



HAL
open science

A test of non-identifying restrictions and confidence regions for partially identified parameters

Alfred Galichon, Marc Henry

► **To cite this version:**

Alfred Galichon, Marc Henry. A test of non-identifying restrictions and confidence regions for partially identified parameters. *Econometrics*, 2009, 152 (2), pp.186. 10.1016/j.jeconom.2009.01.010 . hal-00558632

HAL Id: hal-00558632

<https://hal.science/hal-00558632>

Submitted on 23 Jan 2011

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Accepted Manuscript

A test of non-identifying restrictions and confidence regions for partially identified parameters

Alfred Galichon, Marc Henry

PII: S0304-4076(09)00014-1
DOI: [10.1016/j.jeconom.2009.01.010](https://doi.org/10.1016/j.jeconom.2009.01.010)
Reference: ECONOM 3148

To appear in: *Journal of Econometrics*



Please cite this article as: Galichon, A., Henry, M., A test of non-identifying restrictions and confidence regions for partially identified parameters. *Journal of Econometrics* (2009), doi:10.1016/j.jeconom.2009.01.010

This is a PDF file of an unedited manuscript that has been accepted for publication. As a service to our customers we are providing this early version of the manuscript. The manuscript will undergo copyediting, typesetting, and review of the resulting proof before it is published in its final form. Please note that during the production process errors may be discovered which could affect the content, and all legal disclaimers that apply to the journal pertain.

A Test of Non-Identifying Restrictions and Confidence Regions for Partially Identified Parameters

Alfred Galichon and Marc Henry

Harvard University and Columbia University

First draft: September 15, 2005

This draft¹: May 16, 2008

Abstract

We propose an easily implementable test of the validity of a set of theoretical restrictions on the relationship between economic variables, which do not necessarily identify the data generating process. The restrictions can be derived from any model of interactions, allowing censoring and multiple equilibria. When the restrictions are parameterized, the test can be inverted to yield confidence regions for partially identified parameters, thereby complementing other proposals, primarily Chernozhukov, Hong, and Tamer (2007).

JEL Classification: C10, C12, C13, C14, C52, C61

Keywords: partial identification, mass transportation, specification test.

¹This research was partly carried out while the first author was visiting the Bendheim Center for Finance, Princeton University and financial support from NSF grant SES 0350770 to Princeton University, from NSF grant SES 0532398, from the Program for Economic Research at Columbia University and from the Conseil Général des Mines is gratefully acknowledged. We are grateful to Victor Chernozhukov, Pierre-André Chiappori, Guido Imbens and Bernard Salanié for encouragement, support and many helpful discussions. We also thank three anonymous referees, whose detailed and insightful comments helped significantly improve the paper, and we thank conference participants at Econometrics in Rio and seminar participants at Berkeley, Chicago, Columbia, École polytechnique, Harvard-MIT, MIT Sloane OR, Northwestern, NYU, Princeton, SAMSI, Stanford, the Weierstrass Institut and Yale for helpful comments (with the usual disclaimer). Correspondence addresses: Department of Economics, Harvard University, Littauer Center, 1805 Cambridge Street, Cambridge, MA 02138, USA. galichon@fas.harvard.edu and Department of Economics, Columbia University, 420 W 118th Street, New York, NY 10027, USA. mh530@columbia.edu.

Introduction

In several rapidly expanding areas of economic research, the identification problem is steadily becoming more acute. In policy and program evaluation (Manski (1990)) and more general contexts with censored or missing data (Molinari (2003), Magnac and Maurin (2005)) and measurement error (Chen, Hong, and Tamer (2005)), ad hoc imputation rules lead to fragile inference. In demand estimation based on revealed preference (Blundell, Browning, and Crawford (2005)) the data is generically insufficient for identification. In the analysis of social interactions (Brock and Durlauf (2005), Manski (2004)), complex strategies to reduce the large dimensionality of the correlation structure are needed. In the estimation of models with complex strategic interactions and multiple equilibria (Tamer (2003), Andrews, Berry, and Jia (2003), Pakes, Porter, Ho, and Ishii (2004)), assumptions on equilibrium selection mechanisms may not be available or acceptable.

More generally, in all areas of investigation with structural data insufficiencies or incompletely specified economic mechanisms, the hypothesized structure fails to identify a unique possible generating mechanism for the data that is actually observed. Hence, when the structure depends on unknown parameters, and even if a unique value of the parameter can still be construed as the true value in some well defined way, it does not correspond in a one-to-one mapping with a probability measure for the observed variables. We then call the structural restrictions non-identifying. In other words, even if we abstract from sampling uncertainty and assume the distribution of the observable variables is perfectly known, no unique parameter but a whole set of parameter values (hereafter called identified set in the terminology of Manski (2005)) will be compatible with it.

Once a theoretical description of an economic system is given, a natural question to consider is whether the structure can be rejected on the basis of data on its observable components. Marschak and Andrews (1944) construct a collection of production functions that are compatible with structural restrictions and are not rejected by the data. We extend this approach within the general formulation of Koopmans and Reiersol (1950), who define a structure as the combination of a binary relation between observed socioeconomic variables (market entry, insurance coverage, winning bids in auctions, etc...) and unobserved ones (productivity shocks, risk level, or risk attitude, valuations or information depending on the auction paradigm, etc...) and a generating mechanism for the unobserved variables. This setup is employed by Roehrig (1988) and Matzkin (1994), who analyze conditions for nonparametric identification of structures where the endogenous observable variables are functions of unobservable variables and exogenous observable ones.

Here, following Jovanovic (1989), we allow the relation between observable and unobservable variables to be many-to-many, thereby including structures with multiple equilibria (when a value of the latent variables is associated with a set of values of the observable variables) and censored endogenous observable variables (where a value of the observable variable is associated with set of values of the latent variables). We do not strive for identification conditions, but rather for the ability to reject such structures that are incompatible with data, as in the original work of Marschak and Andrews (1944).

We show that such a goal can be attained in all generality (ie. for any structure, involving discrete as well as continuous observable variables), through an appeal to the duality of mass transportation (see Villani (2003) for a comprehensive account of the theory). Given any set of (possibly non-identifying) restrictions on the relation between latent and observable variables, and given the distribution ν of latent variables, the structure thus defined is compatible with the true distribution P of the observable variables if and only if there exists a joint distribution with marginals P and ν and such that the restrictions are almost surely respected. Otherwise, the data could not have been generated in a such a way. We show that the latter condition can be formulated as a mass transportation problem (the problem of transporting a given distribution of mass from an initial location to a different distribution of mass in a final location while minimizing a certain cost of transportation, as originally formulated by Monge (1781)). We show that this optimization problem has a dual formulation, an empirical version of which is a generalized Kolmogorov-Smirnov test statistic. We base a test of the restrictions in the structure on this statistic, whose asymptotic distribution we derive, and approximate using the bootstrapped empirical process.

Once we have a test of the structure, we can form confidence regions for unknown parameters using the methodology of Anderson and Rubin (1949), which consists in collecting all parameter values for which the structure is not rejected by the test at the desired significance level. The construction of such confidence regions has been the focus of much research lately (see for instance the thorough literature review in Chernozhukov, Hong, and Tamer (2007)). Unlike much of the econometric research on this issue, we do not restrict the analysis to models defined by moment inequalities. On the other hand, we consider structures in the sense of Koopmans and Reiersol (1950), and hence parametric distributions for the latent variables. This, however, is a common assumption in empirical work with game theoretic models, as exemplified by Andrews, Berry, and Jia (2003), Ciliberto and Tamer (2006), and more generally Akerberg, Benkard, Berry, and Pakes (2007).

The paper is organized as follows. The next section is divided in four subsections. The first describes the setup; the second defines the hypothesis of compatibility of the structure with

the data; the third explains how to construct a confidence region for the identified set, and the fourth reviews the related literature. The second section is divided in three subsections. The first subsection describes and justifies the generalized Kolmogorov-Smirnov test of compatibility of the structure with the data; the second shows consistency of the test, and the third investigates size properties of the test in a Monte Carlo experiment. The last section concludes.

1 Incomplete model specifications

1.1 Description of the framework

Consider the model of an economy which is composed of an observed variable Y and a latent, unobserved variable U . Formally, (Y, U) is a pair of random vectors defined on a common probability space. The pair (Y, U) has probability law π which is unknown. Y represents the variables that are observable, and U the variables that are unobservable. Y may have discrete and continuous components. Y may include variables of interest in their own right, and randomly censored or otherwise transformed versions of variables of interest. We call the law of the observable variables P . It is unknown, but the data available is a sample of independent and identically distributed vectors (Y_1, \dots, Y_n) with law P . U includes random shocks and other unobserved heterogeneity components. The law π of (Y, U) can be decomposed into the unconditional distribution P of Y and the conditional distribution of U given Y , namely $\pi_{U|Y}$. Throughout the paper it is supposed that $\pi_{U|Y}$ is unknown but fixed across observations.

The distribution of U is parameterized by a vector $\theta_1 \in \Theta_1$, where Θ_1 is an open subset of \mathbb{R}^{d_1} , and the law of U is denoted ν_{θ_1} . Finally, an economic model is given to us in the form of a set of restrictions on the vector (Y, U) , which can be summarized without loss of generality by the relation $U \in \Gamma_{\theta_2}(Y)$ where Γ_{θ_2} is a many-to-many mapping, which is completely given except for the vector of structural parameters $\theta_2 \in \Theta_2$, where Θ_2 is an open subset of \mathbb{R}^{d_2} . θ_1 and θ_2 may contain common components. We call θ the combination of the two, so that $\theta \in \Theta$, with Θ an open subset of \mathbb{R}^{d_θ} , and $d_\theta \leq d_1 + d_2$. From now on, we shall therefore denote the distribution of U by ν_θ and the many-to-many mapping by Γ_θ . In all that follows, we assume that Γ_θ is measurable (a very weak requirement which is defined in the appendix), and has non-empty and closed values.

We are interested in testing the compatibility of the observed variables Y with the model

described by (Γ, ν) . A related question is set-inference in a parametric model $(\Gamma_\theta, \nu_\theta)$: a confidence region for θ can be obtained by inverting the specification test, namely retaining the values of θ which are not rejected. Note that if $\theta_2 = (\beta, \eta)$, where β are the parameters of interest and $\eta \in H$ are nuisance parameters, we can redefine the economic model restrictions as $U \in \Gamma_\beta(Y)$ where Γ_β is defined by $\Gamma_\beta(y) = \bigcup_{\eta \in H} \Gamma_{(\beta, \eta)}(y)$ for all $y \in \mathbb{R}^{d_y}$. Hence we can assume again without loss of generality that θ_2 is indeed the parameter of interest. As the main focus of the present paper is to derive a specification test, whenever there is no ambiguity we shall implicitly fix the parameter θ and drop it from our notations.

