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A NEW APPROACH TO QUANTITATIVE PROPAGATION OF CHAOS FOR DRIFT, DIFFUSION AND JUMP PROCESSES

S. MISCHLER, C. MOUHOT, AND B. WENNBERG

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ABSTRACT. This paper is devoted to the study of the mean-field limit for many-particle systems undergoing jump, drift or diffusion processes, as well as combinations of them. The main results are quantitative estimates on the decay of fluctuations around the deterministic limit and of correlations between particles, as the number of particles goes to infinity. To this end we introduce a general functional framework which reduces this question to the one of proving a purely functional estimate on some abstract generator operators (*consistency estimate*) together with fine stability estimates on the flow of the limiting non-linear equation (*stability estimates*). Then we apply this method to a Boltzmann collision jump process (for Maxwell molecules), to a McKean-Vlasov drift-diffusion process and to an inelastic Boltzmann collision jump process with (stochastic) thermal bath. To our knowledge, our approach yields the first such quantitative results for a combination of jump and diffusion processes.

Mathematics Subject Classification (2000): 76P05 Rarefied gas flows, Boltzmann equation [See also 82B40, 82C40, 82D05], 76T25 Granular flows [See also 74C99, 74E20], 60J75 Jump processes, 60J60 Diffusion processes [See also 58J65].

Keywords: mean-field limit; quantitative; fluctuations; Boltzmann equation; McKean-Vlasov equation; drift-diffusion; inelastic collision; granular gas.

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1. INTRODUCTION AND MAIN RESULTS

1.1. Mean-field limit for particles systems and main result. The aim of this paper is to investigate the propagation of chaos for various particle systems undergoing jump, drift and diffusion processes. Here and below, E denotes the states space of one particle, assumed to be a separable and locally compact metric space (however in the applications we always take $E = \mathbb{R}^d$, $d \geq 1$). Moreover given any metric space Z , we define $C_b(Z)$ the space of continuous and bounded functions on Z and $P(Z)$ the space of Borel probabilities on Z .

The notion of *chaoticity*, as introduced by Kac in [15] is our starting point in this paper. A sequence $(f^N)_{N \geq 1}$, $f^N \in P_{\text{sym}}(E^N)$ (the set of symmetric probability measures on E^N , see later for a precise definition) is said to be f -chaotic for some given one-particle probability $f \in P(E)$, if for any $\ell \in \mathbb{N}^*$ and any $\varphi \in C_b(E)^{\otimes \ell}$ there holds

$$\lim_{N \rightarrow \infty} \langle f^N, \varphi \otimes \mathbf{1}^{N-\ell} \rangle = \langle f^{\otimes \ell}, \varphi \rangle.$$

Here, the brackets $\langle \cdot, \cdot \rangle$ stand for the duality bracket associated to $P(E^N)$ and $C_b(E^N)$ and to $P(E^\ell)$ and $C_b(E^\ell)$ respectively. In other words, and more precisely, the above convergence means

$$\int_{E^N} \varphi(V) (f^N - f^{\otimes N})(dV) \xrightarrow{N \rightarrow \infty} 0,$$

where $V = (v_1, \dots, v_N) \in E^N$, $f^{\otimes N}(dV) = f(dv_1) \cdots f(dv_N)$ and for any $\varphi(V) = \varphi_1(v_1) \cdots \varphi_\ell(v_\ell)$, $\varphi_j \in C_b(E)$. Roughly speaking it means that $f^N \sim f^{\otimes N}$ when $N \rightarrow \infty$, or in other (probabilistic) words f^N is the law of a sequence of stochastic variables which are asymptotically independents. However note that the sense of this convergence is to be understood in terms of the weak topology for any projection over a *finite number* of marginals.

In the present paper, we will deal with *quantified chaoticity*, in the sense that we prove the above limit but also measure precisely its rate of convergence. Namely, we say that $(f^N)_{N \geq 1}$ is f -chaotic with rate $\varepsilon(N)$, where $\varepsilon(N) \rightarrow 0$ when $N \rightarrow \infty$ (typically $\varepsilon(N) = N^{-r}$, $r > 0$), in the duality sense associated to some normed space of *smooth functions* $\mathcal{F} \subset C_b(E)$, if for any $\ell \in \mathbb{N}^*$ there exists $K_\ell \in (0, \infty)$ such that for any $\varphi \in \mathcal{F}^{\otimes \ell}$, $\|\varphi\|_{\mathcal{F}} \leq 1$, there holds

$$(1.1) \quad \forall N \geq \ell, \quad |\langle f^N - f^{\otimes N}, \varphi \otimes \mathbf{1}^{N-\ell} \rangle| \leq K_\ell \varepsilon(N).$$

Roughly speaking, our main results read as follows. We consider

- an initial distribution of one (typical) particle $f_{in} \in P(E)$ (with possibly additional moments bounded condition),
- an N -particle system evolution $f_t^N \in P_{\text{sym}}(E^N)$ associated to the factorized (and then f_{in} -chaotic) initial data $f_{in}^N = f_{in}^{\otimes N}$,
- a one particle evolution $f_t \in P(E)$ associated to the initial datum f_{in} .

We prove the following *quantified chaoticity estimate of the N -particle system towards its mean-field limit*:

$$(1.2) \quad \sup_{t \in [0, T]} |\langle f_t^N - f_t^{\otimes N}, \varphi \otimes \mathbf{1}^{N-\ell} \rangle| \leq \varepsilon(N)$$

where

- $T \in (0, +\infty)$,
- $\varphi = \varphi_1 \otimes \cdots \otimes \varphi_\ell$, $\varphi_j \in \mathcal{F} \subset C_b(E)$, $\|\varphi_j\|_{\mathcal{F}} \leq 1$,
- $N \geq 2\ell$,
- $\varepsilon(N) = K_\ell \bar{\varepsilon}(N) \rightarrow 0$ when $N \rightarrow \infty$.

Estimate (1.2) is thus a quantitative estimate on the amplitude of the fluctuations around the deterministic limit as time evolves, in the many-particle asymptotic. In probabilistic words, it is a quantitative version of the weak law of large numbers for an underlying stochastic process associated to the N -particle dynamic. This estimate implies the propagation of chaos as stated above, provided that \mathcal{F} is dense in $C_b(E)$ with respect to the topology of uniform convergence on compact sets.

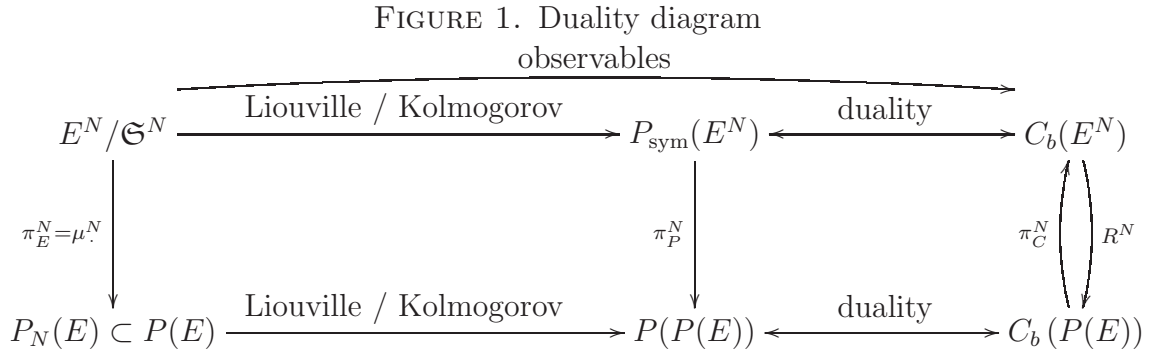
We first establish that quantitative estimate in a simple abstract framework by making general assumptions on both the N -particle dynamic and the (nonlinear) mean-field limit dynamic of one typical particle. In this way, the problem is then reduced to proving

- (A) a purely functional estimate on the dual generator G^N of the N -particle dynamics which establishes that at a first order G^N is linked to the (possibly nonlinear) mean-field limit generator Q (*consistency estimate*);
- (B) together with some fine stability estimates on the flow of the mean field limit equation (*stability estimates*).

Point (A) of our method is largely inspired and generalized from the “duality viewpoint of Grünbaum’s paper [14]. Point (B) and the combination of points (A) and (B) is completely new to our knowledge, and motivates the development of new (again, to our knowledge) stability estimates for the limiting nonlinear equations. Note also that on the contrary to many (probabilistic) approaches we make minimal assumptions on the N -particle dynamics, at the price of stronger (but realistic in many situations!) assumptions on the nonlinear dynamics of one particle in the mean-field limit.

1.2. Short review of results of propagation of chaos and novelty of this paper. We first illustrate our propagation of chaos approach on the Boltzmann model for maxwellian molecules and Grad’s angular cut-off. For such a bounded kernel case the result is well-known since the pioneering works of Kac [15, 16] and McKean [18] (who prove the propagation of chaos without any rate) and from the works by Graham and Méléard [11, 12, 13, 19] (who establish the propagation of chaos with rate $\mathcal{O}(1/\sqrt{N})$). In these papers, the cornerstone of the proof is a combinatorial argument applied to the equation on the law (Wild sum expansion) or to the stochastic flow (stochastic tree). These approaches are restricted to a constant (or at least bounded) collision rate.

We next illustrate our method on the McKean-Vlasov model. For such a model again, propagation of chaos is well-known and has been extensively studied. One of the most popular and efficient approach to deal with this model is the so-called “*coupling method*” introduced in the 1970’s, which yields the optimal convergence rate $\mathcal{O}(1/\sqrt{N})$. We refer to the lecture notes [24, 19] as well as to the references therein for a detailed discussion of that method.



As a third example, we illustrate our method on a mixed jump-diffusive equation which arises from granular gas modeling. For such a model, it seems that both the “*combinatory method*” and the “*coupling method*” fail while our present method is robust enough to apply and yield quantitative chaos estimates. Let us also emphasize that the *BBGKY method* and the *nonlinear martingale method* (see again [24, 19] or [20]) would probably apply but would give a propagation of chaos without any rate since they are based on compactness arguments. In a companion paper [21], we develop our the abstract method in a more general framework in order to (1) apply it to Boltzmann collision models associated to unbounded collisions rates, (2) develop a theory of *uniform in time* propagation of chaos estimates.

1.3. Plan of the paper. In Section 2, we present the method in an abstract framework and we establish the abstract quantitative propagation of chaos Theorem 2.13. In Section 3, we apply the method to the Boltzmann equation associated to the Maxwell molecules collision kernel with Grad’s cut-off. In Section 4, we apply the method to the McKean-Vlasov equation, and finally, in Section 5, it is applied to some mixed jump and diffusion equations which come from the modeling of granular gases.

2. THE ABSTRACT METHOD

In this section we introduce the abstract framework and then state and prove the key abstract result. The assumptions are stated in each subsections where the new notions are introduced.

2.1. The general functional framework of the duality approach. The figure 1 sums up the duality viewpoint (norms and duality brackets are defined in Subsections 2.3).

In this diagram:

- \mathfrak{S}^N denotes the N -permutation group.
- $P_{\text{sym}}(E^N)$ denotes the set of *symmetric probabilities on E^N* : For a given permutation $\sigma \in \mathfrak{S}^N$, a vector $V = (v_1, \dots, v_N) \in E^N$, a function $\phi \in C_b(E^N)$ and a probability measure $f^N \in P(E^N)$ we define $V_\sigma = (v_{\sigma(1)}, \dots, v_{\sigma(N)}) \in E^N$ and $\phi_\sigma \in C_b(E^N)$ by setting $\phi_\sigma(V) = \phi(V_\sigma)$ and finally $f_\sigma^N \in P(E^N)$ by

setting $\langle f_\sigma^N, \phi \rangle = \langle f^N, \phi_\sigma \rangle$; we say that f^N is symmetric if it is invariant under permutations, i.e. $f_\sigma^N = f^N$ for any permutation $\sigma \in \mathfrak{S}^N$.

- For any $V \in E^N$ the probability measure μ_V^N denotes the associated *empirical measure*:

$$(2.1) \quad \mu_V^N = \frac{1}{N} \sum_{i=1}^N \delta_{v_i}, \quad V = (v_1, \dots, v_N)$$

where $\delta_{v_i} \in P(E)$ denotes the Dirac mass on E at point v_i .

- $P_N(E)$ denotes the subset $\{\mu_V^N, V \in E^N\}$ of $P(E)$.
- $P(P(E))$ denotes the set of probabilities $P(E)$ (which is well-defined since $P(E)$ is a Polish space, i.e. a separable completely metrizable topological space, see [2, Theorem 6.8] for instance).
- $C_b(P(E))$ denotes the space of continuous and bounded functions on $P(E)$, this latter being endowed with the weak or strong topologies (see Subsection 2.3).
- The arrow pointing from E^N/\mathfrak{S}^N to $P_N(E)$ denotes the map π_E^N defined by

$$\forall V \in E^N/\mathfrak{S}^N, \quad \pi_E^N(V) := \mu_V^N.$$

- The arrow pointing from $C_b(P(E))$ to $C_b(E^N)$ denotes the following map π_C^N

$$\forall \Phi \in C_b(P(E)), \forall V \in E^N, \quad (\pi_C^N \Phi)(V) := \Phi(\mu_V^N).$$

- The opposite arrow, pointing from $C_b(E^N)$ to $C_b(P(E))$ denotes the transformation R^N defined by:

$$\forall \phi \in C_b(E^N), \forall \rho \in P(E), \quad R^N[\phi](\rho) = R_\phi^N(\rho) := \langle \rho^{\otimes N}, \phi \rangle.$$

- The arrow pointing from $P_{\text{sym}}(E^N)$ to $P(P(E))$ denotes the following transformation: consider $f^N \in P_{\text{sym}}(E^N)$ and define $\pi_P^N f^N \in P(P(E))$ by setting

$$\forall \varphi \in \Phi \in C_b(P(E)), \quad \langle \pi_P^N f^N, \Phi \rangle = \langle f^N, \pi_C^N \Phi \rangle,$$

where the first bracket means corresponds to the duality $(P(P(E)), C_b(P(E)))$ and the second bracket corresponds to the duality $(P(E^N), C_b(E^N))$.

- The arrows pointing from the first column to the second one denote the procedure of either writing the *Liouville* transport equation associated with the set of ODEs of a particle system, or writing the *Kolmogorov* equation for the evolution of the law of a stochastic Markov process of a particles system.
- Finally the dual spaces of the spaces of probabilities on the phase space can be interpreted as the spaces of observables on the original systems. We shall discuss this point later.

Remark 2.1. *In statistics, the simplest notion of empirical measure corresponds to a random measure of the form (2.1) where the v_i are i.i.d. random variables with a distribution $m \in P(E)$; in this sense, an empirical measure is a random map from $P(E)$ to $P_N(E)$. More generally, if V is a random variable $V \in E^N$ with law $f^N \in P(E^N)$, then $\mu_V^N \in P(E)$ is a random measure with law $\pi_P^N f^N \in P(P(E))$. Specialising to $f^N = f^{\otimes N}$, a tensor product, we recover the first notion. In this paper V is time dependent, describing the dynamics of the N -particle system, and*

although it may be natural to assume that its components v_i are independent initially this is certainly not true for positive times. Indeed, the main purpose of the paper is to investigate how independence of a finite number of components is recovered as $N \rightarrow \infty$.

2.2. The evolution semigroups. The evolution of the N -particle systems as well as the mean-field dynamics are described in terms of semigroups, which are defined here. Let $N \geq 1$.

Step 1. Consider a process (\mathcal{V}_t^N) on E^N which describes the trajectories of the particles (Lagrangian viewpoint). The evolution can correspond to stochastic ODEs (Markov process), or deterministic ODEs (deterministic Hamiltonian flow). We make the fundamental assumption that this flow commutes with permutations: for any $\sigma \in \Sigma^N$, the solution at time t starting from $(\mathcal{V}_0^N)_\sigma$ is $(\mathcal{V}_t^N)_\sigma$. This reflects mathematically the fact that particles are indistinguishable.

Step 2. From this flow on E^N we derive a *linear* semigroup S_t^N acting on $P_{\text{sym}}(E^N)$ for the distribution of particles in the phase space E^N . This corresponds to a linear evolution equation

$$(2.2) \quad \partial_t f^N = A^N f^N, \quad f^N \in P_{\text{sym}}(E^N),$$

which can be interpreted as the forward Kolmogorov equation on the law in case where (V_t) is a Markov process, or the Liouville equation on the probability density in case of a Hamiltonian process. As a consequence of the assumption that the flow (\mathcal{V}_t^N) commutes with permutation, S_t^N acts on $P_{\text{sym}}(E^N)$. In other words, if the law f_0^N of \mathcal{V}_0^N belongs to $P_{\text{sym}}(E^N)$, then for any times the law f_t^N of \mathcal{V}_t^N belongs to $P_{\text{sym}}(E^N)$.

Step 3. The *dual* semigroup T_t^N of S_t^N which acts on functions $\phi \in C_b(E^N)$ is defined by

$$\forall f^N \in P(E^N), \phi \in C_b(E^N), \quad \langle f^N, T_t^N(\phi) \rangle := \langle S_t^N(f^N), \phi \rangle.$$

It corresponds to the following *linear* evolution equation (with generator denoted by G^N):

$$(2.3) \quad \partial_t \phi = G^N(\phi), \quad \phi \in C_b(E^N).$$

This is the semigroup of the *observables* on the evolution system (\mathcal{V}_t^N) on E^N .

(A1) Assumption on the N -particle system. We assume that T_t^N and its generator G^N are well defined in $P(E^N)$ and satisfy the symmetry condition introduced in Step 2 above.

Step 4. We consider a semigroup $f_t = S_t^{NL}(f)$ (in general nonlinear) for the 1-particle distribution, corresponding to an evolution equation (in general nonlinear)

$$(2.4) \quad \partial_t f_t = Q(f_t)$$

which is referred to as *the limiting equation* (in our case a *kinetic* equation). The assumptions ensuring that this is indeed the limiting equation in the mean-field limit shall be stated later on, since more abstract objects are needed. Note that this limit can be framed either as f_t^N is f_t -chaotic on some time interval, or

(since the N -particle dynamics is in one-to-one correspondence with the dynamics of the corresponding empirical measures $\mu_{V_t}^N(t)$) as $\mu_{V_t}^N$ converges to f_t on some time interval.

Step 5. From this nonlinear semigroup we derive the *pushforward* semigroup T_t^∞ acting on $C_b(P(E))$ which is defined by:

$$\forall f \in P(E), \Phi \in C_b(P(E)), \quad T_t^\infty[\Phi](f) := \Phi(S_t^{NL}(f)).$$

Note that T_t^∞ is always *linear* as a function of Φ (although of course $T_t^\infty[\Phi](f)$ is not linear as a function of f). We denote its generator by G^∞ , which corresponds to the following *linear* evolution equation on $C_b(P(E))$:

$$(2.5) \quad \partial_t \Phi = G^\infty(\Phi).$$

Remark 2.2. The “Liouville” (resp. “Kolmogorov”) label on the arrows in Figure 1 refers to the relation between a deterministic dynamical system and the Liouville equation (resp. a Markov process and the Kolmogorov equation). Let us explain now the name “observable” and the interpretation of the pushforward semigroup.

