



HAL
open science

Cramér asymptotics for finite time first passage probabilities of general Lévy processes

Zbigniew Palmowski, Martijn Pistorius

► **To cite this version:**

Zbigniew Palmowski, Martijn Pistorius. Cramér asymptotics for finite time first passage probabilities of general Lévy processes. *Statistics and Probability Letters*, 2009, 79 (16), pp.1752. 10.1016/j.spl.2009.04.014 . hal-00558699

HAL Id: hal-00558699

<https://hal.science/hal-00558699>

Submitted on 24 Jan 2011

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Accepted Manuscript

Cramér asymptotics for finite time first passage probabilities of general Lévy processes

Zbigniew Palmowski, Martijn Pistorius

PII: S0167-7152(09)00163-1
DOI: 10.1016/j.spl.2009.04.014
Reference: STAPRO 5410

To appear in: *Statistics and Probability Letters*

Received date: 27 May 2008
Revised date: 26 April 2009
Accepted date: 27 April 2009

Please cite this article as: Palmowski, Z., Pistorius, M., Cramér asymptotics for finite time first passage probabilities of general Lévy processes. *Statistics and Probability Letters* (2009), doi:10.1016/j.spl.2009.04.014

This is a PDF file of an unedited manuscript that has been accepted for publication. As a service to our customers we are providing this early version of the manuscript. The manuscript will undergo copyediting, typesetting, and review of the resulting proof before it is published in its final form. Please note that during the production process errors may be discovered which could affect the content, and all legal disclaimers that apply to the journal pertain.



* Manuscript

[Click here to view linked References](#)

Cramér asymptotics for finite time first passage probabilities of general Lévy processes

Zbigniew Palmowski* Martijn Pistorius†

April 26, 2009

Abstract

We derive the exact asymptotics of $P(\sup_{u \leq t} X(u) > x)$ if x and t tend to infinity with x/t constant, for a general Lévy process X that admits exponential moments. The proof is based on a renewal argument and a two-dimensional renewal theorem of Höglund [9].

1 Introduction

The study of boundary crossing probabilities of Lévy processes has applications in many fields, including ruin theory (see e.g. Rolski et al. [13] and Asmussen [2]), queueing theory (see e.g. Borovkov [6] and Prabhu [11]), statistics (see e.g. Siegmund [15]) and mathematical finance (see e.g. Roberts and Shortland [12]).

As in many cases closed form expressions for (finite time) first passage probabilities are either not available or intractable, a good deal of the literature has been devoted to logarithmic or exact asymptotics for first passage probabilities, using different techniques. Martin-Löf [10] and Collamore [7] derived large deviation results for first passage probabilities of a general class of processes. Employing two-dimensional renewal theory and asymptotic properties of ladder processes, respectively, Höglund [9] and von Bahr [3] obtained exact asymptotics for ruin probabilities of the classical risk process (see also Asmussen [2]). Bertoin and Doney [5] generalised the classical Cramér-Lundberg approximation (of the perpetual ruin probability of a classical risk process) to general Lévy processes.

*University of Wrocław, pl. Grunwaldzki 2/4, 50-384 Wrocław, Poland, E-mail: zpalma@gmail.com

†Imperial College London, Department of Mathematics, South Kensington Campus, London SW7 2AZ, UK, E-mail: m.pistorius@imperial.ac.uk

In this paper we obtain the exact asymptotics of the finite time ruin probability $P(\tau(x) \leq t)$, where $\tau(x) = \inf\{t \geq 0 : X(t) > x\}$, for a general Lévy process $X(t)$ ($X(0) = 0$), if x and t jointly tend to infinity in fixed proportion, generalising Arfwedson [1] and Höglund [9] who treated the case of a classical risk process. The proof is based on an embedding of the ladder process of X and a two-dimensional renewal theorem of Höglund [9].

The remainder of the paper is organized as follows. In Section 2 the main result is presented, and its proof is given in Section 3.

