{\rm bd}} |\sigma| v_{\kappa\sigma}^{n-}(\rho_\kappa^{n+1}-\rho_\sigma^{in,n+1}) = 0,$$

  which is a more usual formulation of the scheme for the pure advection equation.
- *We only consider here the implicit version of the scheme in order to avoid the introduction of a stability CFL condition but all the results given below are valid for the explicit scheme.*

2.3. Outline of the paper. In Section 3, we prove existence and uniqueness of the approximate solution, for small enough time steps (this condition on δt being independent of the mesh $\mathcal{T}$), then we establish *a priori* estimates on those solutions and their traces: a uniform L^∞-bound and a weak $L^2(H^1)$ estimate, which will be useful in the convergence analysis. In Section 4, we prove the weak-$\star$ convergence in L^∞ of the approximate solution towards the unique weak solution of the problem, as well as for the traces. In Section 5, we prove that this convergence is in fact strong in $L^\infty(]0,T[,L^p(\Omega))$, for any $p<+\infty$, together with a suitable strong convergence result for the traces. The proof of this result is based on the same tools than those used to prove existence and uniqueness of weak solutions for the problem in the framework of renormalized solutions, namely the Friedrichs commutator lemma. Note that the strong convergence of the approximate solutions in $L^p(]0,T[\times\Omega)$ is easier to obtain; the difficult point here is to prove a convergence which is uniform in time. We conclude the paper by some extensions and remarks concerning the scheme under study.

2.4. A technical result. We conclude this introduction by a technical result that will be useful in the sequel.

LEMMA 2.1.

1. *For any $1\le p<+\infty$, $f\in L^p(]0,T[\times\Gamma)$ we have*

$$\sum_{n=0}^{N-1}\sum_{\sigma\in\mathcal{E}_{\rm bd}}\int_{t^n}^{t^{n+1}}\int_\sigma \left|f(t,x)-f_\sigma^{n+1}\right|^p\,dx\,dt \xrightarrow[(\delta t,h_\mathcal{T})\to 0]{} 0,$$

   where f_σ^{n+1} is the mean-value of f on $]t^n,t^{n+1}[\times\sigma$.
2. *For any $v\in L^1(]0,T[,(W^{1,1}(\Omega))^d)$, we have*

$$\sum_{n=0}^{N-1}\sum_{\substack{\sigma\in\mathcal{E}_{\rm bd}\\ v_{\kappa\sigma}^n\le 0}}\int_{t^n}^{t^{n+1}}\int_\sigma (v\cdot\boldsymbol{\nu}_\sigma)^+\,dx\,dt + \sum_{n=0}^{N-1}\sum_{\substack{\sigma\in\mathcal{E}_{\rm bd}\\ v_{\kappa\sigma}^n\ge 0}}\int_{t^n}^{t^{n+1}}\int_\sigma (v\cdot\boldsymbol{\nu}_\sigma)^-\,dx\,dt \xrightarrow[(\delta t,h_\mathcal{T})\to 0]{} 0.$$

Proof.

1. This is a classical result of the approximation theory: the claim is clear when f is smooth and the general case is obtained by density, since the left-hand side term can be bounded by $2^p\|f\|^p_{L^p(]0,T[\times\Gamma)}$, uniformly with respect to δt and $h_\mathcal{T}$.
2. We observe that, if $v_{\kappa\sigma}^n\le 0$ then we have

$$\begin{aligned}\int_{t^n}^{t^{n+1}}\int_\sigma (v\cdot\boldsymbol{\nu}_\sigma)^+\,dt\,dx &= \frac{1}{2}\delta t|\sigma|v_{\kappa\sigma}^n + \frac{1}{2}\int_{t^n}^{t^{n+1}}\int_\sigma |v\cdot\boldsymbol{\nu}_\sigma|\,dt\,dx\\ &\le \underbrace{\frac{1}{2}\delta t|\sigma|v_{\kappa\sigma}^n + \frac{1}{2}\delta t|\sigma||v_{\kappa\sigma}^n|}_{=0} + \frac{1}{2}\int_{t^n}^{t^{n+1}}\int_\sigma |v\cdot\boldsymbol{\nu}_\sigma - v_{\kappa\sigma}^n|\,dt\,dx \le \frac{1}{2}\int_{t^n}^{t^{n+1}}\int_\sigma |v\cdot\boldsymbol{\nu}_\sigma - v_{\kappa\sigma}^n|\,dt\,dx,\end{aligned}$$

   so that, by the first part of the Lemma applied to $f=(v\cdot\boldsymbol{\nu})$, we get

$$\sum_{n=0}^{N-1}\sum_{\substack{\sigma\in\mathcal{E}_{\rm bd}\\ v_{\kappa\sigma}^n<0}}\int_{t^n}^{t^{n+1}}\int_\sigma (v\cdot\boldsymbol{\nu}_\sigma)^+\,dx\,dt \le \frac{1}{2}\sum_{n=0}^{N-1}\sum_{\sigma\in\mathcal{E}_{\rm bd}}\int_{t^n}^{t^{n+1}}\int_\sigma |v\cdot\boldsymbol{\nu}_\sigma - v_{\kappa\sigma}^n|\,dt\,dx \xrightarrow[(\delta t,h_\mathcal{T})\to 0]{} 0.$$

   The other term being treated similarly, the claim is proven.

□

3. Existence and uniqueness. *A priori* estimates.

3.1. Existence and uniqueness. First properties. Since the scheme we are studying is implicit in time, it is needed to prove that the approximate solution actually exists and is unique. This is the goal of the first result of this paper.

THEOREM 3.1. *Assume that* (1.2),(1.3a) *and* (1.4a) *hold. There exists* $\delta t_{\max} > 0$ *(depending only on* $(c + \operatorname{div} v)^-$*) such that for any initial and boundary data* $\rho^0 \in L^\infty(\Omega)$, $\rho^{in} \in L^\infty(]0,T[\times\Gamma)$, *any mesh* $\mathcal{T}$ *and any time step such that*

$$\delta t \leq \delta t_{\max},$$

there exists an unique solution of the scheme (2.4).

Moreover in that case, the scheme is monotone, that is

$$\big(\rho^0 \geq 0 \text{ and } \rho^{in} \geq 0\big) \implies \big(\rho_\kappa^n \geq 0, \forall \kappa \in \mathcal{T}, \forall n \in [\![0,N]\!]\big).$$

Finally, the following L^∞ *bound holds:*

$$|\rho_\kappa^n| \leq \max(\|\rho^0\|_{L^\infty}, \|\rho^{in}\|_{L^\infty}) \exp\left(2\int_0^{t^n} \|(c+\operatorname{div} v)^-\|_{L^\infty}\, dt\right), \quad \forall \kappa \in \mathcal{T}, \forall n \in [\![0,N]\!].$$

It will be clear in the proof that, in the case where $c + \operatorname{div} v \geq 0$ (in particular for the pure advection equation where $c = -\operatorname{div} v$), we have $\delta t_{\max} = +\infty$.

Proof. The initial data $(\rho_\kappa^0)_{\kappa\in\mathcal{T}}$ is directly defined from ρ^0. Assume now that $(\rho_\kappa^n)_{\kappa\in\mathcal{T}}$ is known at time t^n, $n \leq N-1$ and let us show that $(\rho_\kappa^{n+1})_{\kappa\in\mathcal{T}}$ is uniquely defined.

The set of equations being linear with the same number of unknowns as that of equations, it is enough to show that, if $\rho^{in} = 0$, and $\rho_\kappa^n = 0$ for any $\kappa \in \mathcal{T}$, then any solution of the system satisfies $\rho_\kappa^{n+1} = 0$ for any $\kappa \in \mathcal{T}$. To this end, we will in fact prove the monotony of the scheme which will imply its well-posedness.

By assumption $t \mapsto \|(c+\operatorname{div} v)^-(t)\|_{L^\infty}$ is integrable on $]0,T[$, hence there exists $\delta t_{\max} > 0$ such that

$$\int_I \|(c+\operatorname{div} v)^-(t)\|_{L^\infty}\, dt \leq \frac{1}{2}, \quad \forall I \subset]0,T[, \text{ s.t. } |I| \leq \delta t_{\max}. \tag{3.1}$$

- **Step 1. Change of variables.** For any $n \in [\![0, N-1]\!]$, let us define

$$\gamma^n = \frac{1}{\delta t}\int_{t^n}^{t^{n+1}} \|(c+\operatorname{div} v)^-\|_{L^\infty}\, dt, \tag{3.2}$$

and $(\alpha^n)_n$ by

$$\alpha^0 = 1, \quad \alpha^{n+1} = (1-\delta t\gamma^n)\alpha^n, \quad \forall n \in [\![0,N-1]\!]. \tag{3.3}$$

Using the following basic inequality

$$\frac{1}{1-x} \leq 1 + 2x \leq e^{2x}, \quad \forall x \in [0,1/2],$$

and the property (3.1) defining $\delta t_{\max}$, it is easily seen that we have

$$\forall n \in [\![0,N-1]\!], \quad 0 \leq \exp\left(-2\int_0^{t^n} \|(c+\operatorname{div} v)^-\|_{L^\infty}\, dt\right) \leq \alpha^n \leq 1. \tag{3.4}$$

We can now perform the following change of variables

$$\tilde\rho_\kappa^n = \alpha^n \rho_\kappa^n, \quad \forall \kappa \in \mathcal{T}, \ \forall n \in [\![0,N]\!],$$

$$\tilde\rho_\sigma^n = \alpha^n \rho_\sigma^n, \text{ and } \tilde\rho_\sigma^{in,n} = \alpha^n \rho_\sigma^{in,n}, \quad \forall \sigma \in \mathcal{E}_{\rm bd}, \ \forall n \in [\![0,N]\!].$$

and we get

$$\begin{cases} |\kappa|\dfrac{\tilde\rho_\kappa^{n+1} - \tilde\rho_\kappa^n}{\delta t} + \dfrac{\alpha^n}{\alpha^{n+1}} \displaystyle\sum_{\sigma\in\mathcal{E}_\kappa\cap\mathcal{E}_{\rm int}} |\sigma|\big(v_{\kappa\sigma}^+\tilde\rho_\kappa^{n+1} - v_{\kappa\sigma}^-\tilde\rho_{\mathcal{L}}^{n+1}\big) + \dfrac{\alpha^n}{\alpha^{n+1}} \displaystyle\sum_{\sigma\in\mathcal{E}_\kappa\cap\mathcal{E}_{\rm bd}} |\sigma| v_{\kappa\sigma}^n \tilde\rho_\sigma^{n+1} \\ \qquad + |\kappa|\dfrac{\alpha^n}{\alpha^{n+1}}(c_\kappa^n + \gamma^n)\tilde\rho_\kappa^{n+1} = 0, \quad \forall n \in [\![0,N-1]\!], \forall \kappa \in \mathcal{T}, \\ \tilde\rho_\kappa^0 = \rho_\kappa^0, \quad \forall \kappa \in \mathcal{T}, \\ \tilde\rho_\sigma^{n+1} = \tilde\rho_\sigma^{in,n+1}, \quad \forall n \in [\![0,N-1]\!], \forall \sigma \in \mathcal{E}_{\rm bd}, \text{ s.t. } v_{\kappa\sigma}^n \leq 0, \\ \tilde\rho_\sigma^{n+1} = \tilde\rho_\kappa^{n+1}, \quad \forall n \in [\![0,N-1]\!], \forall \sigma \in \mathcal{E}_{\rm bd}, \text{ s.t. } v_{\kappa\sigma}^n > 0. \end{cases} \tag{3.5}$$

By using the formula $v^{n+}_{\kappa\sigma} = v^{n}_{\kappa\sigma} + v^{n-}_{\kappa\sigma}$, and the boundary conditions we consider in the scheme, we write

$$\sum_{\sigma\in\mathcal{E}_\kappa\cap\mathcal{E}_{\rm int}} |\sigma| v^{n+}_{\kappa\sigma}\tilde\rho^{n+1}_\kappa + \sum_{\sigma\in\mathcal{E}_\kappa\cap\mathcal{E}_{\rm bd}} |\sigma| v^{n}_{\kappa\sigma}\tilde\rho^{n+1}_\sigma = \sum_{\sigma\in\mathcal{E}_\kappa\cap\mathcal{E}_{\rm int}} |\sigma| v^{n}_{\kappa\sigma}\tilde\rho^{n+1}_\kappa + \sum_{\sigma\in\mathcal{E}_\kappa\cap\mathcal{E}_{\rm int}} |\sigma| v^{n-}_{\kappa\sigma}\tilde\rho^{n+1}_\kappa + \sum_{\sigma\in\mathcal{E}_\kappa\cap\mathcal{E}_{\rm bd}} |\sigma| v^{n}_{\kappa\sigma}\tilde\rho^{n+1}_\sigma$$
$$= \underbrace{\sum_{\sigma\in\mathcal{E}_\kappa} |\sigma| v^{n}_{\kappa\sigma}\tilde\rho^{n+1}_\kappa}_{=|\kappa|(\mathrm{div}\, v)^n_\kappa\tilde\rho^{n+1}_\kappa} + \sum_{\sigma\in\mathcal{E}_\kappa\cap\mathcal{E}_{\rm int}} |\sigma| v^{n-}_{\kappa\sigma}\tilde\rho^{n+1}_\kappa + \sum_{\sigma\in\mathcal{E}_\kappa\cap\mathcal{E}_{\rm bd}} |\sigma| \underbrace{(v^{n}_{\kappa\sigma}\tilde\rho^{n+1}_\sigma - v^{n}_{\kappa\sigma}\tilde\rho^{n+1}_\kappa)}_{=-v^{n-}_{\kappa\sigma}(\tilde\rho^{n+1}_\sigma-\tilde\rho^{n+1}_\kappa)}.$$

Hence we may write the first equation in (3.5) in the following equivalent form

$$|\kappa|\frac{\tilde\rho^{n+1}_\kappa-\tilde\rho^n_\kappa}{\delta t} + \frac{\alpha^n}{\alpha^{n+1}}\sum_{\sigma\in\mathcal{E}_\kappa\cap\mathcal{E}_{\rm int}} |\sigma| v^{n-}_{\kappa\sigma}(\tilde\rho^{n+1}_\kappa - \tilde\rho^{n+1}_\mathcal{L}) + \frac{\alpha^n}{\alpha^{n+1}}\sum_{\sigma\in\mathcal{E}_\kappa\cap\mathcal{E}_{\rm bd}} |\sigma| v^{n-}_{\kappa\sigma}(\tilde\rho^{n+1}_\kappa - \tilde\rho^{in,n+1}_\sigma)$$
$$+ |\kappa|\frac{\alpha^n}{\alpha^{n+1}}(c^n_\kappa + (\mathrm{div}\, v)^n_\kappa + \gamma^n)\tilde\rho^{n+1}_\kappa = 0. \quad (3.6)$$

- **Step 2. Monotonicity. Existence and uniqueness.** We assume that $\rho^0 \geq 0$ and $\rho^{in} \geq 0$. By induction we assume that n is such that $\rho^n_\kappa \geq 0, \forall \kappa \in \mathcal{T}$ and we want to show that $\rho^{n+1}_\kappa \geq 0, \forall \kappa \in \mathcal{T}$. By contradiction, let us assume that there is $\kappa \in \mathcal{T}$ such that

$$\rho^{n+1}_\kappa = \min_{\mathcal{L}\in\mathcal{T}} \rho^{n+1}_\mathcal{L} < 0.$$

In formula (3.6) for this particular control volume κ, we see that the two sums over edges are non-positive since $\tilde\rho^{n+1}_\kappa \leq \tilde\rho^{n+1}_\mathcal{L}$ for any $\mathcal{L} \in \mathcal{T}$, and since $\tilde\rho^{in,n+1}_\sigma \geq 0$ and $\tilde\rho^{n+1}_\kappa < 0$. Furthermore, by the definition of γ^n in formula (3.2), we see that

$$c^n_\kappa + (\mathrm{div}\, v)^n_\kappa + \gamma^n \geq 0. \quad (3.7)$$

Since we assumed that $\tilde\rho^{n+1}_\kappa < 0$, it finally remains the inequality $\tilde\rho^{n+1}_\kappa \geq \tilde\rho^n_\kappa$. This implies $\tilde\rho^n_\kappa < 0$, which is impossible since we assumed that the approximate solution is non-negative at time t^n. The scheme (2.4) being a linear set of equations with the same number of equations as that of unknowns, it is well known that the monotonicity property implies existence and uniqueness of the approximate solution for any data.

- **Step 3. L^∞-bound.** We will first show that

$$\|\tilde\rho^{n+1}_\mathcal{T}\|_{L^\infty} \leq \max(\|\rho^0\|_{L^\infty}, \|\rho^{in}\|_{L^\infty}), \quad \forall n \in [\![0, N-1]\!]. \quad (3.8)$$

Let us consider $\kappa \in \mathcal{T}$ such that $|\tilde\rho^{n+1}_\kappa| = \max_{\mathcal{L}\in\mathcal{T}} |\tilde\rho^{n+1}_\mathcal{L}|$. Let $\varepsilon \in \{-1, 1\}$ be the sign of $\tilde\rho^{n+1}_\kappa$. We multiply (3.6) by ε and we obtain

$$|\kappa|\frac{|\tilde\rho^{n+1}_\kappa| - \varepsilon\tilde\rho^n_\kappa}{\delta t} + \frac{\alpha^n}{\alpha^{n+1}}\sum_{\sigma\in\mathcal{E}_\kappa\cap\mathcal{E}_{\rm int}} |\sigma| v^{n-}_{\kappa\sigma}(|\tilde\rho^{n+1}_\kappa| - \varepsilon\tilde\rho^{n+1}_\mathcal{L}) + \frac{\alpha^n}{\alpha^{n+1}}\sum_{\sigma\in\mathcal{E}_\kappa\cap\mathcal{E}_{\rm bd}} |\sigma| v^{n-}_{\kappa\sigma}(|\tilde\rho^{n+1}_\kappa| - \varepsilon\tilde\rho^{in,n+1}_\sigma)$$
$$+ |\kappa|\frac{\alpha^n}{\alpha^{n+1}}(c^n_\kappa + (\mathrm{div}\, v)^n_\kappa + \gamma^n)|\tilde\rho^{n+1}_\kappa| = 0. \quad (3.9)$$

Two cases have to be considered:

  - First case : there is some $\sigma \in \mathcal{E}_\kappa \cap \mathcal{E}_{\rm bd}$ such that $|\tilde\rho^{n+1}_\kappa| < \varepsilon\tilde\rho^{in,n+1}_\sigma$. In that case, using (3.4), we have

$$\|\tilde\rho^{n+1}_\mathcal{T}\|_{L^\infty} = |\tilde\rho^{n+1}_\kappa| \leq |\tilde\rho^{in,n+1}_\sigma| \leq \|\rho^{in}\|_{L^\infty}\alpha^{n+1} \leq \|\rho^{in}\|_{L^\infty}. \quad (3.10)$$

  - Second case : for any $\sigma \in \mathcal{E}_\kappa \cap \mathcal{E}_{\rm bd}$ we have $|\tilde\rho^{n+1}_\kappa| \geq \varepsilon\tilde\rho^{in,n+1}_\sigma$. Hence the two sums in (3.9) are non-negative and we recall that the choice of γ^n leads to (3.7). We thus deduce

$$|\kappa|\frac{|\tilde\rho^{n+1}_\kappa| - \varepsilon\tilde\rho^n_\kappa}{\delta t} \leq 0,$$

    which leads to

$$\|\tilde\rho^{n+1}_\mathcal{T}\|_{L^\infty} \leq \|\tilde\rho^n_\mathcal{T}\|_{L^\infty}. \quad (3.11)$$

Gathering (3.10) and (3.11), we easily obtain (3.8) by induction. By definition of $\tilde\rho^n_\mathcal{T}$, (3.8) implies

$$\|\rho^n_\mathcal{T}\|_{L^\infty} \leq \frac{1}{\alpha^n}\max(\|\rho^0\|_{L^\infty}, \|\rho^{in}\|_{L^\infty}),$$

and the claim follows with (3.4).

□

REMARK 3.1. *In the particular case of the so-called mass conservation equation, that is for $c = 0$, we can prove (by a duality argument) existence, uniqueness and monotonicity without any additional assumption on $(c + \operatorname{div} v)^- = (\operatorname{div} v)^-$ and without any condition on the time-step. These conditions are however mandatory, even in that case, to obtain the L^∞ bound on the approximate solution.*

We shall now define the approximate solution to be the piecewise constant function $\rho_{\mathcal{T},\delta t} \in L^\infty(]0,T[\times\Omega)$ defined as follows

$$\rho_{\mathcal{T},\delta t} = \sum_{n=0}^{N-1} \sum_{\kappa\in\mathcal{T}} \rho_\kappa^{n+1} \mathbb{1}_{]t^n,t^{n+1}[\times\kappa}.$$

We define the trace $\gamma\rho_{\mathcal{T},\delta t} \in L^\infty(]0,T[\times\Gamma)$ of this approximate solution as follows

$$\gamma\rho_{\mathcal{T},\delta t} = \sum_{n=0}^{N-1} \sum_{\sigma\in\mathcal{E}_{\rm bd}} \rho_\kappa^{n+1} \mathbb{1}_{]t^n,t^{n+1}[\times\sigma},$$

where in this sum κ is the unique control volume such that $\sigma \in \mathcal{E}_\kappa$. Note that this definition is nothing but the trace, in the BV sense, of the function $\rho_{\mathcal{T},\delta t}$.

