# DIVERGENCE-FREE AND CURL-FREE WAVELETS ON THE SQUARE FOR NUMERICAL SIMULATIONS

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# DIVERGENCE-FREE AND CURL-FREE WAVELETS ON THE SQUARE FOR NUMERICAL SIMULATIONS

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We present a construction of divergence-free and curl-free wavelets on the square, that could satisfy suitable boundary conditions. This construction is based on the existence of biorthogonal multiresolution analyses (BMRA) on $[0, 1]$, linked by differentiation and integration. We introduce new BMRAs and wavelets for the spaces of divergence-free and curl-free vector functions on the square. The interest of such constructions is illustrated on examples including the Helmholtz-Hodge decomposition of vector flows and the Stokes problem.

*Keywords*: Divergence-free and curl-free wavelets ; Helmholtz-Hodge decomposition ; Stokes problem

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## 1. Introduction

In many physical problems, like the numerical simulation of incompressible flows or in electromagnetism, the solution has to fulfill a divergence-free condition. For the numerical treatment of the relevant equations (Navier-Stokes equation in fluid mechanism or Maxwell's equation in electromagnetism) it is helpful to have at hand bases satisfying a divergence-free or a curl-free condition. In the context of solution schemes for Partial Differential Equations, wavelet bases provide very efficient algorithms, characterized by a reduced computational complexity, with respect to standard methods[7]. Divergence-free wavelet bases on $\mathbb{R}^d$, with compact support, were originally defined by Lemarié-Rieusset in 1992[20] and applied by Urban to the numerical solution of

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the Stokes-problem[25]. In the periodic case, anisotropic divergence-free wavelets have been constructed in[15], and firstly used to compute the numerical solution of the Navier-Stokes equation in velocity-pressure formulation[14]. Such numerical scheme requires, at each time-step, the Helmholtz-Hodge decomposition of the nonlinear term, which is no more divergence-free. In Fourier space, this decomposition writes explicitly, whereas in wavelet domain, it can be computed using divergence-free and curl-free wavelets [26,13]. In the general case with physical boundary conditions, it is the key of Navier-Stokes numerical simulations to have at hand an explicit and efficient procedure to compute the Helmholtz-Hodge decomposition of the nonlinear term.

Precisely, the Helmholtz-Hodge decomposition of a vector field $\mathbf{u}$ on the square $\Omega = [0,1]^2$ consists in splitting $\mathbf{u}$ into a divergence-free part and a curl-free part[16]. A first formulation leading to an orthogonal splitting is the following : there exist a stream function $\psi$ and a potential field $q$ such that :

$$\begin{cases} \mathbf{u} = \mathbf{curl}\ \psi + \nabla q \text{ in } \Omega \\ \mathbf{curl}\ \psi \cdot \vec{\nu} = 0 \qquad \text{on } \Gamma = \partial\Omega \end{cases} \tag{1.1}$$

where $\vec{\nu}$ is the outward normal to $\Gamma$. This decomposition corresponds to the orthogonal splitting of the space $(L^2(\Omega))^2$ :

$$(L^2(\Omega))^2 = \mathcal{H}_{div,\Gamma}(\Omega) \oplus \mathcal{H}^{\perp}_{div}(\Omega) \tag{1.2}$$

where

$$\mathcal{H}_{div,\Gamma}(\Omega) = \{\mathbf{u} \in (L^2(\Omega))^2 : \mathbf{div}(\mathbf{u}) = 0,\ \mathbf{u} \cdot \vec{\nu}|_{\Gamma} = 0\}$$

is the divergence-free function space with velocity tangent to the boundary. It can also be seen as the "curl" space :

$$\mathcal{H}_{div}(\Omega) = \{\mathbf{u} = \mathbf{curl}\ \psi\ ;\ \psi \in H^1_0(\Omega)\}$$

On the other side the space of gradient functions

$$\mathcal{H}^{\perp}_{div}(\Omega) = \{\nabla q\ ;\ q \in H^1(\Omega)\}$$

corresponds to a curl-free function space[16].

Other types of boundary conditions for the divergence-free space can be considered. For instance, the decomposition :

$$(L^2(\Omega))^2 = \mathcal{H}_{div}(\Omega) \oplus \{\nabla q\ ;\ q \in H^1_0(\Omega)\} \tag{1.3}$$

where now

$$\mathcal{H}_{div}(\Omega) = \{\mathbf{u} \in (L^2(\Omega))^2 : \mathbf{div}(\mathbf{u}) = 0\} = \{\mathbf{u} = \mathbf{curl}\ \psi\ ;\ \psi \in H^1(\Omega)\}$$

does not incorporate boundary condition on $\Gamma$. In fluid mechanism, this type of boundary condition is less considered since in general $\Gamma$ corresponds to a physical wall that cannot be crossed by fluid particles (except for porous media). A more useful condition corresponds to the homogeneous Dirichlet boundary condition on $\Gamma$, which leads to :

$$(H_0^1(\Omega))^2 = \mathcal{H}_{div,0}(\Omega) \oplus \mathcal{H}_{div,0}^{\perp}(\Omega) \tag{1.4}$$

where now

$$\mathcal{H}_{div,0}(\Omega) = \mathcal{H}_{div}(\Omega) \cap (H_0^1(\Omega))^2$$

while $\mathcal{H}_{div,0}^{\perp}(\Omega)$ is a subspace of $\mathcal{H}_{div}^{\perp}(\Omega) \cap (H_0^1(\Omega))^2$, see[16] for details. For sake of simplicity, we will focus in this article to the divergence-free spaces being involved in decompositions (1.3,1.4).