Given a dynamical system defined by an autonomous system of ordinary differential equations,

$$(2.6) \quad \frac{dV_t}{dt} = F(V_t), \quad V_0 = V_{in},$$

we may write the Liouville transport equation:

$$(2.7) \quad \partial_t \rho + \nabla_v \cdot (F \rho) = 0,$$

which is equivalent to (2.6) under suitable assumptions on F . We denote the (unique) solution to the system (2.6) as $V_t = V_t(v_0)$. Then, if the initial data V_{in} are distributed according to a probability density $\rho_0(V) dV$, the solution V_t at time t is distributed according to a density $\rho_t(V) dV = \rho_0(V_{-t}(v)) dV$; this assumes of course that V_{-t} is well defined. As long as $V_t(v)$ is continuous in v , one can solve the dual equation

$$(2.8) \quad \partial_t \phi_t - F \cdot \nabla_v \phi_t = 0$$

and we call it the equation of observables. The solution is given $\phi_t(v) = \phi_0(V_t(v))$ and satisfies

$$\langle \phi_t, \rho_0 \rangle = \langle V_t^* \phi_0, \rho_0 \rangle = \langle \phi_0, V_{-t}^* \rho_0 \rangle = \langle \phi_0, \rho_t \rangle.$$

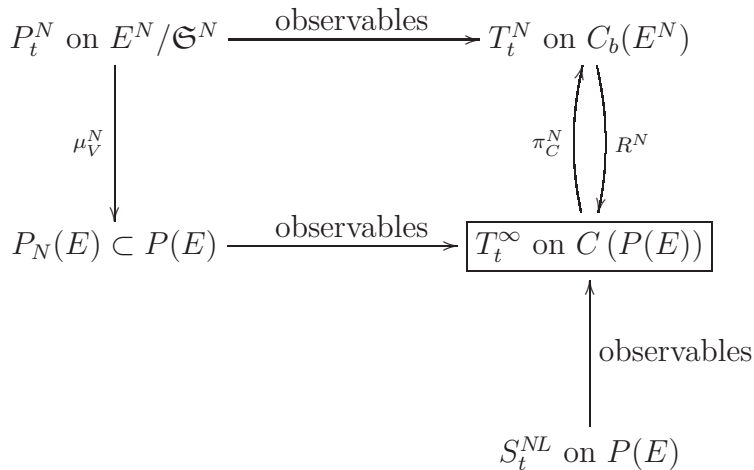
Suppose now that instead of starting with an finite dimensional dynamical system we start with an equation on $P(E)$:

$$(2.9) \quad \frac{\partial f}{\partial t} = Q(f), \quad f_0 = f_{in}.$$

What now plays the role of the application V_t is the semigroup S_t^{NL} . Then formally if f_{in} is distributed according to μ_{in} in $P(P(E))$, the solution f_t at time t will be distributed according to $\mu_t = \mu_{in}(S_{-t}^{NL}(\cdot))$, as long as this is well defined, i.e. μ_t satisfies an infinite dimensional Liouville equation

$$\partial_t \mu + \nabla_f \cdot (Q(f) \mu) = 0, \quad \mu_0 = \mu_{in}$$

FIGURE 2. Connexion between the two dynamics



(everything is formal at the level). And by formal analogy with the finite dimensional case, one may always write the dual equation on $C_b(P(E))$:

$$\partial_t \Phi + Q(f) \cdot \nabla_f \Phi = 0, \quad \Phi_0 = \Phi_{in}$$

(everything is formal again). This is the semigroup of observables of equation (2.9), with $\Phi_t(f) = \Phi_{in}(S_t^{NL} f_{in})$ which is nothing but the pushforward semigroup T^∞ defined in Step 5 above. This dual formulation is convenient since for equations like Boltzmann equation, S_{-t}^{NL} may not be defined due to the irreversibility of the equation.

Solutions to this infinite dimensional Liouville equations are known as statistical solutions to (2.9), and have been studied in e.g. [1, 9] for the Boltzmann equation.

Summing up we obtain the diagram in Figure 2 for the semigroups. Hence a key point of our construction is that, through the evolution of *observables* over the nonlinear limit equation and the N -particle system, one can “interface” the two evolution systems *via* the maps π_C^N and R^N . From now we shall denote $\pi^N = \pi_C^N$.

2.3. The metric issue. $P(E)$ is our fundamental “state space”, where we shall compare the marginals of the N -particle density f_t^N and the chaotic infinite particle dynamics $f_t^{\otimes N}$ through their observables.

Since we shall work with continuous functions on $P(E)$, the choice of topology is important. There are canonical choices (which determine two different sets $C(P(E))$): (1) the strong topology (associated to the total variation norm) and (2) the weak topology (that is the trace on $P(E)$ of the weak topology $\sigma(M^1(E), C_b(E))$ where M^1 denotes the space of Radon measures on E with finite mass).

On the one hand, for a given locally compact and separable metric space Z , the space $M^1(Z)$ of finite Borel measures on Z is a Banach space when it is endowed with the total variation norm:

$$\begin{aligned}
 \forall f \in M^1(Z), \quad \|f\|_{TV} &:= f^+(Z) + f^-(Z) \\
 &= \sup_{\phi \in C_b(Z), \|\phi\|_\infty \leq 1} \langle f, \phi \rangle = \sup_{\phi \in C_0(Z), \|\phi\|_\infty \leq 1} \langle f, \phi \rangle,
 \end{aligned}$$

where $f = f^+ - f^-$ stands for the Hahn decomposition and the equality between the two last terms comes from the fact that Z is locally compact and separable. We recall that $f_k \xrightarrow{TV} f$ (strong topology) when (f_k) and f belongs to $M^1(Z)$ and $\|f_k - f\|_{TV} \rightarrow 0$ when $k \rightarrow \infty$. We also recall (f_k) in $M^1(Z)$ converges weakly to $f \in M^1(Z)$, written $f_k \rightharpoonup f$, if

$$\forall \varphi \in C_b(Z) \quad \langle f, \varphi \rangle = \lim_{k \rightarrow \infty} \langle f_k, \varphi \rangle,$$

and we denote by $\sigma(M^1(Z), C_b(Z))$ the associated topology. The key observation for the sequel is that this notion of convergence is metrizable with different, non-equivalent metrics, and which such metric we choose plays an important role as soon as one wants to perform *differential calculus* on $P(Z)$.

The set $C_b(P(E))$ depends on the choice of topology on $P(E)$. In the sequel, we will denote $C_b(P(E), w)$ the space of continuous and bounded functions on $P(E)$ endowed with the weak topology, and $C_b(P(E), TV)$ the similar space on $P(E)$ endowed with the total variation norm. It is clear that $C_b(P(E), w) \subset C_b(P(E), TV)$ since $f_k \xrightarrow{TV} f$ implies $f_k \rightharpoonup f$.

However, the supremum norm $\|\Phi\|_{L^\infty(P(E))}$ does not depend on the choice of topology on $P(E)$, and endows $C_b(P(E))$ with a Banach space topology. The transformations π^N and R^N satisfy:

$$(2.10) \quad \|\pi^N \Phi\|_{L^\infty(E^N)} \leq \|\Phi\|_{L^\infty(P(E))} \quad \text{and} \quad \|R^N[\phi]\|_{L^\infty(P(E))} \leq \|\phi\|_{L^\infty(E^N)}.$$

The transformation π^N is well defined from $C_b(P(E), w)$ to $C_b(E^N)$, but it does not map $C_b(P(E), TV)$ into $C_b(E^N)$.

In the other way round, the transformation R^N is well defined from $C_b(E^N)$ to $C_b(P(E), w)$, and therefore also from $C_b(E^N)$ to $C_b(P(E), TV)$: for any $\phi \in C_b(E^N)$ and for any sequence $f_k \rightharpoonup f$ weakly, we have $f_k^{\otimes N} \rightharpoonup f^{\otimes N}$ weakly, and then $R^N[\phi](f_k) \rightarrow R^N[\phi](f)$.

The different metric structures associated with the weak topology are not seen at the level of $C_b(P(E), w)$. However any norm (or semi-norm) “more regular” than the uniform norm on $C_b(P(E))$ (in the sense of controlling some modulus of continuity or some differential) strongly depends on this choice, as the abstract Lipschitz spaces defined below illustrate.

Definition 2.3. For given functions $m_G : E \rightarrow \mathbb{R}_+$ we define the following weighted subspaces of probabilities

$$P_G := \{f \in P(E); \langle f, m_G \rangle < \infty\},$$

and the corresponding vectorial spaces of “increments” are defined as

$$\mathcal{I}P_G := \{f_1 - f_2 ; f_1, f_2 \in P_G\}.$$

Then the distance $dist_G$ we consider can be either be directly defined on $P_G(E)$, or (as in many situations we shall consider) it can be moreover inherited from a larger normed vector space structure in the following sense: there is a vector space $\mathcal{G} \supset \mathcal{I}P_G$ endowed with a norm $\|\cdot\|_{\mathcal{G}}$ such that $dist_G$ is defined on P_G by

$$\forall f_1, f_2 \in P_G, \quad dist_G(f_1, f_2) := \|f_1 - f_2\|_{\mathcal{G}}.$$

We say that $P_{\mathcal{G}}$ has a **bounded diameter** if there exists $K_{\mathcal{G}} > 0$ such that

$$\forall f \in P_{\mathcal{G}}, \quad \text{dist}_{\mathcal{G}}(f, g) \leq K_{\mathcal{G}}$$

for some given fixed $g \in P_{\mathcal{G}}$.

Finally, we say that two metrics d_0 and d_1 on $P_{\mathcal{G}}$ are **topologically uniformly equivalent on bounded sets** if there exists $\kappa \in (0, \infty)$ and for any $a \in (0, \infty)$ there exists $C_a \in (0, \infty)$ such that

$$\forall i = 0, 1 \quad \forall f_1, f_2 \in \mathcal{B}P_{\mathcal{G},a} \quad d_i(f_1, f_2) \leq C_a [d_{1-i}(f_1, f_2)]^{\kappa}$$

where

$$\mathcal{B}P_{\mathcal{G},a} := \{f \in P_{\mathcal{G}} ; \langle f, m_{\mathcal{G}} \rangle \leq a\}.$$

If such that d_0 and d_1 are resulting from some normed spaces \mathcal{G}_0 and \mathcal{G}_1 , we (slightly abusively) say that \mathcal{G}_0 and \mathcal{G}_1 are **topologically uniformly equivalent (on bounded sets)**.

We also define the vector space $UC(P_{\mathcal{G}}(E); \mathbb{R})$ of uniformly continuous and bounded function $\Psi : P_{\mathcal{G}} \rightarrow \mathbb{R}$, where the continuity is related the metric topology on $P_{\mathcal{G}}$ defined by $\text{dist}_{\mathcal{G}}$ above. Observe that this is a Banach space when endowed with the supremum norm.

Example 2.4. With the choice $m_{\mathcal{G}} := 1$, $\|\cdot\|_{\mathcal{G}} := \|\cdot\|_{TV}$ and $P_{\mathcal{G}}(E) = P(E)$.

Example 2.5. There exist many ways to define distances on $P(E)$ which are topologically equivalent to the weak topology of measures, see for instance [23, 7]. In subsection 2.12 below, a selection of such distances of particular interest are presented. They are all topologically uniformly equivalent (at least on bounded sets of $P(E)$, when the bounded sets are defined as above thanks to a suitable weight function $m_{\mathcal{G}}$).

2.4. Differential calculus for functions of probability measures. We start with a definition of Lipschitz regularity, for which a mere metric structure is sufficient.

Definition 2.6. For metric spaces $\tilde{\mathcal{G}}_1$ and $\tilde{\mathcal{G}}_2$ we denote by $C^{0,1}(\tilde{\mathcal{G}}_1, \tilde{\mathcal{G}}_2)$ the space of functions from $\tilde{\mathcal{G}}_1$ to $\tilde{\mathcal{G}}_2$ with Lipschitz regularity, i.e. the set of functions $\Psi : \tilde{\mathcal{G}}_1 \rightarrow \tilde{\mathcal{G}}_2$ such that there exists a constant $C > 0$ so that

$$(2.11) \quad \forall f, g \in \tilde{\mathcal{G}}_1 \quad \text{dist}_{\tilde{\mathcal{G}}_2}(\Psi(g), \Psi(f)) \leq C \text{dist}_{\tilde{\mathcal{G}}_1}(g, f).$$

We then define the semi-norm $[\cdot]_{C^{0,1}(\tilde{\mathcal{G}}_1, \tilde{\mathcal{G}}_2)}$ on $C^{0,1}(\tilde{\mathcal{G}}_1, \tilde{\mathcal{G}}_2)$ as the infimum of the constants $C > 0$ such that (2.11) holds.

Let us now define a higher order differential calculus; this is where our assumption that our distances are inherited from a normed vector space structure shall play a role.

Definition 2.7. Consider some normed spaces \mathcal{G}_1 and \mathcal{G}_2 , and some metric sets $\tilde{\mathcal{G}}_1$ and $\tilde{\mathcal{G}}_2$ such that $\tilde{\mathcal{G}}_i - \tilde{\mathcal{G}}_i \subset \mathcal{G}_i$, and some $k \in \mathbb{N}$. We define $C^{k,1}(\tilde{\mathcal{G}}_1; \tilde{\mathcal{G}}_2)$, the space of functions k -times continuously differentiable from $\tilde{\mathcal{G}}_1$ to $\tilde{\mathcal{G}}_2$, with bounded derivatives, and such that the k -th derivative is Lipschitz continuous in the sense of Definition 2.6.

More explicitly, this is the set of functions $\Psi : \tilde{\mathcal{G}}_1 \rightarrow \tilde{\mathcal{G}}_2$ continuous such that there exists $D^j \Psi : \tilde{\mathcal{G}}_1 \rightarrow \mathcal{B}^j(\mathcal{G}_1, \mathcal{G}_2)$ continuous and bounded, where $\mathcal{B}^j(\mathcal{G}_1, \mathcal{G}_2)$ is the space of bounded j -multilinear applications from \mathcal{G}_1 to \mathcal{G}_2 (endowed with its canonical norm) for $j = 1, \dots, k$, and some constants $C_j > 0$, $j = 1, \dots, k$, so that for any $j = 1, \dots, k$

$$(2.12) \quad \forall f, g \in \tilde{\mathcal{G}}_1 \quad \left\| \Psi(g) - \sum_{i=0}^j \langle D^i \Psi(f), (g-f)^{\otimes i} \rangle \right\|_{\mathcal{G}_2} \leq C_j \|g-f\|_{\mathcal{G}_1}^{j+1}$$

(with the convention $D^0 \Psi = \Psi$).

We define the following seminorms on $C^{k,1}(\tilde{\mathcal{G}}_1, \tilde{\mathcal{G}}_2)$

$$[\Psi]_{j,0} := \sup_{f \in \tilde{\mathcal{G}}_1} \|D^j \Psi(f)\|_{\mathcal{B}^j(\mathcal{G}_1, \mathcal{G}_2)}, \quad j = 1, \dots, k,$$

with

$$\|L\|_{\mathcal{B}^j(\mathcal{G}_1, \mathcal{G}_2)} := \sup_{h_i, \|h_i\|_{\mathcal{G}_1} \leq 1, 1 \leq i \leq j} \|L(h_1, \dots, h_j)\|_{\mathcal{G}_2},$$

and

$$[\Psi]_{j,1} := \sup_{f, g \in \tilde{\mathcal{G}}_1} \frac{\left\| \Psi(g) - \sum_{i=0}^j \langle D^i \Psi(f), (g-f)^{\otimes i} \rangle \right\|_{\mathcal{G}_2}}{\|g-f\|_{\mathcal{G}_1}^{j+1}}.$$

Finally we combine these semi-norms into the norm

$$\|\Psi\|_{C^{k,1}(\tilde{\mathcal{G}}_1, \tilde{\mathcal{G}}_2)} = \sum_{j=1}^k [\Psi]_{j,0} + [\Psi]_{k,1}.$$

The following lemma confirms that this differential calculus is well-behaved for composition, which seems a minimal requirement for further applications.

Lemma 2.8. Consider $\mathcal{U} \in C^{k,1}(\tilde{\mathcal{G}}_1, \tilde{\mathcal{G}}_2)$ and $\mathcal{V} \in C^{k,1}(\tilde{\mathcal{G}}_2, \tilde{\mathcal{G}}_3)$.

Then the composition $\Psi := \mathcal{V} \circ \mathcal{U}$ belongs to $C^{k,1}(\tilde{\mathcal{G}}_1, \tilde{\mathcal{G}}_3)$.

Moreover the following chain rule holds at first order $k = 1$

$$(2.13) \quad \forall f \in \tilde{\mathcal{G}}_1, \quad D\Psi[f] = D\mathcal{V}[\mathcal{U}(f)] \circ D\mathcal{U}[f],$$

with the estimates

$$[\Psi]_{0,1} \leq [\mathcal{V}]_{0,1} [\mathcal{U}]_{0,1}, \quad [\Psi]_{1,0} \leq [\mathcal{V}]_{1,0} [\mathcal{U}]_{1,0}, \quad [\Psi]_{1,1} \leq [\mathcal{V}]_{1,0} [\mathcal{U}]_{1,1} + [\mathcal{V}]_{1,1} [\mathcal{U}]_{0,1}^2.$$

At second order $k = 2$ one also has the chain rule

$$(2.14) \quad \forall f \in \tilde{\mathcal{G}}_1, \quad D^2\Psi[f] = D^2\mathcal{V}[\mathcal{U}(f)] \circ (D\mathcal{U}[f] \otimes D\mathcal{U}[f]) + D\mathcal{V}[\mathcal{U}(f)] \circ D^2\mathcal{U}[f].$$

Proof of Lemma 2.8. It is straightforward by writing and compounding the expansions of \mathcal{U} and \mathcal{V} provided by Definition 2.7. \square

2.5. Identification of the generator of the pushforward semigroup. As a first example of application of this differential calculus, we compute the generator of the pushforward limiting semigroup, under the following assumptions:

(A2) Nonlinear semigroup. Consider a probability space $P_{\mathcal{G}_1}(E)$ (defined in Definition 2.3) associated to a weight function $m_{\mathcal{G}_1}$, endowed with the metric induced from a normed space \mathcal{G}_1 , and with bounded diameter. Assume that for any $\tau > 0$ we have:

(i) The equation (2.4) generates a continuous semigroup S_t^{NL} on $P_{\mathcal{G}_1}$ which is furthermore uniformly Lipschitz continuous: there exists C_τ such that

$$\forall f, g \in P_{\mathcal{G}_1} \quad \sup_{t \in [0, \tau]} \text{dist}_{\mathcal{G}_1}(S_t^{NL} f, S_t^{NL} g) \leq C_\tau \text{dist}_{\mathcal{G}_1}(f, g).$$

(ii) There exists $\delta \in (0, 1]$ such that the (possibly nonlinear) generator Q (introduced in equation (2.4)) is δ -Hölder continuous from $P_{\mathcal{G}_1}$ into \mathcal{G}_1 :

$$\exists L \in (0, +\infty) \text{ s.t. } \forall f, g \in P_{\mathcal{G}_1}, \quad \|Q(f) - Q(g)\|_{\mathcal{G}_1} \leq L \|f - g\|_{\mathcal{G}_1}^\delta.$$

Then we have the following lemma.

Lemma 2.9. *Under assumption (A2) the pushforward semigroup T_t^∞ is a contraction semigroup on the Banach space $UC(P_{\mathcal{G}_1}(E); \mathbb{R})$ and its generator G^∞ is an unbounded linear operator on $UC(P_{\mathcal{G}_1}(E); \mathbb{R})$ with domain $\text{Dom}(G^\infty)$ containing $C^{1,1}(P_{\mathcal{G}_1}(E); \mathbb{R})$. It is defined by*

$$(2.15) \quad \forall \Phi \in C^{1,1}(P_{\mathcal{G}_1}(E); \mathbb{R}), \quad \forall \rho \in P_{\mathcal{G}_1}(E), \quad (G^\infty \Phi)(\rho) = \langle D\Phi[\rho], Q(\rho) \rangle.$$

Proof of Lemma 2.9. The proof is split into several steps.

Step 1. We claim that for any $f_{in} \in P_{\mathcal{G}_1}$ and $\tau > 0$ the map

$$\mathcal{S}(f_{in}) : [0, \tau] \rightarrow P_{\mathcal{G}_1}, \quad t \mapsto \mathcal{S}_t^{NL}(f_{in})$$

is right-differentiable at $t = 0^+$ with $\mathcal{S}(f_{in})'(0) = Q(f_{in})$. Denote $f_t := S_t^{NL} f_{in}$. First, $Q(f_t)$ is bounded in \mathcal{G}_1 uniformly on $t \in [0, \tau]$ and $f_{in} \in P_{\mathcal{G}_1}$ since

$$\begin{aligned} \sup_{[0, \tau]} \|Q(f_t)\|_{\mathcal{G}_1} &\leq \sup_{[0, \tau]} \|Q(f_t) - Q(f_{in})\|_{\mathcal{G}_1} + \|Q(f_{in})\|_{\mathcal{G}_1} \\ &\leq \sup_{[0, \tau]} L \|f_t - f_{in}\|_{\mathcal{G}_1} + \|Q(f_{in})\|_{\mathcal{G}_1} =: K, \end{aligned}$$

and the first term is bounded because $t \mapsto f_t$ is bounded thanks to (A2i). We deduce that uniformly on $f_{in} \in P_{\mathcal{G}_1}$

$$(2.16) \quad \|f_t - f_{in}\|_{\mathcal{G}_1} = \left\| \int_0^t Q(f_s) ds \right\|_{\mathcal{G}_1} \leq K t,$$

and then using **(A2ii)** and that inequality (2.16)

$$\begin{aligned} \|f_t - f_{in} - tQ(f_{in})\|_{\mathcal{G}_1} &= \left\| \int_0^t (Q(f_s) - Q(f_{in})) ds \right\|_{\mathcal{G}_1} \\ &= L \int_0^t \|f_s - f_{in}\|_{\mathcal{G}_1}^\delta ds \\ &\leq L \int_0^t (Ks)^\delta ds = L K^\delta \frac{t^{1+\delta}}{1+\delta}, \end{aligned}$$

which clearly implies the claim.

