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Three-dimensional General Solutions for Thermoporoelastic Media and Its Application

Xiang-Yu LI^{1*}, W. Q. Chen² and Hui-Ying Wang³

¹UPMC Sisyphé, Boîte 105, 4 Place Jussieu, 75252 Paris Cedex 05, France

²Department of Engineering Mechanics, Zhejiang University, Yuquan Campus, Hangzhou 310027, P. R. China

³E.N.S.M.A., University of Poitiers, B.P., B.P. 109-Chasseneuil du Poitou 86960 Futuroscope Cedex, France

Abstract

This paper presents a set of 3D general solutions for thermoporoelastic media for the steady-state problem. By introducing two displacement functions, the equations governing the elastic, pressure and temperature fields are simplified. The operator theory and superposition principle are then employed to express all the physical quantities in terms of two functions, one of which satisfies a quasi-Laplace equation and the other satisfies a differential equation of the eighth order. The generalized Almansi's theorem is used to derive the displacements, pressure and temperature in terms of five quasi-harmonic functions for various cases of material characteristic roots. To show its practical significance, an infinite medium containing a penny-shaped crack subjected to mechanical, pressure and temperature loads on the crack surface is given as an example. A potential theory method is employed to solve the problem. One integro-differential equation and two integral equations are derived, which bear the same structures to those reported in literature. For a penny shaped crack subjected to uniformly distributed loads, exact and complete solutions in terms of elementary functions are obtained, which can serve as a benchmark for various kinds of numerical codes and approximate solutions.

Keywords: Thermoporoelasticity, Transversely isotropic, General solutions, Potential theory method, Penny-shaped crack.

1. Introduction

The theory of poroelasticity was originally developed by Biot (1941) to consider the interaction between two phases composing a continuum. Now poroelasticity has many applications in various fields in science and engineering such as soil consolidation (Yue and Selvadurai, 1996), filtration (Barry et al., 1997) and

* Corresponding author.

E-mail address: zjuparis6@hotmail.com (Xiang-Yu, LI)

bio-science (Oomens et al., 1987; Cowin, 1999).

It is well known that geomaterials are typically anisotropic (Cheng, 1997). In fact, the simple natural deposition of sedimentary rocks exhibit transverse isotropy and modeling anisotropic materials as an equivalent isotropic one can yield unexpected and erroneous results which has been demonstrated by analytical and numerical solutions (Abousleiman et al., 1996; Cui et al., 1996). Hence, it is natural to extend the isotropic theory to an anisotropic one, which was initially achieved by Biot (1955), based on the formality of the generalized Hooke's law. To consider the effect of temperature, which arises in deep drilling and nuclear storage facilities, thermoporoelasticity model was founded (Coussy, 1995; Abousleiman and Ekbote, 2000). Accordingly, analytical solutions of fundamental problems for thermoporoelastic media with material anisotropy play an import role in obtaining a deep insight in the interaction between the phases involved.

Potential theory is the tool for studying problems governed by Laplace equations in several areas of physics (Kellogg, 1929). Fabrikant (1989, 1991) developed a so called potential theory method by creatively representing the inverse of the distance of two points in Euclidean space in term of an integral (Chen and Ding, 2004a). Based on the general solutions proposed by Elliott (1948) for pure elasticity, Fabrikant (1989, 1991) then performed 3D exact analyses of punch and crack problems by employing the method. Chen and his co-authors extended Fabrikant's potential theory method to solve mixed boundary value problems mainly stemming from three-dimensional crack or contact problems of multi-field coupled media, for example, piezoelectricity (Chen and Shioya, 2000; Chen, 2000), thermoelasticity (Chen et al., 2004b), magneto-electro-thermo-elasticity (Chen, 2004c). It is noted that all the works by Chen and his coauthors were also based on the corresponding general solutions (Chen and Ding, 2004a). To the authors' best knowledge, no parallel work in the domain of thermoporoelasticity has been reported yet in literature, mainly due to the lack of the general solution which should be expressed in terms of quasi-harmonic functions as that for purely elastic materials.

The purpose of the present paper is to construct a 3D thermoporoelastic general

solution expressed by quasi-harmonic functions only. Two displacement functions are introduced in order for simplifying the basic 3D equations for transversely isotropic thermoporoelastic media in a steady-state. Then, a general solution expressed by two functions is derived by employing the operator theory and the superposition principle. One function involved in the solution satisfies a quasi-Laplace equation and the other satisfies an eight-order partial differential equation. The general solution is further expressed in terms of five quasi-harmonic functions by virtue of the generalized Almansi's theorem. Four possible cases of material characteristic roots are considered. To show its practical significance, an infinite medium containing a plane crack subjected to symmetric mechanical, pore pressure and temperature loads is considered for instance. The problem can be transformed to the corresponding mixed boundary-value problem of a half space and thus the potential theory method developed by Fabrikant can be utilized. One integro-differential equation and two integral equations are derived, which bear the same structures as those reported in the literature. For a penny shaped crack subjected to uniformly distributed loads, exact and complete solutions in terms of elementary functions are obtained. The present solution is compared with the results from the literature and good agreement is achieved. The solution presented in this paper can serve as a benchmark for various kinds of numerical codes and approximate analyses.

2. Governing Equations

In Cartesian coordinates (x, y, z) , the constitutive equations for transversely isotropic thermoporoelastic media with the z -axis coincident with the axis of rotational material symmetry are listed as follows (Coussy, 1995; Abousleiman and Ekbote, 2000)

$$\sigma_x = c_{11} \frac{\partial u}{\partial x} + (c_{11} - 2c_{66}) \frac{\partial v}{\partial y} + c_{13} \frac{\partial w}{\partial z} - \alpha_1 P - \beta_1 T, \quad \sigma_{yz} = c_{44} \left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z} \right),$$

$$\sigma_y = (c_{11} - 2c_{66}) \frac{\partial u}{\partial x} + c_{11} \frac{\partial v}{\partial y} + c_{13} \frac{\partial w}{\partial z} - \alpha_1 P - \beta_1 T, \quad \sigma_{zx} = c_{44} \left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} \right), \quad (1)$$

$$\sigma_z = c_{13} \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) + c_{33} \frac{\partial w}{\partial z} - \alpha_3 P - \beta_3 T, \quad \sigma_{xy} = c_{66} \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right),$$

$$P = M \left[\xi - \alpha_1 \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) - \alpha_3 \frac{\partial w}{\partial z} + \beta_m T \right], \quad (2)$$

in which $u(v, w)$ and σ_{ij} are components of displacement and stress, respectively; P and T are changes in the pore pressure and temperature, respectively; $P = 0$ and $T = 0$ correspond to the stress-free state; ξ is variation of fluid content; c_{ij} , $\alpha_i (\alpha_3, M)$ and $\beta_i (\beta_3, \beta_m)$ are elastic moduli, Biot's effective stress coefficients and thermal constants, respectively. It is noted that c_{ij} , α_i and β_i can be expressed in terms of engineering constants such as Young's moduli, Poisson's ratio, etc., which are shown in Appendix A (Kanj and Abousleiman, 2005).

In the following analysis, uncoupled thermoporoelastic theory is adopted by assuming that the thermoporoelastic loading varies slowly in time and the rate of both fluid mass content and entropy vanishes. Consequently, if the medium is in a steady-state, the pressure and temperature fields are constant in time and governed respectively by the following two quasi-Laplace equations

$$\left(\kappa_{11} \Delta + \kappa_{33} \frac{\partial^2}{\partial z^2} \right) P = 0, \quad (3)$$

$$\left(\lambda_{11} \Delta + \lambda_{33} \frac{\partial^2}{\partial z^2} \right) T = 0, \quad (4)$$

where $\Delta = \partial^2 / \partial x^2 + \partial^2 / \partial y^2$ is the planar Laplacian; $\kappa_{11} (\kappa_{33})$ and $\lambda_{11} (\lambda_{33})$ are coefficients of permeability and thermal conductivity.

The equations governing the displacement field can be easily derived by inserting Eqn. (1) into the equilibrium equations in terms of stress components and are shown to be

$$\begin{aligned}
& \left(c_{11} \frac{\partial^2}{\partial x^2} + c_{66} \frac{\partial^2}{\partial y^2} + c_{44} \frac{\partial^2}{\partial z^2} \right) u + (c_{12} + c_{66}) \frac{\partial^2 v}{\partial x \partial y} + (c_{13} + c_{44}) \frac{\partial^2 w}{\partial x \partial z} - \alpha_1 \frac{\partial P}{\partial x} - \beta_1 \frac{\partial T}{\partial x} = 0, \\
& (c_{12} + c_{66}) \frac{\partial^2 u}{\partial x \partial y} + \left(c_{66} \frac{\partial^2}{\partial x^2} + c_{11} \frac{\partial^2}{\partial y^2} + c_{44} \frac{\partial^2}{\partial z^2} \right) v + (c_{13} + c_{44}) \frac{\partial^2 w}{\partial y \partial z} - \alpha_1 \frac{\partial P}{\partial y} - \beta_1 \frac{\partial T}{\partial y} = 0, \quad (5) \\
& (c_{13} + c_{44}) \frac{\partial}{\partial z} \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) + \left(c_{44} \Delta + c_{33} \frac{\partial^2}{\partial z^2} \right) w - \alpha_3 \frac{\partial P}{\partial z} - \beta_3 \frac{\partial T}{\partial z} = 0,
\end{aligned}$$

where $c_{12} = c_{11} - 2c_{66}$.