Accordingly, the objective of the present paper is to provide multiresolution analyses and wavelet bases of the spaces $\mathcal{H}_{div}(\Omega)$ and $\mathcal{H}_{div}^{\perp}(\Omega)$. We present in the next section a new construction, based on wavelets on the interval $[0, 1]$ that can satisfy homogeneous boundary conditions, as in [21]. Indeed, the key to our construction lies on the definition of a couple of wavelet bases on the interval, linked by differentiation, as in the theoretical approach of Jouini-Lemarié-Rieusset[18]. The construction of divergence-free and curl-free approximation spaces and wavelets satisfying suitable boundary conditions are then straightforward. Moreover, our method extends readily to the cube $[0.1]^d$ by tensor product [22].

The outline of the paper is as follows. Section 2 details the principles of the construction of divergence-free and curl-free BMRAs and wavelets on the square. Section 3 is dedicated to the description of the divergence-free fast wavelet transform. Finally, section 4 illustrates applications of these new wavelets for numerical simulations : nonlinear approximation of a divergence-free vector field, the Helmholtz decomposition and the Stokes problem.

## 2. Divergence-free and curl-free wavelets on the square

Divergence-free wavelets on the whole space $\mathbb{R}^d$ have been firstly constructed by Battle-Federbush [2] in the orthogonal case. Since these

previous functions don't have compact support they where not implemented, contrarily to the biorthogonal bases arising from the construction proposed by Lemarié-Rieusset in [20]. Urban was the first who used them in a practical framework, the Stokes problem [25]. Later, Urban proposed to extend this construction to derive curl-free wavelets [26]. An alternative fast decomposition into divergence-free wavelets was proposed by Deriaz-Perrier, based on anisotropic (tensor-product) wavelets in the periodic case [15]. The objective below is to extend these constructions to the square $[0,1]^2$, following the construction principle already exposed in the theoretical work of Jouini-Lemarié-Rieusset [18].

### 2.1. *Construction principle*

The construction of divergence-free wavelets on the cube $[0,1]^d$ uses the same arguments as in the whole domain $\mathbb{R}^d$ [20,18]. The key ingredient is to have at hand two one-dimensional multiresolution analyses $(V_j^1)$ and $(V_j^0)$ of $L^2(0,1)$ linked by differentiation :

$$\frac{d}{dx} V_j^1 = V_j^0 \tag{2.1}$$

On the interval $[0,1]$, following [18], the biorthogonal spaces should satisfy :

$$\tilde{V}_j^0 = H_0^1(0,1) \cap \int_0^x \tilde{V}_j^1 = \Big\{ f \ : \ f' \in \tilde{V}_j^1 \text{ and } \ f(0) = f(1) = 0 \Big\} \tag{2.2}$$

The existence of such spaces follows from the fundamental proposition of Lemarié-Rieusset, used at the beginning to construct divergence-free wavelets on the whole space $\mathbb{R}^d$ [20] :

**Proposition 2.1.**
*Let $(V_j^1(\mathbb{R}))$ be a multiresolution analysis (MRA) of $L^2(\mathbb{R})$, with differentiable and compactly supported scaling function $\varphi^1$ and associated wavelet $\psi^1$. Then there exists a MRA $(V_j^0(\mathbb{R}))$, with associated scaling function $\varphi^0$ and wavelet $\psi^0$, such that :*

$$(\varphi^1)'(x) \ = \ \varphi^0(x) \ - \ \varphi^0(x-1) \qquad \text{and} \qquad (\psi^1)'(x) \ = \ 4 \ \psi^0(x) \tag{2.3}$$

*Similar relations hold for the dual functions $\left(\tilde{\varphi}^1, \tilde{\psi}^1\right)$ and $\left(\tilde{\varphi}^0, \tilde{\psi}^0\right)$ of the primal ones $\left(\varphi^1, \psi^1\right)$ and $\left(\varphi^0, \psi^0\right)$ :*

$$\int_x^{x+1} \tilde{\varphi}^1(t) \ dt = \tilde{\varphi}^0(x) \qquad \text{and} \qquad (\tilde{\psi}^0)'(x) \ = \ -4 \ \tilde{\psi}^1(x) \tag{2.4}$$

In [18], Jouini-Lemarié-Rieusset prove that from the scaling functions $(\varphi^1, \tilde{\varphi}^1)$, $(\varphi^0, \tilde{\varphi}^0)$ and wavelets $(\psi^1, \tilde{\psi}^1)$, $(\psi^0, \tilde{\psi}^0)$ of proposition 2.1, it is possible to exhibit multiresolution analyses on the interval satisfying the relations (2.1,2.2).

Our objective in the present paper is first to provide an effective construction of such multiresolution analyses, which enable boundary conditions, fast wavelet algorithms, approximation results and practical computations. This is done in section 2.2.

Then the construction of biorthogonal MRAs and wavelets bases of $\mathcal{H}_{div}(\Omega)$ (with suitable boundary conditions) and $\mathcal{H}^{\perp}_{div}(\Omega)$, are obtained by considering resp. the **curl** of $(V_j^1 \otimes V_j^1)$, and the **grad** of $(V_j^1 \otimes V_j^1)$. This will be described in section 2.3.

### 2.2. *Construction of biorthogonal MRAs on* $[0,1]$ *linked by differentiation*

We detail now the construction of spaces $(V_j^1, \tilde{V}_j^1)$ and $(V_j^0, \tilde{V}_j^0)$ satisfying (2.1) and (2.2). We proceed in two steps. The first step (construction of $(V_j^1, \tilde{V}_j^1)$, section 2.2.1) is classical and based on biorthogonal multiresolution analyses on the interval reproducing polynomials [9,11,21,17,4], but it is needed to introduce the sequels. The second step (section 2.2.2) introduces a practical and new way to provide spaces $(V_j^0, \tilde{V}_j^0)$ and associated wavelets. It is based on proposition 2.1.