Example 1. *A prominent example for this set-up is provided by the class of models defined by a static game of interaction. Consider a game where the payoff function for player j , $j = 1, \dots, J$ is given by $\Pi_j(S_j, S_{-j}, X_j, U_j; \theta)$, where S_j is player j 's strategy and S_{-j} is their opponents' strategies. X_j is a vector of observable characteristics of player j and U_j a vector of unobservable determinants of the payoff. Finally θ is a vector of parameters. Pure strategy equilibrium conditions define a many-to-many mapping Γ_θ from unobservable player characteristics U to observable variables $Y = (S, X)$. More precisely, $\Gamma_\theta(s, x) = \{u \in \mathbb{R}^J : \Pi_j(s_j, s_{-j}, x_j, u_j; \theta) \geq \Pi_j(s, s_{-j}, x_j, u_j; \theta), \text{ for all } S \text{ and all } j\}$. When the strategies are discrete, this is the set-up considered by Andrews, Berry, and Jia (2003), Pakes, Porter, Ho, and Ishii (2004), and Ciliberto and Tamer (2006).*

A special case of the latter example is given in Jovanovic (1989) and will serve as our first illustrative example:

Pilot Example 1. *The payoff functions are $\Pi_1(Y_1, Y_2, U_1, U_2) = (\theta Y_2 - U_2)1_{\{Y_1=1\}}$ and $\Pi_2(Y_1, Y_2, U_1, U_2) = (\theta Y_1 - U_1)1_{\{Y_2=1\}}$, where $Y_i \in \{0, 1\}$ is firm i 's action, and $U = (U_1, U_2)'$ are exogenous costs. The firms know their costs; the analyst, however, knows only that U is uniformly distributed on $[0, 1]^2$, and that the structural parameter θ is in $(0, 1]$. There are two pure strategy Nash equilibria. The first is $Y_1 = Y_2 = 0$ for all $U \in [0, 1]^2$. The second is $Y_1 = Y_2 = 1$ for all $U \in [0, \theta]^2$ and zero otherwise. Since the two firms' actions are perfectly correlated, we shall denote them by a single binary variable $Y = Y_1 = Y_2$. Hence the structure is described by the many-to-many mapping: $\Gamma_\theta(1) = [0, \theta]$ and $\Gamma_\theta(0) = [0, 1]$. In this case, since Y is Bernoulli, we can characterize P with the probability p of observing a 1.*

A second example illustrates the case with continuous observable variables:

Pilot Example 2. *Tinbergen (1951) first spelt out the implications of skill and job requirement heterogeneity on the distribution of wages. We adopt a simplified version of the*

skill-requirements relation for illustrative purposes. Suppose one observes available jobs in an economy, each characterized by a set of characteristics Y with distribution P . Worker's skills are unobserved, and are assumed for illustrative purposes to be characterized by an index $U \in \mathbb{R}$. Fulfillment of job Y is known to require a range of skills $\Gamma_\theta(Y) = [\underline{s}_\theta(Y), \bar{s}_\theta(Y)]$. The distribution of skills is parameterized by ν_θ .

1.2 Partial Identification

Identification of the parameter θ would require the correspondence between the law of the observations P and the parameter vector θ to be a function. Compared to the setup described in Roehrig (1988), there is the added complexity of the possibility that the observable variables have discrete components, and that the structure allows multiple equilibria. Conditions ensuring identification are likely to prove complicated and restrictive, and will often rule out multiple equilibria, which is the norm rather than the exception in example 1. We therefore eschew identification, and allow the relation between P and θ to be many-to-many. Our objective is to conduct inference on the set Θ_I of parameter values that are compatible with the true law of the observable variables P .

Let us formally define compatibility of a given value θ_0 of the parameter vector with a law P for the observable variables Y . When θ_0 is fixed, all the elements in the model are completely known. We therefore have a structure in the terminology of Koopmans and Reiersol (1950) extended by Jovanovic (1989). The structure is given by the law ν_{θ_0} for U , and the many-to-many mapping Γ_{θ_0} linking Y and U . We denote this structure by the triple $(P, \Gamma_{\theta_0}, \nu_{\theta_0})$. Consider now the restrictions that $(P, \Gamma_{\theta_0}, \nu_{\theta_0})$ imposes on the unknown π , the law of the vector of variables (Y, U) .

- Its marginal with respect to Y is P ,
- Its marginal with respect to U is ν_{θ_0} ,
- The economic restrictions $U \in \Gamma_{\theta_0}(Y)$ hold π almost surely.

A probability law π that satisfies the restrictions above may or may not exist. If and only if it does, we say that the structure $(P, \Gamma_{\theta_0}, \nu_{\theta_0})$ is internally consistent, or simply that the value θ_0 of the parameter is compatible with the law P of the observable variables. If no value θ_0 is found such that the structure is internally consistent, then the model restrictions are rejected.

Definition 1. A structure (P, Γ, ν) for (Y, U) given by a probability law P for Y , a probability law ν for U and a set of restrictions $U \in \Gamma(Y)$ is called internally consistent if there exists a law π for the vector (Y, U) with marginals P and ν such that $\pi(\{U \in \Gamma(Y)\}) = 1$.

We can now define the identified set as the set of values of the parameters that achieve this internal consistency. They are observationally equivalent, since even though they may correspond to different π 's, they correspond to the same P .

Definition 2. The identified set $\Theta_I = \Theta_I(P)$ is the set of values θ of the parameter vector such that the structure $(P, \Gamma_\theta, \nu_\theta)$ is internally consistent.

We illustrate the previous definitions with our pilot example:

Pilot example 1 continued For a given value of θ , the structure $(P, \Gamma_\theta, \nu_\theta)$ is defined by p , Γ_θ and the uniform distribution ν_θ on $[0, 1]^2$. $(P, \Gamma_\theta, \nu_\theta)$ is internally consistent if there exists a probability on $\{0, 1\} \times [0, 1]^2$ with marginal frequency p of observing a $Y = 1$, and uniform marginal distribution for the costs U such that $Y = 1 \Rightarrow U \leq \theta$ almost surely (where the last inequality is meant coordinate by coordinate).

The previous example illustrates the fact that definition 1 is not very easy to apply to derive the identified set in specific problems. We therefore propose a characterization of internal consistency which will prove more practical, and which, as we shall see in the next section, will motivate the construction of the statistic to test internal consistency.

Proposition 1. A structure (P, Γ, ν) is internally consistent if and only if $\sup_{A \in \mathcal{B}} [P(A) - \nu(\Gamma(A))] = 0$ where \mathcal{B} is the collection of measurable sets in the space of realizations of Y .

This proposition shows that checking internal consistency of a structure is equivalent to checking that the P -measure of a set is always dominated by the ν -measure of the image of this set by Γ (recall that the image of a set A by a many-to-many mapping is defined by $\Gamma(A) = \bigcup_{a \in A} \Gamma(a)$). Note that it is relatively easy to show necessity, i.e. that the existence of π satisfying the constraints (the definition of internal consistency) implies that $\sup_{A \in \mathcal{B}} [P(A) - \nu(\Gamma(A))] = 0$. Indeed, the definition of internal consistency implies that $Y \in A \Rightarrow U \in \Gamma(A)$, so that $1_{\{Y \in A\}} \leq 1_{\{U \in \Gamma(A)\}}$, π -almost surely. Taking expectation, we have $\mathbb{E}_\pi(1_{\{Y \in A\}}) \leq \mathbb{E}_\pi(1_{\{U \in \Gamma(A)\}})$, which yields the result, since π has marginals P and ν . The converse (proved in the appendix) is far more involved, as it relies on mass transportation duality, where mass P is transported into mass ν with 0-1 cost of transportation associated with violations of the restrictions $U \in \Gamma(Y)$.

Pilot example 1 continued For a given θ , it is now very easy to derive the condition for internal consistency of the structure. Indeed, all we need to check is that $\sup_{A \in 2^{\{0,1\}}} [P(A) - \nu_\theta(\Gamma_\theta(A))] = 0$ (where 2^B is the collection of all subsets of a set B), which only constrains $P(\{1\}) \leq \nu_\theta([0, \theta]^2)$, hence $p \leq \theta^2$. So the identified set for the structural parameter is $\Theta_I = [\sqrt{p}, 1]$.

Remark 1. *Further dimension reduction requires the determination of classes of sets A on which to check the inequality between $P(A)$ and $\nu(\Gamma(A))$. This is needed for instance when the observable variables are discrete and take many different values, since checking the inequality for all subsets of the set of possible values would involve a very large number of operations. Galichon and Henry (2006b) addresses this issue with a theory of core determining classes.*

Pilot example 2 continued Fixing θ (and dropping it from the notation), the necessary and sufficient condition for internal consistency of the structure is that $P(A) \leq \nu(\Gamma(A))$ for any measurable set A . Suppose for expositional purposes that the jobs are characterized by a real valued random variable Y , and that required skills are monotone in the sense that \underline{s} and \bar{s} are nondecreasing. As shown in Galichon and Henry (2006b), the inequality needs to be checked only on sets of the form $A = (-\infty, y]$ and $A = (y, +\infty)$, for $y \in \mathbb{R}$, so that a necessary and sufficient condition for internal consistency of the structure is that $F_\nu(\underline{s}(y)) \leq F(y) \leq F_\nu(\bar{s}(y))$, where F is the cumulative distribution function of jobs Y , and F_ν is the cumulative distribution function of skills U .

1.3 Inference on the identified set

Given a sample (Y_1, \dots, Y_n) of independently and identically distributed realizations of Y , our objective is to construct a sequence of random sets Θ_n^α such that for all $\theta \in \Theta_I$, $\lim_{n \rightarrow \infty} \Pr(\theta \in \Theta_n^\alpha) = 1 - \alpha$. In other words, we are concerned with constructing a region Θ_n^α that covers each value of the identified set, as opposed to a region $\tilde{\Theta}$ that covers the identified set uniformly, i.e. such that $\Pr(\Theta_I \subseteq \tilde{\Theta}) = 1 - \alpha$. We do so by including in Θ_n^α all the values of θ such that we fail to reject a test of internal consistency of $(P, \Gamma_\theta, \nu_\theta)$ with asymptotic level $1 - \alpha$. We shall demonstrate the construction of a test statistic $T_n(\theta)$ and a sequence $c_n^\alpha(\theta)$ such that, conditionally on the structure $(P, \Gamma_\theta, \nu_\theta)$ being internally consistent, the probability that $T_n(\theta) \leq c_n^\alpha(\theta)$ is $1 - \alpha$ asymptotically, i.e.

$$\lim_{n \rightarrow \infty} \Pr(T_n(\theta) \leq c_n^\alpha(\theta) \mid (P, \Gamma_\theta, \nu_\theta) \text{ is internally consistent}) = 1 - \alpha. \quad (1)$$

Hence we define our confidence region in the following way.