Step 2. We claim that (T_t^∞) is a C_0 -semigroup of linear and bounded operators on $UC(P_{\mathcal{G}_1}(E); \mathbb{R})$. Indeed, first for any $\Phi \in UC(P_{\mathcal{G}_1}(E); \mathbb{R})$ and denoting by ω_Φ the modulus of continuity of Φ , we have

$$\begin{aligned} |(T_t^\infty \Phi)(g) - (T_t^\infty \Phi)(f)| &= |\Phi(S_t^{NL}(g)) - \Phi(S_t^{NL}(f))| \\ &\leq \omega_\Phi(\text{dist}_{\mathcal{G}_1}(S_t^{NL}(g), S_t^{NL}(f))) \\ &\leq \omega_\Phi(C_\tau \text{dist}_{\mathcal{G}_1}(f, g)) \end{aligned}$$

so that $T_t^\infty \Phi \in UC(P_{\mathcal{G}_1}(E); \mathbb{R})$ for any $t \geq 0$. Next, we have

$$\|T_t^\infty\| = \sup_{\|\Phi\| \leq 1} \|T_t^\infty \Phi\| = \sup_{\|\Phi\| \leq 1} \sup_{f \in P_{\mathcal{G}_1}} |\Phi(S_t^{NL}(f))| \leq 1, \quad \|\Phi\| = \sup_{h \in P_{\mathcal{G}_1}} |\Phi(h)|.$$

Finally, from (2.16), for any $\Phi \in UC(P_{\mathcal{G}_1}(E); \mathbb{R})$, we have

$$\|T_t^\infty \Phi - \Phi\| = \sup_{f \in P_{\mathcal{G}_1}} |\Phi(S_t^{NL}(f)) - \Phi(f)| \leq \omega_\Phi(Kt) \rightarrow 0 \quad \text{as } t \rightarrow 0^+.$$

As a consequence, Hille-Yosida Theorem (see for instance [22, Theorem 3.1]) implies that (T_t^∞) is associated to a closed generator G^∞ with dense domain $\text{dom}(G^\infty) \subset UC(P_{\mathcal{G}_1}(E); \mathbb{R})$.

Step 3. Let us propose a candidate for this generator (at least defined on a subset of its domain). Let \tilde{G}^∞ be defined by

$$\forall \Phi \in C^{1,1}(P_{\mathcal{G}_1}; \mathbb{R}), \forall f \in P_{\mathcal{G}_1}, \quad (\tilde{G}^\infty \Phi)(f) := \langle D\Phi[f], Q(f) \rangle.$$

The RHS is well defined since $D\Phi(f) \in \mathcal{B}(\mathcal{G}_1, \mathbb{R}) = \mathcal{G}'_1$ and $Q(f) \in \mathcal{G}_1$ by assumption. Moreover, since both $f \mapsto D\Phi[f]$ and $f \mapsto Q(f)$ are uniformly continuous, so is the map $f \mapsto (\tilde{G}^\infty \Phi)(f)$. It yields $\tilde{G}^\infty \Phi \in UC(P_{\mathcal{G}_1}(E); \mathbb{R})$.

Step 4. Finally, by composition,

$$\forall f \in P_{\mathcal{G}_1}(E), \quad t \mapsto T_t^\infty \Phi(f) = \Phi \circ S_t^{NL}(f)$$

is right-differentiable at $t = 0^+$ and

$$\begin{aligned} \frac{d}{dt}(T_t^\infty \Phi)(\rho)|_{t=0} &:= \frac{d}{dt}(\Phi \circ \mathcal{S}(\rho)(t))|_{t=0} \\ &= \langle D\Phi(\mathcal{S}(\rho)(0)), \frac{d}{dt}\mathcal{S}(\rho)(0) \rangle \\ &= \langle D\Phi[\rho], Q(\rho) \rangle = (\tilde{G}^\infty \Phi)(\rho), \end{aligned}$$

which implies $\Phi \in \text{Dom}(G^\infty)$ and (2.15). \square

2.6. Polynomial functions in $C_b(E)$. There is a natural notion of polynomial functions in the space $C_b(E)$.

Definition 2.10. We call **polynomial over probabilities** a function $R^\ell[\varphi]$ with $\varphi = \varphi_1 \times \cdots \times \varphi_\ell$, $\varphi_i \in C_b(E)$ and $\ell \in \mathbb{N}$, and where R^ℓ is the transformation already defined:

$$\forall f \in P(E), \quad \langle R^\ell[\varphi], f \rangle := \int \varphi f.$$

Observe that the set of polynomial defines an algebra (as expected from such a notion).

Let us now study the smoothness of such particular elements of $C_b(P(E))$ in the sense of the preceding differential calculus (depending on the smoothness of φ used to construct the polynomial). We need first some preliminary definitions.

Definition 2.11. (i) We say that a pair $(\mathcal{F}, \mathcal{G})$ of normed vector spaces such that $\mathcal{F} \subset C_b(E)$ and $P(E) - P(E) \subset \mathcal{G}$ is “in duality” if

$$(2.17) \quad \forall f, g \in P(E), \forall \phi \in \mathcal{F} \quad | \langle (f - g), \phi \rangle | \leq C \|f - g\|_{\mathcal{G}} \|\phi\|_{\mathcal{F}}.$$

(ii) More generally we say that a pair $(\mathcal{F}, P_{\mathcal{G}})$ of a normed vector space $\mathcal{F} \subset C_b(E)$ endowed with the norm $\|\cdot\|_{\mathcal{F}}$ and a probability space $P_{\mathcal{G}} \subset P(E)$ endowed with a metric $d_{\mathcal{G}}$ are in duality if

$$(2.18) \quad \forall f, g \in P_{\mathcal{G}}, \forall \phi \in \mathcal{F} \quad | \langle g - f, \phi \rangle | \leq C d_{\mathcal{G}}(f, g) \|\phi\|_{\mathcal{F}}.$$

Lemma 2.12. If the situation (i) above holds the polynomial function $R^\ell[\varphi]$ is of class $C^{k,1}(P_{\mathcal{G}}, \mathbb{R})$ for any $k \geq 0$.

In the more general situation (ii) above the polynomial function $R^\ell[\varphi]$ is at least of class $C^{0,1}(P_{\mathcal{G}}, \mathbb{R})$.

Proof. The proof mainly follows from the multilinearity of R . Indeed, defining

$$\mathcal{G} \rightarrow \mathbb{R}, \quad h \mapsto DR_\varphi^\ell[f](h) := \sum_{i=1}^{\ell} \left(\prod_{j \neq i} \langle \varphi_j, f \rangle \right) \langle \varphi_i, h \rangle,$$

we may write

$$R_\varphi^\ell(f_2) - R_\varphi^\ell(f_1) = \sum_{i=1}^{\ell} \left(\prod_{1 \leq k < i} \langle \varphi_k, f_2 \rangle \right) \langle \varphi_i, f_2 - f_1 \rangle \left(\prod_{i < k \leq \ell} \langle \varphi_k, f_1 \rangle \right),$$

and

$$\begin{aligned} & R_\varphi^\ell(f_2) - R_\varphi^\ell(f_1) - DR_\varphi^\ell[f_1](f_2 - f_1) = \\ &= \sum_{1 \leq j < i \leq \ell} \left(\prod_{1 \leq k < j} \langle \varphi_k, f_2 \rangle \right) \langle \varphi_j, f_2 - f_1 \rangle \left(\prod_{j < k < i} \langle \varphi_k, f_1 \rangle \right) \langle \varphi_i, f_2 - f_1 \rangle \left(\prod_{i < k \leq \ell} \langle \varphi_k, f_1 \rangle \right). \end{aligned}$$

We deduce then

$$\begin{aligned} |R_\varphi^\ell(f_2) - R_\varphi^\ell(f_1)| &\leq \|\varphi\|_{1, \mathcal{F} \otimes (L^\infty)^{\ell-1}} \|f_2 - f_1\|_{\mathcal{G}}, \\ |DR_\varphi^\ell[f_1](h)| &\leq \|\varphi\|_{1, \mathcal{F} \otimes (L^\infty)^{\ell-1}} \|h\|_{\mathcal{G}}, \\ |R_\varphi^\ell(f_2) - R_\varphi^\ell(f_1) - DR_\varphi^\ell[f_1](f_2 - f_1)| &\leq \|\varphi\|_{1, \mathcal{F}^2 \otimes (L^\infty)^{\ell-2}} \|f_2 - f_1\|_{\mathcal{G}}^2, \end{aligned}$$

where

$$\begin{aligned} \|\varphi\|_{1, \mathcal{F}^k \otimes (L^\infty)^{\ell-k}} &:= \sum_{i_1, \dots, i_k \text{ distinct in } \llbracket 1, \ell \rrbracket} \|\varphi_{i_1}\|_{\mathcal{F}} \cdots \|\varphi_{i_k}\|_{\mathcal{F}} \prod_{j \neq (i_1, \dots, i_k)} \|\varphi_j\|_{L^\infty(E)} \\ &\leq \begin{cases} \ell \|\varphi\|_{\infty, \mathcal{F} \otimes (L^\infty)^{\ell-1}} & \text{for } k = 1, \\ \frac{\ell(\ell-1)}{2} \|\varphi\|_{\infty, \mathcal{F}^2 \otimes (L^\infty)^{\ell-2}} & \text{for } k = 2, \end{cases} \end{aligned}$$

and we have defined

$$\begin{aligned} \|\varphi\|_{\infty, \mathcal{F}^k \otimes (L^\infty)^{\ell-k}} &:= \max_{i_1, \dots, i_k \text{ distinct in } \llbracket 1, \ell \rrbracket} \|\varphi_{i_1}\|_{\mathcal{F}} \cdots \|\varphi_{i_k}\|_{\mathcal{F}} \prod_{j \neq (i_1, \dots, i_k)} \|\varphi_j\|_{L^\infty(E)} \\ &\leq \|\varphi\|_{\mathcal{F}^{\otimes \ell}}, \end{aligned}$$

since $\|\cdot\|_{L^\infty(E)} \leq \|\cdot\|_{\mathcal{F}}$. □

2.7. Assumptions. We now state the remaining assumptions of the main abstract theorem.

First we assume that **(A1)** and **(A2)** hold. Hence the semigroups S_t^N , T_t^N , S_t^{NL} and T_t^∞ are well defined as well as the generators G^N and G^∞ .

(A3) Convergence of the generators. Let $P_{\mathcal{G}_1}$ be the probability space endowed with a metric considered in **(A2)**. We assume that there is $k \geq 0$, and a function $\varepsilon(N)$ with $\lim_{N \rightarrow \infty} \varepsilon(N) = 0$ such that the generators G^N satisfy for any $N \in \mathbb{N}$:

$$(2.19) \quad \forall \Phi \in C^{k,1}(P_{\mathcal{G}_1}; \mathbb{R}), \quad \|G^N(\pi_N \Phi) - \pi_N \langle Q, D\Phi \rangle\|_{L^\infty(\mathbb{E}_N)} \leq \varepsilon(N) \|\Phi\|_{C^{k,1}(P_{\mathcal{G}_1})}.$$

(A4) Differential stability of the limiting semigroup. We assume that the flow S_t^{NL} is $C^{k,1}(P_{\mathcal{G}_1}, P_{\mathcal{G}_2})$ in the sense that there exists $C_{4,T} > 0$ such that

$$(2.20) \quad \int_0^T \|S_t^{NL}\|_{C^{k,1}(P_{\mathcal{G}_1}, P_{\mathcal{G}_2})} dt \leq C_{4,T}$$

where $P_{\mathcal{G}_2}$ is the same subset of probabilities as $P_{\mathcal{G}_1}$, endowed with the norm associated to a normed space $\mathcal{G}_2 \supset \mathcal{G}_1$ possibly larger than \mathcal{G}_1 .

(A5) Weak stability of the limiting semigroup. We assume that, for some probability space $P_{\mathcal{G}_3}(E)$ (associated to a weight function $m_{\mathcal{G}_3}$ and some metric $\text{dist}_{\mathcal{G}_3}$) and every $T > 0$ there exists a constant $C_{5,T} \in (0, \infty)$ such that

$$(2.21) \quad \forall f_1, f_2 \in P_{\mathcal{G}_3}(E), \quad \sup_{[0,T]} \text{dist}_{\mathcal{G}_3}(S_t^{NL}(f_1), S_t^{NL}(f_2)) \leq C_{5,T} \text{dist}_{\mathcal{G}_3}(f_1, f_2).$$

2.8. Statement of the result.

Theorem 2.13 (Fluctuation estimate). *Consider a process \mathcal{V}_t^N in E^N/\mathfrak{S}_N , and the related semigroups S_t^N and T_t^N as defined above. Assume that there are normed spaces $\mathcal{F}_i \subset C_b(E)$ and some metric spaces $P_{\mathcal{G}_i} \subset P(E)$ as defined above, $i = 1, 2, 3$, in duality, and such that the assumptions **(A1)**-**(A2)**-**(A3)**-**(A4)**-**(A5)** are satisfied. Let $f_{in} \in P(E)$, and consider a hierarchy of N -particle solutions $f_t^N = S_t^N(f_{in}^{\otimes N})$, and a solution $f_t = S_t^{NL}(f_{in})$ to the limiting equation.*

*Then there is some absolute constant $C \in (0, \infty)$ and, for any $\ell \in \mathbb{N}$, there is a constant $C_{k,\ell} \in (0, \infty)$ (depending on ℓ and k in **(A3)**-**(A4)**) such that for any $N, \ell \in \mathbb{N}^*$, with $N \geq 2\ell$, and for any*

$$\varphi = \varphi_1 \otimes \cdots \otimes \varphi_\ell \in \mathcal{F}^{\otimes \ell}, \quad \mathcal{F} := \mathcal{F}_1 \cap \mathcal{F}_2 \cap \mathcal{F}_3, \quad \|\varphi_j\|_{\mathcal{F}} \leq 1,$$

we have

$$(2.22) \quad \sup_{[0,T]} \left| \left\langle \left(S_t^N(f_{in}^{\otimes N}) - (S_t^{NL}(f_{in}))^{\otimes N} \right), \varphi \otimes \mathbf{1}^{N-\ell} \right\rangle \right| \\ \leq C \left[\frac{\ell^2}{N} + C_{k,\ell} C_{4,T} \varepsilon(N) + C_{5,T} \ell \Omega_N^{\mathcal{G}_3}(f_{in}) \right]$$

with

$$(2.23) \quad \Omega_N^{\mathcal{G}_3}(f_{in}) := \int_{E^N} \text{dist}_{\mathcal{G}_3}(\mu_V^N, f_{in}) f_{in}^{\otimes N}(dV).$$

2.9. Remarks.

- (1) This theorem immediately implies the propagation of chaos as soon as \mathcal{F} is dense in $C_b(E)$ in the topology of uniform convergence on compact sets. This condition is satisfied in all examples given below.
- (2) The key idea of our method is to treat the N -particle system as a perturbation (in a very degenerated sense) of the limiting problem, and thus to lower as much as possible assumptions on the particle system in order to avoid the complications of dynamics in high dimensions.
- (3) It is worth emphasizing that in the applications the last term in the RHS of (2.22) always has the worst rate of decay. This is the term which measures the chaoticity of the initial data. So typically our theorem does not “worsen” the fluctuation estimate around chaos along time.
- (4) With this abstract result at hands, the difficulty in proving the propagation of chaos is reduced to (a) the “consistency” estimate (2.19) (which is natural whenever the construction is done properly!) and (2) the two “stability” estimates (2.20) and (2.21) on the limit problem. The second estimate (2.21) turns out to be a natural requirement in order to be able to handle empirical measures for the limiting semigroup S_t^{NL} . The first estimate (2.20) is at the core of the proof: the fact that the limiting semigroup correctly propagates the norm $C^{k,1}(P_{\mathcal{G}_1})$ intuitively means that it propagates a bound on the fluctuations around the tensorial case, robust enough so that one can apply the consistency estimate. Its identification as well as its proof in some physical cases are the second main novelty of the method in this paper and in [21].

- (5) For $f_{in} \in P_{d+5}(\mathbb{R}^d)$ the space of probability measures with moment of order $d + 5$ bounded, the quantity $\Omega_N^{\mathcal{G}_3}(f_{in})$ goes to zero thanks to quantitative functional law of large numbers due to Rachev and Rüschendorf [23]. For more general initial data $f_{in} \in P(E)$ the quantity $\Omega_N^{\mathcal{G}_3}(f_{in})$ also goes to zero as a consequence of Hewitt and Savage theorem, see [21].

2.10. Proof of Theorems 2.13. For a given function $\varphi \in (\mathcal{F}_1 \cap \mathcal{F}_2 \cap \mathcal{F}_3)^{\otimes \ell}$, we break up the term to be estimated into four parts:

$$\begin{aligned} & \left| \left\langle S_t^N(f_{in}^{\otimes N}) - (S_t^{NL}(f_{in}))^{\otimes N}, \varphi \otimes \mathbf{1}^{\otimes N-\ell} \right\rangle \right| \leq \\ & \leq \left| \left\langle S_t^N(f_{in}^N), \varphi \otimes \mathbf{1}^{\otimes N-\ell} \right\rangle - \left\langle S_t^N(f_{in}^N), R^\ell[\varphi] \circ \mu_V^N \right\rangle \right| \\ & + \left| \left\langle f_{in}^N, T_t^N(R^\ell[\varphi] \circ \mu_V^N) \right\rangle - \left\langle f_{in}^N, (T_t^\infty R^\ell[\varphi]) \circ \mu_V^N \right\rangle \right| \\ & + \left| \left\langle f_{in}^{\otimes N}, (T_t^\infty R^\ell[\varphi]) \circ \mu_V^N \right\rangle - \left\langle (S_t^{NL}(f_{in}))^{\otimes \ell}, \varphi \right\rangle \right| =: \mathcal{T}_1 + \mathcal{T}_2 + \mathcal{T}_3. \end{aligned}$$

We deal separately with each term in this expression:

- \mathcal{T}_1 is estimated by purely combinatorial arguments;
- \mathcal{T}_2 is estimated thanks to the consistency estimate **(A3)** on the generators plus the fine stability assumption **(A4)** on the limit semigroup;
- \mathcal{T}_3 is estimated in terms of the function $\Omega_N^{\mathcal{G}_3}(f_{in})$ (measuring how good f_{in} can be approximated in weak \mathcal{G}_3 distance by empirical measures) and thanks to the weak measure stability assumption **(A5)**.

Step 1: Estimate of the first term \mathcal{T}_1 . We prove that for any $t \geq 0$ and any $N \geq 2\ell$ there holds

$$(2.24) \quad \mathcal{T}_1 := \left| \left\langle S_t^N(f_{in}^N), \varphi \otimes \mathbf{1}^{\otimes N-\ell} \right\rangle - \left\langle S_t^N(f_{in}^N), R_\varphi^\ell \circ \mu_V^N \right\rangle \right| \leq \frac{2\ell^2 \|\varphi\|_{L^\infty(E^\ell)}}{N}.$$

Since $S_t^N(f_{in}^N)$ is a symmetric probability measure, estimate (2.24) is a direct consequence of the following lemma.