#### 2.2.1. *Construction of* $(V_j^1, \tilde{V}_j^1)$ *on* $[0,1]$ *with polynomial reproduction* $(r, \tilde{r})$

We first recall the definition of a biorthogonal multiresolution analysis (BMRA) on $[0,1]$ [18,8] :

**Definition 2.1.**
The sequence $(V_j^1, \tilde{V}_j^1)$, $j \geq j_{min}$ $(j_{min} \in \mathbb{N}^*)$ is a biorthogonal MRA of approximation order $(r, \tilde{r})$ on the interval $[0,1]$ associated to the generators $(\varphi^1, \tilde{\varphi}^1)$, if it satisfies :
(i) $V_j^1 \subset V_{j+1}^1$, $\tilde{V}_j^1 \subset \tilde{V}_{j+1}^1$ and $\overline{\cup_{j \geq j_{min}} V_j^1} = \overline{\cup_{j \geq j_{min}} \tilde{V}_j^1} = L^2(0,1)$.
(ii) $V_j^{1,int} \subset V_j^1 \subset V_j^1(\mathbb{R})|_{[0,1]}$, $\tilde{V}_j^{1,int} \subset \tilde{V}_j^1 \subset \tilde{V}_j^1(\mathbb{R})|_{[0,1]}$.
(iii) $V_j^1$ and $\tilde{V}_j^1$ are finite dimensional biorthogonal spaces spanned by biorthogonal bases $\{\varphi_{j,k}^1 : \ k \in \Delta_j\}$ and $\{\tilde{\varphi}_{j,k}^1 : \ k \in \Delta_j\}$ :

$< \varphi^1_{j,k}, \tilde{\varphi}^1_{j,k'} >= \delta_{k,k'},\ \forall\ k, k' \in \Delta_j$.
(iv) $V^1_j$ and $\tilde{V}^1_j$ have respectively $r$ and $\tilde{r}$ polynomial exactness.

In point $(ii)$, $V^1_j(\mathbb{R})|_{[0,1]}$ means the restriction of $V^1_j(\mathbb{R})$-functions to the interval $[0,1]$, whereas $V^{1,int}_j$ means interior functions of $V^1_j(\mathbb{R})$, as introduced below (definition 2.2), and same for the biorthogonal spaces. The dimension $\Delta_j \approx 2^j$ of spaces $V^1_j$ and $\tilde{V}^1_j$ will be explicited later.

To construct such spaces $(V^1_j, \tilde{V}^1_j)$, as described in the numerous and now classical approaches [9,11,21,17,4], we start with generators $(\varphi^1, \tilde{\varphi}^1)$, that are biorthogonal scaling functions of a BMRA on $\mathbb{R}$. We suppose that $\varphi^1$ is compactly supported on $[n_{min}, n_{max}]$ and reproduces polynomials up to degree $r-1$ :

$$0 \leq \ell \leq r-1, \quad \frac{x^\ell}{\ell!} = \sum_{k=-\infty}^{+\infty} \tilde{p}^1_\ell(k)\ \varphi^1(x-k), \quad \forall\ x \in \mathbb{R} \tag{2.5}$$

with $\tilde{p}^1_\ell(k) = \langle \frac{x^\ell}{\ell!}, \tilde{\varphi}^1(x-k) \rangle$. We suppose also that $\tilde{\varphi}^1$ reproduces polynomials up to degree $\tilde{r}-1$ :

$$0 \leq \ell \leq \tilde{r}-1, \quad \frac{x^\ell}{\ell!} = \sum_{k=-\infty}^{+\infty} p^1_\ell(k)\ \tilde{\varphi}^1(x-k), \quad \forall\ x \in \mathbb{R} \tag{2.6}$$

with $p^1_\ell(k) = \langle \frac{x^\ell}{\ell!}, \varphi^1(x-k) \rangle$.

To define a BMRA on $[0,1]$ we first define the set of *interior scaling functions* :

**Definition 2.2.**
Let $\delta_\flat,\ \delta_\sharp \in \mathbb{N}$ be two fixed parameters. For $j \geq 0$, *interior scaling functions* of $V^1_j$ are defined as scaling functions $\varphi^1_{j,k}(x) = 2^{j/2}\varphi^1(2^j x - k)$ whose supports are included into $[\frac{\delta_\flat}{2^j}, 1 - \frac{\delta_\sharp}{2^j}] \subset [0,1]$.
If $supp\ \varphi^1 = [n_{min}, n_{max}]$, they correspond to indices $k$ such that :

$$\delta_\flat - n_{min} \leq k \leq 2^j - \delta_\sharp - n_{max}$$

The space generated by *interior scaling functions* is then given by :

$$V^{1,int}_j = span\{\varphi^1_{j,k}\ ;\ k = k_\flat,\ 2^j - k_\sharp\}$$

with $k_\flat = \delta_\flat - n_{min}$ and $k_\sharp = \delta_\sharp + n_{max}$.

Similarly, let $\tilde{\delta}_\flat,\ \tilde{\delta}_\sharp \in \mathbb{N}$ be two parameters. Interior scaling functions of $\tilde{V}^1_j$ are defined as scaling functions $\tilde{\varphi}^1_{j,k}(x) = 2^{j/2}\tilde{\varphi}^1(2^j x - k)$ whose

supports are included into $[\frac{\tilde{\delta}_\flat}{2^j}, 1-\frac{\tilde{\delta}_\sharp}{2^j}]$. The space generated by *interior scaling functions* is then given by :

$$\tilde{V}_j^{1,int} = span\{\tilde{\varphi}_{j,k}^1 \ ; \ k=\tilde{k}_\flat, \ 2^j-\tilde{k}_\sharp\}$$

with $\tilde{k}_\flat = \tilde{\delta}_\flat - \tilde{n}_{min}$ and $\tilde{k}_\sharp = \tilde{\delta}_\sharp + \tilde{n}_{max}$, if the support of $\tilde{\varphi}^1$ is $[\tilde{n}_{min}, \tilde{n}_{max}]$.