We also need to introduce the discretization of the initial data

$$\rho_{\mathcal{T}}^0 = \sum_{\kappa\in\mathcal{T}} \rho_\kappa^0 \mathbb{1}_\kappa.$$

REMARK 3.2. *By standard approximation arguments, we know that $\rho_{\mathcal{T}}^0$ converges towards ρ^0 in $L^\infty(\Omega)$ weak-$\star$ and in $L^p(\Omega)$-strong for any $p < +\infty$.*

With these notations, the L^∞ bound given in Theorem 3.1 leads to the inequalities

$$\|\rho_{\mathcal{T},\delta t}\|_{L^\infty(]0,T[\times\Omega)} \le \rho_{\max}, \quad \|\gamma\rho_{\mathcal{T},\delta t}\|_{L^\infty(]0,T[\times\Gamma)} \le \rho_{\max}, \tag{3.12}$$

where $\rho_{\max}$ does not depend on δt and $\mathcal{T}$ and is defined by

$$\rho_{\max} = \max(\|\rho^0\|_{L^\infty(\Omega)}, \|\rho^{in}\|_{L^\infty(]0,T[\times\Gamma)}) e^{2\int_0^T \|(c+\operatorname{div} v)^-\|_{L^\infty(\Omega)}\, dt}.$$

Notice that, by (1.6), we know that the exact solution ρ satisfies similar estimates

$$\|\rho\|_{L^\infty(]0,T[\times\Omega)} \le \rho_{\max}, \quad \|\gamma\rho\|_{L^\infty(]0,T[\times\Gamma,|d\mu_v|)} \le \rho_{\max}.$$

3.2. Weak $L^2(H^1)$ estimate. In the following proposition, we derive a kind of energy estimate for the solution of the finite volume scheme.

PROPOSITION 3.2. *Assume that* (1.2), (1.3a) *and* (1.4a) *hold. There exists $M > 0$ depending only on c, v, ρ^0 and ρ^{in}, such that for any $\delta t \le \delta t_{\max}$ and any mesh $\mathcal{T}$ we have the following bound*

$$\sum_{n=0}^{N-1} \delta t \sum_{\sigma\in\mathcal{E}_{\rm bd}} |\sigma| |v_{\kappa\sigma}^n| (\rho_\kappa^{n+1} - \rho_\sigma^{n+1})^2 + \sum_{n=0}^{N-1} \delta t \sum_{\substack{\sigma\in\mathcal{E}_{\rm int}\\ \sigma=\kappa|\mathcal{L}}} |\sigma| |v_{\kappa\mathcal{L}}^n| (\rho_\mathcal{L}^{n+1} - \rho_\kappa^{n+1})^2 \le M. \tag{3.13}$$

This estimate can be understood as a weak $L^2(]0,T[,H^1(\Omega))$ estimate since, if the mesh is quasi-uniform, we can write (for the interior edges for instance)

$$\sum_{n=0}^{N-1} \delta t \sum_{\substack{\sigma\in\mathcal{E}_{\rm int}\\ \sigma=\kappa|\mathcal{L}}} |\sigma| d_{\kappa\mathcal{L}} |v_{\kappa\mathcal{L}}^n| \left| \frac{\rho_\mathcal{L}^{n+1} - \rho_\kappa^{n+1}}{d_{\kappa\mathcal{L}}} \right|^2 \le \frac{M}{h_\mathcal{T}},$$

where $d_{\kappa\mathcal{L}}$ is the distance between x_κ and $x_\mathcal{L}$. Hence, for a smooth exact solution ρ, if we think ρ_κ^{n+1} as an approximation of $\rho(t^{n+1}, x_\kappa)$ (which it is not !), then the left-hand side of this inequality looks like the square of a weighted discrete $L^2(H^1)$ norm, the weight being proportional to the mean-value of the flow across each edge. In particular, this estimate provide useful information only on the parts of $]0,T[\times\Omega$ where the vector-field v does not vanish. Such kind of property is also known as a weak BV estimate, in the framework of nonlinear scalar conservation laws, see [CGH93, EGH00].

As shown in [Boy05, BF11], if one consider the following parabolic approximation of the problem

$$\partial_t \rho_\varepsilon + \operatorname{div}(\rho_\varepsilon v) + c\rho_\varepsilon - \varepsilon\Delta\rho_\varepsilon = 0, \tag{3.14}$$

with the initial data $\rho_\varepsilon(0) = \rho^0$ and the Fourier boundary condition $\varepsilon\frac{\partial\rho_\varepsilon}{\partial\boldsymbol{\nu}} + (\rho_\varepsilon - \rho^{in})(v\cdot\boldsymbol{\nu})^- = 0$, the corresponding estimate reads

$$\|\rho_\varepsilon\|_{L^2(]0,T[,H^1(\Omega))} \leq \frac{C}{\sqrt{\varepsilon}}.$$

Here, the size of mesh $h_\mathcal{T}$ plays the role of the approximation parameter ε and, moreover, the numerical diffusion tensor is isotropic and heterogeneous.

It was shown in [Boy05, BF11] that the solution to (3.14) strongly converges towards the solution ρ of the transport equation in $L^\infty(]0,T[,L^p(\Omega))$ for any $p<+\infty$. We will use the same kind of idea in the present paper in order to show the uniform in time strong convergence of our finite volume approximate solution in Section 5.

Proof. First of all, for any interior edge $\sigma=\kappa|\mathcal{L}\in\mathcal{E}_{\rm int}$ we define $\rho_\sigma^{n+1}=(\rho_\kappa^{n+1}+\rho_\mathcal{L}^{n+1})/2$. Recall that for boundary edges the value of ρ_σ^{n+1} is already given in the definition of the scheme.

By simple algebraic manipulations, the finite volume scheme (2.4) then reads

$$|\kappa|\frac{\rho_\kappa^{n+1}-\rho_\kappa^n}{\delta t}+\sum_{\sigma\in\mathcal{E}_\kappa}|\sigma|v_{\kappa\sigma}^n\rho_\sigma^{n+1}+|\kappa|c_\kappa^n\rho_\kappa^{n+1}-\sum_{\sigma\in\mathcal{E}_\kappa\cap\mathcal{E}_{\rm int}}|\sigma||v_{\kappa\sigma}^n|\frac{\rho_\mathcal{L}^{n+1}-\rho_\kappa^{n+1}}{2}=0,\quad\forall\kappa\in\mathcal{T},\forall n\in[\![0,N-1]\!]. \tag{3.15}$$

We multiply (3.15) by $\delta t\rho_\kappa^{n+1}$, we sum over n and κ and we finally use the algebraic identity $ab=\frac12a^2+\frac12b^2-\frac12(a-b)^2$, to obtain

$$\begin{aligned}\sum_{n=0}^{N-1}\sum_{\kappa\in\mathcal{T}}|\kappa|(\rho_\kappa^{n+1}-\rho_\kappa^n)\rho_\kappa^{n+1}+\sum_{n=0}^{N-1}\delta t\sum_{\kappa\in\mathcal{T}}|\kappa|c_\kappa^n|\rho_\kappa^{n+1}|^2-\sum_{n=0}^{N-1}\delta t\sum_{\kappa\in\mathcal{T}}\sum_{\sigma\in\mathcal{E}_\kappa\cap\mathcal{E}_{\rm int}}|\sigma||v_{\kappa\sigma}^n|\frac{\rho_\mathcal{L}^{n+1}-\rho_\kappa^{n+1}}{2}\rho_\kappa^{n+1}\\ +\sum_{n=0}^{N-1}\delta t\sum_{\kappa\in\mathcal{T}}\sum_{\sigma\in\mathcal{E}_\kappa}|\sigma|v_{\kappa\sigma}^n\left(\frac12(\rho_\sigma^{n+1})^2+\frac12(\rho_\kappa^{n+1})^2-\frac12(\rho_\sigma^{n+1}-\rho_\kappa^{n+1})^2\right)=0.\end{aligned}$$

Note that we have

$$v_{\kappa\sigma}^n=-v_{\mathcal{L}\sigma}^n=v_{\kappa\mathcal{L}}^n,\ \text{and}\ (\rho_\sigma^{n+1}-\rho_\kappa^{n+1})^2=(\rho_\sigma^{n+1}-\rho_\mathcal{L}^{n+1})^2,\ \forall\sigma=\kappa|\mathcal{L}\in\mathcal{E}_{\rm int},$$

so that, reorganizing the sums on the edges and using (2.2), we get

$$\begin{aligned}\sum_{n=0}^{N-1}\sum_{\kappa\in\mathcal{T}}|\kappa|(\rho_\kappa^{n+1}-\rho_\kappa^n)\rho_\kappa^{n+1}+\sum_{n=0}^{N-1}\delta t\sum_{\kappa\in\mathcal{T}}|\kappa|\left(c_\kappa^n+\frac12(\operatorname{div}v)_\kappa^n\right)|\rho_\kappa^{n+1}|^2+\frac12\sum_{n=0}^{N-1}\delta t\sum_{\sigma\in\mathcal{E}_{\rm int}}|\sigma||v_{\kappa\sigma}^n|(\rho_\mathcal{L}^{n+1}-\rho_\kappa^{n+1})^2\\ +\sum_{n=0}^{N-1}\delta t\sum_{\sigma\in\mathcal{E}_{\rm bd}}|\sigma|v_\sigma^n\left(\frac12(\rho_\sigma^{n+1})^2-\frac12(\rho_\sigma^{n+1}-\rho_\kappa^{n+1})^2\right)=0.\end{aligned}$$

By definition of the scheme, for any time iteration n and any boundary edge σ such that $v_{\kappa\sigma}^n>0$ we have $\rho_\sigma^{n+1}=\rho_\kappa^{n+1}$ and for the other boundary edges (when $v_{\kappa\sigma}^n\leq0$) the value of $\rho_\sigma^{n+1}=\rho_\sigma^{in,n+1}$ is prescribed by the boundary data. Thus, we obtain

$$\begin{aligned}\frac12\|\rho_\mathcal{T}^N\|_{L^2}^2+\frac12\sum_{n=0}^{N-1}\delta t\|\rho_\mathcal{T}^{n+1}-\rho_\mathcal{T}^n\|_{L^2}^2+\underbrace{\sum_{n=0}^{N-1}\delta t\sum_{\kappa\in\mathcal{T}}|\kappa|\left(c_\kappa^n+\frac12(\operatorname{div}v)_\kappa^n\right)|\rho_\kappa^{n+1}|^2}_{=T_1}\\ \underbrace{+\frac12\sum_{n=0}^{N-1}\delta t\sum_{\substack{\sigma\in\mathcal{E}_{\rm bd}\\ v_\sigma^n<0}}|\sigma||v_\sigma^n|(\rho_\sigma^{in,n+1}-\rho_\kappa^{n+1})^2+\frac12\sum_{n=0}^{N-1}\delta t\sum_{\sigma=\kappa|\mathcal{L}\in\mathcal{E}_{\rm int}}|\sigma||v_{\kappa\mathcal{L}}^n|(\rho_\mathcal{L}^{n+1}-\rho_\kappa^{n+1})^2}_{=T_2}\\ +\frac12\sum_{n=0}^{N-1}\delta t\sum_{\substack{\sigma\in\mathcal{E}_{\rm bd}\\ v_\sigma^n\geq0}}|\sigma|v_\sigma^n(\rho_\sigma^{n+1})^2=\frac12\|\rho_\mathcal{T}^0\|_{L^2}^2+\frac12\sum_{n=0}^{N-1}\delta t\sum_{\substack{\sigma\in\mathcal{E}_{\rm bd}\\ v_\sigma^n<0}}|\sigma||v_\sigma^n|(\rho_\sigma^{in,n+1})^2.\end{aligned} \tag{3.16}$$

All the terms in this identity are non-negative, except possibly the term T_1. Nevertheless, using the L^∞ bound on $\rho_{\mathcal{T},\delta t}$, we can bound this term as follows

$$|T_1|\leq(\rho_{\max})^2\left\|\left(c+\frac12\operatorname{div}v\right)^-\right\|_{L^1(]0,T[\times\Omega)}.$$

We thus obtain a bound on the term T_2, which is exactly (3.13) with

$$M = \frac{1}{2}\|\rho^0\|_{L^2}^2 + \frac{\rho_{\max}^2}{2}\int_0^T\int_\Gamma |v\cdot\boldsymbol{\nu}|\,dt\,dx + \rho_{\max}^2\left\|\left(c+\frac{1}{2}\mathrm{div}\,v\right)^-\right\|_{L^1(]0,T[\times\Omega)}.$$

□

4. Weak convergence result. In this section, we are going to prove the weak convergence of the solution of the finite volume scheme towards the unique weak solution of the initial and boundary value problem (1.1).

THEOREM 4.1. *Assume that* (1.2), (1.3) *and* (1.4) *hold. Let* $\mathrm{reg}_{\max} > 0$ *be given and consider a family of meshes and time steps, such that* $(\delta t, h_{\mathcal{T}}) \to 0$ *and satisfying the bound*

$$\max\left(\mathrm{reg}(\mathcal{T}), \max_{\kappa\in\mathcal{T}}\frac{\delta t}{d_\kappa}\right) \le \mathrm{reg}_{\max}. \tag{4.1}$$

Then, we have

$$\rho_{\mathcal{T},\delta t} \xrightarrow[(\delta t,h_{\mathcal{T}})\to 0]{} \rho, \quad \textit{in } L^\infty(]0,T[\times\Omega) \textit{ weak-}\star,$$

$$\gamma\rho_{\mathcal{T},\delta t} \xrightarrow[(\delta t,h_{\mathcal{T}})\to 0]{} \gamma\rho, \quad \textit{in } L^\infty(]0,T[\times\Gamma, |d\mu_v|) \textit{ weak-}\star,$$

where ρ *and* $\gamma\rho$ *solves* (1.5).

Proof. Notice first that assumption (1.4b) is only used to ensure uniqueness of the weak solution and of its trace (see Theorem 1.1) and is not directly used in the following computations.

Thanks to the L^∞ bounds (3.12), we can find subsequences of $(\rho_{\mathcal{T},\delta t})$ and $(\gamma\rho_{\mathcal{T},\delta t})$ which weak-$\star$ converge in the spaces given above. In fact, up to another extraction of a subsequence, we may also assume that $\gamma\rho_{\mathcal{T},\delta t}$ weak-$\star$ converges in $L^\infty(]0,T[\times\Gamma)$. We denote by ρ and g the respective limits of these two subsequences. We will show that ρ and g satisfy the weak formulation of the problem. Since this weak solution is unique, we will then deduce the claim.

Trace identification. Let us first show that $g = \rho^{in}$, $d\mu_v^-$-almost everywhere by writing

$$\begin{aligned}
&\int_0^T\int_\Gamma |\gamma\rho_{\mathcal{T},\delta t} - \rho^{in}|(v\cdot\boldsymbol{\nu})^-\,dt\,dx \\
&\qquad\le 2\rho_{\max}\sum_{n=0}^{N-1}\sum_{\substack{\sigma\in\mathcal{E}_{\mathrm{bd}}\\ v_{\kappa\sigma}^n>0}}\int_{t^n}^{t^{n+1}}\int_\sigma (v\cdot\boldsymbol{\nu}_\sigma)^-\,dx\,dt + \sum_{n=0}^{N-1}\sum_{\substack{\sigma\in\mathcal{E}_{\mathrm{bd}}\\ v_{\kappa\sigma}^n\le 0}}\int_{t^n}^{t^{n+1}}\int_\sigma \left|\rho_\sigma^{in,n+1} - \rho^{in}\right|(v\cdot\boldsymbol{\nu}_\sigma)^-\,dx\,dt, \\
&\qquad\le 2\rho_{\max}\sum_{n=0}^{N-1}\sum_{\substack{\sigma\in\mathcal{E}_{\mathrm{bd}}\\ v_{\kappa\sigma}^n>0}}\int_{t^n}^{t^{n+1}}\int_\sigma (v\cdot\boldsymbol{\nu}_\sigma)^-\,dx\,dt + \|v\cdot\boldsymbol{\nu}\|_{L^\alpha(]0,T[\times\Gamma)}\left(\sum_{n=0}^{N-1}\sum_{\sigma\in\mathcal{E}_{\mathrm{bd}}}\int_{t^n}^{t^{n+1}}\int_\sigma \left|\rho_\sigma^{in,n+1} - \rho^{in}\right|^{\alpha'}\,dx\,dt\right)^{\frac{1}{\alpha'}}.
\end{aligned}$$

The two terms in the right-hand side tend to zero when δt and $h_{\mathcal{T}}$ tend to zero by Lemma 2.1, so that we proved that $\gamma\rho_{\mathcal{T},\delta t}$ strongly converges towards ρ^{in} in $L^1(]0,T[\times\Gamma, d\mu_v^-)$ which implies, in particular, that the weak-$\star$ limit g of $\gamma\rho_{\mathcal{T},\delta t}$ coincides with ρ^{in} $d\mu_v^-$-almost everywhere.

Weak formulation. Let $\phi \in \mathcal{C}_c^\infty([0,T[\times\overline{\Omega})$ be a smooth test function. We want to show that the weak limits ρ and g obtained above solve the weak formulation of the problem.

We define $\phi_\kappa^n = \phi(t^n, x_\kappa)$, where we recall that $x_\kappa \in \kappa$ is a point arbitrarily chosen in each control volume. We multiply the first equation of (2.4) by $\delta t\phi_\kappa^n$ and we sum over $n \in [\![0, N-1]\!]$ and $\kappa\in\mathcal{T}$. It follows

$$\begin{aligned}
&\underbrace{\sum_{n=0}^{N-1}\sum_{\kappa\in\mathcal{T}}|\kappa|(\rho_\kappa^{n+1}-\rho_\kappa^n)\phi_\kappa^n}_{=T_1} + \underbrace{\sum_{n=0}^{N-1}\delta t\sum_{\kappa\in\mathcal{T}}|\kappa|c_\kappa^n\rho_\kappa^{n+1}\phi_\kappa^n}_{=T_2} \\
&\qquad + \underbrace{\sum_{n=0}^{N-1}\delta t\sum_{\kappa\in\mathcal{T}}\sum_{\sigma\in\mathcal{E}_\kappa\cap\mathcal{E}_{\mathrm{int}}}|\sigma|(v_{\kappa\sigma}^{n+}\rho_\kappa^{n+1} - v_{\kappa\sigma}^{n-}\rho_{\mathcal{L}}^{n+1})\phi_\kappa^n + \sum_{n=0}^{N-1}\delta t\sum_{\sigma\in\mathcal{E}_{\mathrm{bd}}}|\sigma|v_{\kappa\sigma}^n\rho_\sigma^{n+1}\phi_\kappa^n}_{=T_3} = 0.
\end{aligned} \tag{4.2}$$

Since ϕ has a compact support in time in $[0,T[$, we have $\phi_\kappa^n = 0$ for any $\kappa \in \mathcal{T}$, for $n = N$. Thus, the term T_1 can be expressed as follows

$$\begin{aligned} T_1 &= \sum_{n=0}^{N-1} \sum_{\kappa\in\mathcal{T}} |\kappa|\rho_\kappa^{n+1}(\phi_\kappa^n - \phi_\kappa^{n+1}) - \sum_{\kappa\in\mathcal{T}} |\kappa|\rho_\kappa^0\phi_\kappa^0 \\ &= -\sum_{n=0}^{N-1} \sum_{\kappa\in\mathcal{T}} \rho_\kappa^{n+1} \int_{t^n}^{t^{n+1}} |\kappa|\partial_t\phi(t,x_\kappa)\,dt - \sum_{\kappa\in\mathcal{T}} |\kappa|\rho_\kappa^0\phi_\kappa^0. \end{aligned}$$

Since ϕ is smooth, and $\|\rho_{\mathcal{T},\delta t}\|_{L^\infty} \le \rho_{\max}$, we get that

$$\left| T_1 + \int_0^T \int_\Omega \rho_{\mathcal{T},\delta t}\partial_t\phi\,dx\,dt + \int_\Omega \rho_{\mathcal{T}}^0(x)\phi(0,x)\,dx \right| \le C_{\phi,\rho_{\max}}(\delta t + h_{\mathcal{T}}).$$

By weak convergence of $\rho_{\mathcal{T},\delta t}$ and $\rho_{\mathcal{T}}^0$, we deduce that

$$T_1 \xrightarrow[(\delta t,h_{\mathcal{T}})\to 0]{} -\int_0^T \int_\Omega \rho\partial_t\phi\,dx\,dt - \int_\Omega \rho^0\phi(0,.)\,dx. \tag{4.3}$$

For the term T_2, we easily see that

$$\left| T_2 - \int_0^T \int_\Omega \rho_{\mathcal{T},\delta t} c\phi\,dx\,dt \right| \le C_{\phi,\rho_{\max}}\|c\|_{L^1}(\delta t + h_{\mathcal{T}}),$$

so that

$$T_2 \xrightarrow[(\delta t,h_{\mathcal{T}})\to 0]{} \int_0^T \int_\Omega \rho c\phi\,dx\,dt. \tag{4.4}$$

Let us now concentrate on the term T_3 in (4.2). We first write $v_{\kappa\sigma}^{n+} = v_{\kappa\sigma}^n + v_{\kappa\sigma}^{n-}$ so that we get

$$T_3 = \underbrace{\sum_{n=0}^{N-1} \delta t \sum_{\kappa\in\mathcal{T}} |\kappa|(\operatorname{div} v)_\kappa^n \rho_\kappa^{n+1}\phi_\kappa^n}_{=T_{31}} + \underbrace{\sum_{n=0}^{N-1} \delta t \sum_{\kappa\in\mathcal{T}} \sum_{\sigma\in\mathcal{E}_\kappa\cap\mathcal{E}_{\rm int}} |\sigma| v_{\kappa\sigma}^{n-}(\rho_\kappa^{n+1} - \rho_{\mathcal{L}}^{n+1})\phi_\kappa^n + \sum_{n=0}^{N-1} \delta t \sum_{\sigma\in\mathcal{E}_{\rm bd}} |\sigma| v_{\kappa\sigma}^n(\rho_\sigma^{n+1} - \rho_\kappa^{n+1})\phi_\kappa^n}_{=T_{32}}$$

The term T_{31} can be treated in the very same way as the term T_2, let us concentrate on the term T_{32}. Reordering the summation on the interior edges by using the conservativity property $v_{\kappa\mathcal{L}}^n = v_{\kappa\sigma}^n = -v_{\mathcal{L}\sigma}^n$, we get

$$T_{32} = \sum_{n=0}^{N-1} \delta t \sum_{\sigma=\kappa|\mathcal{L}\in\mathcal{E}_{\rm int}} |\sigma| \left(v_{\kappa\mathcal{L}}^{n-}(\rho_\kappa^{n+1} - \rho_{\mathcal{L}}^{n+1})\phi_\kappa^n + v_{\kappa\mathcal{L}}^{n+}(\rho_{\mathcal{L}}^{n+1} - \rho_\kappa^{n+1})\phi_{\mathcal{L}}^n \right) + \sum_{n=0}^{N-1} \delta t \sum_{\sigma\in\mathcal{E}_{\rm bd}} |\sigma| v_{\kappa\sigma}^n(\rho_\sigma^{n+1} - \rho_\kappa^{n+1})\phi_\kappa^n.$$

We will now stress on the fact that, the test function ϕ and the solution $\rho_{\mathcal{T},\delta t}$ of the finite volume scheme being fixed, the term T_{32} actually depends on the velocity field v, so that we shall in fact denote this term by $T_{32}(v)$.