In the steady-state, Eqns. (1-4) indicate that the fluid content ξ is not an independent field variable and can be determined once the pressure, temperature and elastic fields have been obtained.

3. Static thermoporoelastic general solution

Introducing the following two displacement functions Ψ_0 and H as

$$u = \frac{\partial \Psi_0}{\partial y} - \frac{\partial H}{\partial x}, \quad v = -\frac{\partial \Psi_0}{\partial x} - \frac{\partial H}{\partial y}, \quad (6)$$

Eqns. (3)-(5) can be transformed into

$$\left(c_{66} \Delta + c_{44} \frac{\partial^2}{\partial z^2} \right) \Psi_0 = 0, \quad (7)$$

$$\mathbf{M} \begin{Bmatrix} H \\ w \\ P \\ T \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}, \quad (8)$$

where \mathbf{M} is the following operator matrix:

$$\mathbf{M} = \begin{bmatrix} c_{11} \Delta + c_{44} \frac{\partial^2}{\partial z^2} & -(c_{13} + c_{44}) \frac{\partial}{\partial z} & \alpha_1 & \beta_1 \\ -(c_{13} + c_{44}) \Delta \frac{\partial}{\partial z} & c_{44} \Delta + c_{33} \frac{\partial^2}{\partial z^2} & -\alpha_3 \frac{\partial}{\partial z} & -\beta_3 \frac{\partial}{\partial z} \\ 0 & 0 & \kappa_{11} \Delta + \kappa_{33} \frac{\partial^2}{\partial z^2} & 0 \\ 0 & 0 & 0 & \lambda_{11} \Delta + \lambda_{33} \frac{\partial^2}{\partial z^2} \end{bmatrix}. \quad (9)$$

The ‘‘determinant’’ of \mathbf{M} reads

$$|\mathbf{M}| = \left(a_0 \frac{\partial^4}{\partial z^4} + b_0 \frac{\partial^2}{\partial z^2} \Delta + c_0 \Delta^2 \right) \left(\kappa_{11} \Delta + \kappa_{33} \frac{\partial^2}{\partial z^2} \right) \left(\lambda_{11} \Delta + \lambda_{33} \frac{\partial^2}{\partial z^2} \right), \quad (10)$$

where

$$a_0 = c_{33}c_{44}, \quad b_0 = c_{11}c_{33} + c_{44}^2 - (c_{13} + c_{44})^2, \quad c_0 = c_{11}c_{44}. \quad (11)$$

The above derivation procedure is similar to that given by Chen et al. (2004b), where the equation of heat conduction and the other two equations governing the displacement functions are combined, instead of solving the temperature field independently and a priori as shown in the traditional general solutions with a particular part (Ashida et al., 1993; Ding et al., 1997). Here, the idea is employed to seek the 3D general solutions for thermoporoelastic media, although the pressure and temperature field can be solved independently.

According to the operator theory, the following general solutions can be obtained

$$H = C_{i1}F, \quad w = C_{i2}F, \quad P = C_{i3}F, \quad T = C_{i4}F, \quad (i=1, 2, 3, 4), \quad (12)$$

where C_{ij} are the algebraic cofactors of \mathbf{M} , and F satisfies

$$\left(\Delta + \frac{\partial^2}{\partial z_1^2} \right) \left(\Delta + \frac{\partial^2}{\partial z_2^2} \right) \left(\Delta + \frac{\partial^2}{\partial z_3^2} \right) \left(\Delta + \frac{\partial^2}{\partial z_4^2} \right) F = 0, \quad (13)$$

where $z_i = s_i z$, $s_3 = \sqrt{\kappa_{11}/\kappa_{33}}$, $s_4 = \sqrt{\lambda_{11}/\lambda_{33}}$, and $s_j (j=1,2)$ is the characteristic root with positive real part of the following algebraic equation (Ding et al., 1997)

$$a_0 s^4 - b_0 s^2 + c_0 = 0. \quad (14)$$

It is seen that if i was set to 1 or 2 in Eqn. (12), one can get two sets of general solutions with $P=0$ and $T=0$, which are actually the same to those for pure elasticity (Elliott, 1948; Ding et al., 1997); $i=3$ corresponds to the general solution \mathbf{X}_1 (say) with $T=0$, which is identical to that for poroelasticity; Taking $i=4$ corresponds to the general solution \mathbf{X}_2 (say) with $P=0$, which is equivalent to the one given by Chen et al. (2004b) for thermoelasticity.

Due to the linear nature of the thermoporoelastic theory adopted in this paper, superposing \mathbf{X}_1 and \mathbf{X}_2 leads to

$$\begin{aligned}
 u &= \frac{\partial \Psi_0}{\partial y} - \left(a_1 \frac{\partial^4}{\partial z^4} + b_1 \frac{\partial^2}{\partial z^2} \Delta + c_1 \Delta^2 \right) \frac{\partial F}{\partial x}, \\
 v &= -\frac{\partial \Psi_0}{\partial x} - \left(a_1 \frac{\partial^4}{\partial z^4} + b_1 \frac{\partial^2}{\partial z^2} \Delta + c_1 \Delta^2 \right) \frac{\partial F}{\partial y}, \\
 w &= \left(a_2 \frac{\partial^4}{\partial z^4} + b_2 \frac{\partial^2}{\partial z^2} \Delta + c_2 \Delta^2 \right) \frac{\partial F}{\partial z}, \\
 P &= \left(a_3 \frac{\partial^6}{\partial z^6} + b_3 \frac{\partial^4}{\partial z^4} \Delta + c_3 \frac{\partial^2}{\partial z^2} \Delta^2 + d_3 \Delta^3 \right) F, \\
 T &= \left(a_4 \frac{\partial^6}{\partial z^6} + b_4 \frac{\partial^4}{\partial z^4} \Delta + c_4 \frac{\partial^2}{\partial z^2} \Delta^2 + d_4 \Delta^3 \right) F,
 \end{aligned} \tag{15}$$

where

$$\begin{aligned}
 a_1 &= (c_{13} + c_{44})(\kappa_{33}\beta_3 + \lambda_{33}\alpha_3) - c_{33}(\kappa_{33}\beta_1 + \lambda_{33}\alpha_1), \\
 b_1 &= (c_{13} + c_{44})(\kappa_{11}\beta_3 + \lambda_{11}\alpha_3) - c_{44}(\kappa_{33}\beta_1 + \lambda_{33}\alpha_1) - c_{33}(\kappa_{11}\beta_1 + \lambda_{11}\alpha_1), \\
 c_1 &= -c_{44}(\kappa_{11}\beta_1 + \lambda_{11}\alpha_1), \quad a_2 = c_{44}(\kappa_{33}\beta_3 + \lambda_{33}\alpha_3), \\
 b_2 &= c_{11}(\kappa_{33}\beta_3 + \lambda_{33}\alpha_3) + c_{44}(\kappa_{11}\beta_3 + \lambda_{11}\alpha_3) - (c_{13} + c_{44})(\kappa_{33}\beta_1 + \lambda_{33}\alpha_1), \\
 c_2 &= c_{11}(\kappa_{11}\beta_3 + \lambda_{11}\alpha_3) - (c_{13} + c_{44})(\kappa_{11}\beta_1 + \lambda_{11}\alpha_1), \\
 a_3 &= a_0\lambda_{33}, \quad b_3 = a_0\lambda_{11} + b_0\lambda_{33}, \quad c_3 = b_0\lambda_{11} + c_0\lambda_{33}, \quad d_3 = c_0\lambda_{11}, \\
 a_4 &= a_0\kappa_{33}, \quad b_4 = a_0\kappa_{11} + b_0\kappa_{33}, \quad c_4 = b_0\kappa_{11} + c_0\kappa_{33}, \quad d_4 = c_0\kappa_{11}.
 \end{aligned} \tag{16}$$

Taking advantage of the generalized Almansi's theorem (Ding et al., 1996), F can be expressed by four quasi-harmonic functions

$$F = \begin{cases} F_1 + F_2 + F_3 + F_4, & \text{for distinct } s_i, & \text{(Case 1)} \\ F_1 + F_2 + F_3 + zF_4, & s_1 \neq s_2 \neq s_3 = s_4 \neq s_1, & \text{(Case 2)} \\ F_1 + F_2 + zF_3 + z^2F_4, & s_1 \neq s_2 = s_3 = s_4, & \text{(Case 3)} \\ F_1 + zF_2 + z^2F_3 + z^3F_4, & s_1 = s_2 = s_3 = s_4, & \text{(Case 4)} \end{cases} \tag{17}$$

where F_i respectively satisfy

$$\left(\Delta + \frac{\partial^2}{\partial z_i^2} \right) F_i = 0, \quad (i = 1, 2, 3, 4). \tag{18}$$