**Remark 2.1.**
The parameters $(\delta_\flat, \ \delta_\sharp, \ \tilde{\delta}_\flat, \ \tilde{\delta}_\sharp)$ are "free" parameters (chosen as small as possible), and chosen in practice to adjust the dimension of the spaces $V_j^1$ and $\tilde{V}_j^1$.

To preserve the polynomial reproduction (2.5, 2.6) on the interval $[0,1]$, we follow the approach of [21,4] and define edge scaling functions at the edge 0 :

**Definition 2.3.**
The *edge scaling functions* at the edge 0 are defined by :

$$0 \le \ell \le r-1, \qquad \Phi_\ell^{1,\flat}(x) = \sum_{k=1-n_{max}}^{k_\flat-1} \tilde{p}_\ell^1(k) \ \varphi_k^1(x) \ \chi_{[0,+\infty[}$$

and for the biorthogonal space :

$$0 \le \ell \le \tilde{r}-1, \qquad \tilde{\Phi}_\ell^{1,\flat}(x) = \sum_{k=1-\tilde{n}_{max}}^{\tilde{k}_\flat-1} p_\ell^1(k) \ \tilde{\varphi}_k^1(x) \ \chi_{[0,+\infty[}$$

At the edge 1, the edge scaling functions $\Phi_{j,\ell}^{1,\sharp}$ are constructed on $]-\infty,1]$ by symmetry, using the transform $Tf(x) = f(1-x)$.

As usual, one define the multiresolution spaces $V_j^1$ on $[0,1]$, by the direct sum :

$$V_j^1 = V_j^{1,\flat} \oplus V_j^{1,int} \oplus V_j^{1,\sharp} \tag{2.7}$$

where :

$$V_j^{1,\flat} = span\{\Phi_{j,\ell}^{1,\flat}(x) = 2^{j/2}\Phi_\ell^{1,\flat}(2^j x) \ ; \ \ell = 0, \cdots, r-1\}$$
$$V_j^{1,\sharp} = span\{\Phi_{j,\ell}^{1,\sharp}(1-x) = 2^{j/2}\Phi_\ell^{1,\sharp}(2^j(1-x)) \ ; \ \ell = 0, \cdots, r-1\}$$

are the *edge spaces*. In practice we have to choose $j \ge j_{min}$ where $j_{min}$ is the smallest integer which verifies :

$$j_{min} > \log_2[n_{max} - n_{min} + \delta_\sharp + \delta_\flat]$$

This condition ensures that the supports of edge scaling functions at 0 do not intersect the supports of edge scaling functions at 1.
The polynomial reproduction in $V_j^1$ is then satisfied since, for $0 \leq \ell \leq r-1$ and $x \in [0,1]$ we have :

$$\frac{2^{j/2}(2^j x)^\ell}{\ell!} = 2^{j/2}\Phi_\ell^{1,\flat}(2^j x) + \sum_{k=k_\flat}^{2^j-k_\sharp} \tilde{p}_\ell^1(k)\ \varphi_{j,k}^1(x) + 2^{j/2}\Phi_\ell^{1,\sharp}(2^j(1-x)) \quad (2.8)$$

Similarly, multiresolution spaces $\tilde{V}_j^1$ are defined with the same structure, allowing the polynomial reproduction up to degree $\tilde{r}-1$ :

$$\tilde{V}_j^1 = span\{\tilde{\Phi}_{j,\ell}^{1,\flat}\}_{\ell=0,\tilde{r}-1} \oplus \tilde{V}_j^{1,int} \oplus span\{\tilde{\Phi}_{j,\ell}^{1,\sharp}\}_{\ell=0,\tilde{r}-1} \quad (2.9)$$

In order to obtain the equality between dimensions of $V_j^1$ and $\tilde{V}_j^1$, we have to adjust the parameters $\tilde{\delta}_\flat = \tilde{k}_\flat - \tilde{n}_{max}$ and $\tilde{\delta}_\sharp = \tilde{k}_\sharp + \tilde{n}_{min}$ in the definition 2.2 such that :

$$k_\flat - r = \tilde{k}_\flat - \tilde{r} \qquad \text{and} \qquad k_\sharp - r = \tilde{k}_\sharp - \tilde{r} \quad (2.10)$$

which leads to :

$$\Delta_j = \dim(V_j^1) = \dim(\tilde{V}_j^1) = 2^j - (\delta_\flat + \delta_\sharp) - (n_{max} - n_{min}) + 2r + 1$$

where $(\delta_\flat,\ \delta_\sharp)$ remain "free" parameters of the construction. Like for $V_j^1$ we have to choose $j \geq \tilde{j}_{min}$ with $\tilde{j}_{min} > \log_2[\tilde{n}_{max} - \tilde{n}_{min} + \tilde{\delta}_\sharp + \tilde{\delta}_\flat]$.

The last point of the construction lies in the biorthogonalization process of the new basis functions, since edge scaling functions of $V_j^1$ and $\tilde{V}_j^1$ are not biorthogonal. Several biorthogonalization methods exist [1,11,17,21], here we apply on one hand the method proposed by Dahmen and al. [11] when using B-spline generators, and on the other hand a Gram-Schmidt process with Daubechies orthogonal generators [21]. In both cases, it requires the inversion of the Gram matrix associated with the two systems, which for orthogonal and B-Spline generators is non singular [11,21].
Finally, the spaces $(V_j^1, \tilde{V}_j^1)_{j \geq \max\{j_{min}, \tilde{j}_{min}\}}$, form a biorthogonal MRA of $L^2(0,1)$ in the sense of definition 2.1.