The behavior of $T_{32}(v)$ when δt and $h_{\mathcal{T}}$ go to 0 is then given by the following lemma and its corollary.

LEMMA 4.2. *Let $v \in L^1(]0,T[,(W^{1,1}(\Omega))^d)$.*

1. *For any $w \in L^1(]0,T[,(W^{1,1}(\Omega))^d)$ we have*

$$|T_{32}(v) - T_{32}(w)| \le C_{\phi,\rho_{\max}}\|v - w\|_{L^1(W^{1,1})}.$$

2. *For any $w \in (\mathcal{C}^\infty([0,T]\times\overline{\Omega}))^d$, there exists $R_{32}(v,w)$ (depending on $v,w,\mathcal{T},\delta t,\rho^{\mathcal{T}},\phi$) such that we have*

$$T_{32}(w) + R_{32}(v,w) \xrightarrow[(\delta t,h_{\mathcal{T}})\to 0]{} -\int_0^T \int_\Omega \rho\operatorname{div}(\phi w)\,dx\,dt + \int_0^T \int_\Gamma g\phi(w\cdot\boldsymbol{\nu})\,dx\,dt, \tag{4.5}$$

and

$$|R_{32}(v,w)| \le C_{\phi,\rho_{\max},\mathrm{reg}_{\max}}\|v - w\|_{L^1(W^{1,1})}. \tag{4.6}$$

COROLLARY 4.3. *For any $v \in L^1(]0,T[,(W^{1,1}(\Omega))^d)$, we have the following convergence*

$$T_{32}(v) \xrightarrow[(\delta t, h_{\mathcal{T}})\to 0]{} -\int_0^T \int_\Omega \rho \mathrm{div}\,(\phi v)\,dx\,dt + \int_0^T \int_\Gamma g\phi(v\cdot\boldsymbol{\nu})\,dx\,dt.$$

Proof of Corollary 4.3. Let $\varepsilon > 0$ be given. By density, we may find $v_\varepsilon \in (\mathcal{C}^\infty([0,T]\times\overline{\Omega}))^d$ such that $\|v-v_\varepsilon\|_{L^1(W^{1,1})} \le \varepsilon$. Using Lemma 4.2, we then write

$$\begin{aligned}
&\left|T_{32}(v) + \int_0^T \int_\Omega \rho \mathrm{div}\,(\phi v)\,dx\,dt - \int_0^T \int_\Gamma g\phi(v\cdot\boldsymbol{\nu})\,dx\,dt\right| \\
&\quad\le |T_{32}(v) - T_{32}(v_\varepsilon)| + \left|R_{32}(v,v_\varepsilon) + T_{32}(v_\varepsilon) + \int_0^T \int_\Omega \rho \mathrm{div}\,(\phi v_\varepsilon)\,dx\,dt - \int_0^T \int_\Gamma g\phi(v_\varepsilon\cdot\boldsymbol{\nu})\,dx\,dt\right| \\
&\qquad + |R_{32}(v,v_\varepsilon)| + \left|\int_0^T \int_\Omega \rho \mathrm{div}\,(\phi(v-v_\varepsilon))\,dx\,dt - \int_0^T \int_\Gamma g\phi((v-v_\varepsilon)\cdot\boldsymbol{\nu})\,dx\,dt\right| \\
&\quad\le C\|v-v_\varepsilon\|_{L^1(W^{1,1})} + \left|R_{32}(v,v_\varepsilon) + T_{32}(v_\varepsilon) + \int_0^T \int_\Omega \rho \mathrm{div}\,(\phi v_\varepsilon)\,dx\,dt - \int_0^T \int_\Gamma g\phi(v_\varepsilon\cdot\boldsymbol{\nu})\,dx\,dt\right|.
\end{aligned}$$

Since ε is fixed and v_ε is smooth, by the second point of Lemma 4.2 the second term above is less than ε for δt and $h_{\mathcal{T}}$ small enough. For such values of δt and $h_{\mathcal{T}}$, we finally obtain a bound of the left-hand side by $(1+C)\varepsilon$ which concludes the proof. □

It remains now to prove the lemma.

Proof of Lemma 4.2.

1. For any $\sigma \in \mathcal{E}$ and any $n \in [\![0, N-1]\!]$ we define an interface value ϕ_σ^n for ϕ, as follows

$$\phi_\sigma^n = \phi(t^n, x_\sigma),$$

where x_σ is an arbitrary point in σ. We can then get

$$\begin{aligned}
T_{32}(w) = &\underbrace{\sum_{n=0}^{N-1} \delta t \sum_{\substack{\sigma\in\mathcal{E}_{\rm int}\\ \sigma=\mathcal{K}|\mathcal{L}}} |\sigma| w_{\mathcal{K}\mathcal{L}}^n (\rho_{\mathcal{L}}^{n+1} - \rho_{\mathcal{K}}^{n+1})\phi_\sigma^n + \sum_{n=0}^{N-1} \delta t \sum_{\sigma\in\mathcal{E}_{\rm bd}} |\sigma| w_{\mathcal{K}\sigma}^n (\rho_\sigma^{n+1} - \rho_{\mathcal{K}}^{n+1})\phi_{\mathcal{K}}^n}_{=T'_{32}(w)} \\
&+ \underbrace{\sum_{n=0}^{N-1} \delta t \sum_{\substack{\sigma\in\mathcal{E}_{\rm int}\\ \sigma=\mathcal{K}|\mathcal{L}}} |\sigma| \left(w_{\mathcal{K}\mathcal{L}}^{n-} (\rho_{\mathcal{K}}^{n+1} - \rho_{\mathcal{L}}^{n+1})(\phi_{\mathcal{K}}^n - \phi_\sigma^n) + w_{\mathcal{K}\mathcal{L}}^{n+} (\rho_{\mathcal{L}}^{n+1} - \rho_{\mathcal{K}}^{n+1})(\phi_{\mathcal{L}}^n - \phi_\sigma^n)\right)}_{=R_1(w)}.
\end{aligned}$$

The term $T'_{32}(w)$ can be written as follows (paying attention to the fact that only interior edges are taken into account in the first sum)

$$\begin{aligned}
T'_{32}(w) = &\underbrace{-\sum_{n=0}^{N-1} \delta t \sum_{\mathcal{K}\in\mathcal{T}} \rho_{\mathcal{K}}^{n+1} \left(\sum_{\sigma\in\mathcal{E}_{\mathcal{K}}} |\sigma| w_{\mathcal{K}\sigma}^n \phi_\sigma^n\right)}_{=T_{321}(w)} \\
&+ \underbrace{\sum_{n=0}^{N-1} \delta t \sum_{\sigma\in\mathcal{E}_{\rm bd}} |\sigma| w_{\mathcal{K}\sigma}^n \rho_{\mathcal{K}}^{n+1} \phi_\sigma^n + \sum_{n=0}^{N-1} \delta t \sum_{\sigma\in\mathcal{E}_{\rm bd}} |\sigma| w_{\mathcal{K}\sigma}^n (\rho_\sigma^{n+1} - \rho_{\mathcal{K}}^{n+1})\phi_{\mathcal{K}}^n}_{=T'_{321}(w)}.
\end{aligned}$$

For the boundary terms $T'_{321}(w)$, we see that we can write

$$T'_{321}(w) = \underbrace{\sum_{n=0}^{N-1} \delta t \sum_{\sigma\in\mathcal{E}_{\rm bd}} |\sigma| w_{\mathcal{K}\sigma}^n \rho_\sigma^{n+1} \phi_\sigma^n}_{=T_{322}(w)} + \underbrace{\sum_{n=0}^{N-1} \delta t \sum_{\sigma\in\mathcal{E}_{\rm bd}} |\sigma| w_{\mathcal{K}\sigma}^n (\rho_\sigma^{n+1} - \rho_{\mathcal{K}}^{n+1})(\phi_{\mathcal{K}}^n - \phi_\sigma^n)}_{=R_2(w)}.$$

Finally the term $T_{32}(w)$ we are studying is written

$$T_{32}(w) = T_{321}(w) + T_{322}(w) + R_1(w) + R_2(w),$$

and we shall analyse each term separately as follows. Notice that each term is linear with respect to the vector field w except $R_1(w)$.

- For each n and κ, by definition of ϕ_κ^n and ϕ_σ^n, and using (2.1) and (2.2), we have

$$\begin{aligned}\left|\sum_{\sigma\in\mathcal{E}_\mathcal{K}} |\sigma| w_{\mathcal{K}\sigma}^n \phi_\sigma^n\right| &\le \left|\sum_{\sigma\in\mathcal{E}_\mathcal{K}} |\sigma| w_{\mathcal{K}\sigma}^n \underbrace{(\phi_\sigma^n - \phi_\kappa^n)}_{|.|\le C_\phi d_\mathcal{K}}\right| + \left|\sum_{\sigma\in\mathcal{E}_\mathcal{K}} |\sigma| w_{\mathcal{K}\sigma}^n \phi_\kappa^n\right| \\ &\le C_\phi d_\mathcal{K} \frac{1}{\delta t}\int_{t^n}^{t^{n+1}}\int_{\partial\mathcal{K}} |w\cdot\boldsymbol{\nu}_\kappa| dx\,dt + |\phi_\kappa^n||\kappa||(\operatorname{div} w)_\kappa^n| \\ &\le C_{\phi,\mathrm{reg}(\mathcal{T})} d_\mathcal{K} \frac{1}{\delta t}\int_{t^n}^{t^{n+1}} \frac{1}{d_\mathcal{K}} \|w(t)\|_{W^{1,1}(\mathcal{K})}\,dt + C_\phi \frac{1}{\delta t}\int_{t^n}^{t^{n+1}} \|\operatorname{div} w(t)\|_{L^1(\mathcal{K})}\,dt,\end{aligned}$$

  so that, multiplying by δt and summing over n and κ lead to the estimate

$$|T_{321}(w)| \le C_{\phi,\rho_{\max},\mathrm{reg}_{\max}} \|w\|_{L^1(W^{1,1})}.$$

- We easily find that

$$|T_{322}(w)| \le \rho_{\max}\|\phi\|_{L^\infty}\int_0^T\int_\Gamma |w\cdot\boldsymbol{\nu}| dx\,dt \le C_{\phi,\rho_{\max}}\|w\|_{L^1(W^{1,1})}.$$

- Since $|\phi_\kappa^n - \phi_\sigma^n| \le C_\phi d_\mathcal{K}$ and thanks to the L^∞ bound (3.12) on the approximate solution $\rho_{\mathcal{T},\delta t}$ we have (by using the fact that the maps $s\in\mathbb{R}\mapsto s^\pm$ are Lipschitz-continuous),

$$\begin{aligned}|R_1(w) - R_1(v)| &\le \|\rho_{\mathcal{T},\delta t}\|_{L^\infty} C_\phi \sum_{n=0}^{N-1}\delta t \sum_{\sigma=\mathcal{K}|\mathcal{L}\in\mathcal{E}_{\mathrm{int}}} |\sigma||v_{\mathcal{K}\mathcal{L}}^n - w_{\mathcal{K}\mathcal{L}}^n|(d_\mathcal{K}+d_\mathcal{L}) \\ &\le C_{\phi,\rho_{\max}} \sum_{n=0}^{N-1}\delta t \sum_{\kappa\in\mathcal{T}}\left(\sum_{\sigma\in\mathcal{E}_\mathcal{K}} |\sigma| d_\mathcal{K} |v_{\mathcal{K}\sigma}^n - w_{\mathcal{K}\sigma}^n|\right) \\ &\le C_{\phi,\rho_{\max}} \sum_{n=0}^{N-1}\delta t \sum_{\kappa\in\mathcal{T}} \frac{1}{\delta t} d_\mathcal{K}\int_{t^n}^{t^{n+1}} \|v(t)-w(t)\|_{L^1(\partial\mathcal{K})}\,dt \\ &\le C_{\phi,\rho_{\max},\mathrm{reg}_{\max}}\|v-w\|_{L^1(W^{1,1})}.\end{aligned}$$

- The bound on $R_2(w)$ is obtained in a similar way as the one for $T_{322}(w)$.

Collecting all the above estimates, and using the linearity of T_{321}, T_{322} and R_2, the first claim of the lemma is proven.

2. In the second part of this lemma, we consider a smooth vector field w and, as we did before, we split the term $T_{32}(w)$ into the same formal four parts $T_{321}(w) + T_{322}(w) + R_1(w) + R_2(w)$, except that we change the definition of the interface values ϕ_σ^n of ϕ. For a given $n\in[\![0,N-1]\!]$ and $\sigma\in\mathcal{E}$, two cases have to be considered:
   - If $(w\cdot\boldsymbol{\nu}_{\mathcal{K}\mathcal{L}})$ has a constant sign on $[t^n,t^{n+1}]\times\sigma$ then, by the mean-value theorem (recall that ϕ and w are smooth) we find that there exists some point $(\xi_\sigma^n, x_\sigma^n)\in[t^n,t^{n+1}]\times\sigma$ such that

$$\int_{t^n}^{t^{n+1}}\int_\sigma \phi(w\cdot\boldsymbol{\nu}_{\mathcal{K}\mathcal{L}})\,dx\,dt = \phi(\xi_\sigma^n, x_\sigma^n)\left(\int_{t^n}^{t^{n+1}}\int_\sigma (w\cdot\boldsymbol{\nu}_{\mathcal{K}\mathcal{L}})\,dx\,dt\right), \tag{4.7}$$

     and we then define

$$\phi_\sigma^n = \phi(\xi_\sigma^n, x_\sigma^n).$$

   - If $(w\cdot\boldsymbol{\nu}_\sigma)$ has not a constant sign on $[t^n,t^{n+1}]\times\sigma$, then we choose

$$\phi_\sigma^n = \phi(t^n, x_\sigma),$$

     for some arbitrary point $x_\sigma\in\sigma$. Notice, in that case, that $(w\cdot\boldsymbol{\nu}_{\mathcal{K}\mathcal{L}})$ necessarily vanishes at some point in $[t^n,t^{n+1}]\times\sigma$ and then we have

$$|w(t,x)\cdot\boldsymbol{\nu}_{\mathcal{K}\mathcal{L}}| \le \mathcal{L}(w)(\delta t + d_\sigma),\quad \forall (t,x)\in[t^n,t^{n+1}]\times\sigma.$$

It follows that

$$\left|\int_{t^n}^{t^{n+1}}\int_\sigma \phi(w\cdot\boldsymbol{\nu}_{\mathcal{K}\mathcal{L}})\,dx\,dt - \phi_\sigma^n\left(\int_{t^n}^{t^{n+1}}\int_\sigma (w\cdot\boldsymbol{\nu}_{\mathcal{K}\mathcal{L}})\,dx\,dt\right)\right| \le C_{\phi,w}\delta t|\sigma|(\delta t^2 + d_\sigma^2). \tag{4.8}$$

With this particular choice of the interface values for ϕ we can now study all the terms T_{321}, T_{322}, R_1 and R_2 as follows.

- By (4.7) and (4.8), for any $\sigma\in\mathcal{E}, n\in[\![0,N-1]\!]$, we have

$$w_{\mathcal{K}\sigma}^n\phi_\sigma^n = \frac{1}{\delta t|\sigma|}\int_{t^n}^{t^{n+1}}\int_\sigma \phi(w\cdot\boldsymbol{\nu}_{\mathcal{K}\sigma})\,dx\,dt + O_{\phi,w}(\delta t^2+d_\sigma^2),$$

the last term being exactly 0 if the sign of $w\cdot\boldsymbol{\nu}_{\mathcal{K}\sigma}$ is constant. It follows, by (4.1), that

$$\begin{aligned}
T_{321}(w) &= -\sum_{n=0}^{N-1}\sum_{\mathcal{K}\in\mathcal{T}}\rho_{\mathcal{K}}^{n+1}\int_{t^n}^{t^{n+1}}\underbrace{\left(\sum_{\sigma\in\mathcal{E}_{\mathcal{K}}}\int_\sigma \phi(w\cdot\boldsymbol{\nu}_{\mathcal{K}\sigma})\,dx\right)}_{=\int_{\mathcal{K}}\mathrm{div}\,(\phi w)\,dx}dt \\
&\qquad + (1+\mathrm{reg}_{\max})\left(\sum_{n=0}^{N-1}\delta t\sum_{\mathcal{K}\in\mathcal{T}}\rho_{\mathcal{K}}^{n+1}\underbrace{\sum_{\sigma\in\mathcal{E}_{\mathcal{K}}}|\sigma|d_{\mathcal{K}}}_{\le \mathrm{reg}(\mathcal{T})|\mathcal{K}|}\right)O(\delta t+h_{\mathcal{T}}) \\
&= -\int_0^T\int_\Omega \rho_{\mathcal{T},\delta t}\mathrm{div}\,(\phi w)\,dx\,dt + O_{\mathrm{reg}_{\max},w,\phi}(\delta t+h_{\mathcal{T}}).
\end{aligned}$$

  Since w and ϕ are fixed, we finally deduce, using the weak-$\star$ convergence of $\rho_{\mathcal{T},\delta t}$, that

$$T_{321}(w) \xrightarrow[(\delta t,h_{\mathcal{T}})\to 0]{} -\int_0^T\int_\Omega \rho\mathrm{div}\,(\phi w)\,dx\,dt.$$

- The term $T_{322}(w)$ can be treated in the very same way (in fact this term is even easier) since we can write

$$T_{322}(w) = \int_0^T\int_\Gamma(\gamma\rho_{\mathcal{T},\delta t})\phi(w\cdot\boldsymbol{\nu})\,dx\,dt + O(h_{\mathcal{T}}(\delta t+h_{\mathcal{T}})),$$

  and then by weak-$\star$ convergence of the trace $\gamma\rho_{\mathcal{T},\delta t}$ we get

$$T_{322}(w) \xrightarrow[(\delta t,h_{\mathcal{T}})\to 0]{} \int_0^T\int_\Gamma g\phi(w\cdot\boldsymbol{\nu})\,dx\,dt.$$

- The term $R_1(w)$: Let us write

$$R_1(w) = R_1(v) + \big(R_1(w)-R_1(v)\big).$$

  – We use the weak $L^2(H^1)$ estimate (3.13) and the regularity assumption (4.1) in order to bound $R_1(v)$

$$\begin{aligned}
|R_1(v)| &= \sum_{n=0}^{N-1}\delta t\sum_{\substack{\sigma\in\mathcal{E}_{\mathrm{int}}\\ \sigma=\mathcal{K}|\mathcal{L}}}|\sigma||v_{\mathcal{K}\mathcal{L}}^n||\rho_{\mathcal{K}}^{n+1}-\rho_{\mathcal{L}}^{n+1}|\left(|\phi_{\mathcal{K}}^n-\phi_\sigma^n|+|\phi_{\mathcal{L}}^n-\phi_\sigma^n|\right) \\
&\le C_{\rho_{\max}}\underbrace{\left(\sum_{n=0}^{N-1}\delta t\sum_{\substack{\sigma\in\mathcal{E}_{\mathrm{int}}\\ \sigma=\mathcal{K}|\mathcal{L}}}|\sigma||v_{\mathcal{K}\mathcal{L}}^n||\rho_{\mathcal{K}}^{n+1}-\rho_{\mathcal{L}}^{n+1}|^2\right)^{\frac12}}_{\le C\text{ by (3.13)}}\left(\sum_{n=0}^{N-1}\delta t\sum_{\mathcal{K}\in\mathcal{T}}\sum_{\sigma\in\mathcal{E}_{\mathcal{K}}}|\sigma||v_{\mathcal{K}\sigma}^n|\underbrace{|\phi_{\mathcal{K}}^n-\phi_\sigma^n|^2}_{\le C_\phi(\delta t+d_{\mathcal{K}})^2}\right)^{\frac12} \\
&\le C_{\phi,\rho_{\max},\mathrm{reg}_{\max}}\left(\sum_{n=0}^{N-1}\delta t\sum_{\mathcal{K}\in\mathcal{T}}\frac{1}{d_{\mathcal{K}}}\|v\|_{W^{1,1}(K)}(\delta t+d_{\mathcal{K}})^2\right)^{\frac12} \\
&\le C_{\phi,\rho_{\max},\mathrm{reg}_{\max}}\|v\|_{L^1(W^{1,1})}^{\frac12}\sqrt{h_{\mathcal{T}}}.
\end{aligned}$$

  Hence, this term tends to zero when $(\delta t, h_{\mathcal{T}})\to 0$.

