

On the counting function of sets with even partition functions

by

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**To Kálmán Győry, Attila Pethő,
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their nice works in number theory.**

Abstract

Let q be an odd positive integer and $P \in \mathbb{F}_2[z]$ be of order q and such that $P(0) = 1$. We denote by $\mathcal{A} = \mathcal{A}(P)$ the unique set of positive integers satisfying $\sum_{n=0}^{\infty} p(\mathcal{A}, n)z^n \equiv P(z) \pmod{2}$, where $p(\mathcal{A}, n)$ is the number of partitions of n with parts in \mathcal{A} . In [5], it is proved that if $A(P, x)$ is the counting function of the set $\mathcal{A}(P)$ then $A(P, x) \ll x(\log x)^{-r/\varphi(q)}$, where r is the order of 2 modulo q and φ is the Euler's function. In this paper, we improve on the constant $c = c(q)$ for which $A(P, x) \ll x(\log x)^{-c}$.

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1 Introduction.

Let \mathbb{N} be the set of positive integers and $\mathcal{A} = \{a_1, a_2, \dots\}$ be a subset of \mathbb{N} . For $n \in \mathbb{N}$, we denote by $p(\mathcal{A}, n)$ the number of partitions of n with parts in \mathcal{A} , i.e. the number of solutions of the equation

$$a_1x_1 + a_2x_2 + \dots = n,$$

in non-negative integers x_1, x_2, \dots . We set $p(\mathcal{A}, 0) = 1$.

Let \mathbb{F}_2 be the field with two elements and $f = 1 + \epsilon_1z + \dots + \epsilon_Nz^N + \dots \in \mathbb{F}_2[[z]]$. Nicolas et al proved (see [13], [4] and [11]) that there is a unique subset $\mathcal{A} = \mathcal{A}(f)$ of \mathbb{N} such that

$$\sum_{n=0}^{\infty} p(\mathcal{A}, n)z^n \equiv f(z) \pmod{2}. \quad (1.1)$$

When f is a rational fraction, it has been shown in [11] that there is a polynomial U such that $\mathcal{A}(f)$ can be easily determined from $\mathcal{A}(U)$. When f is a general power series, nothing about the behaviour of $\mathcal{A}(f)$ is known. From now on, we shall restrict ourselves to the case $f = P$, where

$$P = 1 + \epsilon_1 z + \dots + \epsilon_N z^N \in \mathbb{F}_2[z]$$

is a polynomial of degree $N \geq 1$.

Let $A(P, x)$ be the counting function of the set $\mathcal{A}(P)$, i.e.

$$A(P, x) = |\{n : 1 \leq n \leq x, n \in \mathcal{A}(P)\}|. \quad (1.2)$$

In [10], it is proved that

$$A(P, x) \geq \frac{\log x}{\log 2} - \frac{\log(N+1)}{\log 2}. \quad (1.3)$$

More attention was paid on upper bounds for $A(P, x)$. In [5, Theorem 3], it was observed that when P is a product of cyclotomic polynomials, the set $\mathcal{A}(P)$ is a union of geometric progressions of quotient 2 and so $A(P, x) = \mathcal{O}(\log x)$.

Let the decomposition of P into irreducible factors over $\mathbb{F}_2[z]$ be

$$P = P_1^{\alpha_1} P_2^{\alpha_2} \dots P_l^{\alpha_l}.$$

We denote by β_i , $1 \leq i \leq l$, the order of $P_i(z)$, that is the smallest positive integer such that $P_i(z)$ divides $1 + z^{\beta_i}$ in $\mathbb{F}_2[z]$; it is known that β_i is odd (cf. [12]). We set

$$q = q(P) = \text{lcm}(\beta_1, \beta_2, \dots, \beta_l). \quad (1.4)$$

If $q = 1$ then $P(z) = 1 + z$ and $\mathcal{A}(P) = \{2^k, k \geq 0\}$, so that $A(P, x) = \mathcal{O}(\log x)$. We may suppose that $q \geq 3$. Now, let

$$\sigma(\mathcal{A}, n) = \sum_{a|n, d \in \mathcal{A}} d = \sum_{d|n} d \chi(\mathcal{A}, d), \quad (1.5)$$

where $\chi(\mathcal{A}, \cdot)$ is the characteristic function of the set \mathcal{A} ,

$$\chi(\mathcal{A}, d) = \begin{cases} 1 & \text{if } d \in \mathcal{A} \\ 0 & \text{otherwise.} \end{cases}$$