Moreover, homogeneous boundary conditions can be simply imposed to $V_j^1$, of the form $f^{(\lambda)}(\alpha) = 0$ at point $\alpha = 0$ or 1, with $\lambda = 0, \cdots, r-1$, by removing the edge scaling function $\Phi_\lambda^{1,\flat}$ if $\alpha = 0$ or $\Phi_\lambda^{1,\sharp}$ if $\alpha = 1$ in the definition 2.3 of edge spaces (see [21] for more details). In such case,

we also remove the function $\tilde{\Phi}^{1,\flat}_{\lambda}$ or $\tilde{\Phi}^{1,\sharp}_{\lambda}$ from $\tilde{V}^1_j$ prior to biorthogonalization, to adjust the dimension of the biorthogonal space.

### 2.2.2. *Construction of* $(V^0_j, \tilde{V}^0_j)$ *on* $[0,1]$ *linked by differentiation /integration with* $(V^1_j, \tilde{V}^1_j)$

We will now construct spaces $(V^0_j, \tilde{V}^0_j)$, related to the spaces $(V^1_j, \tilde{V}^1_j)$ of section 2.2.1 by the relations (2.1,2.2) of differentiation/integration. Given $(\varphi^1, \tilde{\varphi}^1)$ biorthogonal scaling functions with approximation orders $(r, \tilde{r})$ (with $r > 1$ and $\tilde{r} \geq r$), and compact supports $[n_{min}, n_{max}]$, $[\tilde{n}_{min}, \tilde{n}_{max}]$, we consider $(\varphi^0, \tilde{\varphi}^0)$ arising from proposition 2.1. Then $(\varphi^0, \tilde{\varphi}^0)$ satisfy some properties on $\mathbb{R}$, that we recall below.

First, the scaling functions $(\varphi^0, \tilde{\varphi}^0)$ are defined such that :

$$\frac{d}{dx}\varphi^1(x) = \varphi^0(x) - \varphi^0(x-1) \quad \text{and} \quad \int_x^{x+1} \tilde{\varphi}^1(t)dt = \tilde{\varphi}^0(x) \tag{2.11}$$

This implies that $\varphi^0$ has for compact support $[n_{min}, n_{max} - 1]$, and reproduces polynomials up to degree $r-2$ :

$$0 \leq \ell \leq r-2, \quad \frac{x^\ell}{\ell!} = \sum_{k=-\infty}^{+\infty} \tilde{p}^0_\ell(k)\ \varphi^0(x-k) \tag{2.12}$$

with $\tilde{p}^0_\ell(k) = \langle \frac{x^\ell}{\ell!}, \tilde{\varphi}^0(x-k) \rangle$.
Equation (2.11) implies : $\tilde{p}^0_\ell(k) = \tilde{p}^1_{\ell+1}(k) - \tilde{p}^1_{\ell+1}(k-1)$ for $\ell = 0, \cdots, r-2$.

In the same way $\tilde{\varphi}^0$ has for compact support $[\tilde{n}_{min} - 1, \tilde{n}_{max}]$, and reproduces polynomials up to degree $\tilde{r}$ :

$$0 \leq \ell \leq \tilde{r}, \quad \frac{x^\ell}{\ell!} = \sum_{k=-\infty}^{+\infty} p^0_\ell(k)\ \tilde{\varphi}^0(x-k) \tag{2.13}$$

with $p^0_\ell(k) = \langle \frac{x^\ell}{\ell!}, \varphi^0(x-k) \rangle$.
Equation (2.11) implies : $p^1_\ell(k) = p^0_{\ell+1}(k+1) - p^0_{\ell+1}(k)$ for $\ell = 1, \cdots, \tilde{r}$.

Like for $V^1_j$, we first define the set of *interior scaling functions* of $V^0_j$ :

**Definition 2.4.**
Let $\delta_\flat,\ \delta_\sharp \in \mathbb{N}$ and $k_\flat = \delta_\flat - n_{min}$ and $k_\sharp = \delta_\sharp + n_{max}$ be the parameters introduced in definition 2.2. For $j \geq 0$, the *interior scaling functions* of

$V_j^0$ are defined as scaling functions $\varphi_{j,k}^0(x) = 2^{j/2}\varphi^0(2^j x - k)$ whose supports are included into $[\frac{\delta_\flat}{2^j}, 1-\frac{\delta_\sharp}{2^j}] \subset [0,1]$. Since $supp\,\varphi^0 = [n_{min}, n_{max}-1]$, the space generated by *interior scaling functions* is given by :

$$V_j^{0,int} = span\{\varphi_{j,k}^0\ ;\ k = k_\flat,\ 2^j - k_\sharp + 1\}$$

Similarly, let $\tilde{\delta}_\flat,\ \tilde{\delta}_\sharp \in \mathbb{N}$, and $\tilde{k}_\flat = \tilde{\delta}_\flat - \tilde{n}_{min}$, $\tilde{k}_\sharp = \tilde{\delta}_\sharp + \tilde{n}_{max}$ be fixed by definition 2.2 and relation (2.10). Interior scaling functions of $\tilde{V}_j^0$ are defined as scaling functions $\tilde{\varphi}_{j,k}^0(x) = 2^{j/2}\varphi^0(2^j x - k)$ whose supports are included into $[\frac{\tilde{\delta}_\flat}{2^j}, 1 - \frac{\tilde{\delta}_\sharp}{2^j}]$. The space generated by *interior scaling functions* is then given by :

$$\tilde{V}_j^{0,int} = span\{\tilde{\varphi}_{j,k}^0\ ;\ k = \tilde{k}_\flat + 1,\ 2^j - \tilde{k}_\sharp\}$$

To preserve the polynomial reproduction (2.12,2.13) on $[0,1]$ in $(V_j^0, \tilde{V}_j^0)$, we define *edge scaling functions* at the edge 0 :

**Definition 2.5.**
The *edge scaling functions* at the edge 0 are defined by :

$$0 \le \ell \le r-2, \qquad \Phi_\ell^{0,\flat}(x) = \sum_{k=2-n_{max}}^{k_\flat - 1} \tilde{p}_\ell^0(k)\ \varphi_k^0(x)\ \chi_{[0,+\infty[}$$

and for the biorthogonal space, they must vanish at 0 :

$$1 \le \ell \le \tilde{r}, \qquad \tilde{\Phi}_\ell^{0,\flat}(x) = \sum_{k=1-\tilde{n}_{max}}^{\tilde{k}_\flat} p_\ell^0(k)\ \tilde{\varphi}_k^0(x)\ \chi_{[0,+\infty[}$$

At the edge 1, the *edge scaling functions* $\Phi_{j,\ell}^{0,\sharp}$ and $\tilde{\Phi}_{j,\ell}^{0,\sharp}$ are constructed by symmetry, using the transform $Tf(x) = f(1-x)$.