Table 1: Summary of the procedure

1. For a given value of θ , calculate $\hat{T}_n(\theta) = \sqrt{n} \sup_{A \in \mathcal{C}_n} [P_n(A) - \nu_\theta(\Gamma_\theta(A))]$, where the collection of sets \mathcal{C}_n is described in table 2, and P_n is the empirical distribution of the sample (Y_1, \dots, Y_n) , so that $P_n(A) = (1/n) \sum_{i=1}^n 1_{\{Y_i \in A\}}$.
2. Choose a large integer B . Draw B bootstrap samples (Y_1^b, \dots, Y_n^b) , $b = 1, \dots, B$ with replacement from the initial sample (Y_1, \dots, Y_n) . For each bootstrap sample, calculate $T_n^b(\theta) = \sup_{A \in \mathcal{C}_{n, h_n}(\theta)} [P^b(A) - P_n(A)]$, where P^b is the empirical distribution of the bootstrap sample, and $\mathcal{C}_{n, h_n}(\theta)$ is described in table 2. Order the $T_n^b(\theta)$'s and call $c_*^\alpha(\theta)$ the $B(1 - \alpha)$ largest.
3. Include θ in Θ_I if and only if $\hat{T}_n(\theta) \leq c_*^\alpha(\theta)$.

Definition 3. The $(1 - \alpha)$ confidence region for Θ_I is $\Theta_n^\alpha = \{\theta \in \Theta : T_n(\theta) \leq c_n^\alpha(\theta)\}$.

The full procedure is summarized in table 1. It is clear from equation 1 and the above definition that our confidence region covers each element of the identified set with probability $1 - \alpha$ asymptotically. Hence, after a section devoted to discussing in detail our contribution within the literature on the topic, the remainder of this paper will be concerned with the construction of the statistic T_n and sequence c_n^α with the required property (1).

Pilot example 1 continued The test statistic is then $T_n(\theta) = \sqrt{n} \sup_{A \in 2^{\{0,1\}}} [P_n(A) - \nu_\theta(\Gamma_\theta(A))]$. Since $P_n(\emptyset) = \nu_\theta(\Gamma_\theta(\emptyset))$ and $P_n(\{1\}) - \nu_\theta(\Gamma_\theta(\{1\})) = p_n - \theta^2$, the test statistic is equal to $T_n(\theta) = \max\{\sqrt{n}(p_n - p) + \sqrt{n}(p - \theta^2), 0\}$ which tends to $\max\{\sqrt{p(1-p)}Z, 0\}$ where Z is a standard normal random variable, if $p = \theta^2$, 0 if $p < \theta^2$, $+\infty$ if $p > \theta^2$. For any θ such that $p \leq \theta^2$, $T_n(\theta)$ has the same limit as $\tilde{T}_n = \sup_{A \in \mathcal{C}_{h_n}} [\sqrt{n}(P_n(A) - P(A))]$ where \mathcal{C}_{h_n} is equal to $\{\emptyset, \{0, 1\}\}$ if $p_n < \theta^2 - h_n$ and $2^{\{0,1\}}$ if $p_n \geq \theta^2 - h_n$. Hence the confidence region Θ_n^α is the set of θ values that are not rejected in a one-sided test of the null hypothesis $p \leq \theta^2$ against the alternative $p > \theta^2$ based on the quantiles of the distribution of $\max\{\sqrt{n}(p^* - p_n), 0\}$ given the sample (where p^* denotes the frequency of 1's in a bootstrap sample).

Table 2: Collection of sets

1. Take the sample (Y_1, \dots, Y_n) . Write $Y_i = (D_i, C_i)$ where D_i includes the discrete components, and C_i the continuous components of the observable variables in the sample. Call \mathfrak{X}_D the set of values taken by D_i . Then, \mathcal{C}_n is the collection of sets of the form $A_D \times [-\infty, C_i]$ or its complement, where $i = 1, \dots, n$, A_D ranges over the subsets of \mathfrak{X}_D , and $[-\infty, C_i]$ denotes the hyper-rectangle bounded above by the components of C_i .
2. Given h_n satisfying $h_n \ln \ln n + h_n^{-1} \sqrt{\ln \ln n/n} \rightarrow 0$ as $n \rightarrow \infty$ (e.g. $h_n = (\ln n)^{-1}$), take $\mathcal{C}_{n, h_n}(\theta) = \{A \in \mathcal{C}_n : P_n(A) \geq \nu_\theta(\Gamma_\theta(A)) - h_n\}$.

1.4 Review of the literature

This paper appears to be the first to cast partial identification as a mass transportation problem. Somewhat related is the specific use of Fréchet-Hoeffding bounds on cell probabilities in Heckman, Smith, and Clements (1997) and Cross and Manski (2002).

The literature on specification testing in econometrics is quite extensive (see the many references in Andrews (1988) for Cramér-von Mises tests and Andrews (1997) for the Kolmogorov-Smirnov type). Jovanovic (1989) proposes to consider testing specifications with multiple equilibria and possible lack of identification with a generalization of the Kolmogorov-Smirnov specification test, which is exceedingly conservative unless the structure is nearly identified. The stochastic dominance tests of McFadden (1989) (see also Linton, Maasoumi, and Whang (2005) and references within) are also related to tests of partially identified structures based on the Kolmogorov-Smirnov statistic. The feasible version of our testing procedure and the use of the bootstrapped empirical process is related to Andrews (1997).

The incompleteness of the structure to be tested raises boundary problems, which appear also in the estimation of models defined by moment inequalities (see Imbens and Manski (2004) and the link drawn by Rosen (2006) with the literature on constrained statistical testing, surveyed in Sen and Silvapulle (2004)) and stochastic dominance testing (see Linton, Maasoumi, and Whang (2005)). Here the asymptotic analysis is carried out via a localization of the empirical processes to treat the boundary problem, which is another major innovation of this paper. Also related is the analysis in Liu and Shao (2003) of the

likelihood ratio test when the likelihood is maximized on a set as opposed to a single point.

The related problem of constructing confidence regions for partially identified structural parameters is the focus of considerable recent research, following the recognition (advocated in Manski (2005)) that ad-hoc identification conditions can considerably weaken inference drawn on their basis. Horowitz and Manski (1998) propose confidence intervals that asymptotically cover interval identified sets with fixed probability. Beyond the interval case, Chernozhukov, Hong, and Tamer (2007) propose a criterion function based method, where the criterion is maximized on a set, as opposed to a single point. The method allows the construction of confidence regions for the identified set and for each parameter value in the identified set. Chernozhukov, Hong, and Tamer (2007) also specialize their method to the case of models defined by moment inequalities, with a quadratic criterion function.

The case of moment inequalities is also considered as a special case by Galichon and Henry (2006a), Romano and Shaikh (2006a) and Romano and Shaikh (2006b) (see also Rosen (2006) and Bugni (2007)). The present paper complements Chernozhukov, Hong, and Tamer (2007) in that it justifies, via a mass transportation argument, the use of a generalized Kolmogorov-Smirnov criterion function in the extended Koopmans and Reiersol (1950) setup presented here. Note that our proposed use of the bootstrap only concerns the empirical process, as in Andrews (2000), so that issues of validity related to bootstrapping the test statistic itself do not arise.

The Anderson and Rubin (1949) approach taken here to construct confidence regions for parameter values within the identified set is also adopted in Chernozhukov, Hong, and Tamer (2007), Andrews, Berry, and Jia (2003), Romano and Shaikh (2006a) among many others. Andrews, Berry, and Jia (2003) work in a similar framework to the present paper (they consider example 1), but restrict their analysis to discrete dependent variables, and use a projection method, so that their inference is likely to be more conservative.

Since confidence regions are asymptotically validated, as emphasized by Imbens and Manski (2004), uniformity of the confidence region for parameter values is a desirable property for small sample accuracy. Andrews and Guggenberger (2006) analyze uniformity of sub-sampling procedures. Romano and Shaikh (2006a) and Romano and Shaikh (2006b) give high level conditions for uniformity of sub-sampling procedures in the criterion-based approach, with specific conditions under which these results hold in case of regression with interval outcomes. Here, we propose to invert a test, which is shown to be asymptotically uniform in level in Galichon and Henry (2008).

In related research, Beresteanu and Molinari (2007) propose a direct analogy to central limit theorem based confidence regions in best linear prediction problems. The confidence region they propose for the identified set, in a problem of best linear prediction with interval outcomes, is the union of a collection of random sets that contain the identified set with pre-specified probability. The latter is obtained from central limit theorems for random sets (see Molchanov (2005) for a comprehensive account of the theory). They propose one-sided and two-sided versions of their test. The Beresteanu and Molinari (2007) two-sided procedure does not suffer from discontinuity at the limit where the identified set is a singleton. However, by construction, Beresteanu and Molinari (2007) only provide confidence regions for the whole set, which are typically larger than identified regions for each point in the identified set.

2 Test of internal consistency

As explained in the previous section, the construction of the confidence region relies on a test of internal consistency of the structure $(P, \Gamma_\theta, \nu_\theta)$ for a fixed θ . We now explain the construction of our test statistic and decision rule, for the hypothesis of internal consistency of a structure (P, Γ, ν) defined by a probability law ν for U and a set of constraints $U \in \Gamma(Y)$. The hypothesis that (P, Γ, ν) is internally consistent is equivalent to the existence of a law π for (Y, U) with marginals P and ν and such that the constraints $U \in \Gamma(Y)$ hold π -almost surely. By proposition 1, this null hypothesis is also equivalent to

$$\mathbb{H}_0 : \sup_{A \in \mathcal{B}} [P(A) - \nu(\Gamma(A))] = 0.$$

2.1 Test statistic and size of the test of internal consistency

We propose the following statistic to test the null described above:

$$T_n = \sqrt{n} \sup_{A \in \mathcal{C}} [P_n(A) - \nu(\Gamma(A))], \quad (2)$$

where P_n is the empirical distribution of the sample (so that for any measurable set A , $P_n(A) = (1/n) \sum_{i=1}^n 1_{\{Y_i \in A\}}$) and where \mathcal{C} is defined in table 3.

This statistic is a generalized Kolmogorov-Smirnov specification test statistic in the sense that when Γ has disjoint images (i.e. Γ^{-1} is a function), T_n is a multivariate Kolmogorov-Smirnov statistic for the test of the hypothesis that the structure is correctly specified, i.e.

Table 3: collections of sets

1. Write $Y = (D, C)$ where D includes the discrete components, and C the continuous components with dimension d_C . Call \mathfrak{X}_D the set of values taken by D . Then, C is the collection of sets of the form $A_D \times [-\infty, c]$ or its complement, where $c \in \mathbb{R}^{d_C}$, A_D ranges over the subsets of \mathfrak{X}_D , and $[-\infty, c]$ is the hyper-rectangle bounded above by the components of c .