Lemma 2.14. *For any $\varphi \in C_b(E^\ell)$ we have*

$$(2.25) \quad \forall N \geq 2\ell, \quad \left| (\varphi \otimes \mathbf{1}^{\otimes N-\ell})_{sym} - \pi_N R_\varphi^\ell \right| \leq \frac{2\ell^2 \|\varphi\|_{L^\infty(E^\ell)}}{N}$$

where for a function $\phi \in C_b(E^N)$, we define its symmetrized version ϕ_{sym} as:

$$(2.26) \quad \phi_{sym} = \frac{1}{|\mathfrak{S}_N|} \sum_{\sigma \in \mathfrak{S}_N} \phi_\sigma.$$

As a consequence for any symmetric measure $f^N \in P(E^N)$ we have

$$(2.27) \quad \left| \left\langle f^N, R_\varphi^\ell(\mu_V^N) \right\rangle - \left\langle f^N, \varphi \right\rangle \right| \leq \frac{2\ell^2 \|\varphi\|_{L^\infty(E^\ell)}}{N}.$$

The lemma is a simple and well known combinatorial computation. We briefly sketch the proof for the sake of completeness.

Proof of Lemma 2.14. For a given $\ell \leq N/2$ we introduce

$$A_{N,\ell} := \{(i_1, \dots, i_\ell) \in [[1, N]]^\ell : \forall k \neq k', i_k \neq i_{k'}\} \quad \text{and} \quad B_{N,\ell} := A_{N,\ell}^c.$$

Since there are $N(N-1)\dots(N-\ell+1)$ ways of choosing ℓ distinct indices among $[[1, N]]$ we get

$$\begin{aligned} \frac{|B_{N,\ell}|}{N^\ell} &= 1 - \left(1 - \frac{1}{N}\right) \dots \left(1 - \frac{\ell-1}{N}\right) = 1 - \exp\left(\sum_{i=0}^{\ell-1} \ln\left(1 - \frac{i}{N}\right)\right) \\ &\leq 1 - \exp\left(-2 \sum_{i=0}^{\ell-1} \frac{i}{N}\right) \leq \frac{\ell^2}{N}, \end{aligned}$$

where we have used

$$\forall x \in [0, 1/2], \quad \ln(1-x) \geq -2x \quad \text{and} \quad \forall x \in \mathbb{R}, \quad e^{-x} \geq 1-x.$$

Then we compute

$$\begin{aligned} R_\varphi^\ell(\mu_V^N) &= \frac{1}{N^\ell} \sum_{i_1, \dots, i_\ell=1}^N \varphi(v_{i_1}, \dots, v_{i_\ell}) \\ &= \frac{1}{N^\ell} \sum_{(i_1, \dots, i_\ell) \in A_{N,\ell}} \varphi(v_{i_1}, \dots, v_{i_\ell}) + \frac{1}{N^\ell} \sum_{(i_1, \dots, i_\ell) \in B_{N,\ell}} \varphi(v_{i_1}, \dots, v_{i_\ell}) \\ &= \frac{1}{N^\ell} \frac{1}{(N-\ell)!} \sum_{\sigma \in \mathfrak{S}_N} \varphi(v_{\sigma(1)}, \dots, v_{\sigma(\ell)}) + \mathcal{O}\left(\frac{\ell^2}{N} \|\varphi\|_{L^\infty}\right) \\ &= \frac{1}{N!} \sum_{\sigma \in \mathfrak{S}_N} \varphi(v_{\sigma(1)}, \dots, v_{\sigma(\ell)}) + \mathcal{O}\left(\frac{2\ell^2}{N} \|\varphi\|_{L^\infty}\right) \end{aligned}$$

and the proof of (2.25) is complete. Next for any $f^N \in P(E^N)$ we have

$$\langle f^N, \varphi \rangle = \left\langle f^N, (\varphi \otimes \mathbf{1}^{\otimes N-\ell})_{sym} \right\rangle,$$

and (2.27) trivially follows from (2.25). \square

Step 2: Estimate of the second term \mathcal{T}_2 .

We prove that for any $t \geq 0$ and any $N \geq 2\ell$ there holds

$$(2.28) \quad \begin{aligned} \mathcal{T}_2 &:= \left| \langle f_{in}^N, T_t^N (R^\ell[\varphi] \circ \mu_V^N) \rangle - \langle f_{in}^N, ((T_t^\infty R^\ell[\varphi]) \circ \mu_V^N) \rangle \right| \\ &\leq C_b(k, \ell) C_{4,T} \varepsilon(N) \|\varphi\|_{\mathcal{F}_1^k \otimes (L^\infty)^{\ell-k}} \end{aligned}$$

for some explicitly given constant $C_b(k, \ell)$ depending only on k and ℓ .

We start from the following identity

$$T_t^N \pi_N - \pi_N T_t^\infty = - \int_0^t \frac{d}{ds} (T_{t-s}^N \pi_N T_s^\infty) ds = \int_0^t T_{t-s}^N [G^N \pi_N - \pi_N G^\infty] T_s^\infty ds.$$

From assumption **(A3)** we have for any $t \in [0, T]$

$$\begin{aligned}
& \left\| (T_t^N \pi_N R^\ell[\varphi] - \pi_N T_t^\infty R^\ell[\varphi]) \right\|_{L^\infty(E^N)} \\
&= \left| \int_0^t \langle S_{t-s}^N(f_{in}^N), [G^N \pi_N - \pi_N G^\infty] (T_s^\infty R^\ell[\varphi]) \rangle ds \right| \\
&\leq \int_0^T \left\| [G^N \pi_N - \pi_N G^\infty] (T_s^\infty R^\ell[\varphi]) \right\|_{L^\infty(E^N)} ds \\
(2.29) \quad &\leq \varepsilon(N) \int_0^T \left\| T_s^\infty R^\ell[\varphi] \right\|_{C^{k,1}(P_{\mathcal{G}_1})} ds.
\end{aligned}$$

Since $T_t^\infty(R^\ell[\varphi]) = R^\ell[\varphi] \circ S_t^{NL}$ with $S_t^{NL} \in C^{k,1}(P_{\mathcal{G}_1}, P_{\mathcal{G}_2})$ thanks to assumption **(A4)** and $R^\ell[\varphi] \in C^{k,1}(P_{\mathcal{G}_2}, \mathbb{R})$ because $\varphi \in \mathcal{F}_2^{\otimes \ell}$ (see subsection 2.6), we obtain with the help of Lemma 2.8 that $T_t^\infty(R^\ell[\varphi]) \in C^{k,1}(P_{\mathcal{G}_1})$ with uniform bound. We hence conclude that

$$(2.30) \quad \int_0^T \left\| T_s^\infty(R^\ell[\varphi]) \right\|_{C^{k,1}(P_{\mathcal{G}_1})} ds \leq C(k) C_{4,T} \left\| R^\ell[\varphi] \right\|_{C^{k,1}(P_{\mathcal{G}_2})}.$$

Going back to the computation (2.29), and plugging (2.30) we deduce (2.28).

Step 3: Estimate of the third term \mathcal{T}_3 .

We claim that for any $t \geq 0$ and any $N \geq 2\ell$,

$$(2.31) \quad \mathcal{T}_3 := \left| \langle f_{in}^{\otimes N}, (T_t^\infty R^\ell[\varphi]) \circ \mu_V^N \rangle - \langle (S_t^{NL}(f_{in}))^{\otimes \ell}, \varphi \rangle \right| \leq \ell C_{5,T} \Omega_N^{\mathcal{G}_3}(f_{in}) \|\varphi\|_{\mathcal{F}_3 \otimes (L^\infty)^{\ell-1}},$$

with $\Omega_N^{\mathcal{G}_3}(f_{in})$ is defined in (2.23).

To prove this, let us write

$$\begin{aligned}
\mathcal{T}_{3,1} &:= \langle f_{in}^{\otimes N}, (T_t^\infty R^\ell[\varphi]) \circ \mu_V^N \rangle = \int_{E^N} R^\ell[\varphi] (S_t^{NL}(\mu_V^N)) f_{in}(dV_1) \dots f_{in}(dV_N) \\
&= \int_{E^N} \left(\prod_{i=1}^{\ell} a_i(V) \right) f_{in}(dV_1) \dots f_{in}(dV_N),
\end{aligned}$$

with

$$a_i = a_i(V) := \int_E \varphi_i(w) S_t^{NL}(\mu_V^N)(dw), \quad i = 1, \dots, \ell.$$

Similarly, we write for the second term

$$\mathcal{T}_{3,2} = \langle (S_t^{NL}(f_{in}))^{\otimes \ell}, \varphi \rangle = \int_{E^N} \left(\prod_{i=1}^{\ell} b_i \right) f_{in}(dV_1) \dots f_{in}(dV_N),$$

with

$$b_i := \int_E \varphi_i(w) S_t^{NL}(f_{in})(dw), \quad i = 1, \dots, \ell.$$

Using the identity

$$\prod_{i=1}^{\ell} a_i - \prod_{i=1}^{\ell} b_i = \sum_{i=1}^{\ell} a_1 \dots a_{i-1} (a_i - b_i) b_{i+1} \dots b_\ell,$$

we get

$$(2.32) \quad \mathcal{T}_3 \leq \sum_{i=1}^{\ell} \left(\prod_{j \neq i} \|\varphi_j\|_{L^\infty(E)} \right) \int_{E^N} |a_i(V) - b_i| f_{in}(dV_1) \dots f_{in}(dV_N).$$

Then by using the duality bracket together with assumption **(A5)** we have

$$(2.33) \quad \begin{aligned} |a_i(V) - b_i| &:= \left| \int_E \varphi_i(w) (S_t^{NL}(f_{in})(dw) - S_t^{NL}(\mu_V^N)(dw)) \right| \\ &\leq \|\varphi_i\|_{\mathcal{F}_3} \text{dist}_{\mathcal{G}_3} (S_t^{NL}(f_{in}), S_t^{NL}(\mu_V^N)) \\ &\leq C_{5,T} \|\varphi_i\|_{\mathcal{F}_3} \text{dist}_{\mathcal{G}_3} (f_{in}, \mu_V^N). \end{aligned}$$

Therefore combining (2.32) and (2.33) (for any $1 \leq i \leq \ell$), we conclude that (2.31) holds. \square

2.11. Remark on assumption (A4). Let us briefly explain how our corner-stone estimate **(A4)** can be obtained in the case of a nonlinear generator Q , which splits into a linear part and a bilinear part:

$$(2.34) \quad \forall f \in P(E), \quad Q(f) = Q_1(f) + Q_2(f, f)$$

with Q_1 linear and Q_2 bilinear symmetric. Assume also for simplicity that $k \leq 2$ in assumptions **(A3)**-**(A4)**.

For two given initial data f_{in} and g_{in} we introduce the following evolution equations

$$\begin{cases} \partial_t g = Q(g) = Q_1(g) + Q_2(g, g), & g|_{t=0} = g_{in}, \\ \partial_t f = Q(f) = Q_1(f) + Q_2(f, f), & f|_{t=0} = f_{in}, \\ \partial_t h = Q'(f)h = Q_1(h) + 2Q_2(f, h), & h|_{t=0} = g_{in} - f_{in}, \\ \partial_t r = Q'(f)r + \frac{1}{2}Q''(f)(h, h) = Q_1(r) + 2Q_2(f, r) + Q_2(h, h), & r|_{t=0} = 0. \end{cases}$$

It is clear that h_t depends linearly on $g_{in} - f_{in}$ (but also nonlinearly on f_{in}) and r_t depends quadratically on h_t (and also nonlinearly on f_{in}), and therefore depends quadratically on $g_{in} - f_{in}$. At least formally, h_t is the linear variation of $f_{in} \mapsto S_t^{NL} f_{in}$ and r_t is its second variation. Let us define $s := f + g$, $d := g - f$, $\omega := g - f - h$, $\psi := g - f - h - r$, for which we get the following evolution equations

$$\begin{cases} \partial_t d = Q_1(d) + Q_2(s, d), & d|_{t=0} = d_{in} = g_{in} - f_{in}, \\ \partial_t \omega = Q_1(\omega) + Q_2(s, \omega) + Q_2(h, d), & \omega|_{t=0} = 0, \\ \partial_t \psi = Q_1(\psi) + Q_2(s, \psi) + Q_2(h, \omega) + Q_2(r, d), & \psi|_{t=0} = 0. \end{cases}$$

Now we can translate the regularity estimates on S_t^{NL} in terms of estimates on these solutions

$$\begin{aligned} \sup_{t \in [0, t]} \|d_t\|_{\mathcal{G}_1} \leq C_T \|d_{in}\|_{\mathcal{G}_1} &\implies S_t^{NL} \in C^{0,1}(P_{\mathcal{G}_1}, P_{\mathcal{G}_1}) \\ \sup_{t \in [0, t]} \|\omega_t\|_{\mathcal{G}_1} \leq C_T \|d_{in}\|_{\mathcal{G}_1}^2 &\implies S_t^{NL} \in C^{1,1}(P_{\mathcal{G}_1}, P_{\mathcal{G}_2}) \\ \sup_{t \in [0, t]} \|\psi_t\|_{\mathcal{G}_1} \leq C_T \|d_{in}\|_{\mathcal{G}_1}^3 &\implies S_t^{NL} \in C^{2,1}(P_{\mathcal{G}_1}, P_{\mathcal{G}_2}). \end{aligned}$$

Such estimates are typically obtained by applying Gronwall lemma to the above mentioned equations satisfied by d , ω and ψ , for well chosen norms $\|\cdot\|_{\mathcal{G}_i}$.

Observe that these computations follow naturally from the key idea of looking for differentiability of the semigroup with respect to the initial data, in some weak measure sense.

2.12. Examples of distances on measures when $E = \mathbb{R}^d$. Here we list some well-known distances on $P(\mathbb{R}^d)$ or on its subsets

$$P_q(\mathbb{R}^d) := \{f \in P(\mathbb{R}^d); \langle f, \langle v \rangle^q \rangle < \infty\}, \quad q \geq 0, \quad \langle v \rangle^2 = 1 + |v|^2.$$

These distances are all topologically uniformly equivalent to the weak topology $\sigma(P(E), C_b(E))$ on the bounded subsets

$$\mathcal{B}P_{q,a}(E) := \{f \in P_q(\mathbb{R}^d), \langle f, \langle v \rangle^q \rangle \leq a\}$$

for any $a \in (0, \infty)$ and for q large enough. A good reference on these matters is [8].

Example 2.15 (Dual-Hölder (or Zolotarev's) distances). *Denote by dist_E a distance on E and fix $v_0 \in E$ (e.g. $v_0 = 0$ when $E = \mathbb{R}^d$ in the sequel). Denote by $\text{Lip}_0(E)$ the set of Lipschitz functions on E vanishing at one arbitrary point $v_0 \in E$ endowed with the norm*

$$[\varphi]_{\text{Lip}} = [\varphi]_1 := \sup_{x, y \in E} \frac{|\varphi(y) - \varphi(x)|}{\text{dist}_E(x, y)}.$$

We then define the dual norm: take $m_{\mathcal{G}} := 1$ and $P_{\mathcal{G}}(E)$ endowed with

$$(2.35) \quad \forall f, g \in P_{\mathcal{G}}, \quad [g - f]_1^* := \sup_{\varphi \in \text{Lip}_0(E)} \frac{\langle g - f, \varphi \rangle}{[\varphi]_1}.$$

Example 2.16 (Monge-Kantorovich-Wasserstein distances). *For $q \in [1, \infty)$, define*

$$P_q(E) = P_q(E) := \{f \in P(E); \langle f, m_{\mathcal{G}} \rangle := \langle f, \text{dist}(\cdot, v_0)^q \rangle < \infty\}$$

and the Monge-Kantorovich-Wasserstein (MKW) distance W_q by

$$(2.36) \quad \forall f, g \in P_q(E), \quad W_q^q(f, g) := \inf_{\pi \in \Pi(f, g)} \int_{E \times E} \text{dist}_E(x, y)^q \pi(dx, dy),$$

where $\Pi(f, g)$ denote the set of probability measures $\Pi \in P(E \times E)$ with marginals f and g ($\Pi(A, E) = f(A)$, $\Pi(E, A) = g(A)$ for any Borel set $A \subset E$). Note that for

$V, Y \in E^N$ and any $q \in [1, \infty)$, one has

$$(2.37) \quad W_q(\mu_V^N, \mu_Y^N) = d_{\ell^q(E^N/\mathfrak{S}_N)}(V, Y) := \min_{\sigma \in \mathfrak{S}_N} \left(\frac{1}{N} \sum_{i=1}^N \text{dist}_E(v_i, y_{\sigma(i)})^q \right)^{1/q},$$

and that

$$(2.38) \quad \forall f, g \in P_1(E), \quad W_1(f, g) = [f - g]_1^* = \sup_{\phi \in \text{Lip}_0(E)} \langle f - g, \phi \rangle$$

as well as

$$(2.39) \quad \forall q \in (1, \infty), \quad \forall f, g \in P_q(\mathbb{R}^d) \quad W_1(f, g) \leq W_q(f, g).$$

We refer to [28] and the references therein for more details on the Monge-Kantorovich-Wasserstein distances and for a proof of these claims.

Example 2.17 (Fourier-based ‘‘Toscani’’ norms). For $E = \mathbb{R}^d$, $m_{\mathcal{G}_1} := 1$, let

$$\forall f \in \mathcal{TP}_{\mathcal{G}_1}, \quad \|f\|_{\mathcal{G}_1} = |f|_s := \sup_{\xi \in \mathbb{R}^d} \frac{|\hat{f}(\xi)|}{\langle \xi \rangle^s}, \quad s > 0.$$

We denote by \mathcal{H}^{-s} (which includes $\mathcal{IP}_{\mathcal{G}_1}$ for s large enough) the Banach space associated to the norm $|\cdot|_s$.

Example 2.18 (Negative Sobolev norms). For $E = \mathbb{R}^d$, $m_{\mathcal{G}_1} := 1$, let

$$\forall f \in \mathcal{TP}_{\mathcal{G}_1}, \quad \|f\|_{\mathcal{G}_1} = \|f\|_{H^{-s}(\mathbb{R}^d)} := \left\| \frac{\hat{f}(\xi)}{\langle \xi \rangle^s} \right\|_{L^2}, \quad s > 0.$$

We denote by H^{-s} (which includes $\mathcal{IP}_{\mathcal{G}_1}$ for s large enough) the Hilbert space associated to the norm $\|\cdot\|_{H^{-s}}$.

2.13. Law of large numbers for measures. The quantitative estimates of propagation of chaos depend on the rate at which $\Omega_N^{g_3}(f_{in})$ converges to zero when $N \rightarrow \infty$. Since in our applications we shall use a common argument, we sketch it now.

First we remark that thanks to (2.38) and (2.39), we have for any $f, g \in P_2(\mathbb{R}^d)$ and $\varphi \in \text{Lip}_0(\mathbb{R}^d)$

$$|\langle g - f, \varphi \rangle| \leq [g - f]_1^* [\varphi]_1 \leq W_1(f, g) [\varphi]_1 \leq W_2(f, g) [\varphi]_1.$$

That precisely means that $\mathcal{F}_3 := \text{Lip}_0(\mathbb{R}^d)$ and $P_{\mathcal{G}_3} := P_2$, endowed with the quadratic MKW distance W_2 , are in duality in the sense of Definition 2.11.

Next, we recall the following classical result.

Theorem 2.19 ([23]). For any $f \in P_{d+5}(\mathbb{R}^d)$ and any $N \geq 1$ there holds

$$(2.40) \quad \Omega_N^{W_2}(f) \leq \text{Cst}(d, M_{d+5}) N^{-\frac{1}{d+4}},$$

for some constant $\text{Cst}(d, M_{d+5})$ which only depends on d and

$$M_{d+5}(f) := \int_{\mathbb{R}^d} |v|^{d+5} f(dv).$$

Combining these two facts, we see that the error term \mathcal{T}_3 in the proof of theorem 2.13 satisfies

$$\mathcal{T}_3 \leq C \ell C_{5,T} \frac{\text{cst}(d, M_{d+5})}{N^{\frac{1}{d+4}}}$$

so long as assumption **(A5)** holds with the present choice of dual functional spaces $(\mathcal{F}_3, P_{\mathcal{G}_3})$ and $f_{in} \in P_{d+5}(\mathbb{R}^d)$.