2 Main result

Let X be a Lévy process with non-monotone paths that satisfies

$$E[e^{\alpha_0 X(1)}] < \infty \quad \text{for some } \alpha_0 > 0, \quad (2.1)$$

and denote by $\tau(x) = \inf\{t \geq 0 : X(t) > x\}$ the first crossing time of x . We exclude the case that X is a compound Poisson process with non-positive infinitesimal drift, as this corresponds to the random walk case which has already been treated in the literature.

The law of X is determined by its Laplace exponent $\psi(\theta) = \log E[e^{\theta X(1)}]$ that is well defined on the maximal domain $\Theta = \{\theta \in \mathbb{R} : \psi(\theta) < \infty\}$. Restricted to the interior Θ° , the map $\theta \mapsto \psi(\theta)$ is convex and differentiable, with derivative $\psi'(\theta)$.¹ Moreover, $\psi'(0+) = E[X(1)]$ if $E[|X(1)|] < \infty$. By the strict convexity of ψ , it follows that ψ' is strictly increasing on $(0, \infty)$ and we denote by $\Gamma : \psi'(0, \infty) \rightarrow (0, \infty)$ its right-inverse function.

Associated to the measure P is the exponential family of measures $\{P^{(c)} : c \in \Theta\}$ defined by their Radon-Nikodym derivatives

$$\left. \frac{dP^{(c)}}{dP} \right|_{\mathcal{F}_t} = \exp(cX(t) - \psi(c)t). \quad (2.2)$$

It is well known that under this change of measure X is still a Lévy process and its new Laplace exponent satisfies

$$\psi^{(c)}(\alpha) = \psi(\alpha + c) - \psi(c). \quad (2.3)$$

Related to X and its running supremum are the local time L of X at its supremum, its right-continuous inverse L^{-1} and the upcrossing ladder

¹For $\theta \in \Theta \setminus \Theta^\circ$, $\psi'(\theta)$ is understood to be $\lim_{\eta \rightarrow \theta, \eta \in \Theta^\circ} \psi'(\eta)$.

process H respectively. The Laplace exponent κ of the bivariate (possibly killed) subordinator (L^{-1}, H) ,

$$e^{-\kappa(\alpha, \beta)t} = E[e^{-\alpha L_t^{-1} - \beta H_t} \mathbf{1}_{(L_t^{-1} < \infty)}], \quad (2.4)$$

is related to ψ via the Wiener-Hopf factorisation identity

$$u - \psi(\theta) = k\kappa(u, -\theta)\widehat{\kappa}(u, \theta), \quad u \geq 0, \theta \in \Theta^o, \quad (2.5)$$

for some constant $k > 0$ where $\widehat{\kappa}$ is the Laplace exponent of the dual ladder process. Refer to Bertoin [4, Ch. VI] for further background on the fluctuation theory of Lévy processes.

Bertoin and Doney [5] showed that, if the Cramér condition holds, that is $\gamma > 0$, where

$$\gamma := \sup\{\theta \in \Theta : \psi(\theta) = 0\}, \quad (2.6)$$

the Cramér-Lundberg approximation remains valid for a general Lévy process:

$$\lim_{x \rightarrow \infty} e^{\gamma x} P[\tau(x) < \infty] = C_\gamma, \quad (2.7)$$

where $C_\gamma \geq 0$ is positive if and only if $E[e^{\gamma X(1)} | X(1)|] < \infty$ and is then given by $C_\gamma = \beta_\gamma / [\gamma m_\gamma]$, where

$$\beta_\gamma = -\log P[H_1 < \infty], \quad m_\gamma = E[e^{\gamma H_1} H_1 \mathbf{1}_{(H_1 < \infty)}].$$

Further, Doob's optional stopping theorem implies the following bound:

$$e^{\gamma x} P(\tau(x) < \infty) = E^{(\gamma)}[e^{-\gamma(X(\tau(x)) - x)} \mathbf{1}_{(\tau(x) < \infty)}] \leq 1. \quad (2.8)$$

The result below concerns the asymptotics of the finite time ruin probability $P(\tau(x) \leq t)$ when x, t jointly tend to infinity in fixed proportion. For a given proportion v the rate of decay is either equal to γvt or to $\psi^*(v)t$, where ψ^* is the convex conjugate of ψ :

$$\psi^*(u) = \sup_{\alpha \in \mathbb{R}} (\alpha u - \psi(\alpha)).$$

We restrict ourselves to Lévy processes satisfying the following condition

$$\sigma > 0 \text{ or the Lévy measure is non-lattice,} \quad (\text{H})$$

where σ denotes the Gaussian coefficient of X . Recall that a measure is called non-lattice if its support is not contained in a set of the form $\{a + bh, h \in \mathbb{Z}\}$, for some $a, b > 0$. Note that (H) is satisfied by any Lévy process whose Lévy measure has infinite mass.