For Case 1, the general solution in Eqn. (15) takes on the simplest form as

$$\begin{aligned}
 u &= \frac{\partial \Psi_0}{\partial y} - \sum_{i=1}^4 \eta_{i1} \frac{\partial^5 F_i}{\partial x \partial z_i^4}, \quad v = -\frac{\partial \Psi_0}{\partial x} - \sum_{i=1}^4 \eta_{i1} \frac{\partial^5 F_i}{\partial y \partial z_i^4}, \\
 w &= \sum_{i=1}^4 \eta_{i2} s_i \frac{\partial^5 F_i}{\partial z_i^5}, \quad P = \eta_{33} \frac{\partial^6 F_3}{\partial z_3^6}, \quad T = \eta_{44} \frac{\partial^6 F_4}{\partial z_4^6},
 \end{aligned} \tag{19}$$

with

$$\begin{aligned}\eta_{ij} &= a_j s_i^4 - b_j s_i^2 + c_j \quad (j=1,2), \quad \eta_{33} = (a_0 s_3^4 - b_0 s_3^2 + c_0)(\lambda_{33} s_3^2 - \lambda_{11}), \\ \eta_{44} &= (a_0 s_3^4 - b_0 s_3^2 + c_0)(\kappa_{33} s_4^2 - \kappa_{11}).\end{aligned}\quad (20)$$

Letting

$$\Psi_i = \eta_{i1} \frac{\partial^4 F_i}{\partial z_i^4}, \quad (i=1,2,3,4) \quad (21)$$

Eqn. (19) can be rewritten as

$$\begin{aligned}u &= \frac{\partial \Psi_0}{\partial y} - \sum_{i=1}^4 \frac{\partial \Psi_i}{\partial x}, \quad v = -\frac{\partial \Psi_0}{\partial x} - \sum_{i=1}^4 \frac{\partial \Psi_i}{\partial y}, \quad w = \sum_{i=1}^4 \mu_{i1} \frac{\partial \Psi_i}{\partial z_i}, \\ P &= \sum_{i=1}^4 \mu_{i2} \frac{\partial^2 \Psi_i}{\partial z_i^2}, \quad T = \sum_{i=1}^4 \mu_{i3} \frac{\partial^2 \Psi_i}{\partial z_i^2},\end{aligned}\quad (22)$$

where $\mu_{i1} = \lambda_{i2} s_i / \lambda_{11}$, $\mu_{ij} = \lambda_{i(j+1)} (\delta_{j2} + \delta_{j3}) (\delta_{i3} + \delta_{i4}) / \lambda_{11}$, ($j=2,3$, $i=1-4$) and δ_{ij} is the Kronecker delta, and

$$\left(\Delta + \frac{\partial^2}{\partial z_i^2} \right) \Psi_i = 0, \quad (i=0, 1, 2, 3, 4), \quad (23)$$

with $z_0 = s_0 z$ and $s_0 = \sqrt{c_{66}/c_{44}}$.

In cylindrical coordinates system (ρ, ϕ, z) , the expressions of w , P and T keep unchanged as given in Eqn. (22), while the expressions of the radial and circumferential displacements turn out to be

$$u_\rho = \frac{\partial \Psi_0}{\rho \partial \phi} - \sum_{i=1}^4 \frac{\partial \Psi_i}{\partial \rho}, \quad u_\phi = -\frac{\partial \Psi_0}{\partial \rho} - \sum_{i=1}^4 \frac{\partial \Psi_i}{\rho \partial \phi}. \quad (24)$$

The general solutions for the other three cases are given in Appendix B for the theoretical completeness.

Once the general solutions are available, it is very convenient to conduct three-dimensional analyses of the mixed boundary-value problems associated with cracks and punches by virtue of the potential theory method proposed by Fabrikant (1989, 1991) as shown in the parallel works by Chen and his coauthors (2000, 2004b, 2004c). In the following section, a flat crack problem is considered as an illustrative example to show the practical significance of the general solution for Case 1.

4. Generalized potential theory method for thermoporoelastic crack problem

To generalize the potential theory method suggested by Fabrikant (1989) to the

case of thermoporoelasticity, the following complex quantities are introduced

$$U = u + iv, \quad \sigma_1 = \sigma_x + \sigma_y, \quad \sigma_2 = \sigma_x - \sigma_y + 2i\sigma_{xy}, \quad \tau_z = \sigma_{zx} + i\sigma_{yz}, \quad (25)$$

where $i = \sqrt{-1}$. Then from Eqns (1) and (22), the expressions of the thermoporoelastic field take on a more compact form

$$\begin{aligned} U &= -\Lambda \left(\sum_{j=1}^4 \Psi_j + i\Psi_0 \right), \quad \sigma_z = \sum_{j=1}^4 \gamma_{j1} \frac{\partial^2 \Psi_0}{\partial z_j^2}, \quad \sigma_1 = \sum_{j=1}^4 \gamma_{j2} \frac{\partial^2 \Psi_0}{\partial z_j^2}, \\ \sigma_2 &= -2c_{66} \Lambda^2 \left(\sum_{j=1}^4 \Psi_j + i\Psi_0 \right), \quad \tau_z = \Lambda \left(\sum_{j=1}^4 \gamma_{j3} \frac{\partial \Psi_j}{\partial z_j} - ic_{44} s_0 \frac{\partial \Psi_0}{\partial z_0} \right), \end{aligned} \quad (26)$$

where $\Lambda = \partial/\partial x + i\partial/\partial y$, and

$$\begin{aligned} \gamma_{j1} &= c_{13} + c_{33}\mu_{j1}s_j - \beta_3\mu_{j3}, \quad \gamma_{j2} = 2[c_{11} - c_{16} + c_{13}\mu_{j1}s_j - \alpha_1\mu_{j2} - \beta_1\mu_{j3}], \\ \gamma_{j3} &= c_{44}(\mu_{j1} - s_j), \quad j = 1-4. \end{aligned} \quad (27)$$

It is note that the identity $\gamma_{j3} = \gamma_{j1}s_j, (j = 1-4)$ holds true, which can be verified by direct substitution.

Now consider an infinite transversely isotropic thermoporoelastic body weakened by a flat crack whose surface is parallel to the plane of isotropy. It is further assumed that the radius of the borehole r_0 is much smaller than the crack length L_0 , so that the borehole effect can be ignored. The coordinates are chosen such that the $x-y$ plane coincides with the crack surface and the origin is located on the surface of the crack. Two symbols S and \bar{S} are introduced to denote the regions on the plane $z=0$, occupied respectively by the crack and its exterior. It may be noted that $S \cup \bar{S} = I$, where I is the entire plane $z=0$ and $S \cap \bar{S} = \emptyset$, implying there is no separation. It is assumed that arbitrarily distributed normal mechanical force $2\pi\Theta_1(x, y)$, pressure $\Theta_2(x, y)$ and temperature $\Theta_3(x, y)$ are symmetrically applied over the upper and lower crack faces. Due to the symmetric condition occupied by the problem, the problem can be transformed to a mixed boundary-value problem of a half space $z \geq 0$ subjected to the following boundary conditions on the plane $z=0$ (Chen, 2004b; Wang, 2000):

$$\begin{aligned}
 (x, y) \in S : \sigma_z &= 2\pi\Theta_1(x, y), P = \Theta_2(x, y), T = \Theta_3(x, y), \\
 (x, y) \in \bar{S} : w &= 0, \frac{\partial P}{\partial z} = 0, \frac{\partial T}{\partial z} = 0, \\
 (x, y) \in I : \tau_z &= 0,
 \end{aligned} \tag{28}$$

where the factor 2π is introduced for the algebraic convenience. In order to extend the potential theory method (Fabrikant, 1989) to thermoporoelasticity, assume that

$$\Psi_0 = 0, \Psi_i = \sum_{j=1}^3 e_{ij} \Omega_j(x, y, z_i), \quad (i = 1-4), \tag{29}$$

where e_{ij} are constants to be determined, and

$$\begin{aligned}
 \Omega_1(x, y, z) &= \iint_S \frac{\omega_1(N_0)}{R(M, N_0)} dS, \\
 \Omega_j(x, y, z) &= \iint_S \omega_j(N) \{z \ln [R(M, N_0) + z] - R(M, N_0)\} dS, \quad (j = 2, 3),
 \end{aligned} \tag{30}$$

where the kernels of the potentials ω_1 , ω_2 and ω_3 are the crack surface displacement $w(x, y, 0)$, pressure gradient $\partial P(x, y, z)/\partial z|_{z=0}$ and temperature gradient $\partial T(x, y, z)/\partial z|_{z=0}$, respectively; $R(M, N_0)$ denotes the distance between the points $M(x, y, z)$ and $N_0(x_0, y_0, 0)$, and $N \in S$.