3. MAXWELL MOLECULE COLLISIONS WITH CUT-OFF

3.1. The model. In this section we assume that $E = \mathbb{R}^d$, $d \geq 2$, and we consider an N -particle system undergoing a space homogeneous random Boltzmann collisions according to a collision kernel $b \in L^1([-1, 1])$ only depending on the deviation angle and locally integrable (Maxwellian molecules with Grad's angular cut-off). We make the normalization hypothesis $\|b\|_{L^1} = \int_{S^{d-1}} b(\sigma_1) d\sigma = 1$. More precisely, given a pre-collisional N -system of velocity particles $V = (v_1, \dots, v_N) \in E^N = (\mathbb{R}^d)^N$, the stochastic runs as follows:

- (i) for any $i' \neq j'$, we draw randomly for the pair of particles $(v_{i'}, v_{j'})$ a random time $T_{i',j'}$ of collision according to an exponential law of parameter 1, and then choose the collision time T_1 and the colliding couple (v_i, v_j) (which is a.s. well-defined) in such a way that

$$T_1 = T_{i,j} := \min_{1 \leq i' \neq j' \leq N} T_{i',j'};$$

- (ii) we then draw $\sigma \in S^{d-1}$ randomly according to the law $b(\cos \theta_{ij})$ where we define the angular deviation θ_{ij} by $\cos \theta_{ij} = \sigma \cdot (v_j - v_i) / |v_j - v_i|$;
- (iii) the new state after collision at time T_1 becomes

$$V^* = V_{ij}^* = R_{ij,\sigma} V = (v_1, \dots, v_i^*, \dots, v_j^*, \dots, v_N),$$

where the rotation $R_{ij,\sigma}$ on the (i, j) pair with vector σ is defined by

$$(3.1) \quad v_i^* = \frac{w_{ij}}{2} + \frac{u_{ij}^*}{2}, \quad v_j^* = \frac{w_{ij}}{2} - \frac{u_{ij}^*}{2},$$

with

$$w_{ij} = v_i + v_j, \quad u_{ij}^* = |u_{ij}| \sigma, \quad u_{ij} = v_i - v_j.$$

The associated Markov process (\mathcal{V}_t) on $(\mathbb{R}^d)^N$ is then built repeating the above construction. After scaling time we denote by f_t^N the law of \mathcal{V}_t , S_t^N the associated semigroup, as well as G^N and T_t^N respectively the associated dual generator and dual semigroup, as in the previous abstract construction.

The **master equation** on the law f_t^N is given in dual form by

$$(3.2) \quad \partial_t \langle f_t^N, \varphi \rangle = \langle f_t^N, G^N \varphi \rangle$$

with

$$(3.3) \quad (G^N \varphi)(V) = \frac{1}{N} \sum_{1 \leq i < j \leq N} \int_{S^{d-1}} b(\cos \theta_{ij}) [\varphi_{ij}^* - \varphi] d\sigma$$

where $\varphi_{ij}^* = \varphi(V_{ij}^*)$ and $\varphi = \varphi(V) \in C_b(\mathbb{R}^{Nd})$. Note that the collision process is invariant under permutation of the velocities, and satisfies the microscopic conservations of momentum and energy at any collision time

$$\forall \alpha = 1, \dots, d, \quad \sum_k v_{k\alpha}^* = \sum_k v_{k\alpha}, \quad |V^*|^2 = |V|^2 := \sum_{k=1}^N |v_k|^2.$$

When $E = \mathbb{R}^d$, we write $V = (v_i)_{1 \leq i \leq N} = (v_1, \dots, v_N) \in E^N$ and $v = (v_\alpha)_{1 \leq \alpha \leq d} \in \mathbb{R}^d$, so that $V = (v_{i\alpha}) \in \mathbb{R}^{Nd}$ with $v_{i\alpha} \in \mathbb{R}$.

As a consequence, for any symmetric initial law $f_{in}^N \in P_{sym}(\mathbb{R}^{Nd})$ the law density f_t^N remains a symmetric probability and conserves momentum and energy

$$\left\{ \begin{array}{l} \forall \alpha = 1, \dots, d, \quad \int_{\mathbb{R}^{dN}} \left(\sum_{k=1}^N v_{k\alpha} \right) f_t^N(dV) = \int_{\mathbb{R}^{dN}} \left(\sum_{k=1}^N v_{k\alpha} \right) f_{in}^N(dV), \\ \forall \phi : \mathbb{R}_+ \rightarrow \mathbb{R}_+, \quad \int_{\mathbb{R}^{dN}} \phi(|V|^2) f_t^N(dV) = \int_{\mathbb{R}^{dN}} \phi(|V|^2) f_{in}^N(dV). \end{array} \right.$$

The formal limit of this N -particle system is the nonlinear homogeneous Boltzmann equation on $P(\mathbb{R}^2)$ defined by

$$(3.4) \quad \partial_t f_t = Q(f_t, f_t)$$

where the quadratic Boltzmann collision operator Q is defined by

$$(3.5) \quad \langle Q(f, f), \varphi \rangle := \int_{\mathbb{R}^{2d} \times S^{d-1}} b(\theta) (\phi(w_2^*) - \phi(w_2)) d\sigma f(dw_1) f(dw_2)$$

for $\varphi \in C_b(\mathbb{R}^d)$ and $f \in P(\mathbb{R}^d)$, with

$$w_1^* = \frac{w_1 + w_2}{2} + \frac{|w_2 - w_1|}{2} \sigma, \quad w_2^* = \frac{w_1 + w_2}{2} - \frac{|w_2 - w_1|}{2} \sigma$$

and $\cos \theta = \sigma \cdot (v - w)/|v - w|$. This equation generates a nonlinear semigroup S_t^{NL} defined by $S_t^{NL} f_{in} := f_t$ for any $f_{in} \in P(\mathbb{R}^d)$, which satisfies conservation of momentum and energy:

$$\forall t \geq 0, \quad \int_{\mathbb{R}^d} v f_t(dv) = \int_{\mathbb{R}^d} v f_{in}(dv), \quad \int_{\mathbb{R}^d} |v|^2 f_t(dv) = \int_{\mathbb{R}^d} |v|^2 f_{in}(dv).$$

3.2. Statement of the result. On the one hand, it is well known that for the Boltzmann collision kernel we have chosen (Maxwellian molecules with angular cut-off, sometimes called *pseudo-Maxwell molecules*) the N -particle Markov process (\mathcal{V}_t) is well defined for any given velocity \mathcal{V}_0 , and in particular, for any given initial law $f_0^N \in P_{sym}((\mathbb{R}^d)^N)$ there exists a unique solution $f_t^N \in P_{sym}((\mathbb{R}^d)^N)$ to equations (3.2)-(3.3) so that the N -particle semigroup S_t^N is well defined, see [16, 17, 25, 19]. On the other hand, it is also well known that for any $f_{in} \in P_m(\mathbb{R}^d)$, $m \geq 0$ the nonlinear Boltzmann equation (3.4)-(3.5) has a unique solution $f_t \in P_m(\mathbb{R}^d)$ and this one conserves momentum and energy when $m \geq 2$, see for instance [25, 26, 10, 27].

Our mean-field limit result then states as follows.

Theorem 3.1 (The Boltzmann equation for Maxwell molecules with Grad's cut-off). *Consider an initial distribution $f_{in} \in P_m(\mathbb{R}^d)$, $m \geq 2$ and the hierarchy of N -particle distributions $f_t^N = S_t^N(f_{in}^{\otimes N})$ following (3.2) and the solution $f_t = S_t^{NL}(f_{in})$ following (3.4).*

Then there are constants $C_T^1, C_T^2 \in (0, \infty)$ only depending on f_{in} and $T \in (0, \infty)$ such that for any

$$\varphi = \varphi_1 \otimes \cdots \otimes \varphi_\ell \in \mathcal{F}^{\otimes \ell}, \quad \mathcal{F} := C_b(\mathbb{R}^d) \cap \text{Lip}(\mathbb{R}^d), \quad \|\varphi_j\|_{\mathcal{F}} \leq 1,$$

we have for $N \geq 2\ell$:

$$(3.6) \quad \sup_{[0, T]} \left| \left\langle \left(S_t^N(f_{in}^N) - (S_t^{NL}(f_{in}))^{\otimes N} \right), \varphi \right\rangle \right| \leq C_T^1 \frac{\ell^2}{N} + C_T^2 \ell \Omega_N^{W_2}(f_{in})$$

where $\Omega_N^{W_2}$ was defined in (2.23) and W_2 is the quadratic MKW distance defined in (2.36).

As a consequence of (3.6) and of the discussion of subsection 2.13, quantitative propagation of chaos holds with rate $\varepsilon(N) \leq C(\ell, T, f_{in}) N^{-\frac{1}{d+4}}$ for any initial data $f_{in} \in P_{d+5}(\mathbb{R}^d)$.

3.3. Proof of Theorem 3.1. We have to establish the assumptions **(A1)**-**(A2)**-**(A3)**-**(A4)**-**(A5)** in order to apply Theorem 2.13. We will do it step by step. In this proof we fix $\mathcal{F}_1 = \mathcal{F}_2 = C_0(\mathbb{R}^d)$ and $\mathcal{F}_3 = \text{Lip}(\mathbb{R}^d)$ and we define $P_{\mathcal{G}_1} = P_{\mathcal{G}_2} := P(\mathbb{R}^d)$ endowed with the total variation norm $\|\cdot\|_{TV}$, $P_{\mathcal{G}_3} := P_2(\mathbb{R}^d)$ endowed with the quadratic MKW distance W_2 . Notice that $(\mathcal{G}_i, \mathcal{F}_i)$, $i = 1, 2$, and $(P_{\mathcal{G}_3}, \mathcal{F}_3)$ are in duality (see subsection 2.13).

Proof of (A1). Taking $m_1 = 1$, the assumption is just a consequence of the fact that (3.2) is well posed for any $f^N(0, \cdot) \in P_{sym}((\mathbb{R}^d)^N)$ so that f_t^N is a probability measure for any $t \geq 0$.

Proof of (A3). We claim that there exists $C_1 \in \mathbb{R}_+$ such that for all $\Phi \in C^{1,1}(P_{\mathcal{G}_1}, \mathbb{R})$

$$(3.7) \quad \left\| G^N(\Phi \circ \mu_V^N) - \langle Q(\mu_V^N, \mu_V^N), D\Phi[\mu_V^N] \rangle \right\|_{L^\infty(E^N)} \leq \frac{C_1}{N} \|\Phi\|_{C^{1,1}(P_{\mathcal{G}_1}, \mathbb{R})},$$

which is nothing but **(A3)** with $k = \eta = 1$ and $\varepsilon(N) = C_1 N^{-1}$.

Take $\Phi \in C^{1,1}(P_{\mathcal{G}_1}, \mathbb{R})$, set $\phi = D\Phi[\mu_V^N]$ and compute

$$\begin{aligned} G^N(\Phi \circ \mu_V^N) &= \frac{1}{N} \sum_{1 \leq i < j \leq N} \int_{\mathbb{S}^{d-1}} b(\theta_{ij}) \left[\Phi(\mu_{V_{ij}^*}^N) - \Phi(\mu_V^N) \right] d\sigma \\ &= \frac{1}{N} \sum_{1 \leq i < j \leq N} \int_{\mathbb{S}^{d-1}} b(\theta_{ij}) \langle \mu_{V_{ij}^*}^N - \mu_V^N, \phi \rangle d\sigma \quad (= I_1(V)) \\ &+ \frac{1}{N} \sum_{1 \leq i < j \leq N} \int_{\mathbb{S}^{d-1}} \mathcal{O} \left(\|\Phi\|_{C^{1,1}} \left\| \mu_{V_{ij}^*}^N - \mu_V^N \right\|_{TV}^2 \right) d\sigma \quad (= I_2(V)). \end{aligned}$$

On the one hand, we have

$$\begin{aligned}
I_1 &= \frac{1}{2N^2} \sum_{i,j=1}^N \int_{\mathbb{S}^{d-1}} b(\theta_{ij}) [\phi(v_i^*) + \phi(v_j^*) - \phi(v_i) - \phi(v_j)] d\sigma \\
&= \frac{1}{2} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \int_{\mathbb{S}^{d-1}} b(\theta) [\phi(v^*) + \phi(w^*) - \phi(v) - \phi(w)] \mu_V^N(dv) \mu_V^N(dw) d\sigma \\
&= \langle Q(\mu_V^N, \mu_V^N), \phi \rangle.
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
I_2(V) &= \frac{1}{2N} \sum_{i,j=1}^N \int_{\mathbb{S}^{d-1}} \mathcal{O} \left(\|\Phi\|_{C^{1,1}} \left(\frac{4}{N} \right)^2 \right) d\sigma \\
&\leq 8 \frac{\|\Phi\|_{C^{1,1}}}{N} \left(\sum_{i,j=1}^N \frac{1}{N^2} \right) \leq 8 \|b\| \frac{\|\Phi\|_{C^{1,1}}}{N}.
\end{aligned}$$

Collecting these two terms we have proved that (3.7) holds.

Proof of (A4). Let us prove that for any $f, h \in P(\mathbb{R}^d)$ and for any $T > 0$

$$(3.8) \quad \sup_{t \in [0, T]} \left\| S_t^{NL}(h) - S_t^{NL}(f) - \mathcal{L}S_t^{NL}[f](h - f) \right\|_{TV} \leq e^{4\|\gamma\|_\infty T} \|h - f\|_{TV}^2,$$

where $\mathcal{L}S_t^{NL}[f]$ is the linearization of S_t^{NL} at f . As a consequence, that implies that (A4) holds with $k = \eta = 1$ and the above definition of $P_{\mathcal{G}_1}$ and $P_{\mathcal{G}_1}$. We denote by f_t, g_t, h_t the solutions to the following equations:

$$\begin{cases} \partial_t f_t = Q(f_t, f_t), & f_{t=0} = f, \\ \partial_t g_t = Q(g_t, g_t), & g_{t=0} = h, \\ \partial_t h_t = 2\tilde{Q}(f_t, h_t) := Q(f_t, h_t) + Q(h_t, f_t), & h_{t=0} = h - f, \end{cases}$$

where \tilde{Q} denotes the symmetrized version of the bilinear collision operator. From standard Gronwall argument, there is existence and uniqueness of such solutions, which moreover satisfy, uniformly on $[0, T]$

$$\|h_t\|_{TV} \leq e^{2T} \|h - f\|_{TV}, \quad \|g_t - f_t\|_{TV} \leq e^{2T} \|h - f\|_{TV}.$$

Now let us denote $r_t := g_t - f_t - h_t$, which satisfies the following equation

$$\partial_t r_t = \tilde{Q}(f_t + g_t, r_t) + \tilde{Q}(g_t - f_t, h_t), \quad r_{in} = 0.$$

Introducing $y_t := \|r_t\|_{TV}$, we have

$$\begin{aligned}
y'_t &\leq \frac{1}{2} \|\tilde{Q}(f_t + g_t, r_t)\|_{TV} + \|\tilde{Q}(g_t - f_t, h_t)\|_{TV} \\
&\leq \|\gamma\|_\infty \|f_t + g_t\|_{TV} \|r_t\|_{TV} + C \|g_t - f_t\|_{TV} \|h_t\|_{TV} \\
&\leq C y_t + C e^{4t} \|h - f\|_{TV}^2,
\end{aligned}$$

from which we deduce

$$\forall t \in [0, T], \quad y_t \leq e^{4T} \|h - f\|_{TV}^2$$

which concludes the proof of (3.8).

Proof of (A2). Assumption (A2i) is clearly a consequence of (A4). For (A2ii) we write

$$\begin{aligned} \|Q(f, f) - Q(g, g)\|_{TV} &= \sup_{\|\varphi\|_{L^\infty} \leq 1} \int_E (Q(f, f) - Q(g, g)) \varphi \, dv \\ &= \sup_{\|\varphi\|_{L^\infty} \leq 1} \int_{E \times E} (f f_* - g g_*) \int_{S^{d-1}} b(\varphi' - \varphi) \, d\sigma \, dv dv_* \\ &\leq 4 \|b\|_{L^1} \|f - g\|_{TV}, \end{aligned}$$

so that the function $f \mapsto Q(f, f)$ is Lipschitz from P_{G_1} to $M^1(\mathbb{R}^d)$.

Proof of (A5). It is known since the seminal work of Tanaka [25] that the nonlinear Boltzmann flow associated to Maxwellian molecules is a contraction for the quadratic MLW distance W_2 : for any $f_{in}, g_{in} \in P_1(\mathbb{R}^d)$ the solutions f_t, g_t to the Boltzmann equation (3.4) satisfy

$$\sup_{[0, T]} W_2(f_t, f g_t) \leq W_2(f_{in}, g_{in}).$$

That immediately implies (A5) in the space P_{G_3} . \square

4. VLASOV AND MCKEAN-VLASOV EQUATIONS

4.1. The model. In this section we assume again that $E = \mathbb{R}^d$ and we consider an N -particle system which undergoes McKean-Vlasov type stochastic dynamics, i.e. a drift deterministic force field combined with diffusion. We refer to the lecture notes [24, 19] and the references therein for more details on the model, and among many references, we highlight the recent paper [6] for recent results and references (using the so-called ‘‘coupling’’ method). The method we shall present here does not rely on any of these references. The results in this section are mostly not new but compare to the latest results of mean-field limit on this equation as far as we know. Indeed we shall make strong smoothness assumptions on the coefficients of the evolution equation in order to avoid technical difficulties and our goal is to advocate for our new method and show its power and ability to deal with very different models.

We assume that the N particles $V^N = (v_1, \dots, v_N)$ satisfies the stochastic differential equation

$$(4.1) \quad dv_{i,t} = \sigma_i \, dB_{i,t} + F_i^N \, dt \quad 1 \leq i \leq N,$$

where the $\sigma_i = \sigma(v_i) \in W^{1,\infty}(\mathbb{R}^d)$, $1 \leq i \leq N$, are the diffusion $d \times d$ -matrices, the $(B_{i,t})_{1 \leq i \leq N}$ are independent standard Wiener processes, and the $(F_i^N)_{1 \leq i \leq N}$ are the force field coordinates. Because of indistinguishability we assume

$$F_i^N(V) := F^N \left(v_i, \mu_{\hat{V}_i^N}^{N-1} \right)$$

with $\hat{V}_i^N := (v_1, \dots, v_{i-1}, v_{i+1}, \dots, v_N)$ and $F^N : \mathbb{R}^d \times P(\mathbb{R}^d) \rightarrow \mathbb{R}^d$. (Note that here and below the latin letters ‘‘ i, j, \dots ’’ label the particles, whereas the greek letters ‘‘ α, β, \dots ’’ label the *coordinates* of the force field or of any particle in \mathbb{R}^d .)

We assume that F^N is uniformly bounded and Lipschitz in both variables (when endowing $P(\mathbb{R}^d)$ with a distance inherited from a negative Sobolev norm). More

precisely, we assume that for any $k > d/2$ there exists $C_{F,k} > 0$ such that for any $v_i \in \mathbb{R}^d$, $f_i \in P(\mathbb{R}^d)$

$$(4.2) \quad \forall N \in \mathbb{N}, \quad |F^N(v_2, f_2) - F^N(v_1, f_1)| \leq C_{F,k} \left[|v_2 - v_1| + \|f_2 - f_1\|_{H^{-k}} \right].$$

The typical example we have in mind is

$$(4.3) \quad F_i^N \left(V, \mu_{\hat{V}_i}^{N-1} \right) = \frac{N}{N-1} F^N \left(v_i, \hat{V}_i \right), \quad F^N \left(v_i, \hat{V}_i \right) := \frac{1}{N} \sum_{j \neq i} b(v_i - v_j)$$

for a smooth vector field $b : \mathbb{R}^d \rightarrow \mathbb{R}^d$, so that

$$F(x, m) = \int_{\mathbb{R}^d} b(x - y) m(dy).$$

It is natural for the limit to exist to assume that there exists a $F : \mathbb{R}^d \times P(\mathbb{R}^d) \rightarrow \mathbb{R}^d$ such that $F^N \rightarrow F$, in the sense that there is a constant $C_{F,lim} > 0$ such that

$$(4.4) \quad \forall N \in \mathbb{N}, \forall v \in \mathbb{R}^d, \forall f \in P(\mathbb{R}^d), \quad |F^N(v, f) - F(v, f)| \leq \frac{C_{F,lim}}{N}.$$

This assumption is trivially satisfied for the example (4.3) for instance.