We write $f \sim g$ if $\lim_{x, t \rightarrow \infty, x=vt+o(t^{1/2})} f(x, t)/g(x, t) = 1$.

Theorem 1 Assume that (H) holds. Suppose that $0 < \psi'(\gamma) < \infty$ and that there exists a $\Gamma(v) \in \Theta^\circ$ such that $\psi'(\Gamma(v)) = v$. If x and t tend to infinity such that $x = vt + o(t^{1/2})$ then

$$P(\tau(x) \leq t) \sim \begin{cases} C_\gamma e^{-\gamma x}, & \text{if } 0 < v < \psi'(\gamma), \\ D_v t^{-1/2} e^{-\psi^*(v)t}, & \text{if } v > \psi'(\gamma), \end{cases}$$

with $C_0 = 1$ and D_v given by

$$D_v = \frac{-v \log E[e^{-\eta_v L_1^{-1}} \mathbf{1}_{(L_1^{-1} < \infty)}]}{\eta_v E[e^{\Gamma(v)H_1 - \eta_v L_1^{-1}} H_1 \mathbf{1}_{(L_1^{-1} < \infty)}]} \times \frac{1}{\Gamma(v) \sqrt{2\pi\psi''(\Gamma(v))}},$$

where $\eta_v = \psi(\Gamma(v))$.

Remark 1 (a) For a spectrally negative Lévy process the joint exponent of the ladder process is given by $\kappa(\alpha, \beta) = \beta + \Phi(\alpha)$ ($\alpha, \beta \geq 0$), where $\Phi(\alpha)$ is the largest root of $\psi(\theta) = \alpha$, and thus

$$D_v = \bar{D}_v := \frac{v}{\psi(\Gamma(v)) \sqrt{2\pi\psi''(\Gamma(v))}}, \quad C_\gamma \equiv 1. \quad (2.9)$$

Indeed,

$$\begin{aligned} D_v &= \bar{D}_v \times \frac{\kappa(\eta_v, 0)}{\Gamma(v) \frac{\partial}{\partial \beta} \kappa(\eta_v, \beta)|_{\beta = -\Gamma(v)} \exp\{-\kappa(\eta_v, -\Gamma(v))\}} \\ &= \bar{D}_v \times \frac{1}{\exp\{-\Phi(\eta_v) + \Gamma(v)\}} = \bar{D}_v \end{aligned}$$

since $\Phi(\eta_v) = \Gamma(v)$.

(b) If X is spectrally positive, $\kappa(\alpha, \beta) = [\alpha - \psi(-\beta)] / [\hat{\Phi}(\alpha) - \beta]$ (see e.g. [4, Thm VII.4]), where $\hat{\Phi}(\alpha)$ is the largest root of $\psi(-\theta) = \alpha$ and we find that

$$D_v = \frac{\Gamma(v) + \tilde{\Gamma}(v)}{\Gamma(v)\tilde{\Gamma}(v)} \frac{1}{\sqrt{2\pi\psi''(\Gamma(v))}}, \quad C_\gamma = \frac{\psi'(0)}{\psi'(\gamma)},$$

where $\tilde{\Gamma}(v) = \sup\{\theta : \psi(-\theta) = \psi(\Gamma(v))\}$, recovering formulas that can be found in Arfwedson [1] and Feller [8] respectively, for the case of a classical risk process.