The zero-shear stress condition at $z = 0$ in Eqn. (28) can be meted by taking

$$\sum_{i=1}^4 \gamma_{i3} e_{ij} = 0, \quad (j = 1, 2, 3). \tag{31}$$

Using the well known property of a simple layer potential, one can arrive at

$$\begin{aligned}
 (x, y) \in \bar{S} : \frac{\partial \Omega_1}{\partial z} \Big|_{z=0} &= 0, \quad \frac{\partial^3 \Omega_j}{\partial z^3} \Big|_{z=0} = 0, \quad (j = 2, 3) \\
 (x, y) \in S : \frac{\partial \Omega_1}{\partial z} \Big|_{z=0} &= -2\pi\omega_1 = -2\pi w(x, y, 0), \\
 (x, y) \in S : \frac{\partial^3 \Omega_2}{\partial z^3} \Big|_{z=0} &= -2\pi\omega_2 = -2\pi \frac{\partial P(x, y, z)}{\partial z} \Big|_{z=0}, \\
 (x, y) \in S : \frac{\partial^3 \Omega_3}{\partial z^3} \Big|_{z=0} &= -2\pi\omega_3 = -2\pi \frac{\partial T(x, y, z)}{\partial z} \Big|_{z=0}.
 \end{aligned} \tag{32}$$

Eqns. (22), (32) and the second condition in Eqn. (28) then lead to

$$\sum_{i=1}^4 \mu_{i1} e_{ij} = -\frac{\delta_{1j}}{2\pi}, \quad \sum_{i=1}^4 \mu_{ik} e_{ij} s_i = -\frac{\delta_{jk}}{2\pi}, \quad (k = 2, 3; j = 1, 2, 3). \quad (33)$$

Thus combining Eqns. (31) and (33) yields

$$\begin{Bmatrix} e_{1j} \\ e_{2j} \\ e_{3j} \\ e_{4j} \end{Bmatrix} = -\frac{1}{2\pi} \begin{bmatrix} \gamma_{13} & \gamma_{23} & \gamma_{33} & \gamma_{43} \\ \mu_{11} & \mu_{21} & \mu_{31} & \mu_{41} \\ \mu_{12}s_1 & \mu_{22}s_2 & \mu_{32}s_3 & \mu_{42}s_4 \\ \mu_{13}s_1 & \mu_{23}s_2 & \mu_{33}s_3 & \mu_{43}s_4 \end{bmatrix}^{-1} \begin{Bmatrix} 0 \\ \delta_{1j} \\ \delta_{2j} \\ \delta_{3j} \end{Bmatrix}, \quad (j = 1, 2, 3). \quad (34)$$

To satisfy the first condition in Eqn. (28), we have

$$m_{11} \Delta \iint_S \frac{\omega_1(N_0)}{R(N_0, N)} dS - \sum_{j=2}^3 m_{1j} \iint_S \frac{\omega_j(N_0)}{R(N_0, N)} dS = -2\pi \Theta_1(N), \quad (35)$$

$$\iint_S \frac{\omega_j(N_0)}{R(N_0, N)} dS = -2\pi s_{j+1} \Theta_j(N), \quad (j = 2, 3), \quad (36)$$

where $m_{1j} = \sum_{i=1}^4 \gamma_{i1} e_{ij}$, ($j = 1, 2, 3$), and $R(N_0, N)$ is the distance between the two points $N_0(x_0, y_0, 0)$ and $N(x, y, 0)$, both of which lie on S . It is noted that m_{11} depends on the elastic constants c_{ij} only; m_{12} is independent of the coefficients (β_i, λ_{ij}) associated with the temperature fields and m_{13} is independent of the coefficients (α_i, κ_{ij}) pertinent to the pressure field. This property can be verified by any standard symbol computation software. Substituting Eqn. (36) into Eqn. (35) gives arise to

$$\Delta \iint_S \frac{\omega_1(N_0)}{R(N_0, N)} dS = -2\pi \bar{\Theta}_1(N), \quad (37)$$

where

$$\bar{\Theta}_1(N) = [\Theta_1(N) + m_{12}s_3\Theta_2(N) + m_{13}s_4\Theta_3(N)]/m_{11}, \quad (38)$$

which can be viewed as the generalized mechanical load. Once $\omega_i(x, y)$ is available, the thermoporoelastic field can be obtained by differentiating the potential function

It is seen that the pressure and thermal loadings can effect the elastic field (see Eqns. 37 and 38), while the mechanical loadings can change neither the thermal nor pressure field (see Eqn. 36); furthermore, no mutual influences exist between the thermal and pressure fields. This is due to the uncoupled theory employed herein.

Consequently, the solutions thus obtained degenerate to those of the crack problems in thermoelasticity (poroelasticity) by setting $\Theta_2 = 0$ ($\Theta_3 = 0$), which provides a way to verify the solutions in this paper by comparing with those in literature.

It is interesting to note that the structure of integral equations in Eqn. (36) is the same as the one for punch problems in elasticity while the structure of the integro-differential equation in Eqn. (37) is identical to that for crack problems in elasticity (Fabrikant, 1989; Chen, 2004b). Due to this similarity, the splendid results given by Fabrikant (1989, 1991) can be directly employed to seek the solutions to the governing equations in some particular cases. This will be shown in the next section, where the cylindrical coordinate system (ρ, ϕ, z) with the origin coincident to the center of the crack is alternatively used for simplicity.

5. A penny-shaped crack subjected to uniformly distributed loads

Consider a penny-shaped crack subjected to uniformly distributed mechanical, pressure and temperature loads

$$\Theta_1(x, y) = \Theta_1^0, \quad \Theta_2(x, y) = \Theta_2^0, \quad \Theta_3(x, y) = \Theta_3^0, \quad \bar{\Theta}_1(x, y) = \bar{\Theta}_1^0, \quad (39)$$

where the quantities with the superscript 0 are prescribed constants. Under this condition, the exact solutions to Eqns. (36) and (37) can be obtained by using the results of Fabrikant (1989) as

$$\omega_1(\rho, \phi) = \frac{2\bar{\Theta}_1^0}{\pi} \sqrt{a^2 - \rho^2}, \quad \omega_j(\rho, \phi) = -\frac{2s_{j+1}}{\pi} \frac{\Theta_j^0}{\sqrt{a^2 - \rho^2}}, \quad (j = 2, 3), \quad (40)$$

where a is the radius of the crack. Inserting Eqn. (39) into Eqn. (30) leads to

$$\begin{aligned} \Omega_1(\rho, \phi, z) &= \bar{\Theta}_1^0 \left[(2a^2 + 2z^2 - \rho^2) \sin^{-1} \left(\frac{a}{l_2} \right) - \frac{2a^2 - 3l_1^2}{a} \sqrt{l_2^2 - a^2} \right], \\ \Omega_j(\rho, \phi, z) &= -2s_{j+1} \Theta_j^0 \left[\left(a^2 + z^2 - \frac{\rho^2}{2} \right) \sin^{-1} \left(\frac{a}{l_2} \right) \right. \\ &\quad \left. - \frac{2a^2 - 3l_1^2}{a} \sqrt{l_2^2 - a^2} + 2az \ln \left(l_2 + \sqrt{l_2^2 - a^2} \right) \right], \quad (j = 2, 3) \end{aligned} \quad (41)$$

where

$$\begin{aligned}
 l_1 &= l_1(\rho, a, z) = \frac{1}{2} \left(\sqrt{(\rho+a)^2 + z^2} - \sqrt{(\rho-a)^2 + z^2} \right), \\
 l_2 &= l_2(\rho, a, z) = \frac{1}{2} \left(\sqrt{(\rho+a)^2 + z^2} + \sqrt{(\rho-a)^2 + z^2} \right).
 \end{aligned} \tag{42}$$

If there are no mechanical and pressure loads, ω_3 and Ω_3 vanish identically, and expressions for ω_j and Ω_j ($j=1,2$) are exactly the same as these given by Chen (2004b). From Eqns. (41)-(42), the thermoporoelastic field is independent of the angular coordinate ϕ because of the axisymmetry pertinent to the problem.