Under the smoothness assumptions (4.2) on the N -particle force fields, for any $N \geq 1$ there exists a Markov process $(\mathcal{V}_t^N)_{t \geq 0}$ which solves the system of stochastic differential equations (4.1), see [24, 19].

The time-dependent law f_t^N of the process \mathcal{V}_t^N satisfies the following linear **master equation** corresponding to (4.1), given in dual form by

$$(4.5) \quad \forall \varphi \in \mathcal{D}(\mathbb{R}^d), \quad \partial_t \langle f_t^N, \varphi \rangle = \langle f_t^N, G^N \varphi \rangle.$$

with

$$\forall V \in R^{dN}, \quad (G^N \varphi)(V) = \sum_{i=1}^N A(v_i) \nabla_i^2 \varphi - \sum_{i=1}^N F^N \left(v_i, \mu_{\hat{V}_i}^{N-1} \right) \cdot \nabla_i \varphi,$$

where we have defined the nonnegative diffusion matrix A , the gradient ∇_i and the Hessian matrix ∇_i^2 associated to the variable $v_i = (v_{i,1}, \dots, v_{i,d}) \in \mathbb{R}^d$ corresponding to the i -th particle by

$$A = \frac{1}{2} \sigma \sigma^* = (A_{\alpha,\beta})_{1 \leq \alpha, \beta \leq d}, \quad A_{\alpha,\beta} = \sum_{\eta=1}^d \sigma_{\alpha,\eta} \sigma_{\beta,\eta},$$

and

$$\nabla_i \varphi = (\partial_{v_{i,\alpha}} \varphi)_{1 \leq \alpha \leq d}, \quad \nabla_i^2 \varphi = \left(\partial_{v_{i,\alpha} v_{i,\beta}}^2 \varphi \right)_{1 \leq \alpha, \beta \leq d}.$$

We also introduce the nonlinear mean-field McKean-Vlasov equation on $P(\mathbb{R}^d)$:

$$(4.6) \quad \frac{\partial f}{\partial t} = Q(f), \quad f(0) = f_{in} \quad \text{in } P(\mathbb{R}^d),$$

with

$$Q(f) = \sum_{\alpha,\beta=1}^d \partial_{\alpha,\beta}^2 (A_{\alpha,\beta} f) - \sum_{\alpha=1}^d \partial_\alpha (F_\alpha(v, f) f).$$

In the sequel, in order to overcome technical difficulties we make some strong structure, smoothness and boundedness assumptions on the coefficients. Namely we assume that

$$(4.7) \quad (A \equiv 0) \quad \text{or} \quad (A \geq \kappa \text{Id}, \quad \kappa > 0, \quad A \in W^{k,\infty}),$$

as well as

$$(4.8) \quad \forall v \in \mathbb{R}^d, \quad \forall m \in P(\mathbb{R}^d) \quad F(v, m) = (b * m)(v), \quad b \in H^{2k},$$

for some $k \in \mathbb{N}$, $k > d/2 + 3$.

4.2. Statement of the result. Our main result in the section is a quantitative propagation of chaos result for McKean-Vlasov equations with smooth coefficients. It states as follows:

Theorem 4.1 (McKean-Vlasov equation). *Consider an initial distribution $f_{in} \in P_m(\mathbb{R}^d)$, $m \geq 2$, and the hierarchy of N -particle distributions $f_t^N = S_t^N(f_{in}^{\otimes N})$ following (4.5) and the nonlinear evolution $f_t = S_t^{NL}(f_{in})$ following (4.6). Assume that (4.7) and (4.8) hold.*

Then there is some constants $C_T^1, C_T^2 \in (0, \infty)$ only depending on f_{in} and $T \in (0, \infty)$ such that for any

$$\varphi = \varphi_1 \otimes \cdots \otimes \varphi_\ell \in \mathcal{F}^{\otimes \ell}, \quad \mathcal{F} := H^k(\mathbb{R}^d) \cap \text{Lip}(\mathbb{R}^d), \quad \|\varphi_j\|_{\mathcal{F}} \leq 1,$$

we have for $N \geq 2\ell$:

$$(4.9) \quad \sup_{[0, T]} \left| \left\langle \left(S_t^N(f_{in}^{\otimes N}) - (S_t^{NL}(f_{in}))^{\otimes N} \right), \varphi \right\rangle \right| \leq C_T^1 \frac{\ell^2}{N} + C_T^2 \ell \Omega_N^{W_2}(f_{in}).$$

As a consequence of (4.9) and of the discussion of subsection 2.13, quantitative propagation of chaos holds with rate $\varepsilon(N) \leq C(\ell, T, f_{in}) N^{-\frac{1}{d+4}}$ for any initial datum $f_{in} \in P_{d+5}(\mathbb{R}^d)$.

Theorem 4.2 (Vlasov equation). *Assuming furthermore that $A \equiv 0$ and $b(0) = 0$, we have (see the proof below) that $\varepsilon(N) = 0$ in assumption **(A3)**.*

Then there is a constant $C \in (0, +\infty)$ such that for any $\varphi \in \text{Lip}(\mathbb{R}^{\ell d})$ and any $N \geq \ell$:

$$(4.10) \quad \sup_{[0, T]} \left| \left\langle \left(S_t^N(f_{in}^{\otimes N}) - (S_t^{NL}(f_{in}))^{\otimes N} \right), \varphi \right\rangle \right| \leq C \|\nabla \varphi\|_{L^\infty(E^\ell)} \left[\frac{\ell}{N} + C_{5,T} \Omega_N^{W_1}(f_{in}) \right].$$

(observe the replacement of W_2 by W_1 in the last term) which in turn implies

$$\sup_{[0, T]} \frac{1}{N} \mathcal{W}_1 \left((S_t^N(f_{in}^{\otimes N}), (S_t^{NL}(f_{in}))^{\otimes N} \right) \leq \frac{C}{N} \left[1 + C_{5,T} \Omega_N^{W_1}(f_{in}) \right].$$

Remark 4.3. *The mean-field classical Vlasov equation reads as*

$$\partial_t f + \xi \cdot \nabla_x f + (\nabla_x V * \rho[f]) \cdot \nabla_\xi f = 0, \quad f = f(t, x, \xi), \quad x, \xi \in \mathbb{R}^d,$$

with $\rho[f](x) = \int f d\xi$, and it falls into our structural assumptions with $v = (x, \xi) \in \mathbb{R}^d \times \mathbb{R}^d$ and with the choice

$$F_x(x, \xi) = \xi, \quad F_\xi(x, \xi) = \nabla_x V * \rho[f]$$

for the limiting system, and (with $X \in (\mathbb{R}^d)^N$ and $\Xi \in (\mathbb{R}^d)^N$)

$$F_X^N = \Xi, \quad (F_\Xi^N)_i = \frac{1}{N} \sum_{j \neq i}^N \nabla_x V(X_i - X_j), \quad i = 1, \dots, N,$$

as soon as $\nabla_x V$ is H^{d+6+0} .

4.3. Proof of Theorem 4.1. As before we shall prove that Theorem 4.1 is a consequence of Theorem 2.13, by proving that assumptions **(A1)**-**(A2)**-**(A3)**-**(A4)**-**(A5)** hold. However in the present model we cannot, as in Section 3, use the total variation norm for the key consistency **(A3)** and differential stability **(A4)** estimates, because the map $V \mapsto \mu_V^N$ is not differentiable from \mathbb{R}^{dN} to $P(\mathbb{R}^d)$ when $P(\mathbb{R}^d)$ is endowed with the total variation norm. In this case, the expression of $G^N \pi^N \Phi$ naturally yields the requirement of being able to define derivatives of $V \mapsto \mu_V^N$. We make therefore another choice of functional spaces:

$$\mathcal{G}_1 := H^{-s_1}, \quad s_1 > \frac{d}{2} + 2, \quad \mathcal{G}_2 := H^{-s_2}, \quad s_2 := s_1 + 2, \quad \mathcal{F}_i = H^{s_i}, \quad i = 1, 2,$$

and $\mathcal{F}_3 = \text{Lip}(\mathbb{R}^d)$, $P_{\mathcal{G}_3} := P_2(E)$ endowed with the quadratic MKW distance W_2 .

Proof of assumption (A1). Taking $m_1 = 1$, the assumption is just a consequence of the fact that (4.5) is well posed for any $f^N(0, \cdot) \in P_{\text{sym}}(E^N)$ so that f_t^N is a probability measure for any $t \geq 0$.

Proof of assumption (A3). We claim that for any $s_1 > d/2 + 2$ there exists a constant C_{s_1} such that for all $\Phi \in C^{2,1}(P_{\mathcal{G}_1}, \mathbb{R})$

$$(4.11) \quad \left\| G^N(\Phi \circ \mu_V^N) - \langle Q(\mu_V^N, \mu_V^N), D\Phi[\mu_V^N] \rangle \right\|_{L^\infty(E^N)} \leq \frac{C_{s_1}}{N} \|\Phi\|_{C^{2,1}(P_{\mathcal{G}_1}, \mathbb{R})},$$

which is nothing but **(A3)** with $k = 2$, $\eta = 1$ and $\varepsilon(N) = C_{s_1} N^{-1}$.

Proof of (4.11). First, the map $\mathbb{R}^{dN} \rightarrow H^{-s_1}(\mathbb{R}^d)$, $V \mapsto \mu_V^N$ is clearly C^2 with

$$\partial_{v_{i,\alpha}} \mu_V^N = \frac{1}{N} \partial_\alpha \delta_{v_i}, \quad \partial_{v_{i,\alpha}, v_{i,\beta}}^2 \mu_V^N = \frac{1}{N^2} \partial_{\alpha\beta}^2 \delta_{v_i}.$$

Take $\Phi \in C_b^{2,1}(P_{\mathcal{G}_1})$. Then the map $\mathbb{R}^{dN} \rightarrow \mathbb{R}$, $V \mapsto \Phi(\mu_V^N)$ is C_b^2 . Indeed, denoting $\phi = \phi_V(\cdot) = D\Phi[\mu_V^N] \in (H^{-s_1}(\mathbb{R}^d))' = H^{s_1}(\mathbb{R}^d)$, we can write:

$$\begin{aligned} \partial_{v_{i,\alpha}} \Phi(\mu_V^N) &= \left\langle D\Phi[\mu_V^N], \frac{1}{N} \partial_\alpha \delta_{v_i} \right\rangle = \frac{1}{N} \partial_\alpha \phi_V(v_i) \\ \partial_{v_{i,\alpha}, v_{i,\beta}}^2 \Phi(\mu_V^N) &= \left\langle D\Phi[\mu_V^N], \frac{1}{N} \partial_{v_{i,\alpha}, v_{i,\beta}}^2 \delta_{v_i} \right\rangle + D^2\Phi[\mu_V^N] \left(\frac{1}{N} \partial_{v_{i,\alpha}} \delta_{v_i}, \frac{1}{N} \partial_{v_{i,\beta}} \delta_{v_i} \right) \\ &= \frac{1}{N} \partial_{\alpha,\beta}^2 \phi_V(v_i) + \frac{1}{N^2} D^2\Phi[\mu_V^N] (\partial_{v_{i,\alpha}} \delta_{v_i}, \partial_{v_{i,\beta}} \delta_{v_i}). \end{aligned}$$

As a consequence, we may compute

$$\begin{aligned} (G^N \pi^N \Phi)(V) &= G^N \Phi(\mu_V^N) \\ &= \sum_{i=1}^N A(v_i) \nabla_i^2 (\Phi(\mu_V^N)) - \sum_{i=1}^N F^N(v_i, \mu_{\hat{V}_i}^{N-1}) \cdot \nabla_i (\Phi(\mu_V^N)) \\ &=: I_1(V) + I_2(V) \end{aligned}$$

with

$$I_1(V) := \frac{1}{N} \sum_{i=1}^N \sum_{\alpha, \beta=1}^d A_{\alpha, \beta}(v_i) \partial_{\alpha, \beta}^2 \phi_V(v_i) + \frac{1}{N} \sum_{i=1}^N \sum_{\alpha=1}^d F_\alpha(v_i, \mu_V^N) \partial_\alpha \phi_V(v_i)$$

and

$$\begin{aligned} I_2(V) &:= \frac{1}{N^2} \sum_{i=1}^N \sum_{\alpha, \beta=1}^d A_{\alpha, \beta}(v_i) D^2 \Phi[\mu_V^N] (\partial_{v_i, \alpha} \delta_{v_i}, \partial_{v_i, \beta} \delta_{v_i}) \\ &\quad + \frac{1}{N} \sum_{i=1}^N \sum_{\alpha=1}^d \left[F_\alpha^N(v_i, \mu_{\hat{V}_i}^{N-1}) - F_\alpha(v_i, \mu_V^N) \right] D\Phi[\mu_V^N] (\partial_{v_i, \alpha} \delta_{v_i}). \end{aligned}$$

On the one hand, using that

$$\left\| \mu_{\hat{V}_i}^{N-1} - \mu_V^N \right\|_{H^{-(s_1-1)}} \leq \frac{C_k}{N}$$

as well as (4.2) and (4.4), we deduce

$$\begin{aligned} I_2(V) &\leq N \frac{1}{N^2} \|A\|_\infty \|D^2 \Phi\|_\infty \|\partial_1 \delta\|_{H^{-(s_1-1)}}^2 \\ &\quad + N d \left(\frac{C_{F,2}}{N} + \frac{C_k C_{F,k}}{N} \right) \frac{1}{N} \|D\Phi\|_\infty \|\partial_1 \delta\|_{H^{-(s_1-1)}} \leq \frac{C_\Phi}{N}. \end{aligned}$$

On the other hand, we just recognize

$$\begin{aligned} I_1(V) &= \left\langle \mu_V^N, \sum_{\alpha, \beta=1}^d A_{\alpha, \beta} \partial_{\alpha, \beta}^2 \phi_V \right\rangle + \left\langle \mu_V^N, \sum_{\alpha=1}^d F_\alpha(\cdot, \mu_V^N) \partial_\alpha \phi_V \right\rangle \\ &= \langle Q(\mu_V^N), \phi_V \rangle = \langle Q(\mu_V^N), D\Phi(\mu_V^N) \rangle = (\pi^N G^\infty \Phi)(V), \end{aligned}$$

thanks to the calculation of the limit dual generator made in subsection 2.5. \square

Proof of assumption (A4). We need here to perform a *second-order* expansion of the limit semigroup.

For any two given initial data $f_{in}, g_{in} \in P(\mathbb{R}^d)$, we denote by

- f_t and g_t the corresponding solutions to the non-linear McKean-Vlasov equation (4.6),
- h_t the solution of the following linearized equation around f_t

$$(4.12) \quad \partial_t h = \nabla^2(Ah) + \nabla[h(b * f) + f(b * h)], \quad h_{t=0} = h_{in} = g_{in} - f_{in},$$

- r_t the solution of the following “second variation” equation around f_t

$$(4.13) \quad \partial_t r = \nabla^2(Ar) + \nabla[r(b * f) + f(b * r)] + \nabla[h(b * h)], \quad r_{|t=0} = r_{in} = 0.$$

Then we shall prove the following a priori estimates.

Lemma 4.4. *For any $s_1 \in \mathbb{N}$, $s_1 > d/2 + 1$ and for any $T > 0$, there exists C_T such that*

$$(4.14) \quad \sup_{[0, T]} \|g - f\|_{H^{-s_1}} \leq C_T \|g_{in} - f_{in}\|_{H^{-s_1}},$$

$$(4.15) \quad \sup_{[0, T]} \|g - f - h\|_{H^{-(s_1+1)}} \leq C_T \|g_{in} - f_{in}\|_{H^{-s_1}}^2,$$

$$(4.16) \quad \sup_{[0, T]} \|g - f - h - r\|_{H^{-(s_1+2)}} \leq C_T \|g_{in} - f_{in}\|_{H^{-s_1}}^3.$$

This shows that the nonlinear semigroup S_t^{NL} associated to the nonlinear McKean-Vlasov equation (4.6) is $C_b^{2,1}(P_{\mathcal{G}_1}, P_{\mathcal{G}_2})$.

Proof of Lemma 4.4. We shall divide the proof into several steps.

Step 1. We shall several times consider the equation

$$(4.17) \quad \partial_t z_t = \nabla^2(A z_t) - \nabla(u_1 z_t + u_2 (b * z_t))$$

with given initial data z_{in} and with some functions $u_1 \in \mathbb{R}^d$, $u_2 \in \mathbb{R}$ to be specified (chosen in order to “match” equations (4.6), (4.12) and (4.13)). We claim that for any $k \in \mathbb{N}$, $k > d/2 + 1$, and any $T > 0$,

$$(4.18) \quad \forall t \in [0, T], \quad \|z_t\|_{H^{-k}} \leq \|z_{in}\|_{H^{-k}} e^{C_k(b, u_1, u_2) T}$$

with

$$C_k(b, u_1, u_2) := C(k) \sup_{t \in [0, T]} \left[\|u_1\|_{W^{k, \infty}} + \|b\|_{H^k} \|u_2\|_{TV} \right].$$

We argue by duality and we consider a smooth solution ζ to the following linear equation (which is the dual equation of (4.17))

$$(4.19) \quad \partial_t \zeta = L_1^* \zeta + L_2^* \zeta, \quad L_1^* \zeta := A \cdot \nabla^2 \zeta, \quad L_2^* \zeta := u_1 \cdot \nabla \zeta + \check{b} * (u_2 \nabla \zeta),$$

with $\check{b}(x) := b(-x)$. For a given $k' \in \mathbb{N}$, $k' \leq k$, we compute

$$\frac{d}{dt} \int |\nabla^{k'} \zeta|^2 = \int (\nabla^{k'} L_1^* \zeta) \nabla^{k'} \zeta + \int (\nabla^{k'} L_2^* \zeta) \nabla^{k'} \zeta =: \mathcal{L}_1 + \mathcal{L}_2.$$

On the one hand in the case $A \not\equiv 0$, we have

$$\mathcal{L}_1 = \sum_{\eta, |\eta|=k'} \int \partial_\eta^{k'} (A_{\alpha\beta} \partial_{\alpha\beta}^2 \zeta) \partial_\eta^{k'} \zeta = \sum_{k''=0}^{k'} \binom{k'}{k''} \sum_{\eta, \gamma} \int (\partial_\gamma^{k''} A_{\alpha\beta}) (\partial_{\eta-\gamma}^{k'-k''} \partial_{\alpha\beta}^2 \zeta) \partial_\eta^{k'} \zeta,$$

where in the last sum the letters η and γ stand for two multi-indices (in \mathbb{N}^d) such that $0 \leq \eta - \gamma$, $|\gamma| = k''$, $|\eta - \gamma| = k' - k''$ (so that $|\eta| = k'$). Then, performing one

integration by part on the term corresponding to $k'' = 0$, we have

$$\begin{aligned} \mathcal{L}_1 &= \sum_{\eta, |\eta|=k'} \left(- \int A_{\alpha\beta} (\partial_\alpha \partial_\eta^{k'} \zeta) (\partial_\beta \partial_\eta^{k'} \zeta) - \int (\partial_\beta A_{\alpha\beta}) (\partial_\eta^{k'} \partial_\alpha \zeta) \partial_\eta^{k'} \zeta \right) \\ &\quad + \sum_{k''=1}^{k'} \binom{k'}{k''} \sum_{\eta, \gamma} \int (\partial_\gamma^{k''} A_{\alpha\beta}) (\partial_{\eta-\gamma}^{k'-k''} \partial_{\alpha\beta}^2 \zeta) \partial_\eta^{k'} \zeta \\ &\leq -\kappa \sum_{|\eta|=k'} \|\nabla \partial_\eta^{k'} \zeta\|_{L^2}^2 + C_{k'} \|A_{\alpha\beta}\|_{W^{k', \infty}} \sum_{\ell, \ell'=0}^{k'} \|\nabla^{\ell+1} \zeta\|_{L^2} \|\nabla^{\ell'} \zeta\|_{L^2}. \end{aligned}$$