In [6] (see also [3] and [2]), it is proved that for all $k \geq 0$, q is a period of the sequence $(\sigma(\mathcal{A}, 2^k n) \bmod 2^{k+1})_{n \geq 1}$, i.e.

$$n_1 \equiv n_2 \pmod{q} \Rightarrow \sigma(\mathcal{A}, 2^k n_1) \equiv \sigma(\mathcal{A}, 2^k n_2) \pmod{2^{k+1}} \quad (1.6)$$

and q is the smallest integer such that (1.6) holds for all k 's. Moreover, if n_1 and n_2 satisfy $n_2 \equiv 2^a n_1 \pmod{q}$ for some $a \geq 0$, then

$$\sigma(\mathcal{A}, 2^k n_2) \equiv \sigma(\mathcal{A}, 2^k n_1) \pmod{2^{k+1}}. \quad (1.7)$$

If m is odd and $k \geq 0$, let

$$S_{\mathcal{A}}(m, k) = \chi(\mathcal{A}, m) + 2\chi(\mathcal{A}, 2m) + \dots + 2^k \chi(\mathcal{A}, 2^k m). \quad (1.8)$$

It follows that for $n = 2^k m$, one has

$$\sigma(\mathcal{A}, n) = \sigma(\mathcal{A}, 2^k m) = \sum_{d|m} d S_{\mathcal{A}}(d, k), \quad (1.9)$$

which, by Möbius inversion formula, gives

$$m S_{\mathcal{A}}(m, k) = \sum_{d|m} \mu(d) \sigma(\mathcal{A}, \frac{n}{d}) = \sum_{d|\bar{m}} \mu(d) \sigma(\mathcal{A}, \frac{n}{d}), \quad (1.10)$$

where μ is the Möbius's function and $\bar{m} = \prod_{p|m} p$ is the radical of m , with $\bar{1} = 1$.

In [7] and [9], precise descriptions of the sets $\mathcal{A}(1 + z + z^3)$ and $\mathcal{A}(1 + z + z^3 + z^4 + z^5)$ are given and asymptotics to the related counting functions are obtained,

$$A(1 + z + z^3, x) \sim c_1 \frac{x}{(\log x)^{\frac{3}{4}}}, \quad x \rightarrow \infty, \quad (1.11)$$

$$A(1 + z + z^3 + z^4 + z^5, x) \sim c_2 \frac{x}{(\log x)^{\frac{1}{4}}}, \quad x \rightarrow \infty, \quad (1.12)$$

where $c_1 = 0.937\dots$, $c_2 = 1.496\dots$. In [1], are considered the sets $\mathcal{A}(P)$ when P is irreducible of prime order q and such that the order of 2 in $(\mathbb{Z}/q\mathbb{Z})^*$ is $\frac{q-1}{2}$. This situation is similar to that of $\mathcal{A}(1 + z + z^3)$, and formula (1.11) can be extended to $A(P, x) \sim c' x (\log x)^{-3/4}$, $x \rightarrow \infty$, for some constant c' depending on P .