2. Given $h > 0$, define

$$\mathcal{C}_b = \{A \in \mathcal{C} : P(A) = \nu(\Gamma(A))\}.$$

$$\mathcal{C}_{b,h} = \{A \in \mathcal{C} : P(A) \geq \nu(\Gamma(A)) - h\}.$$

$$\mathcal{C}_h = \{A \in \mathcal{C} : P_n(A) \geq \nu(\Gamma(A)) - h\}.$$

that the probability law $A \mapsto \nu(\Gamma(A))$ is indeed equal to the true law P generating the observable variables Y . In the general case where Γ is a many-to-many mapping, $A \mapsto \nu(\Gamma(A))$ is no longer a probability measure, since two sets A and B may be disjoint, and yet their images $\Gamma(A)$ and $\Gamma(B)$ are not, so that $\nu(\Gamma(A \cup B))$ may be strictly smaller than $\nu(\Gamma(A)) + \nu(\Gamma(B))$. This introduces significant complications in the asymptotic analysis of the statistic T_n as explained in the following discussion.

We can write

$$T_n = \sqrt{n} \sup_{A \in \mathcal{C}} [P_n(A) - \nu(\Gamma(A))] = \sup_{A \in \mathcal{C}} \{\mathbb{G}_n(A) + \sqrt{n}[P(A) - \nu(\Gamma(A))]\} \quad (3)$$

where $\mathbb{G}_n(A) := \sqrt{n}[P_n(A) - P(A)]$ is the empirical process. In the case of the classical Kolmogorov-Smirnov statistic (i.e. if Γ^{-1} were a function), the term $P(A) - \nu(\Gamma(A))$ would vanish under the null hypothesis. Here, however, under the null we only have $P(A) \leq \nu(\Gamma(A))$, so that the term $\sqrt{n}[P(A) - \nu(\Gamma(A))]$ will also contribute. Indeed, for any set $A \in \mathcal{C}$ such that $P(A) = \nu(\Gamma(A))$ (i.e. $A \in \mathcal{C}_b$ as defined in table 3), the only remaining term in the right-hand-side of equation (3) is the empirical process. On the other hand, for any set $A \in \mathcal{C}$ such that $P(A) < \nu(\Gamma(A))$, $\sqrt{n}[P(A) - \nu(\Gamma(A))]$ will take increasingly large negative values and eventually dominate the expression inside the supremum in the right-hand-side of equation (3) and such a set A will not contribute to the supremum. We show in the proof of theorem 1 that under a very mild assumption on the structure, the limit will only involve a supremum over sets in \mathcal{C}_b . Since \mathcal{C}_b depends on P , it is unknown, and needs to be approximated by a data dependent class \mathcal{C}_{h_n} defined in table 3 (namely \mathcal{C}_h with $h = h_n$).

Definition 4. The test statistic T_n is given by equation (2), and c_n^α is the $1 - \alpha$ quantile of $\tilde{T}_n := \sup_{A \in \mathcal{C}_{h_n}} \mathbb{G}_n(A)$ (with \mathcal{C}_h defined in table 3), i.e. $c_n^\alpha = \inf\{c : \mathbb{P}(\tilde{T}_n \leq c) \geq 1 - \alpha\}$.

Assumption 1. There exists $K > 0$ and $0 < \eta < 1$ such that for all $A \in \mathcal{C}_{b,h}$, for $h > 0$ sufficiently small, there exists an $A_b \in \mathcal{C}_b$ such that $A_b \subseteq A$ and $d_H(A, A_b) \leq Kh^\eta$. (\mathcal{C}_b and \mathcal{C}_h are defined in table 3, and d_H denotes the Hausdorff metric, defined in the appendix.)

Remark 2. Assumption 1 is very mild, in the sense that it fails only in pathological cases, such as the case where $y \in \mathbb{R}$ and $y \mapsto P((-\infty, y]) - \nu(\Gamma((-\infty, y]))$ is C^∞ with all derivatives equal to zero at some $y = y_0$ such that $(-\infty, y_0] \in \mathcal{C}_b$.

Assumption 2. h_n satisfies $h_n \ln \ln n + h_n^{-1} \sqrt{\ln \ln n / n} \rightarrow 0$ as $n \rightarrow \infty$.

Remark 3. Note that assumption 2 is extremely mild, and it is satisfied for instance in case $h_n = (\ln n)^{-1}$ or in case h_n satisfies $h_n n^\eta + h_n^{-1} n^{\eta-1/2} \rightarrow 0$, as $n \rightarrow \infty$ for any $1/2 > \eta > 0$, however small.

Theorem 1. Suppose Y either takes values in a finite set or has density with respect to Lebesgue measure. Under assumption 1 and 2, and using the notations of definition 4, we have

$$\lim_{n \rightarrow \infty} \mathbb{P}(T_n \leq c_n^\alpha \mid (P, \Gamma, \nu) \text{ is internally consistent}) = 1 - \alpha.$$

Theorem 1 is not applicable directly for two reasons:

1. The quantile sequence c_n^α given in definition 4 is infeasible in that the statistic \tilde{T}_n involves the empirical process $\mathbb{G}_n = \sqrt{n}[P_n - P]$ with P unknown.
2. The statistics T_n and \tilde{T}_n are defined as suprema over infinite collections of sets \mathcal{C} and \mathcal{C}_h (with \mathcal{C} and \mathcal{C}_h defined in table 3).

We show now that T_n can be replaced by \hat{T}_n defined in table 2, and that c_n^α can be replaced by c_*^α , which is the $1 - \alpha$ quantile of $T^* := \sup_{A \in \mathcal{C}_{n,h_n}} \mathbb{G}^*(A)$, where $\mathbb{G}^* := \sqrt{n}[P^* - P_n]$ is the bootstrapped empirical process. We thereby justify the fully implementable procedure described in table 1. This feasible version of the test mirrors the feasible version of the conditional Kolmogorov-Smirnov test proposed by Andrews (1997), albeit in generalized form (multivariate and incompletely specified).

To that end, we need a large support assumption and a log concavity assumption for the distribution of observable variables and a continuity assumption on the mapping Γ to ensure that \hat{T}_n has the same limit as T_n .

Assumption 3. *In case P has density with respect to Lebesgue measure, the density is bounded away from zero, absolutely continuous and log concave (note that log concave densities include the uniform, normal, beta, exponential and extreme value distributions).*

Assumption 4. *The functions $y \mapsto \nu(\Gamma((-\infty, y]))$ and $y \mapsto \nu(\Gamma((-\infty, y]^c))$ are Lipschitz, i.e. there exists some $k > 0$ such that $|\nu(\Gamma((-\infty, y])) - \nu(\Gamma((-\infty, y']))| \leq k\|y - y'\|$, and identically for $(-\infty, y]^c$.*

Theorem 2. *Under the assumptions of theorem 1 and assumptions 3 and 4, we have*

$$\lim_{n \rightarrow \infty} \mathbb{P}(\hat{T}_n \leq c_*^\alpha \mid (P, \Gamma, \nu) \text{ is internally consistent}) = 1 - \alpha$$

almost surely, conditionally on the sample.

Remark 4. *The conditions for the validity of the bootstrap procedure are no more restrictive than the conditions for theorem 1. The additional assumptions, which are more high level, are needed only to justify using the data driven class of sets \mathcal{C}_n instead of \mathcal{C} . This follows the proposal in Andrews (1997) in order to simplify the testing procedure as much as possible. However, an alternative feasible version of the test relies on a regular discretization $(y_k)_{k=1}^N$ of the space of continuous observable variables (thereby replacing \mathcal{C}_n by the class of sets of the form $(-\infty, y_k]$, $(-\infty, y_k]^c$, $k = 1, \dots, N$).*

2.2 Consistency of the test

To complete the analysis of the test of internal consistency we give conditions under which the test is consistent. The class of alternatives we consider is the following:

$$\mathbb{H}_a : \sup_{A \in \mathcal{C}} [P(A) - \nu(\Gamma(A))] \neq 0,$$

where \mathcal{C} is defined in table 3. We choose this class of alternatives since it simplifies to the set of alternatives in a multivariate Kolmogorov-Smirnov goodness-of-fit test when P is absolutely continuous with respect to Lebesgue measure and when Γ^{-1} is a function.

We have

Theorem 3. *Under \mathbb{H}_a and the assumptions of theorem 1, $\lim_{n \rightarrow \infty} \mathbb{P}(T_n \geq c_n^\alpha) = 1$.*

Remark 5. *Notice that the validity of this consistency test is completely general, and, unlike theorem 1, the proof is a straightforward extension of the proof of consistency of the traditional Kolmogorov-Smirnov specification test (see for instance page 526 of Lehmann and Romano (2005)).*

2.3 Small sample investigation of the properties of the test of internal consistency

We investigate the small sample properties of our test, and compare it to the properties of the Kolmogorov-Smirnov specification test in the identified case in a small Monte Carlo experiment based on a special case of illustrative example 2.

We consider the following setup illustrated in figure 1: the structure is given by the correspondence $\Gamma(Y) = [\underline{s}(Y), \bar{s}(Y)]$ with $\underline{s}(Y) = \max(0, Y + s)$ and $\bar{s}(Y) = \min(1, Y + s)$, $s = 0.15$, and the latent variable U has law ν , which is the uniform distribution over $[0, 1]$. Y has cumulative distribution function defined on $[0, 1]$ by

$$\begin{aligned} F(y) &= 0 \text{ for } 0 \leq y < s, \\ &= y - s \text{ for } s \leq y < \frac{1+s}{3}, \\ &= \frac{(1+4s)y - 3s}{1-2s} \text{ for } \frac{1+s}{3} \leq y < \frac{2-s}{3}, \\ &= y + s \text{ for } \frac{2-s}{3} \leq y < 1-s, \\ &= 1 \text{ for } 1-s \leq y \leq 1. \end{aligned}$$

We perform 1000 repetitions of the following testing procedure, and we report the proportions of rejections out of these 1000 repetitions. We first generate¹ a sample (U_1, \dots, U_n) of iid uniform $[0, 1]$, with $n = 100, 500, 1000$ and compute the sample of observable variables (Y_1, \dots, Y_n) as $(F^{-1}(U_1), \dots, F^{-1}(U_n))$. P_n is the empirical law of (Y_1, \dots, Y_n) , and \mathcal{C}_{n, h_n} is the collection of sets of the form $[0, Y_i]$, $i = 1, \dots, n$ with $P_n[0, Y_i] = (1/n) \sum_{j=1}^n 1_{\{Y_j \leq Y_i\}} \geq \nu(\Gamma([0, Y_i])) - h_n = \min[1, Y_i + s] - h_n$ or $[Y_i, 1]$, $i = 1, \dots, n$ with $P_n[Y_i, 1] \geq \nu(\Gamma([Y_i, 1])) - h_n = \min[1, 1 - Y_i + s] - h_n$.