We deduce that for some numerical constant $C_k > 0$ there holds

$$\sum_{k'=0}^k \int (\nabla^{k'} L_1^* \zeta) \nabla^{k'} \zeta \leq C_k \|A_{\alpha\beta}\|_{W^{k, \infty}} \sum_{\ell=0}^k \|\nabla^\ell \zeta\|_{L^2}^2.$$

On the other hand, we have

$$\begin{aligned} \mathcal{L}_2 &= \sum_{\eta, |\eta|=k'} \sum_{\alpha=1}^d \int u_{1\alpha} (\partial_\alpha \partial_\eta^{k'} \zeta) \partial_\eta^{k'} \zeta + \sum_{k''=1}^{k'} \binom{k'}{k''} \sum_{\alpha=1}^d \sum_{\eta, \gamma} \int \partial_\gamma^{k''} u_{1\alpha} (\partial_\alpha \partial_{\eta-\gamma}^{k'-k''} \zeta) \partial_\eta^{k'} \zeta \\ &\quad + (-1)^{k'} \iint (\nabla^{k'} b) (y-x) u_2(y) \nabla \zeta(y) \nabla^{k'} \zeta(x) dx dy =: \mathcal{L}_{2,1} + \mathcal{L}_{2,2} + \mathcal{L}_{2,3}. \end{aligned}$$

Integrating by part the first term $\mathcal{L}_{2,1}$, we have

$$|\mathcal{L}_{2,1}| = \left| -\frac{1}{2} \sum_{\eta, |\eta|=k'} \int (\operatorname{div} u_1) |\partial_\eta^{k'} \zeta|^2 \right| \leq \|\operatorname{div} u_1\|_{L^\infty} \|\nabla^{k'} \zeta\|_{L^2}^2.$$

For the the second term $\mathcal{L}_{2,2}$ we have

$$|\mathcal{L}_{2,2}| = \left| \sum_{k''=1}^{k'} \binom{k'}{k''} \int \nabla^{k''} u_{1\alpha} \nabla^{k'-k''} \partial_\alpha \zeta \nabla^{k'} \zeta \right| \leq C_{k'} \sum_{k''=1}^{k'} \left\| \nabla^{k''} u_1 \right\|_{L^\infty} \|\zeta\|_{H^{k'}}^2.$$

Finally using the Cauchy-Schwartz inequality in the x variable and the Sobolev embedding $W_y^{1, \infty} \subset H_y^k$ (because $k > d/2 + 1$), the third term is controlled by

$$\begin{aligned} |\mathcal{L}_{2,3}| &= \left| \iint (\nabla^{k'} b) (y-x) u_2(y) \nabla \zeta(y) \nabla^{k'} \zeta(x) dx dy \right| \\ &\leq \sup_y \left\| \nabla^{k'} b(y - \cdot) \right\|_{L^2} \|\nabla^{k'} \zeta\|_{L^2} \int |u_2(y)| |\nabla \zeta(y)| dy \\ &\leq \|\nabla^{k'} b\|_{L^2} \|u_2\|_{TV} \|\zeta\|_{H^k} \|\nabla^{k'} \zeta\|_{L^2}. \end{aligned}$$

Gathering and summing all theses terms between $k' = 0$ to $k' = k$, we conclude that

$$\forall t \in [0, T], \quad \|\zeta_t\|_{H^k} \leq \|\zeta_{in}\|_{H^k} e^{C_k(b, u_1, u_2)T}.$$

Denoting by U_t the linear semigroup associated to (4.17), the associated dual semigroup U_t^* is nothing but the one generated by (4.19). As a consequence, for any $z_{in} \in H^k$, we have

$$\langle h_t, z_{in} \rangle = \langle h_{in}, U_t^* z_{in} \rangle \leq \|h_{in}\|_{H^{-k}} \|U_t^* z_{in}\|_{H^k} \leq C_T \|h_{in}\|_{H^{-k}} \|z_{in}\|_{H^k},$$

and we conclude that (4.18) holds.

Step 2. Proof of (4.14). The equation satisfied by the difference $\delta_t = g_t - f_t$ is

$$(4.20) \quad \partial_t \delta = \nabla^2(A\delta) + \nabla(\delta(b*f) + g(b*\delta)), \quad \delta|_{t=0} = \delta_{in} = g_{in} - f_{in},$$

which is nothing but (4.17) with $u_1 := b*f$ and $u_2 = g$. Now, since

$$\|\nabla^k(b*f)\|_{L^2} = \|(\nabla^k b)*f\|_{L^2} \leq \|\nabla^k b\|_{L^2}$$

we conclude that $C_k(b, f*b, g) \leq C \|b\|_{H^k}$ and that (4.14) holds. Proceeding exactly in the same way for the function h we end up with

$$(4.21) \quad \sup_{[0, T]} \|h_t\|_{H^{-k_1}} \leq C_T \|g_{in} - f_{in}\|_{H^{-k_1}},$$

for any $k_1 \in \mathbb{N}$, $k_1 > d/2 + 1$.

Step 3. Second preliminary. First, for $S \in H^{-k}$, $b \in H^k$, we have $S*b \in L^\infty$ with $\|S*b\|_\infty \leq \|b\|_{H^k} \|S\|_{H^{-k}}$. Indeed, for any $\varphi \in L^1$, we compute

$$\begin{aligned} \int (S*b)\varphi &= \int S(\check{b}* \varphi) \\ &\leq \|S\|_{H^{-k}} \|\check{b}* \varphi\|_{H^k} \\ &\leq \|S\|_{H^{-k}} (\|\check{b}* \varphi\|_{L^2}^2 + \dots + \|(\nabla^k \check{b})* \varphi\|_{L^2}^2)^{1/2} \\ &\leq \|S\|_{H^{-k}} \|b\|_{H^k} \|\varphi\|_{L^1}. \end{aligned}$$

Next, for $S \in H^{-k}$, $b \in H^{2k}$ we deduce, iterating the preceding estimate, that $S*b \in W^{k, \infty}$ with $\|S*b\|_{W^{k, \infty}} \leq \|b\|_{H^{2k}} \|S\|_{H^{-k}}$. Moreover, for $S \in H^{-k}$, $\psi \in W^{k, \infty}$, we have $S\psi \in H^{-k}$ with $\|S\psi\|_{H^{-k}} \leq C_k \|S\|_{H^{-k}} \|\psi\|_{W^{k, \infty}}$. Indeed, for any $\varphi \in H^k$, we compute

$$\begin{aligned} |\langle S\psi, \varphi \rangle| &= |\langle S, \psi\varphi \rangle| \\ &\leq \|S\|_{H^{-k}} \|\psi\varphi\|_{H^k} \\ &\leq \|S\|_{H^{-k}} \left(\sum_{k'=0}^k \binom{k}{k'} \|\nabla^{k'} \psi \nabla^{k-k'} \varphi\|_{L^2}^2 \right)^{1/2} \\ &\leq \|S\|_{H^{-k}} \|\psi\|_{W^{k, \infty}} \|\varphi\|_{H^k}. \end{aligned}$$

Step 4. Proof of (4.15). Let $\omega_t := g_t - f_t - h_t = \delta_t - h_t$, which satisfies the equation

$$(4.22) \quad \partial_t \omega = L\omega + \Sigma, \quad \omega|_{t=0} = 0,$$

with

$$L\omega := \frac{1}{2} \nabla^2(A\omega) + \nabla(\omega(b*f) + f(b*\omega)), \quad \Sigma_t = \nabla(\delta_t(b*\delta_t)).$$

Denoting by $\Theta_{s,t} w$ the unique solution of the linear, non-autonomous equation

$$\partial_t w_t = Lw_t, \quad w_s = w,$$

the Duhamel formula for equation (4.22) yields

$$\omega_t = \int_0^t \Theta_{s,t} \Sigma_s ds.$$

Therefore we obtain using (4.18) and the estimates established in the Step 3 that for any $t \in [0, T]$

$$\begin{aligned} \|\omega_t\|_{H^{-k}} &\leq C_T \int_0^t \|\nabla(\delta_s(b * \delta_s))\|_{H^{-k}} ds \\ &\leq C_{T,k} \int_0^t \{ \|\nabla \delta_s\|_{H^{-k}} \|b * \delta_s\|_{W^{k,\infty}} + \|\delta_s\|_{H^{-k}} \|b * (\nabla \delta_s)\|_{W^{k,\infty}} \} ds \\ &\leq C_{T,k} \|b\|_{H^{2k}} \int_0^t \|\delta_s\|_{H^{-(k-1)}} \|\delta_s\|_{H^{-k}} ds, \end{aligned}$$

which together with (4.21) for the $H^{-(k-1)}$ norm implies (4.15).

Step 5. Proof of (4.16). For the second variation r which satisfies the equation

$$\partial_t r = L r + R, \quad r_{in} = 0,$$

with L as above and

$$R_t = \nabla(h_t(b * h_t)),$$

we proceed as in Step 4, taking advantage of the bound (4.21), and we obtain

$$(4.23) \quad \sup_{[0,T]} \|r_t\|_{H^{-k}} \leq C_T \|g_{in} - f_{in}\|_{H^{-(k-1)}}^2.$$

Finally we introduce $\psi_t := g_t - f_t - h_t - r_t$, which satisfies the equation

$$(4.24) \quad \partial_t \psi = L \psi + \Psi, \quad \Psi_t = \nabla(h_t(b * \omega_t) + \omega_t(b * h_t)) \quad \psi_{in} = 0.$$

Therefore, we deduce

$$\begin{aligned} \forall t \in [0, T], \quad \|\psi_t\|_{H^{-k}} &\leq \left\| \int_0^t \Theta_{s,t} \Psi_s ds \right\|_{H^{-k}} \\ &\leq C_T \int_0^t \|h_s\|_{H^{-(k-1)}} \|\omega_s\|_{H^{-(k-1)}} ds, \end{aligned}$$

which together with (4.21) and (4.15) implies (4.16). \square

Proof of (A2). First property **(A2-i)** is a consequence of (4.14) in lemma 4.4. Second we claim that for any probabilities $f_i \in P(\mathbb{R}^d)$, we have

$$(4.25) \quad \|Q(f_i)\|_{H^{-k}} \leq C_{b,1} \quad \text{and} \quad \|Q(f_2) - Q(f_1)\|_{H^{-k}} \leq C_{b,2} \|f_2 - f_1\|_{H^{-(k-2)}}.$$

We write

$$Q(f) = Q_1(f) + Q_2(f) \quad \text{with} \quad Q_1(f) = \nabla^2(Af), \quad Q_2(f) = -\nabla((b * f)f).$$

The linear term Q_1 clearly satisfies (4.25) thanks to Step 3 in the proof of **(A4)** and because $A \in W^{k-2,\infty}$. Concerning the quadratic term Q_2 , using estimates proved in Step 3 of the proof of **(A4)**, we get

$$\begin{aligned} \|Q_2(f)\|_{H^{-k}} &\leq \|b * f\|_{W^{k-1,\infty}} \|f\|_{H^{-(k-1)}} \\ &\leq C_k \|b\|_{H^{2(k-1)}} \|f\|_{H^{-(k-1)}}^2 \leq C_k \|b\|_{H^{2(k-1)}} \end{aligned}$$

and

$$\begin{aligned} \|Q_2(f_2) - Q_2(f_1)\|_{H^{-k}} &\leq \\ &\leq \|b * (f_2 - f_1)\|_{W^{k-1,\infty}} \|f_2\|_{H^{-(k-1)}} + \|b * f_1\|_{W^{k-1,\infty}} \|f_2 - f_1\|_{H^{-(k-1)}} \\ &\leq C_k \|b\|_{H^{2(k-1)}} \|f_2 - f_1\|_{H^{-(k-1)}}. \end{aligned}$$

Then, since $k-2 > d/2$, thanks to an interpolation argument, there exists $\delta \in (0, 1)$ such that

$$(4.26) \quad \begin{aligned} \|Q(f_2) - Q(f_1)\|_{H^{-k}} &\leq C_{b,2} \|f_2 - f_1\|_{H^{-(k-2+d/2)/2}}^{1-\delta} \|f_2 - f_1\|_{H^{-k}}^\delta \\ &\leq C_{b,3} \|f_2 - f_1\|_{H^{-k}}^\delta, \end{aligned}$$

which concludes the proof of **(A2ii)**.

Proof of (A5). We use the well known following estimate (see [24]): for any $p \geq 1$, $f_{in}, g_{in} \in P_p(\mathbb{R}^d)$ and $T > 0$ there exists C_T such that

$$\sup_{t \geq 0} W_q(S_t^{NL} f_{in}, S_t^{NL} g_{in}) \leq C_T W_q(f_{in}, g_{in}),$$

that we use with $p = 2$. Alternatively, estimate (4.14) precisely says that assumption **(A5)** holds in P_{G_1} .

5. INELASTIC COLLISIONS WITH THERMAL BATH

5.1. The model. In this section we consider an N -particle system fully characterized by the velocities of each particle in $E = \mathbb{R}^d$, $d \geq 2$, and undergoing random inelastic Boltzmann collisions together with Brownian motion. The velocities (\mathcal{V}_t) satisfy a mixed jump and diffusion Markov process that we define through its Kolmogorov equation: we assume that the probability law f_t^N of (\mathcal{V}_t) in \mathbb{R}^{dN} satisfies the following **master equation**

$$(5.1) \quad \partial_t \langle f_t^N, \varphi \rangle = \langle f_t^N, G^N \varphi \rangle$$

with generator $G^N = G_1^N + G_2^N$, where G_1^N is associated to an **inelastic** Boltzmann collision process with collision kernel $b \in L^1(-1, 1)$ only depending of the deviation angle as in Section 3, satisfying the normalization assumption

$$\|b\|_{L^1} = \int_{\mathbb{S}^{d-1}} b(\sigma_1) d\sigma = 1,$$

and G_2^N is the generator associated to the Brownian motion.

Namely, we define

$$(5.2) \quad (G_1^N \varphi)(V) = \frac{1}{N} \sum_{i,j=1}^N \int_{\mathbb{S}^{d-1}} b(\cos \theta_{ij}) [\varphi(V_{ij}^*) - \varphi(V)] d\sigma,$$

with $\cos \theta_{ij} = \sigma \cdot (v_j - v_i) / |v_j - v_i|$, $V_{ij}^* = (v_1, \dots, v_{i-1}, v_i^*, v_{i+1}, \dots, v_{j-1}, v_j^*, v_{j+1}, \dots, v_N)$ and as in equation (3.1)

$$(5.3) \quad v_i^* = \frac{w_{ij}}{2} + \frac{u_{ij}^*}{2}, \quad v_j^* = \frac{w_{ij}}{2} - \frac{u_{ij}^*}{2},$$

but here with

$$w_{ij} = v_i + v_j, \quad u_{ij}^* = \left(\frac{1 - \alpha}{2} \right) u_{ij} + \left(\frac{1 + \alpha}{2} \right) |u_{ij}| \sigma, \quad u_{ij} = v_i - v_j$$

for $\alpha \in (0, 1]$.

Observe that for $\alpha = 1$, one recovers the elastic collision process introduced in Section 3.

We also define

$$(5.4) \quad (G_2^N \varphi)(V) = \nu \sum_{i=1}^N \Delta_i \varphi,$$

where $\nu > 0$, $V_i := (v_{i,1}, \dots, v_{i,d})$ and Δ_i denotes the Laplacian in \mathbb{R}^d associated to the i -th particle:

$$\Delta_i := \sum_{\alpha=1}^d \partial_{v_{i,\alpha}, v_{i,\alpha}}^2.$$

In the mean-field limit of infinite number of particles, we consider the following **Boltzmann equation for diffusively excited granular media** on the distribution $f(t, v) \geq 0$, $v \in \mathbb{R}^d$ of particles:

$$(5.5) \quad \frac{\partial f_t}{\partial t} = Q(f_t), \quad f(0) = f_{in} \quad \text{in } P(\mathbb{R}^d),$$

with

$$Q(f) = Q_\alpha(f, f) + \nu \Delta f,$$

where the quadratic Boltzmann collision kernel Q_α is defined thanks to the following dual formulation

$$(5.6) \quad \langle Q_\alpha(f, f), \varphi \rangle := \int_{\mathbb{R}^{2d} \times \mathbb{S}^{d-1}} b(\cos \theta) (\phi(w_2^*) - \phi(w_2)) d\sigma f(dw_1) f(dw_2)$$

for any $\varphi \in C_0(\mathbb{R}^d)$, $f \in P(\mathbb{R}^d)$, and with $\cos \theta = \sigma \cdot (w_2 - w_1) / |w_2 - w_1|$ and

$$w_2^* = \frac{w_1 + w_2}{2} + \frac{u^*}{2}, \quad u^* = \left(\frac{1 - \alpha}{2} \right) (w_1 - w_2) + \left(\frac{1 + \alpha}{2} \right) |w_2 - w_1| \sigma.$$

It is clear from [4, 3, 5], that this equation generates a nonlinear semigroup $S_t^{NL} f_{in} := f_t$ for any $f_{in} \in P_m(\mathbb{R}^d)$, $m \geq 2$. Notice that unlike the classical Boltzmann equation the kinetic energy is not conserved. We finally emphasize that because of the normalizations $\|b\|_{L^1(\mathbb{S}^{d-1})} = 1$ and $f \in P(\mathbb{R}^d)$, the bilinear operator Q_α splits into a quadratic part and a linear part

$$Q(f) = Q_\alpha^+(f, f) - f + \Delta f,$$

where Q^+ is defined through the positive part of the expression (5.6) and where for the sake of simplification of notation we take $\nu = 1$ here and below.

5.2. Statement of the result. The main result in this section is a quantitative estimate of propagation of chaos for the mixed collision and diffusion model introduced above.

Theorem 5.1. *Consider an initial distribution $f_{in} \in P_m(\mathbb{R}^d)$, $m \geq 2$ and the hierarchy of N -particle distributions $f_t^N = S_t^N(f_{in}^{\otimes N})$ following the evolution (5.1), as well the nonlinear semigroup $f_t = SNL_t(f_{in})$ following the evolution (5.5).*

Then there are constants $C_T^1, C_T^2 \in (0, \infty)$ only depending on f_{in} and $T \in (0, \infty)$ such that for any

$$\varphi = \varphi_1 \otimes \cdots \otimes \varphi_\ell \in \mathcal{F}^{\otimes \ell}, \quad \mathcal{F} := W^{9,1}(\mathbb{R}^d) \cap W^{1,\infty}(\mathbb{R}^d), \quad \|\varphi_j\|_{\mathcal{F}} \leq 1,$$

we have for $N \geq 2\ell$:

$$(5.7) \quad \sup_{[0,T]} \left| \left\langle \left(S_t^N(f_{in}^N) - (S_t^{NL}(f_{in}))^{\otimes N} \right), \varphi \right\rangle \right| \leq C_T^1 \frac{\ell^2}{N} + C_T^2 \ell \Omega_N^{W_2}(f_{in}).$$

As a consequence of (5.7) and of the discussion of subsection 2.13, quantitative propagation of chaos holds with rate $\varepsilon(N) \leq C(\ell, T, f_{in}) N^{-\frac{1}{d+4}}$ for any initial datum $f_{in} \in P_{d+5}(\mathbb{R}^d)$.

5.3. Proof of Theorem 5.1. We shall prove that Theorem 5.1 is a consequence of Theorem 2.13 by proving that the assumptions **(A1)**-**(A2)**-**(A3)****(A4)**-**(A5)** hold for the following choice of functional spaces

$$\mathcal{G}_1 := \mathcal{H}^{-s_1}, \quad s_1 := 3, \quad \mathcal{G}_2 := \mathcal{H}^{-s_2}, \quad s_2 := 3s_1 = 9, \quad \mathcal{F}_i = W^{s_i,1}, \quad i = 1, 2,$$

where the Fourier based space \mathcal{H}^{-s} is defined in example 2.17 and $\mathcal{F}_3 = \text{Lip}(\mathbb{R}^d)$, $P_{\mathcal{G}_3} := P_2(E)$ endowed with the quadratic MKW distance W_2 .

Proof of (A1). The well-posedness of equation (5.1)-(5.2) does not raise any difficulty: it is a variation on the well-posedness result for equation (5.5) as obtained in [3, 5].