Remark 2 Heuristically, in the case $v > \psi'(\gamma)$, the asymptotics in Thm. 1 can be regarded as a consequence of the central limit theorem, that is, under the tilted measure $P^{\Gamma(v)}$, asymptotically

$$\frac{\tau(x) - x/v}{\omega\sqrt{x}}$$

follows a standard normal distribution, where by (2.3) and choice of $\Gamma(v)$,

$$\omega^2 = \frac{\text{Var}^{\Gamma(v)}[X_1]}{(E^{\Gamma(v)}[X_1])^3} = \frac{\psi^{\Gamma(v)''}(0)}{(\psi^{\Gamma(v)'}(0))^3} = \frac{\psi''(\Gamma(v))}{v^3}.$$

This explains why the asymptotics remain valid if x deviates $o(x^{1/2}) = o(t^{1/2})$ from the line vt .

In the boundary case $v = \psi'(\gamma)$, in which case $E^{\Gamma(v)}[\tau(x)] = t$, the exact asymptotics of $P(\tau(x) \leq t)$ may depend on the way in which x/t tends to v . Note that this case is excluded from Theorem 1.

Remark 3 In the case $0 < v < \psi'(\gamma)$, the asymptotics in Theorem 1 are a consequence of the law of large numbers. To see why this is the case, note that $e^{\gamma x}P(\tau(x) \leq t) = e^{\gamma x}P(\tau(x) < \infty) - e^{\gamma x}P(t < \tau(x) < \infty)$, where the first term tends to C_γ in view of (2.7), while for the second term the Markov property and (2.8) imply that

$$\begin{aligned} & e^{\gamma x}P(t < \tau(x) < \infty) \\ &= \int_{-\infty}^x P(\tau(x) > t, X(t) \in dy)e^{\gamma y}e^{\gamma(x-y)}P(\tau(x-y) < \infty) \\ &\leq \int_{-\infty}^x P(X(t) \in dy)e^{\gamma y} = P^{(\gamma)}(X(t) \leq x), \end{aligned}$$

which tends to 0 as t tends to infinity in view of the law of large numbers since $E^{(\gamma)}[X(t)] = t\psi'(\gamma) > x$. The proof below deals with the case that $v > \psi'(\gamma)$.

3 Proof of Theorem 1

The idea of the proof is to lift asymptotic results that have been established for random walks by Höglund [9] and Arfwedson [1] to the setting of Lévy processes by considering suitable random walks embedded in the Lévy process (more precisely, in its ladder process). We first briefly recall these results following the Höglund [9] formulation.

3.1 Review of Höglund's random walk asymptotics

Let $(S, R) = \{(S_i, R_i), i = 1, 2, \dots\}$ be a (possibly killed) random walk starting from $(0, 0)$ whose components S and R have non-negative increments, and consider the crossing probabilities

$$\begin{aligned} G_{a,b}(x, y) &= P(N(x) < \infty, S_{N(x)} > x + a, R_{N(x)} \leq y + b), \\ K_{a,b}(x, y) &= P(N(x) < \infty, S_{N(x)} > x + a, R_{N(x)} \geq y + b), \end{aligned}$$

where $a \geq 0, b \in \mathbb{R}$ and $N(x) = \min\{n : S_n > x\}$. Let F denote the (possibly defective) distribution function of the increments of the random walk with joint Laplace transform ϕ and set $F_{(u,v)}(dx, dy) = e^{-ux-vy} F(dx, dy)/\phi(u, v)$. Let

$$V(\zeta) = E_\zeta[(R_1 E_\zeta[S_1] - S_1 E_\zeta[R_1])^2]/E_\zeta[S_1]^3$$

for $\zeta = (\xi, \eta)$ where E_ζ denotes the expectation w.r.t. F_ζ .