For each sample, we draw 1000 bootstrap samples (Y_1^b, \dots, Y_n^b) , and call P^b the law of the bootstrap sample. For each bootstrap sample, we calculate the maximum of the quantities $P^b[0, Y_i] - P_n[0, Y_i]$ for all i such that $[0, Y_i] \in \mathcal{C}_{n, h_n}$ and $P^b[Y_i, 1] - P_n[Y_i, 1]$ for all i such that $[Y_i, 1] \in \mathcal{C}_{n, h_n}$, and call this maximum $\max \mathbb{G}^b$. Order the $\max \mathbb{G}^b$ obtained for all bootstrap draws, and call c_*^α the $(1 - \alpha)1000$ largest, for $\alpha = 0.01, 0.05, 0.1$. Reject if c_*^α is smaller than the maximum of the quantities $P_n[0, Y_i]$ and $P_n[Y_i, 1]$ for $i = 1 \dots, n$.

The results are given in table 4 for the partially identified case ($s = 0.15$) and in table 5, we give the benchmark of the exactly identified case ($s = 0$ and $h_n = 1$), so that the test

¹We use MATLAB version 7.1 with random seed 777.

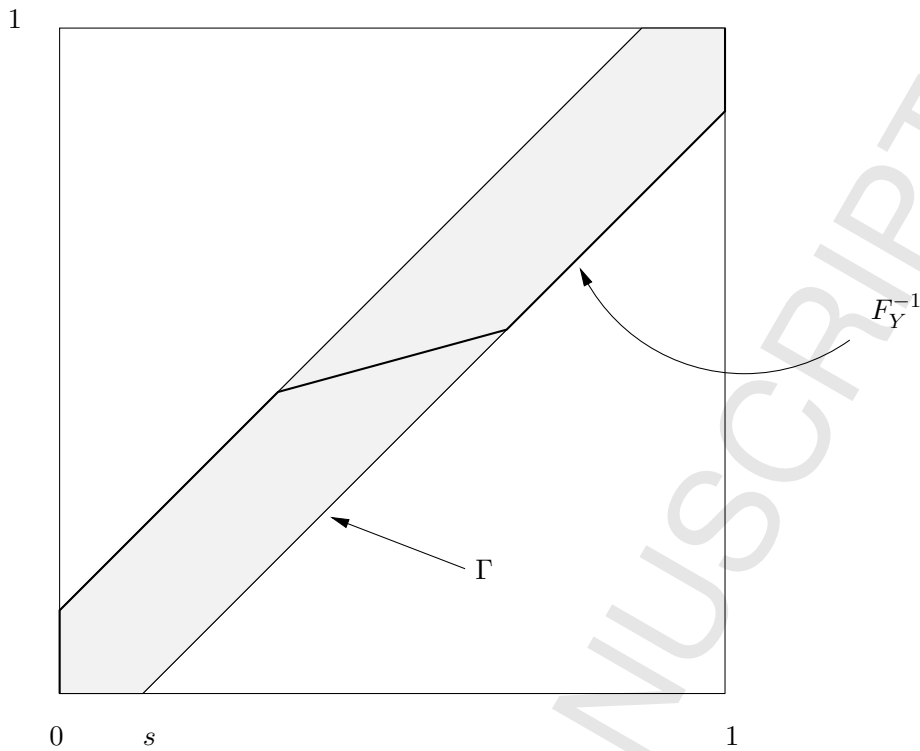


Figure 1: The correspondence Γ is given by the shaded area, and the thick lines trace the inverse cumulative distribution function of Y .

Table 4: Rejection levels for the partially identified case.

Sample Size	100	500	1000
$\alpha = 0.01$	0.001	0.007	0.008
$\alpha = 0.05$	0.010	0.024	0.029
$\alpha = 0.10$	0.029	0.049	0.066

Table 5: Rejection levels for the exactly identified case

Sample Size	100	500	1000
$\alpha = 0.01$	0.019	0.024	0.014
$\alpha = 0.05$	0.074	0.079	0.050
$\alpha = 0.10$	0.138	0.135	0.105

Table 6: Sensitivity of rejection levels to the choice of tuning parameters

Sample Size Tuning	100		500		1000	
	$h_n = 0.05$	$h_n = 0.15$	$h_n = 0.02$	$h_n = 0.10$	$h_n = 0.01$	$h_n = 0.07$
$\alpha = 0.01$	0.004	0	0.012	0.002	0.019	0.005
$\alpha = 0.05$	0.026	0.006	0.049	0.017	0.058	0.022
$\alpha = 0.10$	0.064	0.020	0.090	0.034	0.111	0.043

is a traditional Kolmogorov-Smirnov specification test. The results are given for h_n on the boundary of the admissible rate, i.e. $h_n = \sqrt{\ln \ln n/n}$. This rate was chosen as a power maximizing rate (the rate that will ensure smaller quantiles, hence larger rejection rates). This is the only justification for a choice of rate that we can provide at this stage, as optimal rate choice is beyond the scope of this paper. In applications, it is recommended to provide results for different choices of rates, as one would typically do in density, nonparametric regression or spectral estimation. The rejection rates are low for small sample sizes and improve sharply when sample size increases. To give a sense of the sensitivity of rejection rates to the choice of the tuning parameter h_n , table 6 reports rejection rates in the case of $\alpha = 0.01, 0.05, 0.1$ and $n = 100, 500, 1000$ and choices of tuning parameter h_n that are significantly above, and significantly below the initial choice of $h_n = \sqrt{\ln \ln n/n}$. For $n = 1000$, $\sqrt{\ln \ln n/n} = 0.044$, so we report results for $h_n = 0.010, 0.070$. For $n = 500$, $\sqrt{\ln \ln n/n} = 0.060$, so we report results for $h_n = 0.020, 0.100$. For $n = 100$, $\sqrt{\ln \ln n/n} = 0.120$, so we report results for $h_n = 0.050, 0.150$. Notice that we decrease the investigated range of tuning parameter with sample size, which corresponds to the fact that the tuning parameter converges to zero. For $n = 100$, the rejection rates are sensitive to the choice of rate within the theoretical range (assumption 2) of tuning parameters. For $n = 500$, there is still sensitivity to the choice of h_n , somewhat less so for $n = 1000$. However, as in the case of bandwidth in kernel estimation or in local spectral estimation of time series, it is highly recommended to report empirical results with a good range of values of the tuning parameter h_n . Figure 2 graphs rejection rates against tuning parameter to give a better sense of this sensitivity for sample size 500 and level 0.05. It is important also to note that higher values of the tuning parameter lead to less filtering, i.e. more sets are used in the computation of the supremum of the bootstrap empirical process, leading to larger quantiles, hence smaller rejection rates. Hence it also shows how crucial the filtering procedure is, since without it, the power of the test would be very poor.

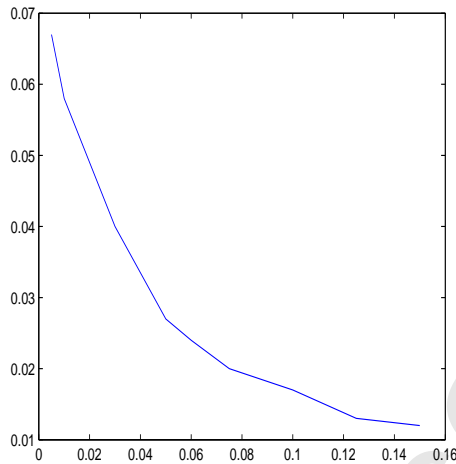


Figure 2: Sensitivity to the tuning parameter. Sample size 500, level 0.05, tuning parameter ranging from 0.005 to 0.15 on the X axis, and rejections rates on the Y axis.

Conclusion

We propose a test of the specification of a structure in the sense of Koopmans and Reiersol (1950), extended by Jovanovic (1989), where observable variables and latent variables are related by a many-to-many mapping, thereby allowing censored observable variables and multiple equilibria. We apply mass transportation duality to derive a simple necessary and sufficient condition for compatibility of such structures and data in complete generality, and to justify the use of a generalized Kolmogorov-Smirnov test statistic. We propose a generically applicable and easily implementable procedure to test compatibility of structure and data, and to construct confidence regions for partially identified parameters specifying the structure. This work therefore complements other proposals, which tend to focus on models defined by moments inequalities. The small sample performance of the test is investigated in a Monte Carlo experiment, and is found to be comparable to the performance of the traditional Kolmogorov-Smirnov specification test statistic.

Appendix

Additional definitions

Definition 5. A many-to-many mapping $\Gamma : \mathbb{R}^{d_1} \rightrightarrows \mathbb{R}^{d_2}$ is called measurable if for each open set $\mathcal{O} \subseteq \mathbb{R}^{d_2}$, $\Gamma^{-1}(\mathcal{O}) = \{x \in \mathbb{R}^{d_1} \mid \Gamma(x) \cap \mathcal{O} \neq \emptyset\}$ is a measurable subset of \mathbb{R}^{d_1} .

Definition 6. Calling d the Euclidean metric, the Hausdorff metric d_H between two sets A_1 and A_2 is defined by

$$d_H(A_1, A_2) = \max \left(\sup_{y \in A_1} \inf_{z \in A_2} d(y, z), \sup_{z \in A_2} \inf_{y \in A_1} d(y, z) \right).$$

Proofs of results in the main text

Proof of proposition 1 : Since Γ is closed valued, $\varphi(y, u) = 1_{\{u \notin \Gamma(y)\}}$ is lower semicontinuous, so that we can apply lemma 1 below to yield

$$\inf_{\pi \in \mathcal{M}(P, \nu)} \pi \varphi = \sup_{f \oplus g \leq \varphi} (Pf + \nu g), \quad (4)$$

where $f \oplus g \leq \varphi$ stands for $f(y) + g(u) \leq \varphi(y, u)$ all y, u . Since the sup-norm of the cost function is 1 (the cost function is an indicator), the supremum in (4) is attained by pairs of functions (f, g) in \mathcal{F} , defined by

$$\mathcal{F} = \{(f, g) \in \mathbb{L}^1(P) \times \mathbb{L}^1(\nu), 0 \leq f \leq 1, -1 \leq g \leq 0, \\ f(y) + g(u) \leq 1_{\{u \notin \Gamma(y)\}}, f \text{ upper semicontinuous}\}.$$

Now, (f, g) can be written as a convex combination of pairs $(1_A, -1_B)$ in \mathcal{F} . Indeed, $f = \int_0^1 1_{\{f \geq x\}} dx$ and $g = \int_0^1 -1_{\{g \leq -x\}} dx$, and for all x , $1_{\{f \geq x\}}(y) - 1_{\{g \leq -x\}}(u) \leq 1_{\{u \notin \Gamma(y)\}}$. Since the functional on the right-hand side of (4) is linear, the supremum is attained on such a pair $(1_A, -1_B)$. Hence, the right-hand side of (4) specializes to

$$\sup_{A \times B \subseteq D} (P(A) - 1 + \nu(B)). \quad (5)$$

For $D = \{(y, u) : u \notin \Gamma(y)\}$, $A \times B \subseteq D$ means that if $y \in A$ and $u \in B$, then $u \notin \Gamma(y)$. In other words $u \in B$ implies $u \notin \Gamma(A)$, which can be written $B \subseteq \Gamma(A)^c$. Hence, the dual problem can be written

$$\sup_{\Gamma(A) \subseteq B^c} (P(A) - 1 + \nu(B)) = \sup_{\Gamma(A) \subseteq B} (P(A) - \nu(B)).$$

and the result follows immediately.