– The term $R_1(w) - R_1(v)$ is bounded as in the first part of the Lemma (even though the choice of ϕ_σ^n is different here), by using the L^∞ bound on $\rho_{\mathcal{T},\delta t}$

$$|R_1(w) - R_1(v)| \leq 2\rho_{\max} \sum_{n=0}^{N-1} \delta t \sum_{\kappa\in\mathcal{T}} \sum_{\sigma\in\mathcal{E}_\kappa} |\sigma||w_{\kappa\sigma}^n - v_{\kappa\sigma}^n| \underbrace{|\phi_\kappa^n - \phi_\sigma^n|}_{\leq C_\phi(\delta t + d_\kappa)}$$
$$\leq C_{\phi,\rho_{\max},\mathrm{reg}_{\max}} \sum_{n=0}^{N-1} \sum_{\kappa\in\mathcal{T}} \int_{t^n}^{t^{n+1}} \|v(t) - w(t)\|_{W^{1,1}(\mathcal{K})}\, dt.$$

This term is uniformly controlled by $C\|v - w\|_{L^1(W^{1,1})}$.

- The term $R_2(w)$:
  We easily find that

$$|R_2(w)| \leq 2\|\rho_{\mathcal{T},\delta t}\|_{L^\infty} C_\phi \|w\|_{L^1(]0,T[\times\Gamma)}(\delta t + h_\mathcal{T}).$$

As a consequence of the previous estimates, we see that properties (4.5) and (4.6) hold with $R_{32}(v,w) = R_1(v) - R_1(w)$.

□

Conclusion. We may now conclude the proof of Theorem 4.1. Indeed, gathering (4.3), (4.4) and the result of Corollary 4.3 we see that the weak limits (ρ, g) solve the weak formulation of the problem and satisfy $g = \rho^{in}$, $d\mu_v^-$-almost everywhere. According to the uniqueness of such a couple in $L^\infty(]0,T[\times\Omega) \times L^\infty(]0,T[\times\Gamma, |d\mu_v|)$, the proof is complete. □

5. Strong convergence result. We want now to show that the approximate solution actually strongly converges towards the weak solution of the problem. We emphasize the fact that the convergence we obtain in Theorem 5.2 is uniform with respect to time.

5.1. An improved Friedrichs-type result. We first need to adapt a little the classical convolution argument (Friedrichs' lemma) used in the renormalized solution theory [DL89, Boy05], then we will state and prove our main result.

LEMMA 5.1. *Assume that* (1.2), (1.3) *and* (1.4a) *hold.*
For any $\boldsymbol{\varepsilon} > 0$, there exists a function $\rho^{\boldsymbol{\varepsilon}} \in W^{1,\infty}(]0,T[\times\Omega)$ satisfying the following properties:

- *We have $\|\rho^{\boldsymbol{\varepsilon}}\|_{L^\infty(]0,T[\times\Omega)} \leq \|\rho\|_{L^\infty(]0,T[\times\Omega)}$.*
- *For any $p < +\infty$, $(\rho^{\boldsymbol{\varepsilon}})_\varepsilon$ converges towards ρ in $\mathcal{C}^0([0,T], L^p(\Omega))$ and there exists $C > 0$, which does not depend on $\boldsymbol{\varepsilon}$ such that*

$$\|\rho^{\boldsymbol{\varepsilon}}\|_{\mathcal{C}^0([0,T],L^p(\Omega))} \leq C\|\rho\|_{L^\infty(]0,T[\times\Omega)}.$$

- *For any $p < +\infty$, the traces $(\gamma\rho^{\boldsymbol{\varepsilon}})_\varepsilon$ (in the usual sense since $\rho^{\boldsymbol{\varepsilon}}$ is continuous up to the boundary) converge towards $\gamma\rho$ in $L^p(]0,T[\times\Gamma, |d\mu_v|)$.*
- *The following equation is satisfied in the distribution sense*

$$\partial_t\rho^{\boldsymbol{\varepsilon}} + \mathrm{div}\,(\rho^{\boldsymbol{\varepsilon}} v) + c\rho^{\boldsymbol{\varepsilon}} = R^{\boldsymbol{\varepsilon}}, \quad \text{in }]0,T[\times\Omega, \tag{5.1}$$

for some $R^{\boldsymbol{\varepsilon}} \in L^1(]0,T[\times\Omega)$ satisfying

$$\|R^{\boldsymbol{\varepsilon}}\|_{L^1(]0,T[\times\Omega)} \xrightarrow[\boldsymbol{\varepsilon}\to 0]{} 0.$$

Notice that $\|\rho^{\boldsymbol{\varepsilon}}\|_{W^{1,\infty}}$ blows up when $\boldsymbol{\varepsilon} \to 0$.

Proof. The usual Friedrichs commutator Lemma, adapted to the case of a non tangential vector field on the boundary in [Boy05, BF11] leads to a family of functions, say $\tilde\rho^{\boldsymbol{\varepsilon}}$, satisfying:

- For any $\varepsilon > 0$, $\tilde\rho^{\boldsymbol{\varepsilon}} \in L^\infty(]0,T[\times W^{1,\infty}(\Omega))$.
- $(\tilde\rho^{\boldsymbol{\varepsilon}})_\varepsilon$ converges to ρ in $\mathcal{C}^0([0,T], L^p(\Omega))$ for any $p < +\infty$.
- The traces $\gamma\tilde\rho^{\boldsymbol{\varepsilon}}$ converges to $\gamma\rho$ in $L^p(]0,T[\times\Gamma, |d\mu_v|)$ for any $p < +\infty$.
- There exists $\tilde R^{\boldsymbol{\varepsilon}} \in L^1(]0,T[\times\Omega)$, such that $\tilde\rho^{\boldsymbol{\varepsilon}}$ solves

$$\partial_t\tilde\rho^{\boldsymbol{\varepsilon}} + \mathrm{div}\,(\tilde\rho^{\boldsymbol{\varepsilon}} v) + c\tilde\rho^{\boldsymbol{\varepsilon}} = \tilde R^{\boldsymbol{\varepsilon}},$$

with $\|\tilde R^{\boldsymbol{\varepsilon}}\|_{L^1(]0,T[\times\Omega)} \xrightarrow[\varepsilon\to 0]{} 0$.

This family of approximations is not exactly the one we need here since it is not smooth enough in the time variable. Thus, $\varepsilon > 0$ being fixed, we need to mollify $\tilde\rho^{\boldsymbol{\varepsilon}}$ in time in order to get the result. Let $\theta \in \mathcal{C}_c^\infty([0,T[)$ such that $\theta = 1$ on $[0, 2\delta_0[$ and $\theta = 0$ on $]T - 2\delta_0, T]$, for $\delta_0 > 0$ small enough. For any $0 < \delta < \delta_0$ we define

$$\tilde\rho^{\boldsymbol{\varepsilon}} \underset{\theta}{\star} \eta_\delta(s,x) = \int_0^T \tilde\rho^{\boldsymbol{\varepsilon}}(t,x)\Big(\theta(s)\eta_\delta(s-t+\delta) + (1-\theta(s))\eta_\delta(s-t-\delta)\Big)\, dt,$$

where $\eta \in \mathcal{C}_c^\infty(]-1,1[)$ is a non-negative mollifying kernel and $\eta_\delta = \frac{1}{\delta}\eta(\cdot/\delta)$. Note that, for any $s \in [0,T]$ and $0 < \delta < \delta_0$ small enough we have

$$\int_0^T \Big(\theta(s)\eta_\delta(s-t+\delta) + (1-\theta(s))\eta_\delta(s-t-\delta)\Big)\,dt = 1,$$

so that we immediately deduce that $\tilde\rho^{\boldsymbol\varepsilon} \underset{\theta}{\star} \eta_\delta$ converges to $\tilde\rho^{\boldsymbol\varepsilon}$ in $\mathcal{C}^0([0,T], L^p(\Omega))$ for any $p < +\infty$ when $\delta \to 0$. Furthermore, since η is smooth, we see that $\tilde\rho^{\boldsymbol\varepsilon} \underset{\theta}{\star} \eta_\delta$ is Lipschitz continuous in all the variables (t,x).

Furthermore, since by assumption (1.3) we have $(v\cdot\boldsymbol\nu) \in L^\alpha(]0,T[\times\Gamma)$, we deduce

$$\|\gamma\tilde\rho^{\boldsymbol\varepsilon} \underset{\theta}{\star} \eta_\delta - \gamma\tilde\rho^{\boldsymbol\varepsilon}\|_{L^1(]0,T[\times\Gamma,|d\mu_v|)} \le \|v\cdot\boldsymbol\nu\|_{L^\alpha(]0,T[\times\Gamma)}\|\gamma\tilde\rho^{\boldsymbol\varepsilon} \underset{\theta}{\star} \eta_\delta - \gamma\tilde\rho^{\boldsymbol\varepsilon}\|_{L^{\alpha'}(]0,T[\times\Gamma)},$$

so that, by standard convolution arguments, we see that $(\gamma\tilde\rho^{\boldsymbol\varepsilon} \underset{\theta}{\star} \eta_\delta)_\delta$ converges towards $\gamma\tilde\rho^{\boldsymbol\varepsilon}$ in $L^1(]0,T[\times\Gamma,|d\mu_v|)$ as δ goes to 0.

We can now observe that $\tilde\rho^{\boldsymbol\varepsilon} \underset{\theta}{\star} \eta_\delta$ solves the following problem

$$\partial_s\big(\tilde\rho^{\boldsymbol\varepsilon} \underset{\theta}{\star} \eta_\delta\big) + \operatorname{div}\big(\tilde\rho^{\boldsymbol\varepsilon} \underset{\theta}{\star} \eta_\delta v\big) + c\tilde\rho^{\boldsymbol\varepsilon} \underset{\theta}{\star} \eta_\delta = R^{\boldsymbol\varepsilon,\delta}, \tag{5.2}$$

with

$$R^{\boldsymbol\varepsilon,\delta}(s,x) = \tilde R^{\boldsymbol\varepsilon} \underset{\theta}{\star} \eta_\delta(s,x) + R_1^{\boldsymbol\varepsilon,\delta}(s,x) + R_2^{\boldsymbol\varepsilon,\delta}(s,x) + R_3^{\boldsymbol\varepsilon,\delta}(s,x),$$

and

$$R_1^{\boldsymbol\varepsilon,\delta}(s,.) = \theta'(s)\int_0^T \tilde\rho^{\boldsymbol\varepsilon}(t,.)\eta_\delta(s-t+\delta)\,dt - \theta'(s)\int_0^T \tilde\rho^{\boldsymbol\varepsilon}(t,.)\eta_\delta(s-t-\delta)\,dt,$$

$$R_2^{\boldsymbol\varepsilon,\delta}(s,.) = \int_0^T \operatorname{div}\big(\tilde\rho^{\boldsymbol\varepsilon}(t,.)(v(s,.)-v(t,.))\big)\big(\theta(s)\eta_\delta(s-t+\delta) + (1-\theta(s))\eta_\delta(s-t-\delta)\big)\,dt,$$

$$R_3^{\boldsymbol\varepsilon,\delta}(s,.) = \int_0^T \tilde\rho^{\boldsymbol\varepsilon}(t,.)(c(s,.)-c(t,.))\big(\theta(s)\eta_\delta(s-t+\delta) + (1-\theta(s))\eta_\delta(s-t-\delta)\big)\,dt.$$

It is clear that $\tilde R^{\boldsymbol\varepsilon} \underset{\theta}{\star} \eta_\delta$ converges to $\tilde R^{\boldsymbol\varepsilon}$ in $L^1(]0,T[\times\Omega)$ when $\delta \to 0$, let us show that the other terms go to zero.

- Observe that

  $$\forall s \in [0,T],\quad \theta'(s)\int_0^T \eta_\delta(s-t+\delta)\,dt = \theta'(s)\int_0^T \eta_\delta(s-t-\delta)\,dt = \theta'(s),$$

  so that, when $\tilde\rho^{\boldsymbol\varepsilon}$ is smooth we have

  $$R_1^{\boldsymbol\varepsilon,\delta}(s,.) = \theta'(s)\int_0^T (\tilde\rho^{\boldsymbol\varepsilon}(t,.)-\tilde\rho^{\boldsymbol\varepsilon}(s,.))\eta_\delta(s-t+\delta)\,dt - \theta'(s)\int_0^T (\tilde\rho^{\boldsymbol\varepsilon}(t,.)-\tilde\rho^{\boldsymbol\varepsilon}(s,.))\eta_\delta(s-t-\delta)\,dt,$$

  so that

  $$|R_1^{\boldsymbol\varepsilon,\delta}(s,.)| \le C\mathcal{L}(\tilde\rho^{\boldsymbol\varepsilon})\delta.$$

  Finally, we have the uniform bound $\|R_1^{\boldsymbol\varepsilon,\delta}\|_{L^1} \le C\|\tilde\rho^{\boldsymbol\varepsilon}\|_{L^1}$ which gives the result by a classical density argument.
- We first observe that $\|R_2^{\boldsymbol\varepsilon,\delta}\|_{L^1} \le C\|\tilde\rho^{\boldsymbol\varepsilon}\|_{L^\infty(W^{1,\infty})}\|v\|_{L^1(W^{1,1})}$, uniformly with respect to δ. We then conclude, by density, observing that the convergence is clear for smooth vector fields v.
- Here we observe that $\|R_3^{\boldsymbol\varepsilon,\delta}\|_{L^1} \le C\|\tilde\rho^{\boldsymbol\varepsilon}\|_{L^\infty}\|c\|_{L^1}$ and that the convergence to zero is clear when c is smooth enough.

Finally we get that $\tilde\rho^{\boldsymbol\varepsilon} \underset{\theta}{\star} \eta_\delta$ solves (5.2) with a right-hand side $R^{\boldsymbol\varepsilon,\delta}$ such that $R^{\boldsymbol\varepsilon,\delta} \to \tilde R^{\boldsymbol\varepsilon}$ when $\delta \to 0$. In particular, for any $\varepsilon > 0$ there exists $\delta(\varepsilon) \to 0$ such that

$$\|\tilde R^{\boldsymbol\varepsilon} - R^{\varepsilon,\delta(\varepsilon)}\|_{L^1} \le \varepsilon,\quad \text{and}\quad \|\gamma\tilde\rho^{\boldsymbol\varepsilon} \underset{\theta}{\star} \eta_{\delta(\varepsilon)} - \gamma\tilde\rho^{\boldsymbol\varepsilon}\|_{L^1(]0,T[\times\Gamma,|d\mu_v|)} \le \varepsilon.$$

Therefore, since $\|\tilde R^{\boldsymbol\varepsilon}\|_{L^1} \to 0$ and $\|\gamma\rho - \gamma\tilde\rho^{\boldsymbol\varepsilon}\|_{L^1(]0,T[\times\Gamma,|d\mu_v|)} \to 0$ as $\varepsilon \to 0$, we get that all the required properties hold if we define

$$\rho^{\boldsymbol\varepsilon} = \tilde\rho^{\boldsymbol\varepsilon} \underset{\theta}{\star} \eta_{\delta(\varepsilon)},\quad \text{and}\quad R^{\boldsymbol\varepsilon} = R^{\varepsilon,\delta(\varepsilon)}.$$

□

5.2. The strong convergence result. Our main result of this section is the following uniform in time strong convergence result.

THEOREM 5.2. *Assume that* (1.2), (1.3) *and* (1.4) *hold.*
Let $\mathrm{reg}_{\max} > 0$ *be given, and consider a family of meshes and time steps, such that* $(\delta t, h_{\mathcal{T}}) \to 0$ *and satisfying* (4.1). *We have the following convergences*

$$\|\rho_{\mathcal{T},\delta t} - \rho\|_{L^\infty(]0,T[,L^p(\Omega))} \xrightarrow[(\delta t,h_{\mathcal{T}})\to 0]{} 0, \quad \forall p < +\infty,$$

$$\|\gamma\rho_{\mathcal{T},\delta t} - \gamma\rho\|_{L^p(]0,T[\times\Gamma,|d\mu_v|)} \xrightarrow[(\delta t,h_{\mathcal{T}})\to 0]{} 0, \quad \forall p < +\infty.$$

REMARK 5.1.

- *The previous theorem implies, in particular, that*

$$\rho_{\mathcal{T}}^N \xrightarrow[(\delta t,h_{\mathcal{T}})\to 0]{} \rho(T), \quad \text{in } L^p(\Omega), \ \forall p < +\infty. \tag{5.3}$$

- *In fact, we can construct an approximate solution which is continuous in time by the formula*

$$\tilde\rho_{\mathcal{T},\delta t}(t,x) = \sum_{n=0}^{N-1}\sum_{\kappa\in\mathcal{T}} \frac{(t^{n+1}-t)\rho_\kappa^n + (t-t^n)\rho_\kappa^{n+1}}{\delta t} \mathbb{1}_{]t^n,t^{n+1}[\times\kappa}.$$

Since $\rho \in \mathcal{C}^0([0,T],L^p(\Omega))$ *for any* $p < +\infty$, *the previous theorem implies that*

$$\|\tilde\rho_{\mathcal{T},\delta t} - \rho\|_{\mathcal{C}^0([0,T],L^p(\Omega))} \xrightarrow[(\delta t,h_{\mathcal{T}})\to 0]{} 0, \quad \forall p < +\infty,$$

In order to simplify the presentation, we will assume in the following proof that

$$c + \frac{1}{2}\mathrm{div}\, v \geq 0, \quad \text{almost everywhere in }]0,T[\times\Omega. \tag{5.4}$$

The general case can be proved by a change of variables similar to the one we used in section 3.1 by using both assumptions (1.4).