**Remark 2.2.**
Following Jouini-Lemarié-Rieusset [18], to preserve the commutation between the derivation and the multiscale projectors, the space $\tilde{V}_j^0$ must satisfy (2.2) : $\tilde{V}_j^0 \subset H_0^1(0,1)$ . Indeed we impose by construction homogeneous Dirichlet boundary conditions to $\tilde{V}_j^0$, since we do not consider $\tilde{\Phi}_0^{0,\flat}$ and and $\tilde{\Phi}_0^{0,\sharp}$ $(\ell = 0)$ in definition 2.5. Nevertheless, in the following we will need to use $\tilde{\Phi}_0^{0,\flat} = \sum_{k=1-\tilde{n}_{max}}^{\tilde{k}_\flat} \tilde{\varphi}_k^0\ \chi_{[0,+\infty[}$ in practical computation.

The multiresolution spaces $V_j^0$ on $[0,1]$ are then defined by :

$$V_j^0 = V_j^{0,\flat} \oplus V_j^{0,int} \oplus V_j^{0,\sharp}$$

where :

$$V_j^{0,\flat} = span\{\Phi_{j,\ell}^{0,\flat}(x) = 2^{j/2}\Phi_\ell^{0,\flat}(2^j x)\ ;\ \ell = 0,\cdots,r-2\}$$
$$V_j^{0,\sharp} = span\{\Phi_{j,\ell}^{0,\sharp}(1-x) = 2^{j/2}\Phi_\ell^{0,\sharp}(2^j(1-x))\ ;\ \ell = 0,\cdots,r-2\}$$

The polynomial reproduction up to degree $r-2$ in $V_j^0$ is then ensured. In practice $j > j_{min}$ where the parameter $j_{min}$ is now adapted to both BMRAs $(V_j^1, \tilde{V}_j^1)$ and $(V_j^0, \tilde{V}_j^0)$ by :

$$j_{min} > \max\{\log_2[n_{max} - n_{min} + \delta_\sharp + \delta_\flat + 1], \log_2[\tilde{n}_{max} - \tilde{n}_{min} + \tilde{\delta}_\sharp + \tilde{\delta}_\flat + 1]\}$$

Multiresolution spaces $\tilde{V}_j^0$ are defined with similar structure, allowing a polynomial reproduction up to degree $\tilde{r}$, and satisfying vanishing boundary conditions at 0 and 1.

$$\tilde{V}_j^0 = span\{\tilde{\Phi}_{j,\ell}^{0,\flat}\}_{\ell=1,\tilde{r}} \oplus \tilde{V}_j^{0,int} \oplus span\{\tilde{\Phi}_{j,\ell}^{1,\sharp}\}_{\ell=1,\tilde{r}} \tag{2.14}$$

A simple calculation shows that :

$$\dim(V_j^0) = 2^j - k_\sharp - k_\flat + 2r \quad and \quad \dim(\tilde{V}_j^0) = 2^j - \tilde{k}_\sharp - \tilde{k}_\flat + 2\tilde{r} \tag{2.15}$$

Since the parameters $k_\flat$, $k_\sharp$, $\tilde{k}_\flat$ and $\tilde{k}_\sharp$ are chosen to satisfy equation (2.10), we obtain : $\dim(V_j^0) = \dim(\tilde{V}_j^0) = \Delta_j - 1$ The following proposition proves that $\frac{d}{dx}V_j^1 = V_j^0$ and $\frac{d}{dx}\tilde{V}_j^0 \subset \tilde{V}_j^1$.

**Proposition 2.2.**
*(i) The interior scaling functions of* $(V_j^1, V_j^0)$ *and* $(\tilde{V}_j^1, \tilde{V}_j^0)$ *introduced in definitions 2.2, 2.4 satisfy :*

$$\frac{d}{dx}(\varphi_{j,k}^1) = 2^j[\varphi_{j,k}^0 - \varphi_{j,k+1}^0],\ \frac{d}{dx}(\tilde{\varphi}_{j,k}^0) = 2^j[\tilde{\varphi}_{j,k-1}^1 - \tilde{\varphi}_{j,k}^1],\ for\ k_\flat \le k \le 2^j - k_\sharp$$

*and*

$$\dim(V_j^{0,int}) = \dim(V_j^{1,int}) + 1, \qquad \dim(\tilde{V}_j^{0,int}) = \dim(\tilde{V}_j^{1,int}) - 1$$

*(ii) The edge scaling functions of* $(V_j^1, V_j^0)$ *introduced in definitions 2.3, 2.5 satisfy : for* $\ell = 1, r-1$,

$$(\Phi_0^{1,\flat})'(x) = -\varphi_{k_\flat}^0, \qquad (\Phi_\ell^{1,\flat})'(x) = \Phi_{\ell-1}^{0,\flat} - \tilde{p}_\ell^1(k_\flat - 1)\ \varphi_{k_\flat}^0$$
$$(\Phi_0^{1,\sharp})'(1-x) = \varphi_{2-k_\sharp}^0,\ (\Phi_\ell^{1,\sharp})'(1-x) = -\Phi_{\ell-1}^{0,\sharp}(1-x) + \tilde{p}_\ell^1(2-k_\sharp)\ \varphi_{2-k_\sharp}^0$$