For our purposes it will suffice to consider random walks that satisfy the following non-lattice assumption (the analogue of the non-lattice assumption in one dimension):

$$\text{The additive group spanned by the support of } F \text{ contains } \mathbb{R}_+^2. \quad (\text{G})$$

Specialised to our setting Prop. 3.2 in Höglund (1990) jointly with the remark given on p. 380 therein read as follows:

Proposition 1 *Assume that (G) holds, and that there exists a $\zeta = (\xi, \eta)$ with $\phi(\zeta) = 1$ such that $v = E_\zeta[S_1]/E_\zeta[R_1]$, where ϕ is finite in a neighbourhood of ζ and $(0, \eta)$. If x, y tend to infinity such that $x = vy + o(y^{1/2}) > 0$ then it holds that*

$$\begin{aligned} G_{a,b}(x, y) &\sim D(a, b)x^{-1/2}e^{x\xi+y\eta} & \text{if } \eta > 0, \\ K_{a,b}(x, y) &\sim D(a, b)x^{-1/2}e^{x\xi+y\eta} & \text{if } \eta < 0, \end{aligned}$$

for $a \geq 0, b \in \mathbb{R}$, where $D(a, b) = C(a, b) \cdot (2\pi V(\zeta))^{-1/2}$, with $V(\zeta) > 0$ and

$$C(a, b) = \frac{1}{|\eta|E_\zeta[S_1]}e^{b\eta} \int_a^\infty P_\zeta(S_1 \geq x)e^{\xi x} dx.$$

3.2 Embedded random walk

Denote by e_1, e_2, \dots a sequence of independent $\exp(q)$ distributed random variables and by $\sigma_n = \sum_{i=1}^n e_i$, with $\sigma_0 = 0$, the corresponding partial sums,

and consider the two-dimensional (killed) random walk $\{(S_i, R_i), i = 1, 2, \dots\}$ starting from $(0, 0)$ with step-sizes distributed according to

$$F^{(q)}(dt, dx) = P(H_{\sigma_1} \in dx, L_{\sigma_1}^{-1} \in dt),$$

and write $G^{(q)}$ for the corresponding crossing probability

$$G^{(q)}(x, y) = G_{0,0}(x, y) = P(N(x) < \infty, R_{N(x)} \leq y).$$

Note that $F^{(q)}$ is a probability measure that is defective precisely if X drifts to $-\infty$, with Laplace transform ϕ given by

$$\phi(u, v) = \iint e^{-ut-vx} F^{(q)}(dt, dx) = \frac{q}{q - \kappa(u, v)}.$$

The key step in the proof is to derive bounds for $P(\tau(x) \leq t)$ in terms of crossing probabilities involving the random walk (S, R) :

Lemma 1 *Let $M, q > 0$. For $x, t > 0$ it holds that*

$$G^{(q)}(x, t) \leq P(\tau(x) \leq t) \leq G^{(q)}(x, t + M)/h(0-, M), \quad (3.1)$$

where $h(0-, M) = \lim_{x \uparrow 0} h(x, M)$, with $h(x, t) := P(H_{\sigma_1} > x, L_{\sigma_1}^{-1} \leq t)$.

Proof: Let $T(x) = \inf\{t \geq 0 : H_t > x\}$ and note that $\tau(x) = L_{T(x)}^{-1}$. By applying the Markov property it follows that

$$\begin{aligned} P(\tau(x) \leq t) &= P(T(x) < \infty, L_{T(x)}^{-1} \leq t) \\ &= \sum_{n=1}^{\infty} P(\sigma_{n-1} \leq T(x) < \sigma_n, L_{T(x)}^{-1} \leq t) \end{aligned} \quad (3.2)$$

$$\begin{aligned} &= \sum_{n=1}^{\infty} P(H_{\sigma_{n-1}} \leq x, H_{\sigma_n} > x, L_{T(x)}^{-1} \leq t) \\ &= \sum_{n=1}^{\infty} \iint P(H_{\sigma_{n-1}} \in dy, L_{\sigma_{n-1}}^{-1} \in ds) \\ &\quad \times P(H_{\sigma_1} > x - y, L_{T(x-y)}^{-1} \leq t - s) \end{aligned} \quad (3.3)$$

$$= \sum_{n=0}^{\infty} F^{(q)*n} \star f(x, t) = (U \star f)(x, t), \quad (3.4)$$

where $U = \sum_{n=0}^{\infty} F^{(q)*n}$, $f(x, t) = P(H_{\sigma_1} > x, L_{T(x)}^{-1} \leq t)$ and \star denotes convolution. Following a similar reasoning it can be checked that