Proof. Notice first that, thanks to the L^∞ bounds (1.6) and (3.12), it is enough to prove the result for $p = 2$.
We now consider the discretization of the family of approximations given by Lemma 5.1 and defined by

$$\rho_\kappa^{\boldsymbol{\varepsilon},n} = \rho^{\boldsymbol{\varepsilon}}(t^n,x_\kappa), \forall\kappa\in\mathcal{T}, \forall n\in[\![0,N]\!], \quad \text{and} \quad \rho_{\mathcal{T},\delta t}^{\boldsymbol{\varepsilon}} = \sum_{n=0}^{N-1}\sum_{\kappa\in\mathcal{T}} \rho_\kappa^{\boldsymbol{\varepsilon},n+1}\mathbb{1}_{]t^n,t^{n+1}[\times\kappa} \in L^\infty(]0,T[\times\Omega).$$

By the triangle inequality, we get

$$\|\rho_{\mathcal{T},\delta t} - \rho\|_{L^\infty(]0,T[,L^2(\Omega))} \leq \|\rho_{\mathcal{T},\delta t} - \rho_{\mathcal{T},\delta t}^{\boldsymbol{\varepsilon}}\|_{L^\infty(]0,T[,L^2(\Omega))} + \|\rho_{\mathcal{T},\delta t}^{\boldsymbol{\varepsilon}} - \rho^{\boldsymbol{\varepsilon}}\|_{L^\infty(]0,T[,L^2(\Omega))} + \|\rho^{\boldsymbol{\varepsilon}} - \rho\|_{L^\infty(]0,T[,L^2(\Omega))}. \tag{5.5}$$

By Lemma 5.1, the third term is known to converge to 0 when $\boldsymbol{\varepsilon}\to 0$ and, since $\rho^{\boldsymbol{\varepsilon}}$ is smooth, the second term can be bounded as follows: for every $t\in]0,T[$, such that $t\in]t^n,t^{n+1}[$ for some n,

$$\|\rho_{\mathcal{T},\delta t}^{\boldsymbol{\varepsilon}}(t) - \rho^{\boldsymbol{\varepsilon}}(t)\|_{L^2(\Omega))}^2 \leq \frac{1}{\delta t}\sum_{\kappa\in\mathcal{T}}\int_{t^n}^{t^{n+1}}\int_\kappa |\rho^{\boldsymbol{\varepsilon}}(t^{n+1},x_\kappa) - \rho^{\boldsymbol{\varepsilon}}(t,x)|^2\,dx \leq C(\delta t^2 + h_{\mathcal{T}}^2)\mathcal{L}(\rho^{\boldsymbol{\varepsilon}})^2. \tag{5.6}$$

Most of the sequel of the proof will consist in estimating the first term in (5.5). To this end, we define interface approximate values of $\rho^{\boldsymbol{\varepsilon}}$ by

$$\rho_\sigma^{\boldsymbol{\varepsilon},n} = \begin{cases} (\rho_\kappa^{\boldsymbol{\varepsilon},n} + \rho_\mathcal{L}^{\boldsymbol{\varepsilon},n})/2, & \forall\sigma = \kappa|\mathcal{L}\in\mathcal{E}_{\mathrm{int}}, \\ \rho_\sigma^{in,n}, & \forall\sigma\in\mathcal{E}_{\mathrm{bd}}, \text{s.t. } v_{\kappa\sigma}^n \leq 0, \\ \rho_\kappa^{\boldsymbol{\varepsilon},n}, & \forall\sigma\in\mathcal{E}_{\mathrm{bd}}, \text{s.t. } v_{\kappa\sigma}^n > 0. \end{cases}$$

For any $\sigma\in\mathcal{E}$, let us choose an arbitrary point $x_\sigma\in\sigma$. By integrating (5.1) over $[t^n,t^{n+1}]\times\kappa$ and putting the result under same form as in (3.15), we obtain

$$|\kappa|\frac{\rho_\kappa^{\boldsymbol{\varepsilon},n+1} - \rho_\kappa^{\boldsymbol{\varepsilon},n}}{\delta t} + \sum_{\sigma\in\mathcal{E}_\kappa} |\sigma| v_{\kappa\sigma}^n \rho_\sigma^{\boldsymbol{\varepsilon},n+1} + |\kappa| c_\kappa^n \rho_\kappa^{\boldsymbol{\varepsilon},n+1} - \sum_{\sigma\in\mathcal{E}_\kappa\cap\mathcal{E}_{\mathrm{int}}} |\sigma| |v_{\kappa\sigma}^n| \frac{\rho_\mathcal{L}^{\boldsymbol{\varepsilon},n+1} - \rho_\kappa^{\boldsymbol{\varepsilon},n+1}}{2}$$
$$= |\kappa|(\delta_\kappa^{\boldsymbol{\varepsilon},n+1} - \delta_\kappa^{\boldsymbol{\varepsilon},n}) + |\kappa| R_\kappa^{\boldsymbol{\varepsilon},n} + \sum_{\sigma\in\mathcal{E}_\kappa} |\sigma|\delta_{\kappa\sigma}^{\boldsymbol{\varepsilon},n} + \sum_{\sigma\in\mathcal{E}_\kappa} |\sigma|\gamma_{\kappa\sigma}^{\boldsymbol{\varepsilon},n}, \quad \forall\kappa\in\mathcal{T}, \forall n\in[\![0,N-1]\!], \tag{5.7}$$

where the remainder terms are defined by

$$\delta_{\mathcal{K}}^{\boldsymbol{\varepsilon},n} = \frac{1}{|\mathcal{K}|}\int_{\mathcal{K}} (\rho^{\boldsymbol{\varepsilon}}(t^n, x_{\mathcal{K}}) - \rho^{\boldsymbol{\varepsilon}}(t^n, x))\, dx,$$

$$\delta_{\mathcal{K}\sigma}^{\boldsymbol{\varepsilon},n} = \frac{1}{\delta t|\sigma|}\int_{t^n}^{t^{n+1}}\int_{\sigma} (v\cdot \boldsymbol{\nu}_{\mathcal{K}\sigma})(\rho^{\boldsymbol{\varepsilon}}(t^{n+1}, x_\sigma) - \rho^{\boldsymbol{\varepsilon}}(t, x))\, dx\, dt,$$

$$\gamma_{\mathcal{K}\sigma}^{\boldsymbol{\varepsilon},n} = \begin{cases} -|v_{\mathcal{K}\sigma}^n| \dfrac{\rho_{\mathcal{L}}^{\boldsymbol{\varepsilon},n+1} - \rho_{\mathcal{K}}^{\boldsymbol{\varepsilon},n+1}}{2} + v_{\mathcal{K}\sigma}^n \left(\rho_\sigma^{\boldsymbol{\varepsilon},n+1} - \rho^{\boldsymbol{\varepsilon}}(t^{n+1}, x_\sigma)\right), & \text{for } \sigma = \mathcal{K}|\mathcal{L} \in \mathcal{E}_{\rm int} \\ v_{\mathcal{K}\sigma}^n \left(\rho_\sigma^{\boldsymbol{\varepsilon},n+1} - \rho^{\boldsymbol{\varepsilon}}(t^{n+1}, x_\sigma)\right), & \text{for } \sigma \in \mathcal{E}_{\rm bd} \end{cases}$$

$$R_{\mathcal{K}}^{\boldsymbol{\varepsilon},n} = \frac{1}{\delta t|\mathcal{K}|}\int_{t^n}^{t^{n+1}}\int_{\mathcal{K}} R^{\boldsymbol{\varepsilon}}\, dx\, dt + \frac{1}{\delta t|\mathcal{K}|}\int_{t^n}^{t^{n+1}}\int_{\mathcal{K}} c(t,x)(\rho^{\boldsymbol{\varepsilon}}(t^n, x_{\mathcal{K}}) - \rho^{\boldsymbol{\varepsilon}}(t,x))\, dx\, dt.$$

Note that for any interior edge $\sigma = \mathcal{K}|\mathcal{L}$ we have the local conservativity properties $\gamma_{\mathcal{K}\sigma}^{\boldsymbol{\varepsilon},n} + \gamma_{\mathcal{L}\sigma}^{\boldsymbol{\varepsilon},n} = 0$ and $\delta_{\mathcal{K}\sigma}^{\boldsymbol{\varepsilon},n} + \delta_{\mathcal{L}\sigma}^{\boldsymbol{\varepsilon},n} = 0$, which are useful to perform the computations which follow. For any $n \in [\![0, N]\!]$, we introduce now the error term

$$E_{\mathcal{K}}^{\boldsymbol{\varepsilon},n} = \rho_{\mathcal{K}}^{\boldsymbol{\varepsilon},n} - \rho_{\mathcal{K}}^n,\ \forall \mathcal{K} \in \mathcal{T}, \quad \text{and} \quad E_{\mathcal{T}}^{\boldsymbol{\varepsilon},n} = \sum_{\mathcal{K}\in\mathcal{T}} E_{\mathcal{K}}^{\boldsymbol{\varepsilon},n} \mathbb{1}_{\mathcal{K}},$$

so that the quantity we want to bound reads

$$\|\rho_{\mathcal{T},\delta t}^{\boldsymbol{\varepsilon}} - \rho_{\mathcal{T},\delta t}\|_{L^\infty(]0,T[,L^2(\Omega))} = \sup_{0\le n\le N} \|E_{\mathcal{T}}^{\boldsymbol{\varepsilon},n}\|_{L^2(\Omega)} = \|E_{\mathcal{T}}^{\boldsymbol{\varepsilon},\tilde{N}}\|_{L^2(\Omega)}, \tag{5.8}$$

for some $\tilde{N} \in [\![0, N]\!]$. Note that, we have the following elementary bounds

$$\sup_{0\le n\le N} \|E_{\mathcal{T}}^{\boldsymbol{\varepsilon},n}\|_{L^\infty(\Omega)} \le 2\rho_{\max}, \tag{5.9}$$

$$\|E_{\mathcal{T}}^{\boldsymbol{\varepsilon},0}\|_{L^2(\Omega)} \le Ch_{\mathcal{T}}\mathcal{L}(\rho^{\boldsymbol{\varepsilon}}) + C\|\rho^{\boldsymbol{\varepsilon}}(0,.) - \rho^0\|_{L^2(\Omega)}. \tag{5.10}$$

We subtract (5.7) and (3.15) then we multiply the result by $E_{\mathcal{K}}^{\boldsymbol{\varepsilon},n+1}$ and we sum over $n = 0, ..., \tilde{N}-1$ and $\kappa \in \mathcal{T}$. Then we do exactly the same computations than the ones we used to obtain (3.16) and we get

$$\begin{aligned}
&\frac{1}{2}\|E_{\mathcal{T}}^{\boldsymbol{\varepsilon},\tilde{N}}\|_{L^2}^2 + \frac{1}{2}\sum_{n=0}^{\tilde{N}-1}\delta t\|E_{\mathcal{T}}^{\boldsymbol{\varepsilon},n+1} - E_{\mathcal{T}}^{\boldsymbol{\varepsilon},n}\|_{L^2}^2 + \sum_{n=0}^{\tilde{N}-1}\delta t\sum_{\mathcal{K}\in\mathcal{T}}|\mathcal{K}|\left(c_{\mathcal{K}}^n + \frac{1}{2}(\operatorname{div} v)_{\mathcal{K}}^n\right)|E_{\mathcal{K}}^{\boldsymbol{\varepsilon},n+1}|^2 \\
&\qquad + \frac{1}{2}\sum_{n=0}^{\tilde{N}-1}\delta t\sum_{\sigma\in\mathcal{E}_{\rm bd}}|\sigma||v_\sigma^n|(E_{\mathcal{K}}^{\boldsymbol{\varepsilon},n+1})^2 + \frac{1}{2}\sum_{n=0}^{\tilde{N}-1}\delta t\sum_{\substack{\sigma\in\mathcal{E}_{\rm int}\\ \sigma=\mathcal{K}|\mathcal{L}}}|\sigma||v_{\mathcal{K}\mathcal{L}}^n|(E_{\mathcal{L}}^{\boldsymbol{\varepsilon},n+1} - E_{\mathcal{K}}^{\boldsymbol{\varepsilon},n+1})^2 \\
&\quad = \frac{1}{2}\|E_{\mathcal{T}}^{\boldsymbol{\varepsilon},0}\|_{L^2}^2 + \sum_{\mathcal{K}\in\mathcal{T}}|\mathcal{K}|\delta_{\mathcal{K}}^{\boldsymbol{\varepsilon},\tilde{N}}E_{\mathcal{K}}^{\boldsymbol{\varepsilon},\tilde{N}} + \sum_{n=0}^{\tilde{N}-1}\delta t\sum_{\mathcal{K}\in\mathcal{T}}|\mathcal{K}|\delta_{\mathcal{K}}^{\boldsymbol{\varepsilon},n}(E_{\mathcal{K}}^{\boldsymbol{\varepsilon},n} - E_{\mathcal{K}}^{\boldsymbol{\varepsilon},n+1}) - \sum_{\mathcal{K}\in\mathcal{T}}|\mathcal{K}|\delta_{\mathcal{K}}^{\boldsymbol{\varepsilon},0}E_{\mathcal{K}}^{\boldsymbol{\varepsilon},0} \\
&\quad + \sum_{n=0}^{\tilde{N}-1}\delta t\sum_{\mathcal{K}\in\mathcal{T}}|\mathcal{K}|R_{\mathcal{K}}^{\boldsymbol{\varepsilon},n}E_{\mathcal{K}}^{\boldsymbol{\varepsilon},n+1} + \sum_{n=0}^{\tilde{N}-1}\delta t\sum_{\substack{\sigma\in\mathcal{E}_{\rm int}\\ \sigma=\mathcal{K}|\mathcal{L}}}|\sigma|\gamma_{\mathcal{K}\sigma}^{\boldsymbol{\varepsilon},n}(E_{\mathcal{K}}^{\boldsymbol{\varepsilon},n+1} - E_{\mathcal{L}}^{\boldsymbol{\varepsilon},n+1}) + \sum_{n=0}^{\tilde{N}-1}\delta t\sum_{\sigma\in\mathcal{E}_{\rm bd}}|\sigma|\gamma_{\mathcal{K}\sigma}^{\boldsymbol{\varepsilon},n}E_{\mathcal{K}}^{\boldsymbol{\varepsilon},n+1} \\
&\qquad + \sum_{n=0}^{\tilde{N}-1}\delta t\sum_{\substack{\sigma\in\mathcal{E}_{\rm int}\\ \sigma=\mathcal{K}|\mathcal{L}}}|\sigma|\delta_{\mathcal{K}\sigma}^{\boldsymbol{\varepsilon},n}(E_{\mathcal{K}}^{\boldsymbol{\varepsilon},n+1} - E_{\mathcal{L}}^{\boldsymbol{\varepsilon},n+1}) + \sum_{n=0}^{\tilde{N}-1}\delta t\sum_{\sigma\in\mathcal{E}_{\rm bd}}|\sigma|\delta_{\mathcal{K}\sigma}^{\boldsymbol{\varepsilon},n}E_{\mathcal{K}}^{\boldsymbol{\varepsilon},n+1}.
\end{aligned} \tag{5.11}$$

We denote by $I_{\mathcal{T},\delta t}^{\boldsymbol{\varepsilon}}$ the left-hand side of this inequality (whose all terms are non-negative thanks to assumption (5.4)) and we have to estimate all the terms $T_i, i = 1, \dots, 9$ in the right-hand side. We want to point out the fact that numerical diffusion terms (fourth and fifth) in the definition of $I_{\mathcal{T},\delta t}^{\boldsymbol{\varepsilon}}$ will be of major importance in the following estimates.

- Term T_1: By (5.9) and (5.10) we get

$$\begin{aligned}|T_1| &\le C_{\rho_{\max}}\|E_{\mathcal{T}}^{\boldsymbol{\varepsilon},0}\|_{L^2(\Omega)} \le C_{\rho_{\max}}\left(\mathcal{L}(\rho^{\boldsymbol{\varepsilon}})h_{\mathcal{T}} + \|\rho^{\boldsymbol{\varepsilon}}(0,.)-\rho^0\|_{L^2(\Omega)}\right)\\ &\le C_{\rho_{\max}}\left(\mathcal{L}(\rho^{\boldsymbol{\varepsilon}})h_{\mathcal{T}} + \|\rho^{\boldsymbol{\varepsilon}}-\rho\|_{L^\infty(]0,T[,L^2(\Omega))}\right).\end{aligned}$$

- Term T_2: We use the Cauchy-Schwarz inequality and the definition of $\delta_{\mathcal{K}}^{\boldsymbol{\varepsilon},n}$

$$|T_2| \le \|E_{\mathcal{T}}^{\boldsymbol{\varepsilon},\tilde{N}}\|_{L^2}\left(\sum_{\mathcal{K}\in\mathcal{T}}|\mathcal{K}||\delta_{\mathcal{K}}^{\boldsymbol{\varepsilon},\tilde{N}}|^2\right)^{\frac{1}{2}} \le \mathcal{L}(\rho^{\boldsymbol{\varepsilon}})h_{\mathcal{T}}|\Omega|\|E_{\mathcal{T}}^{\boldsymbol{\varepsilon},\tilde{N}}\|_{L^2} \le C\mathcal{L}(\rho^{\boldsymbol{\varepsilon}})h_{\mathcal{T}}\sqrt{I_{\mathcal{T},\delta t}^{\boldsymbol{\varepsilon}}}.$$

- Term T_3: By using similar arguments as for T_2 we get

$$|T_3| \le \left(\sum_{n=0}^{\tilde{N}-1}\delta t\|E_{\mathcal{T}}^{\boldsymbol{\varepsilon},n+1}-E_{\mathcal{T}}^{\boldsymbol{\varepsilon},n}\|_{L^2(\Omega)}^2\right)^{\frac{1}{2}}\left(\sum_{n=0}^{\tilde{N}-1}\delta t\sum_{\mathcal{K}\in\mathcal{T}}|\mathcal{K}||\delta_{\mathcal{K}}^{\boldsymbol{\varepsilon},n}|^2\right)^{\frac{1}{2}} \le C\mathcal{L}(\rho^{\boldsymbol{\varepsilon}})h_{\mathcal{T}}\sqrt{I_{\mathcal{T},\delta t}^{\boldsymbol{\varepsilon}}}.$$

- Term T_4: By (5.9) and the definition of $\delta_{\mathcal{K}}^{\boldsymbol{\varepsilon},0}$ we get

$$|T_4| \le \mathcal{L}(\rho^{\boldsymbol{\varepsilon}})h_{\mathcal{T}}\|E_{\mathcal{T}}^{\boldsymbol{\varepsilon},0}\|_{L^2} \le 2\rho_{\max}\mathcal{L}(\rho^{\boldsymbol{\varepsilon}})h_{\mathcal{T}}.$$

- Term T_5: By definition of $R_{\mathcal{K}}^{\boldsymbol{\varepsilon},n}$ and (5.9) we have

$$|T_5| \le 2\rho_{\max}\sum_{n=0}^{\tilde{N}-1}\delta t\sum_{\mathcal{K}\in\mathcal{T}}|\mathcal{K}||R_{\mathcal{K}}^{\boldsymbol{\varepsilon},n}| \le C_{\rho_{\max}}\|R^{\boldsymbol{\varepsilon}}\|_{L^1(]0,T[\times\Omega)} + C_{\rho_{\max}}\|c\|_{L^1}\mathcal{L}(\rho^{\boldsymbol{\varepsilon}})(\delta t + h_{\mathcal{T}}).$$

- Term T_6: Observing that $|\gamma_{\mathcal{K}\sigma}^{\boldsymbol{\varepsilon},n}| \le (d_{\mathcal{K}}+d_{\mathcal{L}})\mathcal{L}(\rho^{\boldsymbol{\varepsilon}})|v_{\mathcal{K}\sigma}^n|$, since we are only concerned with interior edges, we first evaluate this term as follows

$$|T_6| \le \mathcal{L}(\rho^{\boldsymbol{\varepsilon}})\sum_{n=0}^{N-1}\delta t\sum_{\sigma\in\mathcal{E}_{\rm int}}|\sigma||v_{\mathcal{K}\sigma}^n|(d_{\mathcal{K}}+d_{\mathcal{L}})|E_{\mathcal{K}}^{\boldsymbol{\varepsilon},n+1}-E_{\mathcal{L}}^{\boldsymbol{\varepsilon},n+1}|.$$

  We then use the Cauchy-Schwarz inequality, (2.1) and the bound (5.9), to obtain

$$\begin{aligned}|T_6| &\le C_{\rho_{\max}}\mathcal{L}(\rho^{\boldsymbol{\varepsilon}})\left(\sum_{n=0}^{\tilde{N}-1}\delta t\sum_{\sigma\in\mathcal{E}_{\rm int}}|\sigma||v_{\mathcal{K}\sigma}^n||E_{\mathcal{K}}^{\boldsymbol{\varepsilon},n+1}-E_{\mathcal{L}}^{\boldsymbol{\varepsilon},n+1}|^2\right)^{\frac{1}{2}}\\ &\quad\times\left(\sum_{n=0}^{\tilde{N}-1}\delta t\sum_{\sigma\in\mathcal{E}_{\rm int}}|\sigma||v_{\mathcal{K}\sigma}^n|(d_{\mathcal{K}}+d_{\mathcal{L}})^2\right)^{\frac{1}{2}}\\ &\le C_{\rho_{\max}}\sqrt{h_{\mathcal{T}}}\mathcal{L}(\rho^{\boldsymbol{\varepsilon}})\sqrt{I_{\mathcal{T},\delta t}^{\boldsymbol{\varepsilon}}}\left(\sum_{n=0}^{\tilde{N}-1}\sum_{\mathcal{K}\in\mathcal{T}}|\sigma|\int_{t^n}^{t^{n+1}}d_{\mathcal{K}}\|v\|_{L^1(\partial\mathcal{K})}\,dx\,dt\right)^{\frac{1}{2}}\\ &\le C_{\rho_{\max},\mathrm{reg}_{\max}}\sqrt{h_{\mathcal{T}}}\mathcal{L}(\rho^{\boldsymbol{\varepsilon}})\sqrt{I_{\mathcal{T},\delta t}^{\boldsymbol{\varepsilon}}}\|v\|_{L^1(]0,T[,W^{1,1}(\Omega))}^{\frac{1}{2}}.\end{aligned}$$

- Term T_7: For the boundary edges such that $v_{\mathcal{K}\sigma}^n \ge 0$, we have $\rho_\sigma^{\boldsymbol{\varepsilon},n} = \rho_{\mathcal{K}}^{\boldsymbol{\varepsilon},n} = \rho^{\boldsymbol{\varepsilon}}(t^{n+1},x_{\mathcal{K}})$ so that the contribution of this term can be treated in the same way as for the term T_6.
  For the boundary edges such that $v_{\mathcal{K}\sigma}^n < 0$, the value of $\rho_\sigma^{\boldsymbol{\varepsilon},n}$ is given by the boundary data and thus we have to adapt the argument. To this end, we write

$$\gamma_{\mathcal{K}\sigma}^{\boldsymbol{\varepsilon},n} = v_{\mathcal{K}\sigma}^n\frac{1}{\delta t|\sigma|}\int_{t^n}^{t^{n+1}}\int_\sigma(\rho^{in}(t,x)-\rho^{\boldsymbol{\varepsilon}}(t,x))\,dx\,dt + v_{\mathcal{K}\sigma}^n\frac{1}{\delta t|\sigma|}\int_{t^n}^{t^{n+1}}\int_\sigma(\rho^{\boldsymbol{\varepsilon}}(t,x)-\rho^{\boldsymbol{\varepsilon}}(t^{n+1},x_\sigma))\,dx\,dt.$$

  The contribution of the second part of this term can be treated just like in the term T_6, using the fact that $\rho^{\boldsymbol{\varepsilon}}$ is Lipschitz continuous. It remains to evaluate the following contribution

$$T_7' = \sum_{n=0}^{\tilde{N}-1}\sum_{\substack{\sigma\in\mathcal{E}_{\rm bd}\\ v_{\mathcal{K}\sigma}^n<0}}v_{\mathcal{K}\sigma}^n\int_{t^n}^{t^{n+1}}\int_\sigma(\rho^{in}(t,x)-\rho^{\boldsymbol{\varepsilon}}(t,x))\,dx\,dt.$$

Let us introduce, for any $v, w \in L^1(]0,T[,(W^{1,1}(\Omega))^d)$ the notation

$$T_7'(v,w) = \sum_{n=0}^{\tilde{N}-1} \sum_{\substack{\sigma\in\mathcal{E}_{\rm bd}\\ v_{\mathcal{K}\sigma}^n<0}} w_{\mathcal{K}\sigma}^n \int_{t^n}^{t^{n+1}}\int_\sigma (\rho^{in}(t,x)-\rho^{\boldsymbol{\varepsilon}}(t,x))\,dx\,dt,$$

so that, the term we are interested in is $T_7'(v,v)$. Note that $T_7'(v,w)$ is linear with respect to w but nonlinear with respect to w.
We consider a smooth vector field w, to be determined later, and we write $T_7'(v,v) = T_7'(v,w) + T_7'(v,v-w)$.