At this stage, one can derive the whole thermoporoelastic field through differentiation

$$\begin{aligned}
 U &= 2\bar{\Theta}_1^0 \rho e^{i\phi} \sum_{j=1}^4 e_{j1} \left[\sin^{-1} \left(\frac{a}{l_{2j}} \right) - \frac{a\sqrt{l_{2j}^2 - a^2}}{l_{2j}^2} \right] \\
 &\quad - 4 \frac{e^{i\phi}}{\rho} \sum_{k=2}^3 s_{k+1} \Theta_k^0 \sum_{j=1}^4 e_{jk} \left[\sqrt{l_{2j}^2 - a^2} \left(a - \frac{l_{1j}^2}{2a} \right) - az_j + \frac{\rho^2}{2} \sin^{-1} \left(\frac{a}{l_{2j}} \right) \right], \\
 w &= 4\bar{\Theta}_1^0 \sum_{j=1}^4 \mu_{j1} e_{j1} \left[z_j \sin^{-1} \left(\frac{a}{l_{2j}} \right) - \sqrt{a^2 - l_{1j}^2} \right] \\
 &\quad - 4 \sum_{k=2}^3 s_{k+1} \Theta_k^0 \sum_{j=1}^4 \mu_{j1} e_{jk} \left[z_j \sin^{-1} \left(\frac{a}{l_{2j}} \right) - \sqrt{a^2 - l_{1j}^2} + a \ln \left(l_{2j} + \sqrt{l_{2j}^2 - \rho^2} \right) \right], \\
 \sigma_z &= 4\bar{\Theta}_1^0 \sum_{j=1}^4 \gamma_{j1} e_{j1} \left[\sin^{-1} \left(\frac{a}{l_{2j}} \right) - \frac{a\sqrt{l_{2j}^2 - a^2}}{l_{2j}^2 - l_{1j}^2} \right] - 4 \sum_{k=2}^3 s_{k+1} \Theta_k^0 \sum_{j=1}^4 \gamma_{j1} e_{jk} \sin^{-1} \left(\frac{a}{l_{2j}} \right), \\
 \sigma_1 &= 4\bar{\Theta}_1^0 \sum_{j=1}^4 \gamma_{j2} e_{j1} \left[\sin^{-1} \left(\frac{a}{l_{2j}} \right) - \frac{a\sqrt{l_{2j}^2 - a^2}}{l_{2j}^2 - l_{1j}^2} \right] - 4 \sum_{k=2}^3 s_{k+1} \Theta_k^0 \sum_{j=1}^4 \gamma_{j2} e_{jk} \sin^{-1} \left(\frac{a}{l_{2j}} \right), \\
 \sigma_2 &= -8c_{66} \bar{\Theta}_1^0 a e^{2i\phi} \sum_{j=1}^4 \frac{e_{j1} l_{1j}^2 \sqrt{l_{2j}^2 - a^2}}{l_{2j}^2 (l_{2j}^2 - l_{1j}^2)} \\
 &\quad + 4c_{66} \frac{e^{2i\phi}}{\rho^2} \sum_{k=2}^3 s_{k+1} \Theta_k^0 \sum_{j=1}^4 e_{jk} (a\sqrt{l_{2j}^2 - a^2} + z_j \sqrt{a^2 - l_{1j}^2} - 2az_j), \\
 \tau_z &= 4\bar{\Theta}_1^0 a^2 \rho e^{i\phi} \sum_{j=1}^4 \frac{\gamma_{j3} e_{j1} \sqrt{a^2 - l_{1j}^2}}{l_{2j}^2 (l_{2j}^2 - l_{1j}^2)} + 4 \frac{e^{i\phi}}{\rho} \sum_{k=2}^3 s_{k+1} \Theta_k^0 \sum_{j=1}^4 \gamma_{j3} e_{jk} (\sqrt{a^2 - l_{1j}^2} - a), \\
 P &= \frac{2}{\pi} \Theta_2^0 \sin^{-1} \left(\frac{a}{l_{23}} \right), \quad T = \frac{2}{\pi} \Theta_3^0 \sin^{-1} \left(\frac{a}{l_{24}} \right),
 \end{aligned} \tag{43}$$

where $l_{ij} = l_{ij}(a, \rho, z_j)$, ($i=1,2, j=1-4$) defined in Eqn. (42). As expected, the

temperature field is identical to that shown in Chen (2004b).

On account of the following property

$$l_1|_{z=0} = l_1(\rho, a, 0) = \min(a, \rho), \quad l_2|_{z=0} = l_2(\rho, a, 0) = \max(a, \rho), \quad (44)$$

one can derive from Eqn.(43) that

$$\begin{aligned} \sigma_z(\rho, \phi, 0) &= \begin{cases} 4\Theta_1^0 \sin^{-1}\left(\frac{a}{\rho}\right) - \frac{4m_{11}\bar{\Theta}_1^0 a}{\sqrt{\rho^2 - a^2}}, & (\rho, \phi) \in \bar{S}, \\ \Theta_1^0, & (\rho, \phi) \in S, \end{cases} \\ P(\rho, \phi, 0) &= \begin{cases} \frac{2}{\pi} \Theta_2^0 \sin^{-1}\left(\frac{a}{\rho}\right), & (\rho, \phi) \in \bar{S}, \\ \Theta_2^0, & (\rho, \phi) \in S, \end{cases} \\ T(\rho, \phi, 0) &= \begin{cases} \frac{2}{\pi} \Theta_3^0 \sin^{-1}\left(\frac{a}{\rho}\right), & (\rho, \phi) \in \bar{S}, \\ \Theta_3^0, & (\rho, \phi) \in S. \end{cases} \end{aligned} \quad (45)$$

It is evident that the boundary conditions have been satisfied. It is also noted that the normal stress σ_z on \bar{S} is the sum of two terms which are singular and non-singular parts on the crack tip, respectively. This is similar to the crack problem in elasticity (Fabrikant, 1989).

If the stress intensity factor is defined as

$$K_I = \lim_{\rho \rightarrow a} \sqrt{2\pi(\rho - a)} \sigma_z|_{z=0}, \quad (46)$$

then from Eqn. (45), one can obtain

$$K_I = -4m_{11}\bar{\Theta}_1^0 \sqrt{\pi a} = -4\sqrt{\pi a} (\Theta_1^0 + s_3 m_{12} \Theta_2^0 + s_4 m_{13} \Theta_3^0). \quad (47)$$

It is interesting to note that if the pressure and temperature loads are absent, the stress intensity factor is independent of the material property and is consistent with the result given by Fabrikant (1989).

If the mechanical and pressure loads are missing and the temperature is solely presented, the forms of the displacements and stresses shown in Eqn (43) are the same as that in Chen (2004b) with an exception of the coefficients. However, numerical calculation reveals that the results presented in this section agrees well with those shown in the previous work. Furthermore, the pressure field in Eqn. (43) can be verified by solving the mixed boundary-value problems of a half space. These two aspects will be presented in the next section.

6. Verification of the solution

As mentioned before, the pressure and temperature fields can be solved independently. Here, the potential theory method was presented to solve the pressure field governed by Eqn. (3) of a half-space subjected to an arbitrarily distributed pressure $\Theta_2(\rho, \phi)$ on a circular domain $S \equiv (\rho \leq a, 0 \leq \phi \leq 2\pi)$ and with $\partial P/\partial z = 0$ on \bar{S} .

Eqn. (3) is satisfied by assuming

$$P(\rho, \phi, z) = \int_0^{2\pi} \int_0^a \frac{\omega'_2(\rho_0, \phi_0)}{R'} \rho_0 d\rho_0 d\phi_0, \quad (48)$$

where

$$\omega'_2 = \frac{1}{2\pi} \frac{\partial P}{\partial z_3} \Big|_{z_3=0}, \quad R' = \sqrt{\rho^2 + \rho_0^2 - 2\rho\rho_0 \cos(\phi - \phi_0) + z_3^2}. \quad (49)$$

It is noted that the boundary condition $\frac{\partial P(\rho, \phi)}{\partial z} = 0, (\rho, \phi) \in \bar{S}$ is automatically meted. The boundary condition on S is also satisfied by letting

$$\int_0^{2\pi} \int_0^a \frac{\omega'_2(\rho_0, \phi_0)}{\sqrt{\rho^2 + \rho_0^2 - 2\rho\rho_0 \cos(\phi - \phi_0)}} \rho_0 d\rho_0 d\phi_0 = \Theta_2(\rho, \phi). \quad (50)$$

By virtue of the results given by Fabrikant (1989), the solution to Eqn. (49) can be written out as

$$\omega'_2(\rho, \phi) = -\frac{1}{\pi^2 \rho} L(\rho) \frac{d}{d\rho} \int_\rho^a \frac{xdx}{\sqrt{x^2 - \rho^2}} L\left(\frac{1}{x^2}\right) \frac{d}{dx} \int_0^x \frac{ydy}{\sqrt{x^2 - y^2}} L(y) \Theta_2(y, \phi), \quad (51)$$

where $L(\bullet)$ is an operator defined as:

$$L(k)f(\phi) = \frac{1}{2\pi} \int_0^{2\pi} \frac{1-k^2}{1+k^2-2k\cos(\phi-\phi_0)} f(\phi_0) d\phi_0, \quad (0 \leq k < 1). \quad (52)$$

Substituting Eqn. (52) into Eqn. (48) yields

$$P(\rho, \phi, z) = \frac{1}{\pi^2} \int_0^{2\pi} \int_0^a \left[\frac{R'}{h} + \tan^{-1} \left(\frac{h}{R'} \right) \right] \frac{z_3}{(R')^3} \Theta_2(\rho_0, \phi_0) \rho_0 d\rho_0 d\phi_0, \quad (53)$$

where $h = \sqrt{a^2 - l_{13}^2} \sqrt{a^2 - \rho_0^2} / a$ and $l_{13} = l_1(z_3)$ is defined in Eqn. (42).