*whereas those of* $(\tilde{V}_j^1, \tilde{V}_j^0)$ *are linked by : for* $\ell = 1, \tilde{r}-1$,

$$(\tilde{\Phi}_\ell^{0,\flat})' = \tilde{\Phi}_{\ell-1}^{1,\flat} - p_\ell^0(\tilde{k}_\flat)\ \tilde{\varphi}_{\tilde{k}_\flat}^1$$
$$(\tilde{\Phi}_\ell^{0,\sharp}(1-x))' = -\tilde{\Phi}_{\ell-1}^{1,\sharp}(1-x) + p_\ell^0(2-\tilde{k}_\sharp)\ \tilde{\varphi}_{1-\tilde{k}_\sharp}^1$$

*Moreover, the function* $\tilde{\Phi}_0^{0,\flat}$ *verifies :* $(\tilde{\Phi}_0^{0,\flat})' = -\tilde{\varphi}_{\tilde{k}_\flat}^1$

**Proof.**
The point (i) follows from (2.11) and since interior functions are defined by $\varphi^1_{j,k}(x) = 2^{j/2}\varphi^1(2^j x - k)$ (and the same for $\varphi^0_{j,k}, \tilde{\varphi}^1_{j,k}, \tilde{\varphi}^0_{j,k}$).
The point (ii) comes directly from definitions 2.3, 2.5 of edge scaling functions. We focus on the first line of equalities : at edge 0 and for $0 \leq \ell \leq r-1$, the edge scaling functions of $(V^1_j)$ are defined by dilation of :

$$\Phi^{1,\flat}_\ell(x) = \sum_{k=1-n_{max}}^{k_\flat - 1} \tilde{p}^1_\ell(k)\ \varphi^1_k(x)\chi_{[0,+\infty[} \quad \text{with} \quad \varphi^1_k(x) = \varphi^1(x-k)$$

Differentiating in $]0,+\infty[$, one obtains for $\ell = 0$ $(\tilde{p}^1_0(k) = 1, \forall k)$ :

$$(\Phi^{1,\flat}_0)' = \sum_{k=1-n_{max}}^{k_\flat - 1} (\varphi^0_k - \varphi^0_{k+1})\ \chi_{]0,+\infty[} = \varphi^0_{1-n_{max}}\ \chi_{]0,+\infty[} - \varphi^0_{k_\flat} = -\varphi^0_{k_\flat}$$

since $supp\ \varphi^0_{1-n_{max}} = [n_{min} - n_{max} + 1, 0]$.
In the same way, for $\ell = 1, r-1$ :

$$\begin{aligned}(\Phi^{1,\flat}_\ell)' &= \sum_{k=1-n_{max}}^{k_\flat - 1} \tilde{p}^1_\ell(k)\ (\varphi^0_k - \varphi^0_{k+1})\ \chi_{]0,+\infty[} \\ &= \sum_{k=2-n_{max}}^{k_\flat - 1} [\tilde{p}^1_\ell(k) - \tilde{p}^1_\ell(k-1)]\ \varphi^0_k\chi_{]0,+\infty[} - p^1_\ell(k_\flat - 1)\ \varphi^0_{k_\flat}\end{aligned}$$

From (2.4), since $\tilde{p}^0_{\ell-1}(k) = \tilde{p}^1_\ell(k) - \tilde{p}^1_\ell(k-1)$ we get :

$$(\Phi^{1,\flat}_\ell)' = \sum_{k=2-n_{max}}^{k_\flat - 1} \tilde{p}^0_{\ell-1}(k)\ \varphi^0_k\chi_{]0,+\infty[} - \tilde{p}^1_\ell(k_\flat - 1)\ \varphi^0_{k_\flat} = \Phi^{0,\flat}_{\ell-1} - \tilde{p}^1_\ell(k_\flat - 1)\ \varphi^0_{k_\flat}$$

This proves the relation between edge scaling functions at 0 of $V^1_j$ and $V^0_j$. The proof for edge scaling functions at edge 1 and in the biorthogonal spaces $\tilde{V}^1_j$ and $\tilde{V}^0_j$ is obtained with similar arguments. □

For easy reading, the two pairs of biorthogonal bases of $(V^1_j, \tilde{V}^1_j)$ and $(V^0_j, \tilde{V}^0_j)$, arising from the biorthogonalization process, will be now denoted by $(\varphi^1_{j,k}, \tilde{\varphi}^1_{j,k})_{k=1,\Delta_j}$ and $(\varphi^0_{j,k}, \tilde{\varphi}^0_{j,k})_{k=1,\Delta_j-1}$ respectively. The oblique projector on $V^1_j$ parallel to $(\tilde{V}^1_j)^\perp$ will be denoted by $\mathcal{P}^1_j$ :

$$\mathcal{P}^1_j : L^2(0,1) \to V^1_j, \quad f \mapsto \mathcal{P}^1_j(f) = \sum_{k=1}^{\Delta_j} \langle f, \tilde{\varphi}^1_{j,k}\rangle\ \varphi^1_{j,k} \tag{2.16}$$

while $\tilde{\mathcal{P}}_j^1$ will denote its adjoint, and $\mathcal{P}_j^0$, $\tilde{\mathcal{P}}_j^0$ the biorthogonal projectors associated with $(V_j^0, \tilde{V}_j^0)$.

The following proposition proves that the constructed biorthogonal MRAs take place in the theoretical framework of Jouini-Lemarié-Rieusset in [18].