Lemma 1. *If $\varphi : \mathcal{Y} \times \mathcal{U} \rightarrow \mathbb{R}$ is bounded, non-negative and lower semicontinuous, then*

$$\inf_{\pi \in \mathcal{M}(P, \nu)} \pi \varphi = \sup_{f \oplus g \leq \varphi} (Pf + \nu g).$$

Proof of lemma 1 : The left-hand side is immediately seen to be always larger than the right-hand side, so we show the reverse inequality. It is a specialization of the Monge-Kantorovich duality to zero-one cost, which can also be proved using Proposition (3.3) page 424 of Kellerer (1984), but we give a direct proof due to N. Belili for completeness.

[a] case where φ is continuous and \mathcal{U} and \mathcal{Y} are compact.

Call G the set of functions on $\mathcal{Y} \times \mathcal{U}$ strictly dominated by φ and call H the set of functions of the form $f + g$ with f and g continuous functions on \mathcal{Y} and \mathcal{U} respectively. Call $s(c) = Pf + \nu g$ for $c \in H$. It is a well defined linear functional, and is not identically zero on H . G is convex and sup-norm open. Since φ is continuous on the compact $\mathcal{Y} \times \mathcal{U}$, we have

$$s(c) \leq \sup f + \sup g < \sup \varphi$$

for all $c \in G \cap H$, which is non empty and convex. Hence, by the Hahn-Banach theorem, there exists a linear functional η that extends s on the space of continuous functions such that

$$\sup_G \eta = \sup_{G \cap H} s.$$

By the Riesz representation theorem, there exists a unique finite non-negative measure π on $\mathcal{Y} \times \mathcal{U}$ such that $\eta(c) = \pi c$ for all continuous c . Since $\eta = s$ on H , we have

$$\begin{aligned} \int_{\mathcal{Y} \times \mathcal{U}} f(y) d\pi(y, u) &= \int_{\mathcal{Y}} f(y) dP(y) \\ \int_{\mathcal{Y} \times \mathcal{U}} g(u) d\pi(y, u) &= \int_{\mathcal{U}} g(u) d\nu(y), \end{aligned}$$

so that $\pi \in \mathcal{M}(P, \nu)$ and

$$\sup_{f \oplus g \leq \varphi} (Pf + \nu g) = \sup_{G \cap H} s = \sup_G \eta = \pi \varphi.$$

[b] \mathcal{Y} and \mathcal{U} are not necessarily compact, and φ is continuous.

For all $n > 0$, there exists compact sets K_n and L_n such that

$$\max(P(\mathcal{Y} \setminus K_n), \nu(\mathcal{U} \setminus L_n)) \leq \frac{1}{n}.$$

Let (a, b) be an element of $\mathcal{Y} \times \mathcal{U}$ and define two probability measures μ_n and ν_n with compact support by

$$\begin{aligned}\mu_n(A) &= P(A \cap K_n) + P(A \setminus K_n)\delta_a(A) \\ \nu_n(B) &= \nu(B \cap L_n) + \nu(B \setminus L_n)\delta_b(B),\end{aligned}$$

where δ denotes the Dirac measure. By [a] above, there exists π_n with marginals μ_n and ν_n such that

$$\pi_n \varphi \leq \sup_{f \oplus g \leq \varphi} (Pf + \nu g) + \frac{\varphi(a, b)}{n}.$$

Since (π_n) has weakly converging marginals, it is weakly relatively compact. Hence it contains a weakly converging subsequence with limit $\pi \in \mathcal{M}(P, \nu)$. By Skorohod's almost sure representation (see for instance theorem 11.7.2 page 415 of Dudley (2002)), there exists a sequence of random variables X_n on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$ with law π_n and a random variable X_0 on the same probability space with law π such that X_0 is the almost sure limit of (X_n) . By Fatou's lemma, we then have

$$\liminf \pi_n \varphi = \liminf \mathbb{E} \varphi(X_n) \geq \mathbb{E} \liminf \varphi(X_n) = \mathbb{E} \varphi(X_0) = \pi \varphi.$$

Hence we have the desired result.

[c] General case.

φ is the pointwise supremum of a sequence of continuous bounded functions, so the result follows from upward σ -continuity of both $\inf_{\pi \in \mathcal{M}(P, \nu)} \pi \varphi$ and $\sup_{f \oplus g \leq \varphi} (Pf + \nu g)$ on the space of lower semicontinuous functions, shown in propositions (1.21) and (1.28) of Kellerer (1984).

Proof of theorem 1 : We show that T_n and \tilde{T}_n converge in distribution (notation \rightsquigarrow) to the same limit, which has a continuous distribution function. Hence, the result follows.

- Case where $Y = D$ discrete. Let A_0 be the subset of \mathfrak{X}_D that achieves the maximum of $\delta(A) = P(A) - \nu(\Gamma(A))$ over $A \in \mathcal{C} \setminus \mathcal{C}_b$. Call $\delta_0 = \delta(A_0)$, and note that $\delta_0 < 0$.

We have

$$\begin{aligned}T_n &= \sup_{A \in 2^{\mathfrak{X}_D}} [\mathbb{G}_n(A) + \sqrt{n}(P(A) - \nu(\Gamma(A)))] \\ &= \max\left\{\sup_{\mathcal{C}_b} \mathbb{G}_n, \sup_{A \in 2^{\mathfrak{X}_D} \setminus \mathcal{C}_b} [\mathbb{G}_n(A) + \sqrt{n}(P(A) - \nu(\Gamma(A)))]\right\}.\end{aligned}$$

The second term in the maximum of the preceding display is dominated by

$$\sup_{2^{\mathbb{X}^D} \setminus \mathcal{C}_b} \mathbb{G}_n + \sqrt{n}\delta_0,$$

whose limsup is almost surely non-positive. Hence $T_n \rightsquigarrow \sup_{\mathcal{C}_b} \mathbb{G}$ follows from the convergence of the empirical process. $\tilde{T}_n \rightsquigarrow \sup_{\mathcal{C}_b} \mathbb{G}$ follows from the fact that, under assumption 2, for all n sufficiently large, \mathcal{C}_{h_n} is almost surely equal to \mathcal{C}_b .

- Case of $Y = C$ absolutely continuous with respect to Lebesgue measure. Consider two sequences of positive numbers l_n and h_n such that they both satisfy assumption 2, $l_n > h_n$ and $(l_n - h_n)^{-1} \sqrt{\frac{\ln \ln n}{n}} \rightarrow 0$. Notice that $\{\emptyset, \mathbb{R}^{d_C}\} \subseteq \mathcal{C}_b, \mathcal{C}_{b,h}, \mathcal{C}_h$ for any $h > 0$. Since $\mathbb{G}_n(\mathbb{R}^{d_C}) = 0$, we therefore have $\sup_{\mathcal{C}_b} \mathbb{G}_n$, $\sup_{\mathcal{C}_{b,l_n}} \mathbb{G}_n$ and $\sup_{\mathcal{C}_{h_n}} \mathbb{G}_n$ non-negative. Hence, calling ζ_n the indicator function of the event $\sup_{\mathcal{C}} \mathbb{G}_n \leq (l_n - h_n)\sqrt{n}$, we can write

$$\begin{aligned} \zeta_n \sup_{\mathcal{C}_b} \mathbb{G}_n &\leq \zeta_n \max \left\{ \sup_{\mathcal{C}_b} [\mathbb{G}_n + \sqrt{n}(P - \nu\Gamma)], \sup_{\mathcal{C} \setminus \mathcal{C}_b} [\mathbb{G}_n + \sqrt{n}(P - \nu\Gamma)] \right\} \\ &\leq \zeta_n T_n \\ &\leq \zeta_n \sup_{\mathcal{C}_{h_n}} \mathbb{G}_n \\ &\leq \zeta_n \sup_{\mathcal{C}_{b,l_n}} \mathbb{G}_n, \end{aligned}$$

where the first inequality holds because the left-hand side is equal to the first term in the right-hand side, the second inequality holds trivially as an equality since $\mathcal{C} = \mathcal{C}_b \cup \mathcal{C} \setminus \mathcal{C}_b$, the third inequality holds because on $\mathcal{C} \setminus \mathcal{C}_{h_n}$, we have by definition $\mathbb{G}_n + \sqrt{n}(P - \nu\Gamma) = \sqrt{n}(P_n - \nu\Gamma) \leq -h_n \leq 0$, and the last inequality holds because on $\{\zeta_n = 1\}$, we have that $A \in \mathcal{C}_{h_n}$ implies $\nu\Gamma(A) \leq P_n(A) + h_n = P(A) + (P_n - P)(A) + h_n \leq P(A) + \sup_{\mathcal{C}} \mathbb{G}_n / \sqrt{n} + h_n \leq P(A) + l_n - h_n + h_n = P(A) + l_n$, which implies that $A \in \mathcal{C}_{b,l_n}$.

By lemma 2 and Theorem 2.5.2 page 127 of van der Vaart and Wellner (1996), we have that both $\sup_{\mathcal{C}_b} \mathbb{G}_n$ and $\sup_{\mathcal{C}_{b,l_n}} \mathbb{G}_n$ converge in distribution to $\sup_{\mathcal{C}_b} \mathbb{G}$. It is shown below that $\zeta_n \rightarrow_p 1$, so that Slutsky's lemma (lemma 2.8 page 11 of van der Vaart (1998)) yields the weak convergence of $\zeta_n \sup_{\mathcal{C}_b} \mathbb{G}_n$ and $\zeta_n \sup_{\mathcal{C}_{b,l_n}} \mathbb{G}_n$ to the same limit, and hence that of $\zeta_n T_n$ and $\zeta_n \sup_{\mathcal{C}_{h_n}} \mathbb{G}_n$. It follows from Slutsky's lemma again that

$$T_n \rightsquigarrow \sup_{\mathcal{C}_b} \mathbb{G} \quad \text{and} \quad \tilde{T}_n \rightsquigarrow \sup_{\mathcal{C}_b} \mathbb{G}.$$

We now prove that $\zeta_n \rightarrow_p 1$. Indeed, for any $\epsilon > 0$, $P(|\zeta_n - 1| > \epsilon) = P(\zeta_n = 0) = P(\sup_{\mathcal{C}} \mathbb{G}_n > (l_n - h_n)\sqrt{n}) \rightarrow 0$ by the Law of the Iterated Logarithm (see 12.5 page 476 of Dudley (2002)), since $(l_n - h_n)\sqrt{n} \gg \sqrt{\ln \ln n}$ by assumption.