Letting $\Theta_2(\rho, \phi) = \Theta_2^0 = \text{constant}$, from Eqn. (51) one can obtain that

$$\omega'_2(\rho, \phi) = \frac{\Theta_2^0}{\pi^2 \sqrt{a^2 - \rho^2}}, \quad (\rho \leq a). \quad (54)$$

This immediately gives

$$P = \frac{2}{\pi} \Theta_2^0 \sin^{-1} \left(\frac{a}{l_{23}} \right), \quad (55)$$

which is identical to the expression in Eqn. (43). The temperature field can be verified in a similar manner.

To validate the solutions presented in the last section, numerical calculation is performed for a penny-shaped crack subjected to a constant thermal load $\Theta_3^0 = 100^\circ\text{C}$ only. The material constants are listed in Table 1 for a deep-water Gulf shale of Mexico (Kanj and Abousleiman, 2005), where the physical meanings are given in Appendix A. Under this situation, the solution in the preceding section should be identical to those given by Chen et al. (2004b). Comparison is made in Table 2 and good agreement is achieved. This further validates the thermoporoelastic fields in Eqn. (43).

Tables 1-2

7. Concluding remarks

The basic equations in static thermoporoelasticity are simplified by introducing two displacement functions. Then, a set of general solutions in terms of five quasi-harmonic functions are derived by using the operator theory and the superposition principle along with the generalized Almansi's theorem. Four possible cases of material characteristic roots are considered for the theoretical completeness. No particular solutions associated with the pressure and temperature fields appear in the general solutions, making it possible to extend the potential theory method to thermoporoelasticity.

Based on the general solution, the potential theory method developed by Fabrikant (1989) for the mixed boundary-value problems was extended to the transversely isotropic thermoporoelastic plane crack problems. Two potentials with the same mathematical structure are introduced to account for the thermal and pressure effects. The governing equations have the same structures as these reported in literatures. For

the problem of a penny-shaped crack under the actions of uniformly distributed mechanical, pressure and temperature loadings, exact and complete thermoporoelastic fields are expressed explicitly in terms of elementary functions. The present solution is verified by comparing to those given by Chen et al. (2004b). Furthermore, the solutions pertinent to the pore pressure are completely new to literature.

It should be emphasized that the thermoporoelastic field is only valid for the materials with distinct characteristic roots. In the case of multiple roots, the general solutions will take on different forms as shown in Appendix B. The potential theory method can also be employed if appropriate potentials are chosen. However, as pointed out by Fabrikant (1989), the corresponding solutions in the case of multiple roots can be derived directly from the solutions, as presented in this paper, with the aid of L'Hospital rule.

Although the loadings on the penny-shaped crack in this paper are independent of the angular coordinate, it is not the limitation inherent to the potential theory method. In fact, the problems of a penny-shaped crack subjected to the point loading can be successfully solved and the Green's functions which are significant to the boundary element method can be derived. This will be reported in another paper.

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Appendix A

The material coefficients in Eqns. (1) can be denoted by the engineering constants as follows

$$\begin{aligned} c_{11} &= \frac{E(E' - E\nu'^2)}{(1 + \nu)(E' - E'\nu - 2E\nu'^2)}, \quad c_{12} = \frac{E(E'\nu + E\nu'^2)}{(1 + \nu)(E' - E'\nu - 2E\nu'^2)}, \\ c_{13} &= \frac{EE'\nu'}{E' - E'\nu - 2E\nu'^2}, \quad c_{33} = \frac{E'^2(1 - \nu)}{E' - E'\nu - 2E\nu'^2}, \quad c_{44} = \frac{E'}{2(1 + \nu')}, \end{aligned} \quad (\text{A.1})$$

where E is the drained Young's modulus in the plane of isotropy ($x - y$ plane), E' the drained Young's modulus perpendicular to that plane (z -direction), respectively. If the assumption of micro-homogeneity and micro-isotropy of the solid skeleton is adopted, α_1 and α_3 can be formulated in terms of c_{ij} and the bulk modulus of the solid grain K_s according to

$$\alpha_1 = 1 - \frac{c_{11} + c_{12} + c_{13}}{3K_s}, \quad \alpha_3 = 1 - \frac{2c_{13} + c_{33}}{3K_s}. \quad (\text{A.2})$$

Furthermore β_1 , β_3 and β_m are related to the linear expansion coefficients in the isotropic and transverse planes of the material (α_1^s and α_3^s) and the volumetric expansion coefficient α_f of the pore fluid as

$$\beta_1 = (c_{11} + c_{12})\alpha_1^s + c_{13}\alpha_3^s, \quad \beta_3 = 2c_{13}\alpha_1^s + c_{33}\alpha_3^s, \quad \beta_m = 2\alpha_1\alpha_1^s + \alpha_3\alpha_3^s + (\alpha_f - 2\alpha_1^s - \alpha_3^s)\phi', \quad (\text{A.3})$$

where ϕ' is the porosity.

Appendix B

The general solutions for the cases of multiple roots of $s_i (i=1-4)$ are presented in the following without a detailed derivation for sake of simplicity.

Case 2 ($s_1 \neq s_2 \neq s_3 = s_4 \neq s_1$): After the same manipulation as Case 1 and by introducing

$$\Psi_i = \eta_{1i} \frac{\partial^4 F_i}{\partial z_i^4}, \quad (i=1,2), \quad \Psi_3 = \eta_{13} \frac{\partial^4 F_3}{\partial z_3^4} + \eta'_{13} s_3 \frac{\partial^3 F_4}{\partial z_3^3}, \quad \Psi_4 = \frac{\eta_{13}}{s_3} \frac{\partial^4 F_3}{\partial z_3^4} \quad (\text{B.1})$$

where $\eta'_{ij} = 4a_i s_i^2 - 2b_i, (i, j = 1-4)$, the general solutions are

$$\begin{aligned} u &= \frac{\partial \Psi_0}{\partial y} - \frac{\partial}{\partial x} (\Psi_1 + \Psi_2 + \Psi_3 + z_3 \Psi_4), \quad v = -\frac{\partial \Psi_0}{\partial x} - \frac{\partial}{\partial y} (\Psi_1 + \Psi_2 + \Psi_3 + z_3 \Psi_4), \\ w &= \mu_{11} \frac{\partial \Psi_1}{\partial z_1} + \mu_{12} \frac{\partial \Psi_2}{\partial z_2} + \mu_{13} \frac{\partial \Psi_3}{\partial z_3} + \mu_{13} z_3 \frac{\partial \Psi_4}{\partial z_3} + \mu_{15} \Psi_4, \quad P = \mu_{25} \frac{\partial \Psi_4}{\partial z_3}, \quad T = \mu_{35} \frac{\partial \Psi_4}{\partial z_3}, \end{aligned} \quad (\text{B.2})$$

where

$$\begin{aligned} \mu_{15} &= \mu_{13} \left(1 - \frac{\eta'_{13}}{\eta_{13}} s_3^2 \right) + \frac{\eta'_{23}}{\eta_{13}} s_3^3, \\ \mu_{25} &= \frac{(6a_3 s_3^6 - 4a_3 s_3^4 + 2c_3 s_3^2)}{\eta_{13}}, \quad \mu_{35} = \frac{(6a_4 s_3^6 - 4a_4 s_3^4 + 2c_4 s_3^2)}{\eta_{13}}. \end{aligned} \quad (\text{B.3})$$

It is noted that $\Psi_i (i=0-4)$ satisfies Eqn. (23) as well as that in Case 1.