$$G^{(q)}(x, t) = U \star h(x, t). \quad (3.5)$$

In view of (3.4) and (3.5), the lower bound in (3.1) follows since

$$f(x, t) \geq h(x, t),$$

taking note of the fact that $H_{\sigma_1} > x$ precisely if $T(x) < \sigma_1$, while the upper bound in (3.1) follows by observing that for fixed $M > 0$,

$$\begin{aligned} h(x, t + M) &\geq P(H_{\sigma_1} > x, L_{T(x)}^{-1} \leq t, L_{\sigma_1}^{-1} - L_{T(x)}^{-1} \leq M) \\ &= P(H_{\sigma_1} > x, L_{T(x)}^{-1} \leq t)P(L_{\sigma_1}^{-1} \leq M) \\ &= f(x, t)h(0-, M), \end{aligned}$$

where we used the strong Markov property of L^{-1} and the lack of memory property of σ_1 . \square

Applying Höglund's asymptotics in Proposition 1 yields the following result:

Lemma 2 *Let the assumptions of Proposition 1 hold true. If $x, t \rightarrow \infty$ such that for $v > \psi'(\gamma)$ we have $x = vt + o(t^{1/2})$ then*

$$G^{(q)}(x, t + M) \sim D_{q,M} t^{-1/2} e^{-\psi^*(v)t}, \quad M \geq 0,$$

where $D_{q,M} = \frac{v}{\sqrt{2\pi\psi''(\Gamma(v))}} C_{q,M}$ with

$$C_{q,M} = e^{\psi(\Gamma(v))M} \frac{\kappa(\psi(\Gamma(v)), 0)}{c_v \psi(\Gamma(v)) \Gamma(v)} \frac{q}{q + \kappa(\psi(\Gamma(v)), 0)},$$

where $c_v = E[e^{\Gamma(v)H_1 - \psi(\Gamma(v))L_1^{-1}} H_1 \mathbf{1}_{(L_1^{-1} < \infty)}]$.

Lemma 2 is a consequence of the following auxiliary identities:

Lemma 3 *Let $u > \gamma$, $u \in \Theta^o$.*

$$\phi(z, -u) = 1 \quad \text{iff } \kappa(z, -u) = 0 \quad \text{iff } \psi(u) = z \quad (3.6)$$

$$\psi'(u) = E^{(u)}[X(1)] = E^{(u)}[H_{\sigma_1}] \cdot (E^{(u)}[L_{\sigma_1}^{-1}])^{-1} \quad (3.7)$$

$$\begin{aligned} \psi''(u) &= E^{(u)}[(H_{\sigma_1} - \psi'(u)L_{\sigma_1}^{-1})^2] \cdot (E^{(u)}[L_{\sigma_1}^{-1}])^{-1} \\ &= \psi'(u)E^{(u)}[(H_{\sigma_1} - \psi'(u)L_{\sigma_1}^{-1})^2] \cdot (E^{(u)}[H_{\sigma_1}])^{-1} \end{aligned} \quad (3.8)$$

$$\psi^*(v) = v\Gamma(v) - \psi(\Gamma(v)) \quad \text{for } v > 0 \text{ with } \Gamma(v) \in \Theta^o. \quad (3.9)$$

Proof: Eq (3.6): Note that for $u, z > 0$ it holds that $\hat{\kappa}(z, u) > 0$. In view of the identity (2.5) the statement follows.

Eq (3.7): Note that if $u > \gamma$ then by the fact that $\psi(0) = \psi(\gamma) = 0$ and the strict convexity of ψ it follows that $\psi(u) > 0$. In view of (2.5) it follows then that $\kappa(\psi(u), -u) = 0$ for $u \in \Theta^o$, $u > \gamma$. Differentiating with respect to u shows that

$$\psi'(u) = \partial_2 \kappa(\psi(u), -u) (\partial_1 \kappa(\psi(u), -u))^{-1}. \quad (3.10)$$