Proof of (A2). First we prove **(A2i)**, and more precisely S_t^{NL} is $C^{0,1}(P_{\mathcal{G}_1}, P_{\mathcal{G}_1})$, which is an immediate consequence of the following result.

Lemma 5.2. *For any $f_{in}, g_{in} \in P(\mathbb{R}^d)$ and any final time $T \in (0, \infty)$, the associated solutions f_t and g_t to the diffusive inelastic Boltzmann equation (5.5) satisfy for $s \geq 0$*

$$(5.8) \quad \sup_{t \in [0, T]} |f_t - g_t|_s \leq e^{2T} |f_{in} - g_{in}|_s.$$

Proof of Lemma 5.2. We recall Bobylev's identity for Maxwellian inelastic collision kernel (see for instance [3])

$$\mathcal{F}(Q_\alpha^+(f, g))(\xi) = \hat{Q}_\alpha^+(F, G)(\xi) =: \frac{1}{2} \int_{S^{d-1}} b(\sigma \cdot \hat{\xi}) [F^+ G^- + F^- G^+] d\sigma,$$

with $F = \hat{f}$, $G = \hat{g}$, $F^\pm = F(\xi^\pm)$, $G^\pm = G(\xi^\pm)$ and

$$\xi^+ = \frac{3-\alpha}{4} \xi + \frac{1+\alpha}{4} |\xi| \sigma, \quad \xi^- = \frac{1+\alpha}{4} (\xi - |\xi| \sigma).$$

Denoting by $D = \hat{g} - \hat{f}$, $S = \hat{g} + \hat{f}$, the following equation holds

$$(5.9) \quad \partial_t D = \int_{S^2} b(\sigma \cdot \hat{\xi}) \left[\frac{D^+ S^-}{2} + \frac{D^- S^+}{2} \right] d\sigma - D - |\xi|^2 D.$$

Using that $\|S\|_\infty \leq 2$ and then $|\xi^\pm| \leq |\xi|$, we deduce in distributional sense

$$\begin{aligned} \frac{d}{dt} \frac{|D|}{\langle \xi \rangle^s} &\leq \left(\sup_{\xi \in \mathbb{R}^d} \frac{|D|}{\langle \xi \rangle^s} \right) \left(\sup_{\xi \in \mathbb{R}^d} \int_{S^{d-1}} b(\sigma \cdot \hat{\xi}) \left\{ \frac{\langle \xi^+ \rangle^s}{\langle \xi \rangle^s} + \frac{\langle \xi^- \rangle^s}{\langle \xi \rangle^s} \right\} d\sigma \right) \\ &\leq 2 \sup_{\xi \in \mathbb{R}^d} \frac{|D|}{\langle \xi \rangle^s}, \end{aligned}$$

from which we conclude that (5.8) holds. \square

Next we prove that **(A2ii)** holds as an immediate consequence of the following result.

Lemma 5.3. *For any $f, g \in P(\mathbb{R}^d)$ and $s \geq 0$, we have*

$$(5.10) \quad |Q_\alpha(f, f)|_s \leq 2$$

and

$$(5.11) \quad |Q_\alpha(f + g, f - g)|_s \leq 3 |f - g|_s.$$

Moreover for any $s > 2$ there exists $\delta \in (0, 1)$ such that

$$(5.12) \quad |\Delta f - \Delta g|_s \leq 2 |f - g|_s^\delta.$$

Proof of Lemma 5.2. We prove the second inequalities (5.11). We write in Fourier:

$$\begin{aligned} \mathcal{F}(Q_\alpha(f + g, f - g)) &= \hat{Q}_\alpha(D, S) \\ &= \frac{1}{2} \int_{\mathbb{S}^{d-1}} b(\sigma \cdot \hat{\xi}) (S(\xi^+) D(\xi^-) + S(\xi^-) D(\xi^+) - 2D(\xi)) \end{aligned}$$

where \hat{Q}_α is the Fourier form the symmetrization of the collision operator Q_α , which yields

$$\frac{|\hat{Q}_\alpha(D, S)|}{\langle \xi \rangle^s} \leq \mathcal{T}_1 + \mathcal{T}_2 + \mathcal{T}_3,$$

with

$$\begin{aligned} \mathcal{T}_1 &:= \left| \frac{1}{2 \langle \xi \rangle^s} \int_{\mathbb{S}^{d-1}} b(\sigma \cdot \hat{\xi}) S(\xi^+) D(\xi^-) d\sigma \right| \\ &\leq \int_{\mathbb{S}^{d-1}} b(\sigma \cdot \hat{\xi}) \frac{|S(\xi^+)|}{2} \frac{|D(\xi^-)|}{\langle \xi^- \rangle^s} \frac{\langle \xi^- \rangle^s}{\langle \xi \rangle^s} d\sigma \leq |D|_s. \end{aligned}$$

Similar estimates hold for the two other terms \mathcal{T}_2 and \mathcal{T}_3 . The proof of the first inequality (5.10) is similar (and simpler): we just use the Fourier representation of $Q_\alpha(f, f)$ and the bound $\|\hat{f}\|_{L^\infty} \leq 1$. We finally prove the last inequality. We compute

$$|\Delta f - \Delta g|_s = \sup_{\xi \in \mathbb{R}^d} |\xi|^2 \frac{|F - G|}{\langle \xi \rangle^s} \leq \sup_{\xi \in \mathbb{R}^d} |F - G|^{1-\delta} \left(\frac{|F - G|}{\langle \xi \rangle^s} \right)^\delta$$

with $\delta := (s - 2)/s$. □

Proof of (A3). We claim that for any $s_1 \geq 3$ there exists $C_1 \in \mathbb{R}_+$ such that for all $\Phi \in C^{2,1}(P_{\mathcal{G}_1}, \mathbb{R})$

$$(5.13) \quad \left\| G^N(\Phi \circ \mu_V^N) - \langle Q(\mu_V^N, \mu_V^N), D\Phi[\mu_V^N] \rangle \right\|_{L^\infty(E^N)} \leq \frac{C_1}{N} \|\Phi\|_{C^{2,1}(P_{\mathcal{G}_1}, \mathbb{R})},$$

which is nothing but (A3) with $k = 2$, $\eta = 1$ and $\varepsilon(N) = C_1 N^{-1}$.

We begin with a technical lemma on the norm $|\cdot|_s$ which shows that it is well-adapted for obtaining differentiability of the empirical measures. It is worth emphasizing that the choice of $s_1 = 3$ (in fact we only need $s_1 > 2$ by modifying slightly the arguments) comes from the need that the function $V \mapsto \Phi(\mu_V^N)$ be C^2 , see the following lemma.

Lemma 5.4. *The map $\mathbb{R}^{Nd} \rightarrow P_{\mathcal{G}_1}$, $V \mapsto \mu_V^N$ is $C^{2,1}$ and $\partial_{i\alpha}(\mu_V^N) = N^{-1} \partial_\alpha \delta_{V_i}$ as well as $\partial_{i\alpha, i\beta}^2(\mu_V^N) = N^{-1} \partial_{\alpha\beta}^2 \delta_{V_i}$.*

Proof of Lemma 5.4. For $v, w \in \mathbb{R}^d$, we have

$$\begin{aligned} |\delta_v - \delta_w|_s &= \sup_{\xi \in \mathbb{R}^d} \frac{|e^{-i v \cdot \xi} - e^{-i w \cdot \xi}|}{\langle \xi \rangle^s} \leq |v - w| \sup_{\xi \in \mathbb{R}^d} \frac{\|\nabla_v e^{-i v \cdot \xi}\|_{L^\infty(\mathbb{R}^d)}}{\langle \xi \rangle^s} \\ &\leq |v - w| \sup_{\xi \in \mathbb{R}^d} \frac{|\xi|}{\langle \xi \rangle^s} \leq |v - w| \end{aligned}$$

which shows that $v \mapsto \delta_v$ is $C^{0,1}$. For the sake of simplicity we present the proof of differentiability when $d = 1$, the case $d > 1$ being similar. For $v \in \mathbb{R}$ and $h \in \mathbb{R}^*$, we have

$$|\delta_{v+h} - \delta_v - h \delta'_v|_s = \sup_{\xi \in \mathbb{R}} \frac{|(e^{-i \xi h} - 1 + i \xi h) e^{-i v \xi}|}{\langle \xi \rangle^s} \leq \sup_{\xi \in \mathbb{R}} \frac{|\xi h|^2}{\langle \xi \rangle^s} \leq |h|^2,$$

from which we deduce that $v \mapsto \delta_v$ is $C^{1,1}$. Similarly we can go to second order:

$$\begin{aligned} \left| \delta_{v+h} - \delta_v - h \delta'_v + \frac{h^2}{2} \delta''_v \right|_s &= \sup_{\xi \in \mathbb{R}} \frac{|(e^{-i \xi h} - 1 + i \xi h - \xi^2 h^2) e^{-i v \xi}|}{\langle \xi \rangle^s} \leq \sup_{\xi \in \mathbb{R}} \frac{|\xi h|^3}{\langle \xi \rangle^s} \leq |h|^3, \end{aligned}$$

and we easily conclude that $v \mapsto \delta_v$ is $C^{2,1}$. When the dimension d is greater than 1, one can perform exactly the same argument for the partial derivatives of the Dirac mass. □

We come back to the proof of (5.13). Take $\Phi \in C^{2,1}(P_{\mathcal{G}_1}, \mathbb{R})$ and compute separately the contributions of G_i^N , $i = 1, 2$. Proceeding as in the proof of (A3) in Theorem 3.1 we have

$$G_1^N(\Phi \circ \mu_V^N) = \langle Q_\alpha(\mu_V^N, \mu_V^N), D\Phi(\mu_V^N) \rangle + I_2(V)$$

with

$$|I_2(V)| \leq \frac{1}{2N} \sum_{i,j=1}^N \int_{\mathbb{S}^{d-1}} b(\cos(\theta_{ij})) \|\Phi\|_{C^{2,1}} \left| \mu_{V_{ij}^*}^N - \mu_V^N \right|_{s_1}^2 d\sigma \leq \frac{8}{N} \|\Phi\|_{C^{2,1}},$$

since for any $i \neq j$

$$\begin{aligned} \left| \mu_{V_{ij}^*}^N - \mu_V^N \right|_{s_1} &= \frac{1}{N} \left| \delta_{v'_i} + \delta_{v'_j} - \delta_{v_i} - \delta_{v_j} \right|_{s_1} \\ &\leq \frac{1}{N} \left(\left| \delta_{v'_i} \right|_{s_1} + \left| \delta_{v'_j} \right|_{s_1} + \left| \delta_{v_i} \right|_{s_1} + \left| \delta_{v_j} \right|_{s_1} \right) = \frac{4}{N}. \end{aligned}$$

On the other hand, as in the proof of assumption **(A3)** in Section 4, the map $\mathbb{R}^{dN} \rightarrow \mathbb{R}$, $V \mapsto \Phi(\mu_V^N)$ is $C^{2,1}$ thanks to Lemma 5.4 and by denoting $\phi_V = D\Phi[\mu_V^N] \in (\mathcal{H}^{s_1}(\mathbb{R}^d))'$, we may compute

$$\begin{aligned} G_2^N(\Phi(\mu_V^N)) &= \sum_{i=1}^N \Delta_i \Phi(\mu_V^N) \\ &= \sum_{i=1}^N \left\{ \frac{1}{N} (\Delta \phi_V)(v_i) + \frac{1}{N^2} \sum_{\alpha=1}^d D^2 \Phi[\mu_V^N] (\partial_{v_i, \alpha} \delta_{v_i}, \partial_{v_i, \alpha} \delta_{v_i}) \right\} \\ &= \langle \Delta \mu_V^N, \phi_V \rangle + \mathcal{O} \left(\frac{\|\Phi\|_{C^2}}{N} \right). \end{aligned}$$

We conclude the proof by combining the previous estimates. \square

Proof of (A4). For $f_{in}, g_{in} \in P(\mathbb{R}^d)$, we define the associated solutions f_t and g_t to the nonlinear Boltzmann equation; we define $h_t := \mathcal{L}_t^{NL}[f_{in}](g_{in} - f_{in})$ the solution of the linearized Boltzmann equation around f_t ; and we define r_t the solution to the “second variation” equation around f_t . More precisely, we define

$$\begin{cases} \partial_t f_t = Q(f_t, f_t) + \Delta f_t, & f_{|t=0} = f_{in} \\ \partial_t g_t = Q(g_t, g_t) + \Delta g_t, & g_{|t=0} = g_{in} \\ \partial_t h_t = 2Q^+(f_t, h_t) - h_t + \Delta h_t, & h_{|t=0} = h_{in} := g_{in} - f_{in} \\ \partial_t r_t = 2Q^+(f_t, r_t) - r_t + \Delta r_t + Q^+(h_t, h_t), & r_{|t=0} = 0. \end{cases}$$

We then define

$$\begin{cases} d_t := g_t - f_t \\ \omega_t := g_t - f_t - h_t = S_t^{NL}(g_{in}) - S_t^{NL}(f_{in}) - \mathcal{L}_t^{NL}[f_{in}](g_{in} - f_{in}) \\ \psi_t := g_t - f_t - h_t - r_t. \end{cases}$$

Lemma 5.5. *Fix $s \geq 0$ and $T \in (0, \infty)$. There exists C_T such that for any $f_{in}, g_{in} \in P(\mathbb{R}^d)$, the following estimates hold*

$$(5.14) \quad \forall t \in [0, T] \quad |h_t|_s \leq C_T |f_{in} - g_{in}|_s,$$

$$(5.15) \quad \forall t \in [0, T] \quad |\omega_t|_{2s} \leq C_T |f_{in} - g_{in}|_s^2,$$

$$(5.16) \quad \forall t \in [0, T] \quad |r_t|_{2s} \leq C_T |f_{in} - g_{in}|_s^2,$$

$$(5.17) \quad \forall t \in [0, T] \quad |\psi_t|_{3s} \leq C_T |f_{in} - g_{in}|_s^3.$$

As a consequence, the operator \mathcal{L}_t^{NL} defined by $\mathcal{L}_t^{NL}[f_{in}](h_{in}) := h_t$ satisfies $\mathcal{L}_t^{NL}[f_{in}] \in \mathcal{M}^1(\mathcal{G}_1, \mathcal{G}_1)$, the operator \mathcal{B}_t^{NL} defined by $\mathcal{B}_t^{NL}[f_{in}](h_{in}, h_{in}) := r_t$ satisfies $\mathcal{B}_t^{NL}[f_{in}] \in \mathcal{M}^2(\mathcal{G}_1, \mathcal{H}^{-2s})$, and finally $S_t^{NL} \in C^{2,1}(P_{\mathcal{G}_1}, P_{\mathcal{G}_2})$.

Proof of Lemma 5.5. We skip the proof of (5.14) because it is similar to the proof of (5.8). We then deal with each term successively. We work in Fourier variable and we introduce the notations $F = \hat{f}$, $S = \hat{f} + \hat{g}$, $D = \hat{d}$, $H = \hat{h}$, $O = \hat{\omega}$, $R = \hat{r}$ and $\Psi = \hat{\psi}$.

Step 1. The evolution equation satisfied by O is

$$(5.18) \quad \partial_t O = 2\hat{Q}^+(O, S) - O - |\xi|^2 O + 2\hat{Q}^+(H, D).$$

We deduce in distributional sense

$$\frac{d}{dt} \frac{|O(\xi)|}{\langle \xi \rangle^{2s}} \leq \mathcal{T}_1 + \mathcal{T}_2,$$

where

$$\begin{aligned} \mathcal{T}_1 &:= \sup_{\xi \in \mathbb{R}^3} \int_{\mathbb{S}^{d-1}} \frac{b(\sigma \cdot \hat{\xi})}{\langle \xi \rangle^{2s}} \left(\left| \frac{O(\xi^+) S(\xi^-)}{2} \right| + \left| \frac{O(\xi^-) S(\xi^+)}{2} \right| \right) d\sigma \\ &\leq \sup_{\xi \in \mathbb{R}^3} \int_{\mathbb{S}^{d-1}} b(\sigma \cdot \hat{\xi}) \left(\frac{|O(\xi^+)| \langle \xi^+ \rangle^{2s}}{\langle \xi^+ \rangle^{2s} \langle \xi \rangle^{2s}} + \frac{|O(\xi^-)| \langle \xi^- \rangle^{2s}}{\langle \xi^- \rangle^{2s} \langle \xi \rangle^{2s}} \right) d\sigma \\ &\leq 2 \sup_{\xi \in \mathbb{R}^3} \frac{|O(\xi)|}{\langle \xi \rangle^{2s}}, \end{aligned}$$

and

$$\begin{aligned} \mathcal{T}_2 &:= \frac{1}{2} \sup_{\xi \in \mathbb{R}^3} \int_{\mathbb{S}^{d-1}} \frac{b(\sigma \cdot \hat{\xi})}{\langle \xi \rangle^{2s}} |H(\xi^+) D(\xi^-) + H(\xi^-) D(\xi^+)| d\sigma \\ &\leq \frac{1}{2} \sup_{\xi \in \mathbb{R}^3} \int_{\mathbb{S}^{d-1}} b(\sigma \cdot \hat{\xi}) \left(\frac{|H(\xi^+)| |D(\xi^-)|}{\langle \xi^+ \rangle^s \langle \xi^- \rangle^s} + \frac{|D(\xi^+)|^2 |H(\xi^-)|^2}{\langle \xi^+ \rangle^s \langle \xi^- \rangle^s} \right) d\sigma \\ &\leq |h_t|_s |d_t|_s \leq C_T |f_{in} - g_{in}|_s^2, \end{aligned}$$

using the estimates (5.8) and (5.14). We then conclude thanks to a Gronwall lemma.

Step 2. The evolution equation satisfied by R is

$$(5.19) \quad \partial_t R = 2\hat{Q}^+(F, R) - R - |\xi|^2 R + 2\hat{Q}^+(H, H), \quad R|_{t=0} = 0.$$

Equation (5.19) being similar to equation (5.18), with the same computations as in Step 1 we deduce that (5.16) holds.

Step 3. The equation satisfied by Ψ is

$$\partial_t \Psi = \hat{Q}^+(S, \Psi) - \Psi - |\xi|^2 \Psi + \hat{Q}^+(H, \Omega) + \hat{Q}^+(R, D), \quad \Psi|_{t=0} = 0.$$

Then we perform similar computations as in Step 1, and we deduce in distributional sense

$$\frac{d}{dt} \frac{|\Psi(\xi)|}{\langle \xi \rangle^{3s}} \leq \mathcal{T}_1 + \mathcal{T}_2 + \mathcal{T}_3,$$

where

$$\begin{aligned} \mathcal{T}_1 &:= \sup_{\xi \in \mathbb{R}^3} \frac{|\hat{Q}^+(S, \Psi)|}{\langle \xi \rangle^{3s}} \leq 2 \sup_{\xi \in \mathbb{R}^3} \frac{|\Psi(\xi)|}{\langle \xi \rangle^{3s}}, \\ \mathcal{T}_2 &:= \sup_{\xi \in \mathbb{R}^3} \frac{|\hat{Q}^+(H, \Omega)|}{\langle \xi \rangle^{3s}} \leq 2 \sup_{\xi \in \mathbb{R}^3} \frac{|H(\xi)|}{\langle \xi \rangle^s} \sup_{\xi \in \mathbb{R}^3} \frac{|\Omega(\xi)|}{\langle \xi \rangle^{2s}}, \\ \mathcal{T}_3 &:= \sup_{\xi \in \mathbb{R}^3} \frac{|\hat{Q}^+(R, D)|}{\langle \xi \rangle^{3s}} \leq 2 \sup_{\xi \in \mathbb{R}^3} \frac{|R(\xi)|}{\langle \xi \rangle^{2s}} \sup_{\xi \in \mathbb{R}^3} \frac{|D(\xi)|}{\langle \xi \rangle^s}. \end{aligned}$$

Finally we then conclude using the already established estimates (5.8), (5.14), (5.15), (5.17), and a Gronwall lemma. \square

Proof of (A5). We know from [5] (see also [3] for a similar result) that

$$\sup_{t \geq 0} W_2(S_t^{NL} f_{in}, S_t^{NL} g_{in}) \leq W_2(f_{in}, g_{in}).$$

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