Let $P = QR$ be the product in $\mathbb{F}_2[z]$ of two coprime polynomials. In [4], the following is given

$$A(P, x) \leq A(Q, x) + A(R, x) \quad (1.13)$$

and

$$|A(P, x) - A(R, x)| \leq \sum_{0 \leq i \leq \frac{\log x}{\log 2}} A(Q, \frac{x}{2^i}). \quad (1.14)$$

As an application of (1.14), choosing $Q = 1 + z + z^3$, $R = 1 + z + z^3 + z^4 + z^5$ and $P = QR$, we get from (1.11)-(1.14),

$$A(P, x) \sim A(R, x) \sim c_2 x (\log x)^{-1/4}, \quad x \rightarrow \infty.$$

In [5], a claim of Nicolas and Sárközy [15], that some polynomials with $A(P, x) \asymp x$ may exist, was disapproved. More precisely, the following was obtained

Theorem 1.1. *Let $P \in \mathbb{F}_2[z]$ be such that $P(0) = 1$, $\mathcal{A} = \mathcal{A}(P)$ be the unique set obtained from (1.1) and q be the odd number defined by (1.4). Let r be the order of 2 modulo q , that is the smallest positive integer such that $2^r \equiv 1 \pmod{q}$. We shall say that a prime $p \neq 2$ is a bad prime if*

$$\exists i, \quad 0 \leq i \leq r-1 \quad \text{and} \quad p \equiv 2^i \pmod{q}. \quad (1.15)$$

- (i) *If p is a bad prime, we have $\gcd(p, n) = 1$ for all $n \in \mathcal{A}$.*
- (ii) *There exists an absolute constant c_3 such that for all $x > 1$,*

$$A(P, x) \leq 7(c_3)^r \frac{x}{(\log x)^{\frac{r}{\varphi(q)}}}, \quad (1.16)$$

where φ is Euler's function.

2 The sets of bad and semi-bad primes.

Let q be an odd integer ≥ 3 and r be the order of 2 modulo q . Let us call "bad classes" the elements of

$$\mathcal{E} = \{1, 2, \dots, 2^{r-1}\} \subset (\mathbb{Z}/q\mathbb{Z})^*.$$

From (1.15), we know that an odd prime p is bad if $p \pmod{q}$ belongs to \mathcal{E} . The set of bad primes will be denoted by \mathcal{B} . The fact that no element of $\mathcal{A}(P)$ is divisible by a bad prime (cf. Theorem 1.1 (i)) has given (cf. [5]) the upper bound (1.16). Another set of primes, which we call semi-bad primes and denote by \mathcal{S} , will be the tool to improve (1.16) cf. Theorem 2.1 below.

Remark 2.1. 2 is not a bad prime although it is a bad class.

Definition 2.1. . A class of $(\mathbb{Z}/q\mathbb{Z})^*$ is said semi-bad if it does not belong to \mathcal{E} and its square does. We denote by \mathcal{E}' the set of semi-bad classes. A prime p is called semi-bad if its class modulo q is semi-bad, i.e.

$$p \in \mathcal{S} \Leftrightarrow p \pmod{q} \in \mathcal{E}'.$$

We denote by $S(q) = |\mathcal{E}'|$ the number of elements of \mathcal{E}' .

Lemma 2.1. *Let q be an odd integer ≥ 3 and r be the order of 2 modulo q . Let us write the euclidean division*

$$r = 2r' + r'', \quad 0 \leq r'' \leq 1,$$

and set

$$q_2 = \begin{cases} 1 & \text{if 2 is a square modulo } q \\ 0 & \text{if not.} \end{cases}$$

Then the number $S(q)$ of semi-bad classes modulo q is given by

$$S(q) = (r' + r'') \left(2^{\omega(q)} - 2 + r'' \right) + q_2 r' \left(2^{\omega(q)} - r'' \right), \quad (2.1)$$

where $\omega(q)$ is the number of distinct prime factors of q .

Proof. We have to count the number of solutions of the r congruences

$$\mathcal{C}_i : x^2 \equiv 2^i \pmod{q}, \quad 0 \leq i \leq r-1,$$

which do not belong to \mathcal{E} . The number of solutions of \mathcal{C}_0 is $2^{\omega(q)}$; among them, one solution ($x = 1$) belongs to \mathcal{E} and if r is even, an extra one ($x = 2^{r/2}$) belongs to \mathcal{E} . Therefore, the contribution of \mathcal{C}_0 is $2^{\omega(q)} - 2 + r''$.