Also, note that $E^{(u)}[H_{\sigma_1}] = q^{-1}E^{(u)}[H_1]$, $E^{(u)}[L_{\sigma_1}^{-1}] = q^{-1}E^{(u)}[L_1^{-1}]$ and

$$E^{(u)}[H_1] = \partial_2 \kappa(\psi(u), -u), \quad E^{(u)}[L_1^{-1}] = \partial_1 \kappa(\psi(u), -u).$$

Eq (3.8) follows as a matter of calculus, by differentiation of (3.10) with respect to u . Finally, Eq. (3.9) follows from the definition of ψ^* . \square

Proof of Lemma 2 The proof follows by an application of Prop. 1 to $G^{(q)}(x, t + M)$ with

$$(S_1, R_1) = (H_{\sigma_1}, L_{\sigma_1}^{-1}) \quad \text{and} \quad \zeta = (-\Gamma(v), \eta_v).$$

Note that, by (3.6) with $u = \Gamma(v)$, $\phi(\zeta) = 1$, and that $\eta_v = \psi(\Gamma(v)) > 0$ if $v > \psi^*(\gamma)$. For this choice of the parameters, $E_\zeta[S_1] = E^{(\Gamma(v))}[H_{\sigma_1}] = c_v/q$, and Eqs. (3.9), (3.7),(3.8) imply that $\xi x + \eta t = -\psi^*(v)t$ and

$$V(\zeta) = \psi''(\Gamma(v))/\psi'(\Gamma(v)) = \psi''(\Gamma(v))/v.$$

To complete the proof we are left to verify the form of the constants. The calculation of the $C_{q,M} = C(0, 0)e^{\eta M}$ goes as follows:

$$\begin{aligned} C_{q,M} &= \frac{qe^{\psi(\Gamma(v))M}}{\psi(\Gamma(v))c_v} \left(\int_0^\infty e^{-\Gamma(v)x} E[e^{\Gamma(v)H_{\sigma_1} - \psi(\Gamma(v))L_{\sigma_1}^{-1}} \mathbf{1}_{(x \leq H_{\sigma_1} < \infty)}] dx \right) \\ &= \frac{qe^{\psi(\Gamma(v))M}}{\psi(\Gamma(v))\Gamma(v)c_v} \left(1 - E[e^{-\psi(\Gamma(v))L_{\sigma_1}^{-1}} \mathbf{1}_{(L_{\sigma_1}^{-1} < \infty)}] \right) \\ &= \frac{qe^{\psi(\Gamma(v))M}}{\psi(\Gamma(v))\Gamma(v)c_v} \left(1 - \frac{q}{q + \kappa(\psi(\Gamma(v)), 0)} \right) \\ &= \frac{qe^{\psi(\Gamma(v))M}}{\psi(\Gamma(v))\Gamma(v)c_v} \frac{\kappa(\psi(\Gamma(v)), 0)}{q + \kappa(\psi(\Gamma(v)), 0)}, \end{aligned}$$

in view of the definition (2.4) of κ . Combining all results completes the proof. \square

As final preparation for the proof of Theorem 1 we show that the non-lattice condition holds:

Lemma 4 *Suppose that (H) holds true. Then $F^{(q)}$ satisfies (G).*

Proof: The assertion is a consequence of the following identity between measures on $(0, \infty)^2$ (which is itself a consequence of the Wiener-Hopf factorisation, see e.g. Bertoin [4, Cor VI.10])

$$P(X_t \in dx)dt = t \int_0^\infty P(L_u^{-1} \in dt, H_u \in dx)u^{-1}du. \quad (3.11)$$

Fix $(y, v) \in (0, \infty)^2$ in the support of $\mu_X(dt, dx) = P(X_t \in dx)dt$ and let B be an arbitrary open ball around (y, v) . Then $\mu_X(B) > 0$; in view of the identity (3.11) it follows that there exists a set A with positive Lebesgue measure such that $P((L_u^{-1}, H_u) \in B) > 0$ for all $u \in A$ and thus $P((L_{\sigma_1}^{-1}, H_{\sigma_1}) \in B) > 0$. Since B was arbitrary we conclude that (y, v) lies in the support of $F^{(q)}$. To complete the proof we next verify that if a Lévy process X satisfies (H) then μ_X satisfies (G). To this end, let X satisfy (H). Suppose first that its Lévy measure ν has infinite mass or $\sigma > 0$. Then $P(X_t = x) = 0$ for any $t > 0$ and $x \in \mathbb{R}$, according to Sato [14, Thm. 27.4]. Thus, the support of $P(X_t \in dx)$ is uncountable for any $t > 0$, so that μ_X satisfies (G). If ν has finite mass then it is straightforward to verify that $P(X_t \in dx)$ is non-lattice for any $t > 0$ if ν is, and that then μ_X satisfies (G). \square