– Since w is smooth, for any $f \in L^1(]0,T[\times\Gamma)$ we have

$$\left|\sum_{n=0}^{\tilde{N}-1}\sum_{\sigma\in\mathcal{E}_{\rm bd}} w_{\mathcal{K}\sigma}^n \int_{t^n}^{t^{n+1}}\int_\sigma f\,dx\,dt - \int_0^{t^{\tilde{N}}}\int_\Gamma (w\cdot\boldsymbol{\nu}) f\,dx\,dt\right| \le \mathcal{L}(w)(\delta t + h_{\mathcal{T}})\|f\|_{L^1(]0,T[\times\Gamma)}.$$

Applying this result to $f = \left(\sum_{n=0}^{N-1}\sum_{\substack{\sigma\in\mathcal{E}_{\rm bd}\\ v_{\mathcal{K}\sigma}^n<0}} \mathbb{1}_{]t^n,t^{n+1}[\times\sigma}\right)(\rho^{in}-\rho^{\boldsymbol{\varepsilon}})$ and using the L^∞ bounds on ρ^{in} and $\rho^{\boldsymbol{\varepsilon}}$, we then have

$$\begin{aligned}
|T_7'(v,w)| &\le \mathcal{L}(w)(\delta t + h_{\mathcal{T}}) \sum_{n=0}^{\tilde{N}-1}\sum_{\substack{\sigma\in\mathcal{E}_{\rm bd}\\ v_{\mathcal{K}\sigma}^n<0}} \int_{t^n}^{t^{n+1}}\int_\sigma \underbrace{|\rho^{in}(t,x)-\rho^{\boldsymbol{\varepsilon}}(t,x)|}_{\le 2\rho_{\max}}\,dx\,dt\\
&\quad + \sum_{n=0}^{N-1}\sum_{\substack{\sigma\in\mathcal{E}_{\rm bd}\\ v_{\mathcal{K}\sigma}^n<0}} \int_{t^n}^{t^{n+1}}\int_\sigma |w\cdot\boldsymbol{\nu}_\sigma||\rho^{in}(t,x)-\rho^{\boldsymbol{\varepsilon}}(t,x)|\,dx\,dt\\
&\le 2\rho_{\max}T|\Gamma|\mathcal{L}(w)(\delta t + h_{\mathcal{T}}) + 2\rho_{\max}\underbrace{\|(v-w)\cdot\boldsymbol{\nu}\|_{L^1(]0,T[\times\Gamma)}}_{\le C\|v-w\|_{L^1(W^{1,1})}}\\
&\quad + \underbrace{\sum_{n=0}^{N-1}\sum_{\substack{\sigma\in\mathcal{E}_{\rm bd}\\ v_{\mathcal{K}\sigma}^n<0}} \int_{t^n}^{t^{n+1}}\int_\sigma |v\cdot\boldsymbol{\nu}_\sigma||\rho^{in}(t,x)-\rho^{\boldsymbol{\varepsilon}}(t,x)|\,dx\,dt}_{=T_7''}.
\end{aligned}$$

By writing $|v\cdot\boldsymbol{\nu}_\sigma| = (v\cdot\boldsymbol{\nu}_\sigma)^+ + (v\cdot\boldsymbol{\nu}_\sigma)^-$, we finally bound the last term as follows

$$T_7'' \le \int_0^T\int_\Gamma (v\cdot\boldsymbol{\nu})^-|\rho^{in}(t,x)-\rho^{\boldsymbol{\varepsilon}}(t,x)|\,dx\,dt + 2\rho_{\max}\sum_{n=0}^{N-1}\sum_{\substack{\sigma\in\mathcal{E}_{\rm bd}\\ v_{\mathcal{K}\sigma}^n<0}} \int_{t^n}^{t^{n+1}}\int_\sigma (v\cdot\boldsymbol{\nu}_\sigma)^+\,dx\,dt.$$

– The second part of $T_7'(v,v)$ is classically bounded as follows

$$|T_7'(v,v-w)| \le C_{\rho_{\max}}\|v-w\|_{L^1(W^{1,1})}.$$

- Term T_8: this term also needs a particular care. We will also denote it by $T_8(v)$, in order to point out the linear dependence of this term with respect to v. We first estimate $\delta_{\mathcal{K}\sigma}^{\boldsymbol{\varepsilon},n}$ as follows

$$|\delta_{\mathcal{K}\sigma}^{\boldsymbol{\varepsilon},n}| \le (\delta t + d_\sigma)\mathcal{L}(\rho^{\boldsymbol{\varepsilon}})\frac{1}{\delta t|\sigma|}\int_{t^n}^{t^{n+1}}\int_\sigma |v\cdot\boldsymbol{\nu}_{\mathcal{K}\sigma}|\,dx\,dt.$$

As we have already seen, the difficulty comes from the fact that the sign of $v\cdot\boldsymbol{\nu}_{\mathcal{K}\sigma}$ may change inside $[t^n,t^{n+1}]\times\sigma$ and then we can not estimate $\delta_{\mathcal{K}\sigma}^{\boldsymbol{\varepsilon},n}$ by using $|v_{\mathcal{K}\sigma}^n|$.
Consider a smooth vector field $w \in \mathcal{C}^1([0,T]\times\overline{\Omega})^d$, and let us write $T_8(v) = T_8(w) + T_8(v-w)$. In the above estimate of $\delta_{\mathcal{K}\sigma}^{\boldsymbol{\varepsilon},n}$ (with w in place of v), since w is smooth we can write

$$|\delta_{\mathcal{K}\sigma}^{\boldsymbol{\varepsilon},n}(w)| \le (\delta t + d_\sigma)\mathcal{L}(\rho^{\boldsymbol{\varepsilon}})\Big(|w_{\mathcal{K}\sigma}^n| + \mathcal{L}(w)(\delta t + d_\sigma)\Big).$$

We recall that $d_\sigma \le d_\mathcal{K}$ for any $\sigma \in \mathcal{E}_\mathcal{K}$ and that, by (4.1), we have $\delta t \le \mathrm{reg}_{\max} d_\mathcal{K}$ for any $\kappa \in \mathcal{T}$. Therefore, using that $|w^n_{\mathcal{K}\sigma}| \le |v^n_{\mathcal{K}\sigma}| + |w^n_{\mathcal{K}\sigma} - v^n_{\mathcal{K}\sigma}|$, we can then write

$$|T_8(w)| \le C_{\mathrm{reg}_{\max}} \mathcal{L}(\rho^{\boldsymbol{\varepsilon}}) \Bigg[\left(\sum_{n=0}^{\tilde{N}-1} \delta t \sum_{\sigma \in \mathcal{E}_{\mathrm{int}}} |\sigma| |v^n_{\mathcal{K}\sigma}| (d_\mathcal{K} + d_\mathcal{L}) |E^{\boldsymbol{\varepsilon},n+1}_\mathcal{K} - E^{\boldsymbol{\varepsilon},n+1}_\mathcal{L}| \right) + \left(\sum_{n=0}^{\tilde{N}-1} \delta t \sum_{\sigma \in \mathcal{E}_{\mathrm{int}}} |\sigma| |v^n_{\mathcal{K}\sigma} - w^n_{\mathcal{K}\sigma}| (d_\mathcal{K} + d_\mathcal{L}) |E^{\boldsymbol{\varepsilon},n+1}_\mathcal{K} - E^{\boldsymbol{\varepsilon},n+1}_\mathcal{L}| \right) + \mathcal{L}(w)(\delta t + h_\mathcal{T}) \left(\sum_{n=0}^{\tilde{N}-1} \delta t \sum_{\sigma \in \mathcal{E}_{\mathrm{int}}} |\sigma| (d_\mathcal{K} + d_\mathcal{L}) |E^{\boldsymbol{\varepsilon},n+1}_\mathcal{K} - E^{\boldsymbol{\varepsilon},n+1}_\mathcal{L}| \right) \Bigg].$$

We use the Cauchy-Schwarz inequality for the first term above and we simply use the bound (5.9) in the other terms. It follows

$$|T_8(w)| \le C_{\mathrm{reg}_{\max}} \mathcal{L}(\rho^{\boldsymbol{\varepsilon}}) \Bigg[\sqrt{h_\mathcal{T}} \sqrt{I^{\boldsymbol{\varepsilon}}_{\mathcal{T},\delta t}} \left(\sum_{n=0}^{N-1} \int_{t^n}^{t^{n+1}} \sum_{\mathcal{K} \in \mathcal{T}} d_\mathcal{K} \|v(t)\|_{L^1(\partial\mathcal{K})} \, dt \right)^{\frac{1}{2}} + \left(\sum_{n=0}^{N-1} \int_{t^n}^{t^{n+1}} \sum_{\mathcal{K} \in \mathcal{T}} d_\mathcal{K} \|v(t) - w(t)\|_{L^1(\partial\mathcal{K})} \, dt \right) + \mathcal{L}(w)(\delta t + h_\mathcal{T}) T |\Omega| \Bigg],$$

and thus

$$|T_8(w)| \le C_{\mathrm{reg}_{\max},\rho_{\max}} \mathcal{L}(\rho^{\boldsymbol{\varepsilon}}) \left(\sqrt{h_\mathcal{T}} \sqrt{I^{\boldsymbol{\varepsilon}}_{\mathcal{T},\delta t}} \|v\|^{\frac{1}{2}}_{L^1(W^{1,1})} + \|v - w\|_{L^1(W^{1,1})} + \mathcal{L}(w) h_\mathcal{T} \right).$$

Finally, the term $T_8(v-w)$ can be estimated just like above by writing

$$|T_8(v-w)| \le C_{\mathrm{reg}_{\max}} \mathcal{L}(\rho^{\boldsymbol{\varepsilon}}) \|v - w\|_{L^1(W^{1,1})}.$$

- Term T_9: This boundary term does not present any new difficulty since, by using (5.9), we can write

$$|T_9| \le C_{\mathrm{reg}_{\max}} (\delta t + h_\mathcal{T}) \mathcal{L}(\rho^{\boldsymbol{\varepsilon}}) \|v \cdot \boldsymbol{\nu}\|_{L^1(]0,T[\times\Gamma)}.$$

Collecting all the above estimates in the inequality (5.11) and using Young's inequality, we get

$$I^{\boldsymbol{\varepsilon}}_{\mathcal{T},\delta t} \le C_{\mathrm{reg}_{\max},\rho_{\max}} \Bigg(\left(\mathcal{L}(\rho^{\boldsymbol{\varepsilon}}) + \mathcal{L}(w) \right) \left(1 + \mathcal{L}(\rho^{\boldsymbol{\varepsilon}}) \right) h_\mathcal{T} + \left(1 + \mathcal{L}(\rho^{\boldsymbol{\varepsilon}}) \right) \|v - w\|_{L^1(W^{1,1})} + \|\rho^{\boldsymbol{\varepsilon}} - \rho\|_{L^\infty(]0,T[,L^2(\Omega))} + \|R^{\boldsymbol{\varepsilon}}\|_{L^1} + \int_0^T \int_\Gamma (v \cdot \boldsymbol{\nu})^- |\rho^{\boldsymbol{\varepsilon}} - \rho^{in}| \, dx \, dt + \sum_{n=0}^{N-1} \sum_{\substack{\sigma \in \mathcal{E}_{\mathrm{bd}} \\ v^n_{\mathcal{K}\sigma} < 0}} \int_{t^n}^{t^{n+1}} \int_\sigma (v \cdot \boldsymbol{\nu}_\sigma)^+ \, dx \, dt \Bigg). \quad (5.12)$$

By definition of $I^{\boldsymbol{\varepsilon}}_{\mathcal{T},\delta t}$, we have $\|E^{\boldsymbol{\varepsilon},\tilde{N}}_\mathcal{T}\|^2_{L^2} \le 2 I^{\boldsymbol{\varepsilon}}_{\mathcal{T},\delta t}$, and then, by the choice (5.8) of $\tilde{N}$, we see that (5.12) gives an estimate on $\|\rho_{\mathcal{T},\delta t} - \rho^{\boldsymbol{\varepsilon}}_{\mathcal{T},\delta t}\|_{L^\infty(]0,T[,L^2(\Omega))}$. Finally, with (5.6) and (5.5), we obtain

$$\|\rho_{\mathcal{T},\delta t} - \rho\|^2_{L^\infty(]0,T[,L^2(\Omega))} \le C_{\mathrm{reg}_{\max},\rho_{\max}} \Bigg(\left(\mathcal{L}(\rho^{\boldsymbol{\varepsilon}}) + \mathcal{L}(w) \right) \left(1 + \mathcal{L}(\rho^{\boldsymbol{\varepsilon}}) \right) h_\mathcal{T} + \left(1 + \mathcal{L}(\rho^{\boldsymbol{\varepsilon}}) \right) \|v - w\|_{L^1(W^{1,1})} + \|\rho^{\boldsymbol{\varepsilon}} - \rho\|_{L^\infty(]0,T[,L^2(\Omega))} + \|R^{\boldsymbol{\varepsilon}}\|_{L^1} + \int_0^T \int_\Gamma (v \cdot \boldsymbol{\nu})^- |\rho^{\boldsymbol{\varepsilon}} - \rho^{in}| \, dx \, dt + \sum_{n=0}^{N-1} \sum_{\substack{\sigma \in \mathcal{E}_{\mathrm{bd}} \\ v^n_{\mathcal{K}\sigma} < 0}} \int_{t^n}^{t^{n+1}} \int_\sigma (v \cdot \boldsymbol{\nu}_\sigma)^+ \, dx \, dt \Bigg). \quad (5.13)$$

Let now $\Delta > 0$ be any small positive number. By Lemma 5.1, we first choose $\boldsymbol{\varepsilon} > 0$ small enough so that

$$\|\rho^{\boldsymbol{\varepsilon}} - \rho\|^2_{L^\infty(]0,T[,L^2(\Omega))} \le \Delta, \quad \|R^{\boldsymbol{\varepsilon}}\|_{L^1} \le \Delta, \text{ and } \|\gamma \rho^{\boldsymbol{\varepsilon}} - \rho^{in}\|_{L^1(]0,T[\times\Gamma, d\mu^-_v)} \le \Delta.$$

Then, $\boldsymbol{\varepsilon} > 0$ being fixed, we can find a smooth vector field w such that $(1 + \mathcal{L}(\rho^{\boldsymbol{\varepsilon}})) \|v - w\|_{L^1(0,T,W^{1,1}(\Omega))} \le \Delta$. This vector field w being now fixed, all the other terms in (5.13) can be made smaller than Δ for $h_\mathcal{T}$ and δt small enough satisfying (4.1) (we use Lemma 2.1 for the last term). This proves the first strong convergence property.

It remains to prove the strong convergence of the traces, for $p = 2$ for instance. Using the triangle inequality, we first write

$$\|\gamma\rho_{\mathcal{T},\delta t}-\gamma\rho\|_{L^2(]0,T[\times\Gamma,|d\mu_v|)} \le \|\gamma\rho_{\mathcal{T},\delta t}-\gamma\rho^{\boldsymbol{\varepsilon}}_{\mathcal{T},\delta t}\|_{L^2(]0,T[\times\Gamma,|d\mu_v|)}+\|\gamma\rho^{\boldsymbol{\varepsilon}}_{\mathcal{T},\delta t}-\gamma\rho^{\boldsymbol{\varepsilon}}\|_{L^2(]0,T[\times\Gamma,|d\mu_v|)}+\|\gamma\rho^{\boldsymbol{\varepsilon}}-\gamma\rho\|_{L^2(]0,T[\times\Gamma,|d\mu_v|)}.$$

The third term goes to zero when $\boldsymbol{\varepsilon} \to 0$ by 5.1 and the second term is bounded by $C_v\mathcal{L}(\rho^{\boldsymbol{\varepsilon}})(\delta t + h_{\mathcal{T}})$. It thus remain to study the first term. To this end, we write

$$\begin{aligned}\int_0^T\int_\Gamma |\gamma\rho_{\mathcal{T},\delta t}-\gamma\rho^{\boldsymbol{\varepsilon}}_{\mathcal{T},\delta t}|^2|v\cdot\boldsymbol{\nu}|\,dx\,dt &= \sum_{n=0}^{N-1}\sum_{\sigma\in\mathcal{E}_{\rm bd}}\int_{t^n}^{t^{n+1}}\int_\sigma |v\cdot\boldsymbol{\nu}||E^{\boldsymbol{\varepsilon},n+1}_{\mathcal{K}}|^2\,dx\,dt\\ &\le 4\rho^2_{\max}\sum_{n=0}^{N-1}\sum_{\sigma\in\mathcal{E}_{\rm bd}}\int_{t^n}^{t^{n+1}}\int_\sigma |v\cdot\boldsymbol{\nu}-v^n_\sigma|\,dx\,dt + \sum_{n=0}^{N-1}\delta t\sum_{\sigma\in\mathcal{E}_{\rm bd}}|\sigma||v^n_\sigma||E^{\boldsymbol{\varepsilon},n+1}_{\mathcal{K}}|^2.\end{aligned}$$

The first term in the right-hand side tends to 0 when δt and $h_{\mathcal{T}}$ tend to 0 by Lemma 2.1. The second term in the right-hand side is one of the terms in $I^{\boldsymbol{\varepsilon}}_{\mathcal{T},\delta t}$ (the left-hand side of (5.11)) when $\tilde{N}$ is replaced by N. The above proof shows that this term can be bounded like in (5.12).

Therefore, we can conclude by the same argument than above: choosing first $\boldsymbol{\varepsilon} > 0$ small enough, then choosing a smooth w close enough to v and finally δt and $h_{\mathcal{T}}$ small enough. □

6. Additional properties of the approximate solutions.

6.1. Lower bound estimate. We prove in this section a lower bound for the approximate solution, provided that the initial and boundary data are bounded from below by a positive number. The result improves the monotony of the scheme that we have already proven. This kind of result is important in the applications, for instance in fluid mechanics, where the fact that the fluid density remains far from zero uniformly in the discretization parameters may be crucial. Note that the assumption on $c + \operatorname{div} v$ we need is stronger than (1.4a).

THEOREM 6.1 (Lower bound estimate). *Assume that* (1.2) *and* (1.3a) *hold and that* $(c + \operatorname{div} v) \in L^1(]0,T[, L^\infty(\Omega))$. *If there exists some* $\rho_{\min} > 0$ *such that*

$$\begin{cases}\rho^0(x) \ge \rho_{\min}, & \text{for a.e. } x \in \Omega,\\ \rho^{in}(t,x) \ge \rho_{\min}, & \text{for } d\mu_v^- \text{-a.e. } (t,x) \in]0,T[\times\Gamma,\end{cases}$$

then for any mesh $\mathcal{T}$ *and any time step* $\delta t \le \delta t_{\max}$, *the unique solution to the finite volume scheme* (2.4) *satisfies*

$$\rho^n_{\mathcal{K}} \ge \exp\left(-\int_0^{t^n}\|(c+\operatorname{div} v)^+\|_{L^\infty}\,dt\right)\rho_{\min}, \quad \forall \mathcal{K}\in\mathcal{T}, \forall n \in [\![0,N]\!].$$

Proof. Since the assumption on $c + \operatorname{div} v$ implies (1.4a), existence and uniqueness of the approximate solution is given by Theorem 3.1. We now perform the same kind of change of variable as in the proof of Theorem 3.1, by letting $\tilde\rho^n_{\mathcal{K}} = \alpha^n\rho^n_{\mathcal{K}}$ and $\tilde\rho^n_\sigma = \alpha^n\rho^n_\sigma$, with α^n defined by (3.3), where γ^n is now defined by

$$\gamma^n = -\frac{1}{\delta t}\int_{t^n}^{t^{n+1}}\|(c+\operatorname{div} v)^+\|_{L^\infty}\,dt.$$

We thus obtain the same equation (3.6) with this new choice of γ^n. We will now show by induction that $\tilde\rho^n_{\mathcal{K}}$ is greater than $\rho_{\min}$ for every $\mathcal{K}\in\mathcal{T}$ and $n \in [\![0,N]\!]$. This is true for $n = 0$, by definition of $\rho_{\min}$, so that we assume now that n is such that $\inf_{\mathcal{K}\in\mathcal{T}}\tilde\rho^n_{\mathcal{K}} \ge \rho_{\min}$, and we are going to show that $\inf_{\mathcal{K}\in\mathcal{T}}\tilde\rho^{n+1}_{\mathcal{K}} \ge \rho_{\min}$.