Lemma 2. *We have*

$$\sup_{A \in \mathcal{C}_{b, h_n}} \mathbb{G}_n(A) \rightsquigarrow \sup_{A \in \mathcal{C}_b} \mathbb{G}(A),$$

Proof of lemma 2 : Take a bandwidth sequence l_n that satisfies assumption 2, and take \mathcal{C}_{b, l_n} as in table 3. Under assumption 1, take $A \in \mathcal{C}_{b, l_n}$ and an $A_b \in \mathcal{C}_b$ such that $d_H(A, A_b) \leq \zeta_n = Kl_n^\eta$ (we suppress the dependence of A_b on A for ease of notation). As $\mathcal{C}_b \subseteq \mathcal{C}_{b, l_n}$, one has

$$\sup_{A \in \mathcal{C}_b} \mathbb{G}_n(A) \leq \sup_{B \in \mathcal{C}_{b, l_n}} \mathbb{G}_n(A) \quad (6)$$

Second, since $A_b \subseteq A$, one has

$$\begin{aligned} \sup_{A \in \mathcal{C}_{b, l_n}} \mathbb{G}_n(A) &= \sup_{A \in \mathcal{C}_{b, l_n}} [\mathbb{G}_n(A_b) + \mathbb{G}_n(A \setminus A_b)] \\ &\leq \sup_{A \in \mathcal{C}_{b, l_n}} [\mathbb{G}_n(A_b)] + \sup_{A \in \mathcal{C}_{b, l_n}} [\mathbb{G}_n(A \setminus A_b)]. \end{aligned}$$

If we have that

$$\sup_{A \in \mathcal{C}_{b, l_n}} |\mathbb{G}_n(A \setminus A_b)| = O_{\text{a.s.}} \left(\sqrt{\zeta_n \ln \ln n} \right),$$

then

$$\sup_{A \in \mathcal{C}_{b, l_n}} \mathbb{G}_n(A) = \sup_{A \in \mathcal{C}_{b, l_n}} [\mathbb{G}_n(A_b)] + O_{\text{a.s.}} \left(\sqrt{\zeta_n \ln \ln n} \right) \quad (7)$$

noting the dependence of A_b on A in the expression above. But since $A_b \in \mathcal{C}_b$, one has $\sup_{A \in \mathcal{C}_{b, l_n}} [\mathbb{G}_n(A_b)] \leq \sup_{A \in \mathcal{C}_b} \mathbb{G}_n(A)$. This fact, along with (6) and (7), yields the result.

We now show that we have indeed that

$$\sup_{A \in \mathcal{C}_{b, l_n}} |\mathbb{G}_n(A \setminus A_b)| = O_{\text{a.s.}} \left(\sqrt{\zeta_n \ln \ln n} \right).$$

This relies on the construction of a local empirical process relative to the thin regions $A \setminus A_b$. First consider such a region. If $A \in \mathcal{C}_b$, the result holds trivially, so that we may assume that $A \in \mathcal{C}_{b, l_n} \setminus \mathcal{C}_b$, so that $A \setminus A_b$ is not empty. We distinguish the case where A is a bounded rectangle, and the cases where A is unbounded.

- (i) A is a bounded rectangle, i.e. of the form $(y_1, z_1) \times \dots \times (y_{d_y}, z_{d_y})$, with $y_1, \dots, y_{d_y}, z_1, \dots, z_{d_y}$ real. Then, since $d_H(A, A_b) \leq \zeta_n$, A_b is also a bounded rectangle, and the $A \setminus A_b$ is the union of at least one (since A and A_b are distinct) and at most $f(d_y)$ (the number of faces of a rectangle in \mathbb{R}^{d_y}) rectangles with at least one dimension bounded by ζ_n .

- (ii) A is an unbounded rectangle, i.e. of the same form as above, except that some of the edges are $+\infty$ or $-\infty$. Then A_b is also an unbounded rectangle, and $A \setminus A_b$ is also the union of a finite number of rectangles with one dimension bounded by ζ_n .

In both cases (i), and (ii), $A \setminus A_b$ is the union of a finite number of rectangles with at least one dimension bounded by ζ_n . Hence if we control the supremum of the empirical process on one of these thin rectangles, when A ranges over \mathcal{C}_{b,l_n} , we can control it on $A \setminus A_b$.

Hence, it suffices to prove that

$$\sup_{A \in \mathcal{C}_{b,l_n}} |\mathbb{G}_n(\varphi_n(A))| = O_{\text{a.s.}} \left(\sqrt{\zeta_n \ln \ln n} \right),$$

where φ_n is the homothety that carries A into one of the thin rectangles described above.

As an homothety, φ_n is invertible and bi-measurable, and since $\varphi_n(A)$ has at least one dimension bounded by ζ_n , and P is absolutely continuous with respect to Lebesgue measure, $P(\varphi_n(A)) = O(\zeta_n)$ uniformly when A ranges over \mathcal{C}_{b,l_n} . Now, for any $A \in \mathcal{C}_{b,l_n}$, we have

$$\begin{aligned} \mathbb{G}_n(\varphi_n(A)) &= \sqrt{n} [P_n(\varphi_n(A)) - P(\varphi_n(A))] \\ &= \frac{1}{\sqrt{n}} \sum_{i=1}^n (1_{\{\varphi_n(A)\}}(Y_i) - \mathbb{E}_P(1_{\{\varphi_n(A)\}}(Y))) \\ &= \frac{1}{\sqrt{n}} \sum_{i=1}^n (1_A(\varphi_n^{-1}(Y_i)) - \mathbb{E}_P(1_A(\varphi_n^{-1}(Y)))) \\ &:= \sqrt{\zeta_n} L_n(1_A, \varphi_n), \end{aligned}$$

where $L_n(1_A, \varphi_n)$ is defined as

$$\frac{1}{\sqrt{n\zeta_n}} \sum_{i=1}^n (1_A(\varphi_n^{-1}(Y_i)) - \mathbb{E}_P(1_A(\varphi_n^{-1}(Y))))$$

to conform with the notation of Einmahl and Mason (1997).

Conditions A(i)-A(iv) of the latter hold for $a_n = b_n = l_n$ and $a = 0$ under assumption 2, and conditions S(i)-S(iii) and F(ii) and F(iv)-F(viii) hold because \mathcal{F} is here the class of indicator functions of \mathcal{C}_{b,l_n} , hence Donsker (see for instance example 2.6.1 page 135 of van der Vaart and Wellner (1996)). Hence Theorem 1.2 of Einmahl and Mason (1997) holds, and

$$\sup_{A \in \mathcal{C}_{b,l_n}} |L_n(1_A, \varphi_n)| = O_{\text{a.s.}} \left(\sqrt{\ln \ln n} \right)$$

so that the desired result holds.

Proof of theorem 2 : By theorem 2.4 page 857 of Giné and Zinn (1990), the bootstrapped empirical process \mathbb{G}^* converges weakly to \mathbb{G} conditionally almost surely, so that

$$\sup_{A \in \mathcal{C}_{h_n}} \mathbb{G}_n(A) \text{ and } \sup_{A \in \mathcal{C}_{h_n}} \mathbb{G}^*(A)$$

have the same continuous limit. There remains to show that T_n and \hat{T}_n have the same limit, and that $\sup_{A \in \mathcal{C}_{n, h_n}} \mathbb{G}^*(A) = \sup_{A \in \mathcal{C}_{h_n}} \mathbb{G}^*(A)$ so that the result follows. The latter derives from the fact that \mathbb{G}^* takes at most n different values over \mathcal{C}_{h_n} which are exhausted on \mathcal{C}_{n, h_n} . We now prove the former. First, notice that $\mathcal{C}_n \subseteq \mathcal{C}$ implies $\hat{T}_n \leq T_n$.

- Case where $Y = D$ discrete. In that case, there is n_0 such that for all $n \geq n_0$, $\mathcal{C}_n = \mathcal{C}$, and the result trivially follows.
- Case where $Y = C \in \mathbb{R}^{d_y}$ has a density with respect to Lebesgue measure. By Theorem 9.14 page 291 of Villani (2003), there is existence of a one-to-one bi-measurable (i.e. both itself and its inverse are measurable) and Lipschitz (with constant 1) function $\phi : [0, 1]^{d_y} \rightarrow \mathbb{R}^{d_y}$ such that $Y = \phi(V)$ and V is distributed uniformly on $[0, 1]^{d_y}$ (ϕ is called a generalized quantile transformation).

Hence, for any set $A \in \mathcal{C}$, we can write

$$P_n(A) = \frac{1}{n} \sum_{i=1}^n 1_{\{Y_i \in A\}} = \frac{1}{n} \sum_{i=1}^n 1_{\{\phi(U_i) \in A\}} = \frac{1}{n} \sum_{i=1}^n 1_{\{U_i \in \phi^{-1}(A)\}} = \lambda_n(\phi^{-1}(A)),$$

where λ_n denotes the empirical law associated with an iid sample of uniformly distributed variables on $[0, 1]^{d_y}$.

We have $\hat{T}_n - T_n = \sup_{A \in \mathcal{C}_n} [P_n(A) - \nu(\Gamma(A))] - \sup_{A \in \mathcal{C}} [P_n(A) - \nu(\Gamma(A))]$. We show that for all $\epsilon > 0$, there is an n_0 such that for all $n > n_0$,

$$\sup_{y \in \mathbb{R}^{d_y}} \inf_{j \in \{1, \dots, n\}} \{(P_n(-\infty, Y_j] - P_n(-\infty, y]) + (\nu(\Gamma(-\infty, y))) - \nu(\Gamma(-\infty, Y_j]))\} < \epsilon$$

and we can proceed similarly for sets of the form $(-\infty, y]^c$. The proof of the latter proceeds in three steps:

- By the results stated in the two paragraphs following equation (1) page 919 of Talagrand (1994), we have for any $\eta > 0$

$$\sup_{y \in [0, 1]^{d_y}} \min_{j \in \{1, \dots, n\}} \|v - V_j\| = O_{\text{a.s.}}(n^{\eta-1/\max(2, d_y)}).$$

Since ϕ is Lipschitz, the latter implies that

$$\sup_{y \in \mathbb{R}^{d_y}} \min_{j \in \{1, \dots, n\}} \|y - Y_j\| = O_{\text{a.s.}}(n^{\eta-1/\max(2, d_y)}).$$

- Consider the mapping $y \mapsto j(y)$ which achieves the minimum of $\|y - Y_{j(y)}\|$.
B assumption 4, we have for n large enough, $\sup_{y \in \mathbb{R}^{d_y}} (\nu(\Gamma((-\infty, Y_{j(y)}))) - \nu(\Gamma((-\infty, y)))) < \epsilon/2$.
- We have $\sup_{y \in \mathbb{R}^{d_y}} (P(-\infty, y) - P(-\infty, Y_{j(y)})) < \epsilon/4$, since the set $(-\infty, y) \setminus (-\infty, Y_{j(y)})$ shrinks uniformly.
- By Theorem 2.3 page 367 of Stute (1984), we have $\sup_{A \in \mathcal{C}} (P_n(A) - P(A)) < \epsilon/4$ for n large enough, and the result follows.