Case 3 ($s_1 \neq s_2 = s_3 = s_4$): Introducing the following functions satisfying

$$\begin{aligned} \Psi_1 &= \eta_{11} \frac{\partial^4 F_1}{\partial z_1^4}, \quad \Psi_2 = \eta_{12} \frac{\partial^4 F_2}{\partial z_2^4} + \eta'_{12} s_3 \frac{\partial^3 F_3}{\partial z_2^3} + \eta''_{12} \frac{\partial^2 F_4}{\partial z_2^2}, \\ \Psi_3 &= \frac{\eta_{12}}{s_2} \frac{\partial^4 F_3}{\partial z_2^4} + 2\eta'_{12} \frac{\partial^3 F_4}{\partial z_2^3}, \quad \Psi_4 = \frac{\eta_{12}}{s_3^2} \frac{\partial^3 F_3}{\partial z_2^3}, \end{aligned} \quad (\text{B.4})$$

where $\eta''_{ij} = 12a_i s_j^2 - 2b_i, (i, j = 1-4)$, then one can obtain the general solutions as

$$\begin{aligned}
u &= \frac{\partial \Psi_0}{\partial y} - \frac{\partial}{\partial x} \left(\Psi_1 + \Psi_2 + z_2 \Psi_3 + z_2^2 \frac{\partial \Psi_4}{\partial z_2} \right), \\
v &= -\frac{\partial \Psi_0}{\partial x} - \frac{\partial}{\partial y} \left(\Psi_1 + \Psi_2 + z_2 \Psi_3 + z_2^2 \frac{\partial \Psi_4}{\partial z_2} \right), \\
w &= \mu_{11} \frac{\partial \Psi_1}{\partial z_1} + \mu_{12} \frac{\partial \Psi_2}{\partial z_2} + \mu_{12} z_2 \frac{\partial \Psi_3}{\partial z_2} + \mu_{12} z_2^2 \frac{\partial^2 \Psi_4}{\partial z_2^2} + \mu_{15} \left(\Psi_3 + 2z_2 \frac{\partial \Psi_4}{\partial z_2} \right) + \mu_{16} \Psi_4, \\
P &= \mu_{25} \left(\frac{\partial \Psi_3}{\partial z_2} + 2z_2 \frac{\partial^2 \Psi_4}{\partial z_2^2} \right) + \mu_{26} \frac{\partial \Psi_4}{\partial z_2}, \quad T = \mu_{35} \left(\frac{\partial \Psi_3}{\partial z_2} + 2z_2 \frac{\partial^2 \Psi_4}{\partial z_2^2} \right) + \mu_{36} \frac{\partial \Psi_4}{\partial z_2},
\end{aligned} \tag{B.5}$$

where

$$\begin{aligned}
\mu_{16} &= \frac{[(\eta_{22}'' + 2\eta_{22}')s_2^3 - \mu_{12}\eta_{12}''s_2^2 - 2\mu_{15}\eta_{12}'s_2^2]}{\eta_{12}}, \\
\mu_{26} &= \mu_{25} \left(1 - 2\frac{\eta_{22}'}{\eta_{12}}s_2^2 \right) + \frac{\eta_{32}''}{\eta_{12}}s_2^4 + 3\frac{\eta_{32}'}{\eta_{12}}s_2^4, \quad \mu_{36} = \mu_{35} \left(1 - 2\frac{\eta_{22}'}{\eta_{12}}s_2^2 \right) + \frac{\eta_{42}''}{\eta_{12}}s_2^4 + 3\frac{\eta_{42}'}{\eta_{12}}s_2^4.
\end{aligned} \tag{B.6}$$

In view of Eqns. (B.4) and (18), it is also seen that Ψ_i ($i = 0-4$) satisfies Eqn. (23)

Case 4 ($s_1 = s_2 = s_3 = s_4$): Introducing

$$\begin{aligned}
\Psi_1 &= \eta_{11} \frac{\partial^4 F_1}{\partial z_1^4} + \eta_{11}' s_1 \frac{\partial^3 F_2}{\partial z_1^3} + \eta_{11}'' \frac{\partial^2 F_3}{\partial z_1^2} + 24a_1 \eta_{11}'' \frac{\partial F_4}{\partial z_1}, \\
\Psi_2 &= \frac{\eta_{11}}{s_1} \frac{\partial^4 F_2}{\partial z_1^4} + 2\eta_{11}' \frac{\partial^3 F_3}{\partial z_1^3} + \frac{3\eta_{11}''}{s_1} \frac{\partial^2 F_4}{\partial z_1^2}, \quad \Psi_3 = \frac{\eta_{11}}{s_1^2} \frac{\partial^3 F_3}{\partial z_1^3} + \frac{3\eta_{11}'}{s_1} \frac{\partial^2 F_4}{\partial z_1^2}, \quad \Psi_4 = \frac{\eta_{12}}{s_1^3} \frac{\partial^2 F_3}{\partial z_1^2},
\end{aligned} \tag{B.7}$$

into Eqn. (15) leads to

$$\begin{aligned}
u &= \frac{\partial \Psi_0}{\partial y} - \frac{\partial}{\partial x} \left(\Psi_1 + z_1 \Psi_2 + z_1^2 \frac{\partial \Psi_4}{\partial z_1} + z_1^3 \frac{\partial^2 \Psi_4}{\partial z_1^2} \right), \\
v &= -\frac{\partial \Psi_0}{\partial x} - \frac{\partial}{\partial y} \left(\Psi_1 + z_1 \Psi_2 + z_1^2 \frac{\partial \Psi_4}{\partial z_1} + z_1^3 \frac{\partial^2 \Psi_4}{\partial z_1^2} \right), \\
w &= \mu_{11} \left(\frac{\partial \Psi_1}{\partial z_1} + z_1 \frac{\partial \Psi_2}{\partial z_2} + z_1^2 \frac{\partial^2 \Psi_4}{\partial z_1^2} + z_1^3 \frac{\partial^3 \Psi_4}{\partial z_1^3} \right) \\
&\quad + \mu_{15} \left(\Psi_2 + 2z_1 \frac{\partial \Psi_3}{\partial z_1} + 3z_1^2 \frac{\partial^2 \Psi_4}{\partial z_1^2} \right) + \mu_{16} \left(\Psi_3 + 3z_1 \frac{\partial \Psi_4}{\partial z_1} \right) + \mu_{17} \Psi_4, \\
P &= \mu_{25} \left(\frac{\partial \Psi_2}{\partial z_1} + 2z_1 \frac{\partial^2 \Psi_3}{\partial z_2^2} + 3z_1^2 \frac{\partial^3 \Psi_3}{\partial z_1^3} \right) + \mu_{26} \left(\frac{\partial \Psi_3}{\partial z_1} + 3z_1 \frac{\partial^2 \Psi_4}{\partial z_1^2} \right) + \mu_{27} \Psi_4, \\
T &= \mu_{35} \left(\frac{\partial \Psi_2}{\partial z_1} + 2z_1 \frac{\partial^2 \Psi_3}{\partial z_2^2} + 3z_1^2 \frac{\partial^3 \Psi_3}{\partial z_1^3} \right) + \mu_{36} \left(\frac{\partial \Psi_3}{\partial z_1} + 3z_1 \frac{\partial^2 \Psi_4}{\partial z_1^2} \right) + \mu_{37} \Psi_4,
\end{aligned} \tag{B.8}$$

where

$$\begin{aligned}
\mu_{17} &= 3 \frac{[(2\eta_{21}'' - \eta_{22}')s_1^3 - 8a_1\mu_{11}s_1^2 - \mu_{15}\eta_{11}''s_1^2 - 2\mu_{16}\eta_{11}'s_1^2]}{\eta_{11}}, \\
\mu_{27} &= 3 \frac{[(\eta_{31}' + 3\eta_{31}'')s_1^4 - \mu_{25}\eta_{11}'s_1^2 - \mu_{26}\eta_{11}'s_1^2]}{\eta_{11}}, \\
\mu_{37} &= 3 \frac{[(\eta_{41}' + 3\eta_{41}'')s_1^4 - \mu_{25}\eta_{11}'s_1^2 - \mu_{26}\eta_{11}'s_1^2]}{\eta_{11}}.
\end{aligned} \tag{B.9}$$

In view of Eqns. (B1.7) and (18), it is again seen that Ψ_i ($i = 0-4$) satisfies Eqn. (23).

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Captions of tables

Table 1: material parameters for thermoporoelastic analysis

Table 2: The dimensionless stress σ_z/E and displacement w/a on I

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Table 1: Material parameters for thermoporoelastic analysis

Parameter	Value	Unit
E (E')	1854 (927)	[MPa]
K_s	12180	[MPa]
ν (ν')	0.22 (0.44)	–
κ_{11} (κ_{33})	0.1 (0.2)	$\left[\frac{\text{D}}{\text{Pa s}} \right]$
λ_{11} (λ_{33})	2.65 (4.00)	$\left[\frac{\text{W}}{^\circ\text{C m}} \right]$
α_1^s (α_3^s)	6×10^{-6} (1.2×10^{-5})	$\left[\frac{1}{^\circ\text{C}} \right]$

Table 2: The dimensionless stress σ_z/E and displacement w/a on I

$\frac{\rho}{a}$	σ_z/E		$\frac{\rho}{a}$	w/a	
	Present	Chen et al.		Present	Chen et al.
1.001	-0.025909095	-0.025909123	0.000	-0.012331027786	-0.012331041131
1.010	-0.008174811	-0.008174820	0.100	-0.012269217734	-0.012269231012
1.020	-0.005766139	-0.005766145	0.200	-0.012081890432	-0.012081903508
1.030	-0.004696422	-0.004696427	0.300	-0.011763050799	-0.011763063529
1.050	-0.003620044	-0.003620048	0.400	-0.011301573645	-0.011301585876
1.100	-0.002529100	-0.002529103	0.500	-0.010678983317	-0.010678994875
1.500	-0.001036622	-0.001036623	0.600	-0.009864822229	-0.009864832905
2.000	-0.000669137	-0.000669138	0.700	-0.008806115238	-0.008806124769
5.000	-0.000236576	-0.000236576	0.800	-0.007398616672	-0.007398624679
10.00	-0.000116482	-0.000116482	0.900	-0.005374970399	-0.005374976216
20.00	-0.000058022	-0.000058022	1.000	0	0
50.00	-0.000023184	-0.000023184	10.00	0	0
100.0	-0.000011590	-0.000011590	100.0	0	0
1000	-0.000001159	-0.000001159	1000	0	0
10^4	-0.000000116	-0.000000116	10^4	0	0

Captions of figures

Fig. 1 Vertical (*a*) and horizontal (*b*) cross sections of a flat crack embedded in an infinite medium.