Let us consider $\mathcal{K}\in\mathcal{T}$, such that $\tilde\rho^{n+1}_{\mathcal{K}} = \inf_{\mathcal{L}\in\mathcal{T}}\tilde\rho^{n+1}_{\mathcal{L}}$. Two cases have to be considered:

- First case : there is some $\sigma \in \mathcal{E}_{\mathcal{K}}\cup\mathcal{E}_{\rm bd}$ such that $\tilde\rho^{n+1}_{\mathcal{K}} \ge \tilde\rho^{in,n+1}_\sigma$. In that case, by (2.3) and the definition of $\rho_{\min}$ we immediately deduce that $\tilde\rho^{n+1}_{\mathcal{K}} \ge \rho_{\min}$.
- Second case : for any $\sigma \in \mathcal{E}_{\mathcal{K}}\cup\mathcal{E}_{\rm bd}$, we have $\tilde\rho^{n+1}_{\mathcal{K}} < \tilde\rho^{in,n+1}_\sigma$. In that case, (3.6) leads to

$$\tilde\rho^{n+1}_{\mathcal{K}}\underbrace{\left(1+\frac{\delta t\alpha_n}{\alpha_{n+1}}\left(c^n_{\mathcal{K}}+(\operatorname{div} v)^n_{\mathcal{K}}+\gamma^n\right)\right)}_{=\beta^n_{\mathcal{K}}} \ge \tilde\rho^n_{\mathcal{K}}.$$

By definition of α_n, we have

$$\beta^n_{\mathcal{K}} = \frac{1+\delta t(c^n_{\mathcal{K}}+(\operatorname{div} v)^n_{\mathcal{K}})}{1-\delta t\gamma^n}.$$

First, since $\delta t \le \delta t_{\max}$ and by (3.1), we have $\delta t(c^n_{\mathcal{K}}+(\operatorname{div} v)^n_{\mathcal{K}}) \ge -1/2$ so that $\beta^n_{\mathcal{K}} \ge 0$ (recall that $\gamma^n \le 0$). Second, by definition of γ^n, we see that $\beta^n_{\mathcal{K}} \le 1$. Finally, we get $\rho^{n+1}_{\mathcal{K}} \ge \frac{1}{\beta^n_{\mathcal{K}}}\tilde\rho^n_{\mathcal{K}} \ge \tilde\rho^n_{\mathcal{K}} \ge \rho_{\min}$, and the proof is complete.

□

6.2. The case of approximate data. We deal here with the case where the data c and v are not exactly known. This situation occurs, for instance, if the transport equation is coupled with other equations involving c and v: we can imagine for instance that v is given through a Darcy-like problem or some momentum conservation equation in the case of the Navier-Stokes system.

We assume that, for each mesh $\mathcal{T}$ and each time step δt, we are given approximate values

$$\left\{c_\kappa^n,\ \kappa\in\mathcal{T},\ n\in[\![0,N-1]\!]\right\},\quad\text{and}\quad\left\{v_{\kappa\sigma}^n,\ \kappa\in\mathcal{T},\sigma\in\mathcal{E}_\kappa,\ n\in[\![0,N-1]\!]\right\}.$$

We define the discrete divergence to be

$$(\operatorname{div} v)_\kappa^n=\frac{1}{|\kappa|}\sum_{\sigma\in\mathcal{E}_\kappa}|\sigma|v_{\kappa\sigma}^n,\quad\forall\kappa\in\mathcal{T},\forall n\in[\![0,N-1]\!].$$

By adapting the proofs given above, we can show that all the results in this paper remain valid provided that we have the following properties:

- Local conservativity :

$$v_{\kappa\sigma}^n=-v_{\mathcal{L}\sigma}^n,\quad\forall\sigma=\kappa|\mathcal{L}\in\mathcal{E}_{\rm int},\quad\forall n\in[\![0,N]\!].\tag{6.1}$$

- Bounds on $(c+\operatorname{div} v)^-$: We assume that there exists $\delta t_{\max}>0$, $0<\gamma<1$ and $M>0$ such that for any mesh and any $\delta t\le\delta t_{\max}$, we have

$$\begin{cases}\delta t\left(\sup\limits_{\kappa\in\mathcal{T}}(c_\kappa^n+(\operatorname{div} v)_\kappa^n)^-\right)\le\gamma,\quad\forall n\in[\![0,N-1]\!],\\ \sum\limits_{n=0}^{N-1}\delta t\left(\sup\limits_{\kappa\in\mathcal{T}}(c_\kappa^n+(\operatorname{div} v)_\kappa^n)^-\right)\le M.\end{cases}$$

  The L^∞ bound we shall obtain will then depend on γ and M.
- L^1 convergence of the reaction coefficient:

$$\sum_{n=0}^N\sum_{\kappa\in\mathcal{T}}c_\kappa^n\mathbb{1}_{]t^n,t^{n+1}[\times\kappa}\xrightarrow[(\delta t,h_\mathcal{T})\to0]{}c,\quad\text{in }L^1(]0,T[\times\Omega).$$

- L^1 convergence of the divergence:

$$\sum_{n=0}^N\sum_{\kappa\in\mathcal{T}}(\operatorname{div} v)_\kappa^n\mathbb{1}_{]t^n,t^{n+1}[\times\kappa}\xrightarrow[(\delta t,h_\mathcal{T})\to0]{}\operatorname{div} v,\quad\text{in }L^1(]0,T[\times\Omega).\tag{6.2}$$

- L^1 convergence of the vector field:

$$\sum_{n=0}^N\sum_{\kappa\in\mathcal{T}}\sum_{\sigma\in\mathcal{E}_\kappa}d_\kappa\left|\delta t|\sigma|v_{\kappa\sigma}^n-\int_{t^n}^{t^{n+1}}\int_\sigma(v\cdot\boldsymbol{\nu}_{\kappa\sigma})\,dx\,dt\right|\xrightarrow[(\delta t,h_\mathcal{T})\to0]{}0.\tag{6.3}$$

- L^1 convergence of the normal trace of the vector field on the boundary:

$$\sum_{n=0}^N\sum_{\sigma\in\mathcal{E}_{\rm bd}}v_\sigma^n\mathbb{1}_{]t^n,t^{n+1}[\times\sigma}\xrightarrow[(\delta t,h_\mathcal{T})\to0]{}(v\cdot\boldsymbol{\nu}),\quad\text{in }L^1(]0,T[\times\Gamma).$$

Assumption (6.3) is not so straightforward to interpret. It really has to be understood as a kind of convergence in $(L^1(]0,T[\times\Omega))^d$ and not in $L^1(]0,T[,(W^{1,1}(\Omega))^d)$, even though it is needed for v to belong to this last space in order for the traces of v on edges to be well defined. We will give an alternative formulation of this assumption in a particular case (yet not too far from generality).

Let us assume that each control volume κ is convex and that we can choose a point $x_\kappa\in\kappa$ such that $\rho_\kappa=\sup\{r>0,B(x_\kappa,r)\subset\kappa\}$ satisfies

$$\sup_{\kappa\in\mathcal{T}}\frac{d_\kappa}{\rho_\kappa}\le\operatorname{reg}_{\max}.$$

This assumption is a usual regularity assumption on the meshes we consider. We then define the so-called half-diamond cell $D_{\mathcal{K}\sigma}$, to be the pyramid (triangle if $d=2$) based on σ and with $x_{\mathcal{K}}$ as a principal vertex. The above assumptions lead to the following properties for any $\kappa \in \mathcal{T}$

$$\kappa = \bigcup_{\sigma\in\mathcal{E}} D_{\mathcal{K}\sigma}, \quad \text{and} \quad |\sigma| d_{\mathcal{K}} \leq C_{\mathrm{reg}_{\max}} |D_{\mathcal{K}\sigma}|, \quad \forall \sigma \in \mathcal{E}_{\mathcal{K}}.$$

We also introduce the diamond cells $D_\sigma = D_{\mathcal{K}\sigma}$, for $\sigma \in \mathcal{E}_{\mathrm{bd}} \cap \mathcal{E}_{\mathcal{K}}$ and $D_\sigma = D_{\mathcal{K}\sigma} \cup D_{\mathcal{L}\sigma}$, for $\sigma = \kappa|\mathcal{L} \in \mathcal{E}_{\mathrm{int}}$.

With these additional notation and assumption on the meshes, we can treat the following two examples

- First of all, by using for instance Lemma 6.2 in [DE06], we get

$$\sum_{n=0}^{N} \sum_{\kappa\in\mathcal{T}} \sum_{\sigma\in\mathcal{E}_{\mathcal{K}}} d_{\mathcal{K}}|\sigma| \left| \frac{1}{|D_\sigma|} \int_{t^n}^{t^{n+1}} \int_{D_\sigma} (v\cdot \boldsymbol{\nu}_{\mathcal{K}\sigma})\, dx\, dt - \frac{1}{|\sigma|} \int_{t^n}^{t^{n+1}} \int_{\sigma} (v\cdot \boldsymbol{\nu}_{\mathcal{K}\sigma})\, dx\, dt \right|$$
$$\leq C h_{\mathcal{T}} \sum_{n=0}^{N} \sum_{\kappa\in\mathcal{T}} \sum_{\sigma\in\mathcal{E}_{\mathcal{K}}} \frac{|\sigma| d_{\mathcal{K}}}{|D_\sigma|} \int_{t^n}^{t^{n+1}} \int_{D_\sigma} |\nabla v|\, dx\, dt \leq C_{\mathrm{reg}_{\max}} h_{\mathcal{T}} \int_0^T \int_\Omega |\nabla v|\, dx\, dt. \quad (6.4)$$

  This proves that the choice $v^n_{\mathcal{K}\sigma} = \dfrac{1}{\delta t |D_\sigma|} \displaystyle\int_{t^n}^{t^{n+1}} \int_{D_\sigma} (v\cdot \boldsymbol{\nu}_{\mathcal{K}\sigma})\, dx\, dt$, satisfies the property (6.3). Furthermore, we also have

$$\sum_{n=0}^{N} \sum_{\kappa\in\mathcal{T}} \sum_{\sigma\in\mathcal{E}_{\mathcal{K}}} |\sigma| \left| \frac{1}{|D_\sigma|} \int_{t^n}^{t^{n+1}} \int_{D_\sigma} (v\cdot \boldsymbol{\nu}_{\mathcal{K}\sigma})\, dx\, dt - \frac{1}{|\sigma|} \int_{t^n}^{t^{n+1}} \int_{\sigma} (v\cdot \boldsymbol{\nu}_{\mathcal{K}\sigma})\, dx\, dt \right|$$
$$\leq C h_{\mathcal{T}} \sum_{n=0}^{N} \sum_{\kappa\in\mathcal{T}} \sum_{\sigma\in\mathcal{E}_{\mathcal{K}}} \frac{|\sigma| d_{\mathcal{K}}}{|D_\sigma|} \int_{t^n}^{t^{n+1}} \int_{D_\sigma} |\nabla v|\, dx\, dt \leq C_{\mathrm{reg}_{\max}} \int_0^T \int_\Omega |\nabla v|\, dx\, dt,$$

  and thus, by a density argument, we see that (6.2) also holds in that case.
- Suppose now that $v^n_{\mathcal{K}\sigma}$ has the following form $v^n_{\mathcal{K}\sigma} = V^n_\sigma \cdot \boldsymbol{\nu}_{\mathcal{K}\sigma}$, for some vector $V^n_\sigma \in \mathbb{R}^d$ (notice that (6.1) is then satisfied). Then, (6.4) shows that (6.3) is satisfied as soon as the following convergence holds

$$\sum_{n=0}^{N} \sum_{\sigma\in\mathcal{E}} V^n_\sigma \mathbb{1}_{]t^n,t^{n+1}[\times D_\sigma} \xrightarrow[(\delta t, h_{\mathcal{T}})\to 0]{} v, \quad \text{in} \quad (L^1(]0,T[\times\Omega))^d.$$

  The fact that (6.2) is satisfied is an additional condition which depend on the particular choice of V^n_σ.

The two examples above, show that for a given $v \in L^1(]0,T[, (W^{1,1}(\Omega))^d)$, the property (6.3) can be interpreted as a strong L^1 convergence property towards v for some sequence of piecewise constant functions.

REMARK 6.1. *The fact that we only need L^1 convergence of the vector field to obtain strong convergence of the associated weak solution of the transport problem is a well-known feature in the study of the stability of such solutions with respect to variations of the data (see for instance [DL07, BF11]).*

6.3. Remark on the renormalization property. The renormalization property (1.7) was proven for smooth functions β. We want here to show that it still holds for piecewise smooth functions.

LEMMA 6.2. *Assume that* (1.2), (1.3) *and* (1.4) *hold and let ρ be the unique bounded weak solution to* (1.1) *for given data ρ^0 and ρ^{in}.*

1. For any $\alpha \neq 0$, we have the following property

$$c + \operatorname{div} v = 0, \quad \textit{for almost every } (t,x) \textit{ in the level set } \{\rho = \alpha\}.$$

2. For any function $\beta : \mathbb{R} \mapsto \mathbb{R}$ which is continuous and piecewise $\mathcal{C}^1$, we have the renormalization property (1.7).

Proof. Notice that the first property of the Lemma can not hold for $\alpha = 0$ (for instance the solution $\rho \equiv 0$ is an obvious counter-example).

1. For any $0 < \varepsilon \leq 1$, we define

$$\beta_\varepsilon(s) = \frac{\sqrt{\varepsilon}(s-\alpha)}{\sqrt{(s-\alpha)^2+\varepsilon}},$$

which is a smooth function satisfying $\|\beta_\varepsilon\|_{L^\infty} \leq 1$ and $\|\beta'_\varepsilon\|_{L^\infty} \leq 1$. We observe that

$$\beta_\varepsilon(s) \xrightarrow[\varepsilon\to 0]{} 0, \quad \forall s \in \mathbb{R},$$

$$s\beta'_\varepsilon(s) \xrightarrow[\varepsilon\to 0]{} \begin{cases} \alpha, & \text{for } s=\alpha, \\ 0, & \text{for } s\neq\alpha. \end{cases}$$

We then apply (1.7) and find that, for any $\phi \in \mathcal{C}_c^\infty(]0,T[\times\Omega)$ we have

$$\int_0^T \int_\Omega \Big(\beta_\varepsilon(\rho)\partial_t\phi + \beta_\varepsilon(\rho) v\cdot\nabla\phi + \beta_\varepsilon(\rho)(\operatorname{div} v)\phi - \rho\beta'_\varepsilon(\rho)(c+\operatorname{div} v)\phi \Big)\, dx\, dt = 0.$$

We now pass to the limit when $\varepsilon \to 0$ by the Lebesgue convergence theorem to obtain

$$\alpha \int_0^T \int_\Omega (c+\operatorname{div} v)\phi \mathbb{1}_{\{\rho=\alpha\}}\, dx\, dt = 0.$$

Since $\alpha \neq 0$, and $c+\operatorname{div} v \in L^1(]0,T[\times\Omega)$ and by weak-$\star$ density of $\mathcal{C}_c^\infty(]0,T[\times\Omega)$ into $L^\infty(]0,T[\times\Omega)$, we deduce that

$$\int_0^T \int_\Omega (c+\operatorname{div} v)\phi \mathbb{1}_{\{\rho=\alpha\}}\, dx\, dt = 0,$$

for any $\phi \in L^\infty(]0,T[\times\Omega)$. Which gives the expected result.

2. Let $\beta : \mathbb{R} \mapsto \mathbb{R}$ be a continuous piecewise $\mathcal{C}^1$ function. Notice that (1.7a) has a (weak) sense even though β' is not well-defined at some points since for such points $\alpha \in \mathbb{R}$ (even for $\alpha = 0$) we have, by the first point of the Lemma,

$$\int_0^T \int_\Omega (c+\operatorname{div} v)\rho \mathbb{1}_{\{\rho=\alpha\}}\phi\, dx\, dt = 0, \quad \forall \phi \in L^\infty(]0,T[\times\Omega).$$

Such a piecewise smooth function β can be written as the sum of a smooth function and of a finite linear combination of functions of the form $s \mapsto |s-\alpha_i|$. It is then enough to prove the claim for $\beta(s) = |s-\alpha|$ for some $\alpha \in \mathbb{R}$. To this end, we define $\beta_\varepsilon(s) = \frac{(s-\alpha)^2}{\sqrt{(s-\alpha)^2+\varepsilon}}$ which is smooth, we apply (1.7) with β_ε and then we pass to the limit when $\varepsilon \to 0$, using the fact that β_ε simply converges to $|s-\alpha|$ and that β'_ε simply converges towards $\operatorname{sgn}(s-\alpha)$.

□

6.4. Discrete renormalization property. As a consequence of the previous remarks, we may now study what happens when one takes a non-linear function of the approximate solution $\rho_{\mathcal{T},\delta t}$. It is well known that such mathematical operations are very important to obtain useful estimates in the theoretical study of many systems (incompressible and compressible Navier-Stokes equations, Saint-Venant equations, and so on). The aim of the main theorem of this section is to show that these algebraic operation can also be performed at the discrete level, the price to pay being a remainder term which strongly converges in L^1 towards 0 when δt and $h_{\mathcal{T}}$ tend to 0.

Let us begin by a straightforward consequence of our strong convergence theorem 5.2 and the Lebesgue dominated convergence theorem.

LEMMA 6.3. *Under the same assumptions as for Theorem 5.2, for any continuous map $\beta : \mathbb{R} \mapsto \mathbb{R}$, we have*

$$\beta(\rho_{\mathcal{T},\delta t}) \xrightarrow[(\delta t, h_{\mathcal{T}})\to 0]{} \beta(\rho), \text{ in } L^p(]0,T[\times\Omega) \text{ for any } p<+\infty \text{ and in } L^\infty(]0,T[\times\Omega) \text{ weak-}\star,$$

$$\beta(\gamma\rho_{\mathcal{T},\delta t}) \xrightarrow[(\delta t, h_{\mathcal{T}})\to 0]{} \beta(\gamma\rho), \text{ in } L^p(]0,T[\times\Gamma, |d\mu_v|) \text{ for any } p<+\infty.$$

The main result of this section is the following.

THEOREM 6.4. *For any $\beta : \mathbb{R} \mapsto \mathbb{R}$ which is continuous and piecewise $\mathcal{C}^1$, the approximate solution $(\rho_\kappa^n)_{\substack{n\in[\![0,N]\!] \\ \kappa\in\mathcal{T}}}$ satisfy the following set of equations*

$$|\kappa|\frac{\beta(\rho_\kappa^{n+1})-\beta(\rho_\kappa^n)}{\delta t} + \sum_{\sigma\in\mathcal{E}_\kappa\cap\mathcal{E}_{\rm int}} |\sigma|\big(v_{\kappa\sigma}^{n+}\beta(\rho_\kappa^{n+1}) - v_{\kappa\sigma}^{n-}\beta(\rho_{\mathcal{L}}^{n+1})\big) + \sum_{\sigma\in\mathcal{E}_\kappa\cap\mathcal{E}_{\rm bd}} |\sigma| v_{\kappa\sigma}^n \beta(\rho_\sigma^{n+1})$$
$$+ |\kappa| c_\kappa^n \beta'(\rho_\kappa^{n+1})\rho_\kappa^{n+1} + |\kappa|(\operatorname{div} v)_\kappa^n\big(\beta'(\rho_\kappa^{n+1})\rho_\kappa^{n+1} - \beta(\rho_\kappa^{n+1})\big) = |\kappa| R_\kappa^{n+1}, \quad \forall n\in[\![0,N-1]\!], \forall\kappa\in\mathcal{T}, \quad (6.5)$$

where the remainder term $R_{\mathcal{T},\delta t} = (R_\kappa^{n+1})_{\substack{n\in[\![0,N-1]\!] \\ \kappa\in\mathcal{T}}}$ satisfy

$$\|R_{\mathcal{T},\delta t}\|_{L^1(]0,T[\times\Omega)} \xrightarrow[(\delta t, h_{\mathcal{T}})\to 0]{} 0.$$

Furthermore, when β is convex we have

$$R_\kappa^{n+1} \leq 0, \quad \forall \kappa \in \mathcal{T}, \forall n \in [\![0, N-1]\!].$$

REMARK 6.2. *Since β is only assumed to be piecewise $\mathcal{C}^1$, we need to make precise the value of β' at the singular points of β. At the present point, we will assume that at each singular point $s \in \mathbb{R}$, the prescribed value $\beta'(s)$ strictly lies between the left and right limits $\beta'(s^\pm)$.*

We will see at the end, that we may choose in fact any value for β' at singular points, see Corollary 6.6.