Proof of theorem 3 : Under \mathbb{H}_a , there is a set A_0 in \mathcal{C} such that $P(A_0) > \nu(\Gamma(A_0))$.
Now the test statistic is

$$\begin{aligned}
T_n &= \sqrt{n} \sup_{A \in \mathcal{C}} [P_n(A) - \nu(\Gamma(A))] \\
&= \sup_{A \in \mathcal{C}} [\mathbb{G}_n(A) + \sqrt{n}(P(A) - \nu(\Gamma(A)))] \\
&\geq \mathbb{G}_n(A_0) + \sqrt{n}[P(A_0) - \nu(\Gamma(A_0))].
\end{aligned} \tag{8}$$

Hence,

$$\begin{aligned}
T_n - \tilde{T}_n &= \sqrt{n} \sup_{A \in \mathcal{C}} [P_n(A) - \nu(\Gamma(A))] - \sup_{A \in \mathcal{C}_{h_n}} \mathbb{G}_n(A) \\
&\geq \sqrt{n} \sup_{A \in \mathcal{C}} [P_n(A) - \nu(\Gamma(A))] - \sup_{A \in \mathcal{C}} \mathbb{G}_n(A) \\
&\geq \mathbb{G}_n(A_0) + \sqrt{n}[P(A_0) - \nu(\Gamma(A_0))] - \sup_{A \in \mathcal{C}} \mathbb{G}_n(A),
\end{aligned}$$

where the first inequality follows from the fact that $\mathcal{C}_{h_n} \subseteq \mathcal{C}$, and the second inequality follows from (8). Since $P(A_0) > \nu(\Gamma(A_0))$, we have $\sqrt{n}[P(A_0) - \nu(\Gamma(A_0))] \rightarrow \infty$. Hence, since $\mathbb{G}_n(A_0) - \sup_{A \in \mathcal{C}} \mathbb{G}_n(A)$ is a tight sequence (this can be derived for instance from exponential bounds in 2.14.9 and 2.14.10 page 246 of van der Vaart and Wellner (1996)), we have $\mathbb{P}(T_n \geq c_n^\alpha) \rightarrow 1$ for all $\alpha > 0$.

References

- ACKERBERG, D., L. BENKARD, S. BERRY, and A. PAKES (2007): “Econometric tools for analyzing market outcomes,” *Handbook of Econometrics*, Volume 6A.
- ANDERSON, T., and H. RUBIN (1949): “Estimation of the parameters of a single equation in a complete system of stochastic equations,” *Annals of Mathematical Statistics*, 20, 46–63.

- ANDREWS, D. (1988): “Chi-squared diagnostic tests for econometric models,” *Econometrica*, 56, 1419–1453.
- ANDREWS, D. (1997): “A conditional Kolmogorov test,” *Econometrica*, 65, 1097–1128.
- ANDREWS, D. (2000): “Inconsistency of the bootstrap when a parameter is on the boundary of the parameter space,” *Econometrica*, 68, 399–405.
- ANDREWS, D., S. BERRY, and P. JIA (2003): “Placing bounds on parameters of entry games in the presence of multiple equilibria,” unpublished manuscript.
- ANDREWS, D., and P. GUGGENBERGER (2006): “The Limit of Finite Sample Size and a Problem with Subsampling,” unpublished manuscript.
- BERESTEANU, A., and F. MOLINARI (2007): “Asymptotic properties for a class of partially identified models,” forthcoming in *Econometrica*.
- BLUNDELL, R., M. BROWNING, and I. CRAWFORD (2005): “Best nonparametric bounds on demand responses,” unpublished manuscript.
- BROCK, B., and S. DURLAUF (2005): “Identification of binary choice models with social interactions,” unpublished manuscript.
- BUGNI, F. (2007): “Bootstrap methods for some Partially Identified Models,” unpublished manuscript.
- CHEN, X., H. HONG, and E. TAMER (2005): “Measurement error models with auxiliary data,” *Review of Economic Studies*, 22, 343–366.
- CHERNOZHUKOV, V., H. HONG, and E. TAMER (2007): “Estimation and Confidence Regions for Parameter Sets in Econometric Models,” *Econometrica*, 75, 1243–1285.
- CILIBERTO, F., and E. TAMER (2006): “Market structure and multiple equilibria in airline markets,” unpublished manuscript.
- CROSS, P., and C. MANSKI (2002): “Regressions, short and long,” *Econometrica*, 70, 357–368.
- DUDLEY, R. (2002): *Real Analysis and Probability*. Cambridge University Press.
- EINMAHL, U., and D. MASON (1997): “Gaussian approximation of local empirical processes indexed by functions,” *Probability Theory and Related Fields*, 107, 283–311.

- GALICHON, A., and M. HENRY (2006a): “Dilation Bootstrap. A methodology for constructing confidence regions with partially identified models,” unpublished manuscript.
- GALICHON, A., and M. HENRY (2006b): “Inference in incomplete models,” Columbia University Discussion Paper 0506-28 available at <http://www.columbia.edu/cu/economics/discpapr/DP0506-28.pdf>.
- GALICHON, A., and M. HENRY (2008): “Universal power of Kolmogorov-Smirnov tests of under-identifying restrictions,” unpublished manuscript.
- GINÉ, E., and S. ZINN (1990): “Bootstrapping general empirical measures,” *Annals of Probability*, 18, 851–859.
- HECKMAN, J., J. SMITH, and N. CLEMENTS (1997): “Making the most out of programme evaluation and social experiments: accounting for heterogeneity in programme impacts,” *Review of Economic Studies*, 64, 487–535.
- HOROWITZ, J., and C. MANSKI (1998): “Censoring of outcomes and regressors due to survey nonresponse: Identification and estimation using weights and imputations,” *Journal of Econometrics*, 84, 37–58.
- IMBENS, G., and C. MANSKI (2004): “Confidence intervals for partially identified parameters,” *Econometrica*, 72, 1845–1859.
- JOVANOVIC, B. (1989): “Observable implications of models with multiple equilibria,” *Econometrica*, 57, 1431–1437.
- KELLERER, H. (1984): “Duality theorems for marginal problems,” *Zeitschrift für Wahrscheinlichkeitstheorie und Verwandte Gebiete*, 67, 399–432.
- KOOPMANS, T., and O. REIERSOL (1950): “The identification of structural characteristics,” *Annals of Mathematical Statistics*, 21, 165–181.
- LEHMANN, E., and J. ROMANO (2005): *Testing Statistical Hypotheses*. Springer: New York.
- LINTON, O., E. MAASOUMI, and Y. WHANG (2005): “Testing for stochastic dominance under general conditions: a subsampling approach,” *Review of Economic Studies*, 71, 735–765.
- LIU, X., and Y. SHAO (2003): “Asymptotics for likelihood ratio tests under loss of identifiability,” *Annals of Statistics*, 31, 807–832.

- MAGNAC, T., and E. MAURIN (2005): “Partial identification in monotone binary models: discrete regressors and interval data,” unpublished manuscript.
- MANSKI, C. (1990): “Nonparametric bounds on treatment effects,” *American Economic Review*, 80, 319–323.
- MANSKI, C. (2004): “Social learning from private experiences: the dynamics of the selection problem,” *Review of Economic Studies*, 71, 443–458.
- MANSKI, C. (2005): “Partial identification in econometrics,” *New Palgrave Dictionary of Economics*, 2nd Edition.
- MARSCHAK, J., and W. ANDREWS (1944): “Random simultaneous equations and the theory of production,” *Econometrica*, 12, 143–203.
- MATZKIN, R. (1994): “Restrictions of economic theory in nonparametric methods,” in *Handbook of Econometrics*, vol 4, R. F. Engel and D. L. McFadden, eds., pp. 1–16. North Holland.
- MCFADDEN, D. (1989): “Testing for stochastic dominance,” in *Studies in the Economics of Uncertainty (in honor of J. Hadar)*, Part II, T. Fomby and T. Seo, eds., pp. 113–134. Springer-Verlag: New York.
- MOLCHANOV, I. (2005): *Theory of Random Sets*. Springer: New York.
- MOLINARI, F. (2003): “Contaminated, corrupted and missing data,” Northwestern University Ph.D.
- MONGE, G. (1781): *Mémoire sur la théorie des déblais et des remblais*. Académie Royale des Sciences de Paris.
- PAKES, A., J. PORTER, K. HO, and J. ISHII (2004): “Moment inequalities and their application,” unpublished manuscript.
- ROEHRIG, C. (1988): “Conditions for identification in parametric and nonparametric models,” *Econometrica*, 56, 433–447.
- ROMANO, J., and A. SHAIKH (2006a): “Inference for identifiable parameters in partially identified econometric models,” forthcoming in the *Journal of Statistical Planning and Inference*.
- ROMANO, J., and A. SHAIKH (2006b): “Inference for the identified set in partially identified econometric models,” unpublished manuscript.

- ROSEN, A. (2006): “Confidence sets for partially identified parameters that satisfy a finite number of moment inequalities,” unpublished manuscript.
- SEN, P., and M. SILVAPULLE (2004): *Constrained Statistical Inference: Inequality, Order and Shape Restrictions*. Wiley-Interscience: New York.
- STUTE, W. (1984): “The Oscillation behaviour of empirical processes: the multivariate case,” *Annals of Probability*, 12, 361–379.
- TALAGRAND, M. (1994): “The transportation cost from the uniform measure to the empirical measure in dimension greater or equal to three,” *Annals of Probability*, 22, 919–959.
- TAMER, E. (2003): “Incomplete simultaneous discrete response model with multiple equilibria,” *Review of Economic Studies*, 70, 147–165.
- TINBERGEN, J. (1951): “Some remarks on the distribution of labour incomes,” in *International Economic Papers 1: Translations prepared for the economic association*, Eds.: Alan T. Peacock et al., pp. 95–207. London: Macmillan.
- VAN DER VAART, A. (1998): *Asymptotic Statistics*. Cambridge University Press.
- VAN DER VAART, A., and J. WELLNER (1996): *Weak Convergence and Empirical Processes*. New York: Springer.
- VILLANI, C. (2003): *Topics in Optimal Transportation*. Providence: American Mathematical Society.