Proof of Theorem 1: Suppose that $v > \psi'(\gamma)$ (the case $v < \psi'(\gamma)$ was shown in Remark 3). Writing $l(t, x) = t^{1/2}e^{\psi^*(v)t}P(\tau(x) \leq t)$, Lemmas 1, 2 and 3 imply that

$$\begin{aligned} s &= \limsup_{x, t \rightarrow \infty, x=tv+o(t^{1/2})} l(t, x) \leq D_{q, M}/h(0-, M), \\ i &= \liminf_{x, t \rightarrow \infty, x=tv+o(t^{1/2})} l(t, x) \geq D_{q, 0}. \end{aligned}$$

By definition of h and $D_{q, M}$ it directly follows that, as $q \rightarrow \infty$,

$$D_{q, 0} \rightarrow D_v, \quad D_{q, M} \rightarrow D_v e^{\psi(\Gamma(v))M} \quad \text{and} \quad h(0-, M) = P(L_{\sigma_1}^{-1} \leq M) \rightarrow 1.$$

Letting $M \downarrow 0$ yields that $s = i = D_v$, and the proof is complete. \square

Acknowledgments

We thank Florin Avram for helpful advice and useful comments. We are grateful to the referee for supplying the short direct proof (Remark 3) for the case $v < \psi'(\gamma)$. This work is partially supported by EPSRC grant EP/D039053/1 and the Ministry of Science and Higher Education of Poland under the grant N N2014079 33 (2007-2009).

References

- [1] Arfwedson, G. (1955) Research in collective risk theory. II. *Skand. Aktuarietidskr.* **38**, 53–100.
- [2] Asmussen, S. (2000) *Ruin Probabilities*. World Scientific, Singapore.
- [3] von Bahr, B. (1974) Ruin probabilities expressed in terms of ladder height distributions, *Scand. Actuar. J.* **57** 190–204.
- [4] Bertoin, J. (1996) *Lévy processes*. Cambridge University Press.
- [5] Bertoin, J. and Doney, R.A. (1994) Cramér’s estimate for Lévy processes. *Stat. Prob. Lett.* **21**, 363–365.
- [6] Borovkov, A.A. (1976) *Stochastic processes in queueing theory*. Springer, Berlin.
- [7] Collamore, J.F. (1996) Large deviations for first passage times. *Ann. Prob.* **24**, 2065–2078.
- [8] Feller, W. (1971) *An introduction to probability theory and its applications. Vol II. 2nd ed.* Wiley.
- [9] Höglund, T. (1990) An asymptotic expression for the probability of ruin within finite time. *Ann. Prob.* **18**, 378–389.
- [10] Martin-Löf, A. (1986) Entropy estimates for the first passage time of a random walk to a time dependent barrier. *Scand J. Statist.* **13**, 221–229.
- [11] Prabhu, N.U. (1997) *Insurance, queues, dams*. Springer.
- [12] G. O. Roberts and C. F. Shortland (1997) *Math. Finance* **7**, 83–93. Pricing Barrier Options with TimeDependent Coefficients
- [13] Rolski, T., Schmidli, H., Schmidt, V. and Teugels, J. (1999) *Stochastic Processes for Insurance and Finance*. Wiley, Chichester.
- [14] Sato, K. (1999) *Lévy processes and infinitely divisible distributions*. Cambridge Studies in Advanced Mathematics, **68**. Cambridge University Press.
- [15] Siegmund, D. (1985) *Sequential Analysis Test and Confidence Intervals*. Springer-Verlag.