Proof. First of all, we observe that only the restriction of β on the interval $[-\rho_{\max}, \rho_{\max}]$ plays a role in formula (6.5). Hence, we may assume that β has a finite number of singular points. In that case, such a function β can be written as follows

$$\beta(s) = \beta_0(s) + \sum_{i=1}^{m} \alpha_i |s - k_i|,$$

where $k_i \in \mathbb{R}, \alpha_i \in \mathbb{R}^*$ and $\beta_0 \in \mathcal{C}^1$.

We write $\beta_0 = \beta_0^+ - \beta_0^-$ where $\beta_0^\pm$ are convex $\mathcal{C}^1$ functions on $[-\rho_{\max}, \rho_{\max}]$. Hence, we can write $\beta = \beta^+ - \beta^-$ where

$$\beta^+(s) = \beta_0^+(s) + \sum_{\substack{1 \leq i \leq m \\ \alpha_i > 0}} \alpha_i |s - k_i|, \quad \text{and} \quad \beta^-(s) = \beta_0^-(s) + \sum_{\substack{1 \leq i \leq m \\ \alpha_i < 0}} |\alpha_i| |s - k_i|.$$

The approximate solution $\rho_{\mathcal{T},\delta t}$ being fixed, the definition of $R_{\mathcal{T},\delta t}$ is linear with respect to β, and we will then denote when necessary $R_{\mathcal{T},\delta t}(\beta)$. It is then enough to prove the result for $\beta = \beta_0^\pm$ and for $\beta(.) = |. - k|$ for any $k \in \mathbb{R}$ (in that case, we conventionally set $\gamma = \beta'(k) \in]-1, 1[$, see Remark 6.2).

- We assume first that β is $\mathcal{C}^1$ and convex. We find that

$$\begin{aligned} |\kappa| R_\kappa^{n+1}(\beta) = \frac{1}{\delta t} |\kappa| \left(\beta(\rho_\kappa^{n+1}) - \beta(\rho_\kappa^n) - (\rho_\kappa^{n+1} - \rho_\kappa^n) \beta'(\rho_\kappa^{n+1}) \right) \\ - \sum_{\sigma \in \mathcal{E}_\kappa \cap \mathcal{E}_{\rm int}} |\sigma| v_{\kappa\sigma}^{n-} \left[(\beta(\rho_\mathcal{L}^{n+1}) - \beta(\rho_\kappa^{n+1})) - \beta'(\rho_\kappa^{n+1})(\rho_\mathcal{L}^{n+1} - \rho_\kappa^{n+1}) \right] \\ - \sum_{\sigma \in \mathcal{E}_\kappa \cap \mathcal{E}_{\rm bd}} |\sigma| v_{\kappa\sigma}^{n-} \left[(\beta(\rho_\sigma^{n+1}) - \beta(\rho_\kappa^{n+1})) - \beta'(\rho_\kappa^{n+1})(\rho_\sigma^{n+1} - \rho_\kappa^{n+1}) \right]. \end{aligned}$$

  By convexity of β, we observe that $R_\kappa^{n+1}(\beta) \leq 0$ for any $n \in [\![0, N]\!], \kappa \in \mathcal{T}$.
- We now consider the case when $\beta(s) = |s - k|$ for some $k \in \mathbb{R}$, with $\beta'(k) = \gamma \in]-1, 1[$. This function can be approximated by the smooth convex function $\beta_{\varepsilon,\gamma}(s) = \varepsilon \log\cosh\left(\frac{s-k}{\varepsilon} + \operatorname{atanh}\gamma\right)$ in such a way that

$$\beta_{\varepsilon,\gamma}(s) \xrightarrow[\varepsilon \to 0]{} |s - k| = \beta(s),$$

$$\beta'_{\varepsilon,\gamma}(s) \xrightarrow[\varepsilon \to 0]{} \operatorname{sgn}(s - k) + \gamma \delta_{s,k}.$$

  Hence, for this particular function β, we still have $R_{\mathcal{T},\delta t}(\beta) = \lim_{\varepsilon \to 0} R_{\mathcal{T},\delta t}(\beta_{\varepsilon,\gamma}) \leq 0$.
- The previous arguments show that, if β is continuous, piecewise $\mathcal{C}^1$ and convex, then $R_{\mathcal{T},\delta t}(\beta) \leq 0$. Therefore, the L^1-norm of $R_{\mathcal{T},\delta t}(\beta)$ can be computed by summing all the equations (6.5) for $n \in [\![0, N-1]\!]$ and $\kappa \in \mathcal{T}$. We get

$$\begin{aligned} -\|R_{\mathcal{T},\delta t}(\beta)\|_{L^1} = \sum_{n=0}^{N-1} \delta t \sum_{\kappa \in \mathcal{T}} |\kappa| R_\kappa^{n+1} = \sum_{\kappa \in \mathcal{T}} |\kappa| \beta(\rho_\kappa^N) - \sum_{\kappa \in \mathcal{T}} |\kappa| \beta(\rho_\kappa^0) \\ + \sum_{n=0}^{N-1} \delta t \sum_{\sigma \in \mathcal{E}_{\rm bd}} |\sigma| v_\sigma^n \beta(\rho_\sigma^{n+1}) + \sum_{n=0}^{N-1} \delta t \sum_{\kappa \in \mathcal{T}} |\kappa| \left((c_\kappa^n + (\operatorname{div} v)_\kappa^n) \beta'(\rho_\kappa^{n+1}) \rho_\kappa^{n+1} - (\operatorname{div} v)_\kappa^n \beta(\rho_\kappa^{n+1}) \right). \quad (6.6) \end{aligned}$$

  Let us prove that all those terms converge towards their continuous counterparts.
  – The convergence of the first two terms is straightforward from (5.3) and Remark 3.2.

– Let us denote the boundary term by

$$S(v) = \sum_{n=0}^{N-1} \delta t \sum_{\sigma \in \mathcal{E}_{\rm bd}} |\sigma| v_\sigma^n \beta(\rho_\sigma^{n+1}).$$

Choosing a smooth vector field w close enough to v, we have

$$S(v) = S(v-w) + S(w).$$

Furthermore, since β is bounded on $[-\rho_{\max}, \rho_{\max}]$ we have

$$|S(v-w)| \le C_{\rho_{\max},\beta} \|v-w\|_{L^1(]0,T[\times\Gamma)} \le C_{\rho_{\max},\beta} \|v-w\|_{L^1(]0,T[,W^{1,1}(\Omega))},$$

and

$$\begin{aligned}
&\left| S(w) - \int_0^T \int_\Gamma (w\cdot\boldsymbol{\nu})\beta(\gamma\rho)\,dx\,dt \right| \\
&\qquad = \left| S(w) - \int_0^T \int_\Gamma (w\cdot\boldsymbol{\nu})\beta(\gamma\rho_{\mathcal{T},\delta t})\,dx\,dt \right| + C_{\rho_{\max},\beta} \int_0^T \int_\Gamma |w\cdot\boldsymbol{\nu}||\gamma\rho_{\mathcal{T},\delta t} - \gamma\rho|\,dx\,dt \\
&\le \left| S(w) - \int_0^T \int_\Gamma (w\cdot\boldsymbol{\nu})\beta(\gamma\rho_{\mathcal{T},\delta t})\,dx\,dt \right| + C_{\rho_{\max},\beta} \int_0^T \int_\Gamma |v\cdot\boldsymbol{\nu}||\gamma\rho_{\mathcal{T},\delta t} - \gamma\rho|\,dx\,dt + C_{\rho_{\max},\beta}\|v-w\|_{L^1(]0,T[\times\Gamma)} \\
&\quad \le C_{\rho_{\max}} \mathcal{L}(w)(\delta t + h_{\mathcal{T}}) + C_{\rho_{\max},\beta} \int_0^T \int_\Gamma |v\cdot\boldsymbol{\nu}||\gamma\rho_{\mathcal{T},\delta t} - \gamma\rho|\,dx\,dt + C_{\rho_{\max},\beta}\|v-w\|_{L^1(]0,T[\times\Gamma)}.
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
&\left| S(v) - \int_0^T \int_\Gamma (v\cdot\boldsymbol{\nu})\beta(\gamma\rho)\,dx\,dt \right| \\
&\qquad \le C_{\rho_{\max},\beta}\|v-w\|_{L^1(]0,T[,W^{1,1}(\Omega))} + C_{\rho_{\max},\beta}\mathcal{L}(w)(\delta t + h_{\mathcal{T}}) + \|\gamma\rho - \gamma\rho_{\mathcal{T},\delta t}\|_{L^1(]0,T[\times\Gamma,|d\mu_v|)},
\end{aligned}$$

and thus $S(v)$ converges towards $\displaystyle\int_0^T \int_\Gamma (v\cdot\boldsymbol{\nu})|\gamma\rho|^2\,dx\,dt$ when $(\delta t, h_{\mathcal{T}}) \to 0$.

– The interior term containing $\beta(\rho_\kappa^{n+1})$ can be written

$$\int_0^T \int_\Omega (\operatorname{div} v)\beta(\rho_{\mathcal{T},\delta t})\,dx\,dt,$$

and then we can perform the limit by Lemma 6.3.

– The terms containing β' need special care since β' is not continuous. Let us denote by A the finite set of discontinuity points of $s \mapsto s\beta'(s)$ in $[-\rho_{\max}, \rho_{\max}]$. Note that $0 \notin A$.
We first remark that the product $\rho_{\mathcal{T},\delta t}\beta'(\rho_{\mathcal{T},\delta t})$ is bounded in $L^\infty(]0,T[\times\Omega)$ and that $\rho_{\mathcal{T},\delta t}$ strongly converges towards ρ in L^1. Therefore, there is a subsequence for which $\rho_{\mathcal{T},\delta t}$ converges almost everywhere and such that $\rho_{\mathcal{T},\delta t}\beta'(\rho_{\mathcal{T},\delta t})$ has a weak-$\star$ limit denoted by G.
By definition of A, we have

$$\rho_{\mathcal{T},\delta t}\beta'(\rho_{\mathcal{T},\delta t})\mathbb{1}_{\{\rho\notin A\}} \xrightarrow[(\delta t, h_{\mathcal{T}})\to 0]{} \rho\beta'(\rho)\mathbb{1}_{\{\rho\notin A\}}, \quad \text{almost everywhere,}$$

and thus, by the Lebesgue convergence theorem, we can identify one part of the L^∞ weak-$\star$ limit, namely we have $G\mathbb{1}_{\{\rho\notin A\}} = \rho\beta'(\rho)\mathbb{1}_{\{\rho\notin A\}}$. We then deduce that

$$\begin{aligned}
&\sum_{n=0}^{N-1} \delta t \sum_{\kappa\in\mathcal{T}} |\kappa|(c_\kappa^n + (\operatorname{div} v)_\kappa^n)\beta'(\rho_\kappa^{n+1})\rho_\kappa^{n+1} = \int_0^T \int_\Omega (c + \operatorname{div} v)\rho_{\mathcal{T},\delta t}\beta'(\rho_{\mathcal{T},\delta t})\,dx\,dt \\
&\xrightarrow[(\delta t, h_{\mathcal{T}})\to 0]{} \int_0^T \int_\Omega (c+\operatorname{div} v)G\,dx dt = \int_0^T \int_\Omega (c+\operatorname{div} v)\rho\beta'(\rho)\mathbb{1}_{\{\rho\notin A\}}\,dx\,dt + \underbrace{\int_0^T \int_\Omega (c+\operatorname{div} v)G\mathbb{1}_{\{\rho\in A\}}\,dx\,dt}_{\substack{=0, \text{ by Lemma 6.2} \\ \text{since } A \text{ is finite and } 0\notin A}} \\
&\qquad = \int_0^T \int_\Omega (c+\operatorname{div} v)\rho\beta'(\rho)\,dx\,dt.
\end{aligned}$$

This computation being true for any convenient subsequence, we deduce that the convergence holds for the whole sequence.

We can now pass to the limit in (6.6) and find that

$$-\|R_{\mathcal{T},\delta t}\|_{L^1} \xrightarrow[(\delta t,h_{\mathcal{T}})\to 0]{} \int_\Omega \beta(\rho(T,.))\,dx - \int_\Omega \beta(\rho^0)\,dx + \int_0^T \int_\Sigma \beta(\gamma\rho)(v\cdot\boldsymbol{\nu})\,dx\,dt + \int_0^T \int_\Omega c\rho\beta'(\rho) + (\operatorname{div} v)(\rho\beta'(\rho)-\beta(\rho))\,dx\,dt.$$

By the renormalization property (1.7) (see also Lemma 6.2) we know that this limit is exactly 0 and then we deduce the strong convergence to 0 in L^1 of the remainder term $R_{\mathcal{T},\delta t}$.

□

Applying the previous result with $\beta(s)=s^2$ shows, in particular, that the numerical diffusion terms in the estimate (3.16) which leads to the weak $L^2(H^1)$ estimate (3.13) are not only bounded but in fact tend to zero when $h_{\mathcal{T}}$ and δt tend to 0.

Corollary 6.5. *Under the same assumptions as for Theorem 5.2, we have the following properties*

$$\sum_{n=0}^{N-1} \delta t \|\rho_{\mathcal{T}}^{n+1}-\rho_{\mathcal{T}}^n\|_{L^2}^2 \xrightarrow[(\delta t,h_{\mathcal{T}})\to 0]{} 0,$$

$$\sum_{n=0}^{N-1}\delta t \sum_{\substack{\sigma\in\mathcal{E}_{\rm bd}\\ v_\sigma^n<0}} |\sigma||v_\sigma^n|(\rho_\sigma^{in,n+1}-\rho_\kappa^{n+1})^2 + \sum_{n=0}^{N-1}\delta t \sum_{\substack{\sigma\in\mathcal{E}_{\rm int}\\ \sigma=\kappa|\mathcal{L}}} |\sigma||v_{\kappa\mathcal{L}}^n|(\rho_{\mathcal{L}}^{n+1}-\rho_\kappa^{n+1})^2 \xrightarrow[(\delta t,h_{\mathcal{T}})\to 0]{} 0.$$

We conclude this section by a corollary of the previous result which is the discrete counterpart of Lemma 6.2.

Corollary 6.6.

1. *For any $\alpha\in\mathbb{R}\setminus\{0\}$, we have*

$$\sum_{n=0}^{N-1}\delta t\sum_{\kappa\in\mathcal{T}} |\kappa||c_\kappa^n + (\operatorname{div} v)_\kappa^n|\mathbb{1}_{\{\rho_\kappa^{n+1}=\alpha\}} \xrightarrow[(\delta t,h_{\mathcal{T}})\to 0]{} 0.$$

2. *The result of Theorem 6.4 still holds for any choice of the value of β' at singular points.*

Proof.

1. We define $\beta_1(s)=|s-\alpha|$ with the choice $\beta_1'(\alpha)=1/2$ and $\beta_2(s)=|s-\alpha|$ with the choice $\beta_2'(\alpha)=-1/2$. We apply the result of Theorem 6.4 to β_1 and β_2, denoting by $R_{\mathcal{T},\delta t}(\beta_1)$ and $R_{\mathcal{T},\delta t}(\beta_2)$ the associated remainder terms. The functions β_1 and β_2 coincide everywhere, and β_1' and β_2' coincide everywhere except in α. We then subtract the two set of equations to obtain for any $n\in[\![0,N-1]\!]$ and $\kappa\in\mathcal{T}$

$$R_\kappa^{n+1}(\beta_1)-R_\kappa^{n+1}(\beta_2) = \begin{cases} \alpha(c_\kappa^n+(\operatorname{div} v)_\kappa^n) & \text{if } \rho_\kappa^{n+1}=\alpha \\ 0 & \text{if } \rho_\kappa^{n+1}\neq\alpha \end{cases}$$

The result follows from the fact that both $R_{\mathcal{T},\delta t}(\beta_1)$ and $R_{\mathcal{T},\delta t}(\beta_2)$ converge to 0 in L^1 when $(\delta t,h_{\mathcal{T}})\to 0$.

2. This is a straightforward consequence of the first point of the Corollary.

□

7. Conclusion. In this paper, we proposed an analysis of the (implicit) upwind finite volume scheme on general unstructured grids in any dimension for initial and boundary value problems of transport type. The framework considered is one of the weaker possible since in particular the regularity of the velocity field is only assumed to be $L^1(]0,T[,(W^{1,1}(\Omega))^d)$. In that case, Cauchy-Lipschitz theory does not apply and our analysis is then directly based on the renormalized solutions theory for the partial differential equation under study. The main result is the strong convergence in $L^\infty(]0,T[,L^p(\Omega))$ of the approximate solutions towards the unique weak solution of the problem.

We conclude by raising two open questions of interest related to this problem:

- Is it possible to prove an error estimate for the upwind finite volume method for such regularities of the data? Introducing regular Lagrangian flows (see [DL07]) instead of usual characteristic flows of the Cauchy-Lipschitz theory should be of some help.
- It is known (since [Amb04]), that most of the theoretical results on renormalized solutions of the transport equation remain valid in the case where the vector field lies in $L^1(]0,T[,(\mathrm{BV}(\Omega))^d)$ (and usual additional boundedness assumptions on its divergence). Does the present analysis, including boundary conditions, remains valid in this more general framework ?

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REFERENCES

[Amb04] L. Ambrosio. Transport equation and Cauchy problem for BV vector fields. *Invent. Math.*, 158(2):227–260, 2004.

[BF11] F. Boyer and P. Fabrie. *Elements of analysis for the study of some models of incompressible viscous flows.* in preparation. Springer-Verlag, 2011.

[BGP05] D. Bouche, J.-M. Ghidaglia, and F. Pascal. Error estimate and the geometric corrector for the upwind finite volume method applied to the linear advection equation. *SIAM J. Numer. Anal.*, 43(2):578–603 (electronic), 2005.

[Boy05] F. Boyer. Trace theorems and spatial continuity properties for the solutions of the transport equation. *Differential Integral Equations*, 18(8):891–934, 2005.

[CGH93] S. Champier, T. Gallouët, and R. Herbin. Convergence of an upstream finite volume scheme for a nonlinear hyperbolic equation on a triangular mesh. *Numer. Math.*, 66(2):139–157, 1993.

[CVV00] Y. Coudière, J.-P. Vila, and P. Villedieu. Convergence d'un schéma volumes finis explicite en temps pour les systèmes hyperboliques linéaires symétriques en domaines bornés. *C. R. Acad. Sci. Paris Sér. I Math.*, 331(1):95–100, 2000.

[DE06] J. Droniou and R. Eymard. A mixed finite volume scheme for anisotropic diffusion problems on any grid. *Numer. Math.*, 105(1):35–71, 2006.

[Des96] B. Desjardins. A few remarks on ordinary differential equations. *Comm. Partial Differential Equations*, 21(11-12):1667–1703, 1996.

[Des04a] B. Després. Convergence of non-linear finite volume schemes for linear transport. In *Notes from the XIth Jacques-Louis Lions Hispano-French School on Numerical Simulation in Physics and Engineering (Spanish)*, pages 219–239. Grupo Anal. Teor. Numer. Modelos Cienc. Exp. Univ. Cádiz, Cádiz, 2004.

[Des04b] B. Després. Lax theorem and finite volume schemes. *Math. Comp.*, 73(247):1203–1234 (electronic), 2004.

[DL89] R.J. DiPerna and P.-L. Lions. Ordinary differential equations, transport theory and Sobolev spaces. *Invent. Math.*, 98(3):511–547, 1989.

[DL07] C. De Lellis. Notes on hyperbolic systems of conservation laws and transport equations. In *Handbook of differential equations: evolutionary equations. Vol. III*, Handb. Differ. Equ., pages 277–382. Elsevier/North-Holland, Amsterdam, 2007.

[DL11] F. Delarue and F. Lagoutière. Probabilistic analysis of the upwind scheme for transport equations. *Archive for Rational Mechanics and Analysis*, 199:229–268, 2011. 10.1007/s00205-010-0322-x.

[EGH00] R. Eymard, T. Gallouët, and R. Herbin. Finite volume methods. In *Handbook of numerical analysis, Vol. VII*, Handb. Numer. Anal., VII, pages 713–1020. North-Holland, Amsterdam, 2000.

[Fet11] A. Fettah. *Analyse de modèles en mécanique des fluides compressibles.* PhD thesis, Université de Provence, 2011.

[Kuz76] N.N. Kuznecov. The accuracy of certain approximate methods for the computation of weak solutions of a first order quasilinear equation. *Ž. Vyčisl. Mat. i Mat. Fiz.*, 16(6):1489–1502, 1627, 1976.

[LeV02] R.J. LeVeque. *Finite volume methods for hyperbolic problems.* Cambridge Texts in Applied Mathematics. Cambridge University Press, Cambridge, 2002.

[Mer08] B. Merlet. L^∞- and L^2-error estimates for a finite volume approximation of linear advection. *SIAM J. Numer. Anal.*, 46(1):124–150, 2007/08.

[MV07] B. Merlet and J. Vovelle. Error estimate for finite volume scheme. *Numer. Math.*, 106(1):129–155, 2007.

[OV06] M. Ohlberger and J. Vovelle. Error estimate for the approximation of nonlinear conservation laws on bounded domains by the finite volume method. *Math. Comp.*, 75(253):113–150 (electronic), 2006.

[VV03] J.-P. Vila and P. Villedieu. Convergence of an explicit finite volume scheme for first order symmetric systems. *Numer. Math.*, 94(3):573–602, 2003.