The contribution of \mathcal{C}_i when i is even is equal to that of \mathcal{C}_0 by the change of variables $x = 2^{i/2}\xi$, so that the total contribution of the \mathcal{C}_i 's for i even is equal to the first term of the right hand-side of (2.1).

The number of odd i 's, $0 \leq i \leq r-1$, is equal to r' , and the contribution of all the \mathcal{C}_i 's for these i 's are equal and vanish if $q_2 = 0$. When 2 is a square modulo q , \mathcal{C}_1 has $2^{\omega(q)}$ solutions in $(\mathbb{Z}/q\mathbb{Z})^*$, no solution in \mathcal{E} when r is even and one solution ($x = 2^{(r+1)/2}$) in \mathcal{E} when r is odd, which explains the last term of (2.1). \square

We shall prove

Theorem 2.1. Let $P \in \mathbb{F}_2[z]$ with $P(0) = 1$, q be the odd integer defined by (1.4), r be the order of 2 modulo q and $S(q)$ be the number of semi-bad classes given by (2.1). We denote by $\mathcal{A}(P)$ the set obtained from (1.1) and by $A(P, x)$ its counting function. When x tends to infinity, we have

$$A(P, x) \ll_q \frac{x}{(\log x)^{\frac{r+S(q)/2}{\varphi(q)}}}. \quad (2.2)$$

This is the best possible. When P is irréducible, q is prime and $r = \frac{q-1}{2}$, it can be seen from [1] since in this case $(\frac{2}{q}) \equiv 2^{\frac{q-1}{2}} \equiv 1 \pmod{q}$, so that 2 is a square modulo q and $q_2 = 1$. Moreover, formula (1.12) gives the optimality of (2.2) for some prime ($q = 31$) satisfying $r = \frac{q-1}{6}$.

Theorem 2.2. Let $P \in \mathbb{F}_2[z]$ be such that $P(0) = 1$ and $P = P_1 P_2 \cdots P_j$, where the P_i 's are irréducible polynomials in $\mathbb{F}_2[z]$. For $1 \leq i \leq j$, we denote by q_i the order of P_i , by r_i the order of 2 modulo q_i and we set $c = \min_{1 \leq i \leq j} \frac{r_i + S(q_i)/2}{\varphi(q_i)}$. When x tends to infinity, we have

$$A(P, x) \ll \frac{x}{(\log x)^c}. \quad (2.3)$$

where the symbol \ll depends on the q_i 's, $1 \leq i \leq j$.

Let us introduce the additive arithmetic function

$$\omega_S(n) = \sum_{p|n, p \in S} 1. \quad (2.4)$$

Lemma 2.2. Let $k \geq 0$ and m be an odd positive integer, not divisible by any bad prime. If $\omega_S(m) \geq k + 2$ then $2^h m \notin \mathcal{A}(P)$ for all h , $0 \leq h \leq k$.

Proof. Let us write $\overline{m} = m' m''$, with $m' = \prod_{p|\overline{m}, p \in S} p$ and $m'' = \prod_{p|\overline{m}, p \notin S} p$. From (1.10), if $n = 2^k m$ then

$$m S_{\mathcal{A}}(m, k) = \sum_{d|\overline{m}} \mu(d) \sigma(\mathcal{A}, \frac{n}{d}) = \sum_{d'|m'} \sum_{d''|m''} \mu(d') \mu(d'') \sigma(\mathcal{A}, \frac{n}{d' d''}). \quad (2.5)$$