Fig. 2 A penny-shaped crack and the coordinate systems.

Fig. 3 The dimensionless temperature $T(\rho, \phi, 0)/\Theta_3^0$ as a function of ρ/a .

Fig. 4 The dimensionless temperature $T(\rho, \phi, z)/\Theta_3^0$ as a function of z/a on the z -axis ($\rho = a$) and various cylindrical surfaces $\rho = a, 2a$ and $10a$.

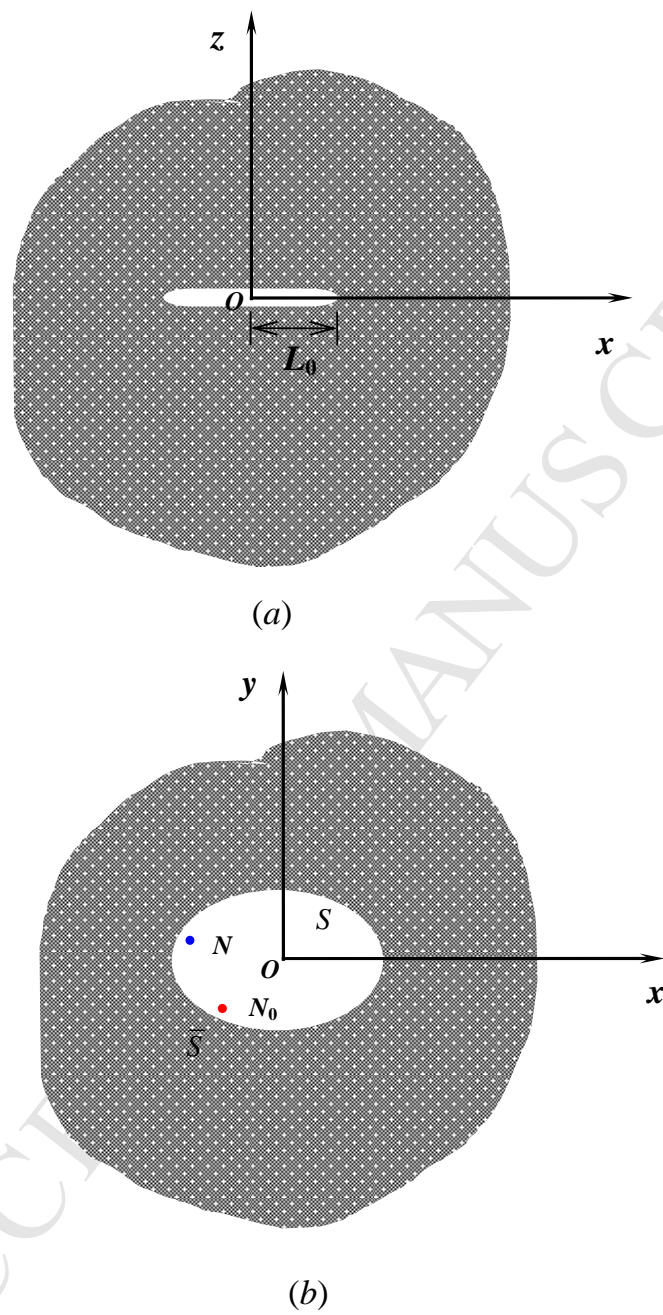


Fig. 1 Vertical (a) and horizontal (b) cross sections of a flat crack embedded in an infinite medium.

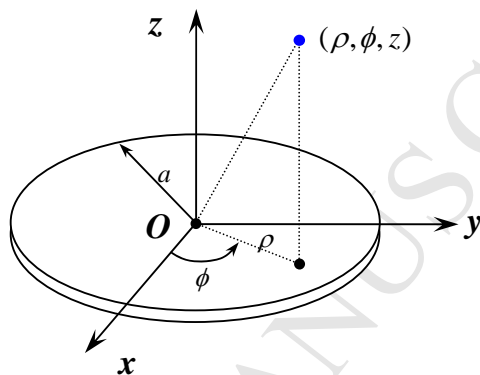


Fig. 2 A penny-shaped crack and the coordinate systems.

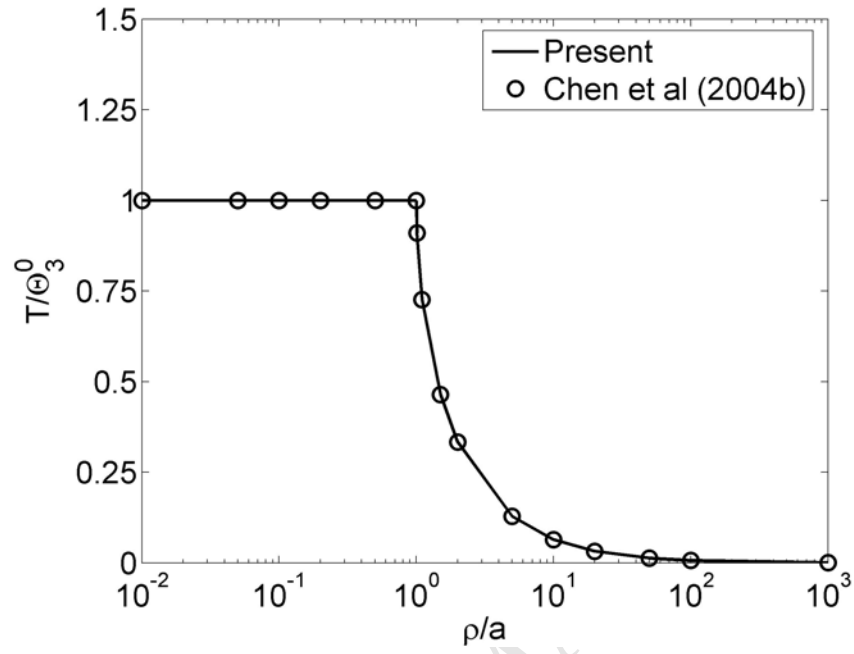


Fig. 3 The dimensionless temperature $T(\rho, \phi, 0)/\Theta_3^0$ as a function of ρ/a .

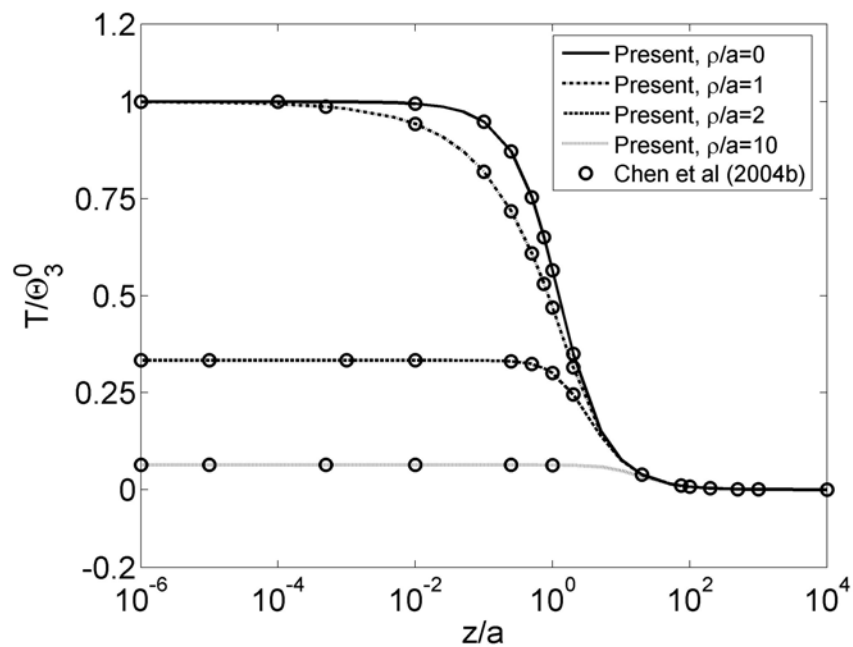


Fig. 4 The dimensionless temperature $T(\rho, \phi, z)/\Theta_3^0$ as a function of z/a on the z -axis ($\rho = 0$) and various cylindrical surfaces $\rho = a$, $2a$ and $10a$.