Let us write $d' = p_{i_1} \cdots p_{i_j}$ and take some p_S from S . If j is even then $\mu(d') = 1$ and $d' \equiv 2^t \pmod{q}$ for some t (depending on d'), $0 \leq t \leq r - 1$. Whereas, if j is odd then $\mu(d') = -1$ and $d' \equiv 2^{t'} p_S \pmod{q}$ for some t' (depending on d'), $0 \leq t' \leq r - 1$. From (1.7), we obtain

$$\mu(d') \sigma(\mathcal{A}, \frac{n}{d' d''}) \equiv \sigma(\mathcal{A}, \frac{n}{d''}) \pmod{2^{k+1}} \text{ if } j \text{ is even,} \quad (2.6)$$

$$\mu(d') \sigma(\mathcal{A}, \frac{n}{d' d''}) \equiv -\sigma(\mathcal{A}, \frac{n p_S}{d''}) \pmod{2^{k+1}} \text{ if } j \text{ is odd.} \quad (2.7)$$

If $\alpha = \omega_S(\overline{m}) \neq 0$, the number of d' with odd j is equal to that with even j and is given by

$$1 + \binom{\alpha}{2} + \binom{\alpha}{4} + \cdots = \binom{\alpha}{1} + \binom{\alpha}{3} + \cdots = 2^{\alpha-1}.$$

From (2.5), we obtain

$$mS_{\mathcal{A}}(m, k) \equiv 2^{\alpha-1} \sum_{d'' \mid m''} \mu(d'') \left(\sigma(\mathcal{A}, \frac{n}{d''}) - \sigma(\mathcal{A}, \frac{np_{\mathcal{S}}}{d''}) \right) \pmod{2^{k+1}}, \quad (2.8)$$

which, when $\alpha = \omega_{\mathcal{S}}(m) \geq k+2$, gives $S_{\mathcal{A}}(m, k) \equiv 0 \pmod{2^{k+1}}$, so that from (1.8),

$$\chi(\mathcal{A}, m) = \chi(\mathcal{A}, 2m) = \cdots = \chi(\mathcal{A}, 2^k m) = 0. \quad (2.9)$$

□

Let us denote by $\mathcal{Q} = \mathcal{Q}(q)$ and \mathcal{N} the sets

$$\mathcal{Q} = \{p \text{ prime}, p \mid q\} \text{ and } \mathcal{N} = \{p \text{ prime}, p \notin \mathcal{B} \cup \mathcal{S} \text{ and } \gcd(p, 2q) = 1\},$$

so that the whole set of primes is equal to $\mathcal{B} \cup \mathcal{S} \cup \mathcal{N} \cup \mathcal{Q} \cup \{2\}$. For $n \geq 1$, let us define the multiplicative arithmetic function

$$\delta(n) = \begin{cases} 1 & \text{if } p \mid n \Rightarrow p \notin \mathcal{B} \text{ (i.e. } p \in \mathcal{S} \cup \mathcal{N} \cup \mathcal{Q} \cup \{2\}) \\ 0 & \text{otherwise.} \end{cases}$$

and for $x > 1$,

$$V(x) = V_q(x) = \sum_{n \geq 1, n2^{\omega_{\mathcal{S}}(n)} \leq x} \delta(n). \quad (2.10)$$

Lemma 2.3. *Under the above notation, we have*

$$V(x) = V_q(x) = \mathcal{O}_q \left(\frac{x}{(\log x)^{\frac{r+S(q)/2}{\varphi(q)}}} \right). \quad (2.11)$$

Proof. To prove (2.11), one should consider, for complex s with $\Re(s) > 1$, the series

$$F(s) = \sum_{n \geq 1} \frac{\delta(n)}{(n2^{\omega_{\mathcal{S}}(n)})^s}, \quad (2.12)$$

This Dirichet series has an Euler's product given by

$$F(s) = \prod_{p \in \mathcal{N} \cup \mathcal{Q} \cup \{2\}} \left(1 - \frac{1}{p^s} \right)^{-1} \prod_{p \in \mathcal{S}} \left(1 + \frac{1}{2^s(p^s - 1)} \right), \quad (2.13)$$

which can be written as

$$F(s) = H(s) \prod_{p \in \mathcal{N}} \left(1 - \frac{1}{p^s} \right)^{-1} \prod_{p \in \mathcal{S}} \left(1 - \frac{1}{p^s} \right)^{-\frac{1}{2^s}}, \quad (2.14)$$

where

$$H(s) = \prod_{p \in \mathcal{Q} \cup \{2\}} \left(1 - \frac{1}{p^s} \right)^{-1} \prod_{p \in \mathcal{S}} \left(1 + \frac{1}{2^s(p^s - 1)} \right) \left(1 - \frac{1}{p^s} \right)^{\frac{1}{2^s}}. \quad (2.15)$$

By applying Selberg-Delange's formula (cf. [8], Théorème 1 and [9], Lemma 4.5), we obtain some constant c_4 such that

$$V(x) = c_4 \frac{x}{(\log x)^{\frac{r+S(q)/2}{\varphi(q)}}} + \mathcal{O}_q \left(\frac{x \log \log x}{\log x} \right). \quad (2.16)$$

The constant c_4 is somewhat complicated, it is given by

$$c_4 = \frac{CH(1)}{\Gamma(1 - \frac{r+S(q)/2}{\varphi(q)}), \quad (2.17)$$

where Γ is the gamma function,

$$H(1) = \frac{2q}{\varphi(q)} \prod_{p \in \mathcal{S}} \left(1 + \frac{1}{2(p-1)}\right) \left(1 - \frac{1}{p}\right)^{\frac{1}{2}} \quad (2.18)$$

and

$$C = \prod_{p \in \mathcal{N}} \left(1 - \frac{1}{p}\right)^{-1} \prod_{p \in \mathcal{S}} \left(1 - \frac{1}{p}\right)^{-\frac{1}{2}} \prod_p \left(1 - \frac{1}{p}\right)^{1 - \frac{r+S(q)/2}{\varphi(q)}},$$

where in the third product, p runs over all primes. \square

3 Proof of the results.

Proof of Theorem 2.1. If $r = \varphi(q)$ then 2 is a generator of $(\mathbb{Z}/q\mathbb{Z})^*$, all primes are bad but 2 and the prime factors of q ; hence by Theorem 2 of [5], $A(P, x) = \mathcal{O}((\log x)^\kappa)$ for some constant κ , so that we may remove the case $r = \varphi(q)$.

Let us write $V(x)$ defined in (2.10) as

$$V(x) = V'(x) + V''(x), \quad (3.1)$$

with

$$V'(x) = \sum_{n \geq 1, n2^{\omega_S(n)} \leq x, \omega_S(n)=0} \delta(n) \quad \text{and} \quad V''(x) = \sum_{n \geq 1, n2^{\omega_S(n)} \leq x, \omega_S(n) \geq 1} \delta(n).$$

Similarly, we write $A(P, x) = \sum_{a \in \mathcal{A}(P), a \leq x} 1 = A' + A''$, with

$$A' = \sum_{a \in \mathcal{A}(P), a \leq x, \omega_S(a)=0} 1 \quad \text{and} \quad A'' = \sum_{a \in \mathcal{A}(P), a \leq x, \omega_S(a) \geq 1} 1.$$

An element a of $\mathcal{A}(P)$ counted in A' is free of bad and semi-bad primes, so that

$$A' \leq V'(x) \leq V'(2x). \quad (3.2)$$

By Lemma 2.2, an element a of $\mathcal{A}(P)$ counted in A'' is of the form $n2^{\omega_S(n)-1}$ with $\omega_S(n) = \omega_S(a) \geq 1$; hence

$$A'' \leq V''(2x). \quad (3.3)$$

Therefore, from (3.1)-(3.3), we get

$$A(P, x) = A' + A'' \leq V'(2x) + V''(2x) = V(2x)$$

and (2.2) follows from Lemma 2.3. \square

Proof of Theorem 2.2. Just use Theorem 2.1 and (1.13). \square

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