**Proposition 2.3.**
*The two pairs of biorthogonal spaces* $(V_j^1, \tilde{V}_j^1)$ *and* $(V_j^0, \tilde{V}_j^0)$ *are related to :*

$$\frac{d}{dx} V_j^1 = V_j^0 \quad \textit{and} \quad \tilde{V}_j^0 = H_0^1 \cap \int_0^x \tilde{V}_j^1$$

**Proof.**
The inclusion $\frac{d}{dx} V_j^1 \subset V_j^0$ is straightforward according to proposition 2.2. The equality of dimensions between spaces leads to the first equality. Moreover proposition 2.2 implies :

$$\int_0^x \tilde{V}_j^1 = \tilde{V}_j^0 \oplus span\{2^{j/2} \tilde{\Phi}_0^{0,\flat}(2^j x) - \tilde{\Phi}_0^{0,\flat}(0)\}$$

Since for $j \geq j_{min}$, $\tilde{V}_j^0 \subset H_0^1(0,1)$ and $2^{j/2}\Phi_0^{0,\flat}(2^j) - \Phi_0^{0,\flat}(0) = -\Phi_0^{0,\flat}(0) \neq 0$ we obtain the second equality. □

We then define the change of bases between the spaces $(\frac{d}{dx} V_j^1, \frac{d}{dx} \tilde{V}_j^0)$ and $(V_j^0, \tilde{V}_j^1)$ as follows.

**Definition 2.6.**
Let $(L_j^1, L_j^0)$ and $(\tilde{L}_j^1, \tilde{L}_j^0)$ be the two pairs of sparse matrices defined by the change of bases between spaces involved in proposition 2.3 :

$$\frac{d}{dx} \varphi_{j,k}^1 = \sum_{n=1}^{\Delta_j - 1} (L_j^1)_{k,n} \ \varphi_{j,n}^0, \qquad \frac{d}{dx} \tilde{\varphi}_{j,k}^0 = \sum_{n=1}^{\Delta_j} (\tilde{L}_j^0)_{k,n} \ \tilde{\varphi}_{j,n}^1 \tag{2.17}$$

and

$$-\int_0^x \varphi_{j,k}^0 = \sum_{m=1}^{\Delta_j} (L_j^0)_{k,m} \ \varphi_{j,m}^1 \tag{2.18}$$

$$-\int_0^x \tilde{\varphi}_{j,k}^1 = a_{j,k} \left( 2^{j/2} \tilde{\Phi}_0^{0,\flat}(2^j x) - \tilde{\Phi}_0^{0,\flat}(0) \right) + \sum_{m=1}^{\Delta_j - 1} (\tilde{L}_j^1)_{k,m} \ \tilde{\varphi}_{j,m}^0 \tag{2.19}$$

**Remark 2.3.**
The matrices $L_j^0$ and $\tilde{L}_j^0$ are rectangular of size $(\Delta_j - 1) \times \Delta_j$. The biorthogonality between $V_j^0$ and $\tilde{V}_j^0$, and the inclusion $\tilde{V}_j^0 \subset H_0^1(0,1)$ lead to :

$$L_j^0(\tilde{L}_j^0)^T = \tilde{L}_j^0(L_j^0)^T = I_{(\Delta_j-1)}$$

where $I_{(\Delta_j-1)}$ denotes the matrix identity of size $(\Delta_j-1)$. These matrices are useful in numerical computations, their explicit form will be given in section 3.

With this definition, we prove the commutation between multiscale projectors and differentiation.

**Proposition 2.4.**
*Let* $(\mathcal{P}_j^1, \tilde{\mathcal{P}}_j^1)$ *and* $(\mathcal{P}_j^0, \tilde{\mathcal{P}}_j^0)$ *be the multiscale projectors defined by* (2.16) *:*

$(i)$ $\forall\ f \in H^1(0,1), \quad \frac{d}{dx} \circ \mathcal{P}_j^1 f = \mathcal{P}_j^0 \circ \frac{d}{dx} f.$

$(ii)$ $\forall\ f \in H_0^1(0,1), \quad \frac{d}{dx} \circ \tilde{\mathcal{P}}_j^0 f = \tilde{\mathcal{P}}_j^1 \circ \frac{d}{dx} f.$

**Proof.**
$(i)$ Since proposition 2.3 is valid, the relation of commutation $\frac{d}{dx} \circ \mathcal{P}_j^1 f = \mathcal{P}_j^0 \circ \frac{d}{dx} f$ follows from [18] in a general setting.
$(ii)$ Let $(\varphi_{j,k}^0, \tilde{\varphi}_{j,k}^0)$ be the biorthogonal scaling functions of $(V_j^0, \tilde{V}_j^0)$. For $f \in H_0^1(0,1)$ we have $\langle f, \varphi_{j,k}^0 \rangle = -\langle \frac{d}{dx} f, \int_0^x \varphi_{j,k}^0 \rangle$, then :

$$\begin{aligned}\frac{d}{dx}\tilde{\mathcal{P}}_j^0(f) &= \sum_k \langle f, \varphi_{j,k}^0 \rangle \frac{d}{dx}\tilde{\varphi}_{j,k}^0 = \sum_n \sum_m \sum_k (L_j^0)_{k,m} (\tilde{L}_j^0)_{k,n} \langle \frac{d}{dx} f, \varphi_{j,m}^1 \rangle\ \tilde{\varphi}_{j,n}^1 \\ &= \sum_n \sum_m \delta_{m,n} \langle \frac{d}{dx} f, \varphi_{j,m}^1 \rangle\ \tilde{\varphi}_{j,n}^1 = \sum_n \langle \frac{d}{dx} f, \varphi_{j,n}^1 \rangle\ \tilde{\varphi}_{j,n}^1 = \tilde{\mathcal{P}}_j^1(\frac{d}{dx} f)\end{aligned}$$

where $L_j^0$ and $\tilde{L}_j^0$ have been introduced in definition 2.6. This ends the proof. □

2.2.3. *Wavelet spaces*

We begin with the construction of wavelet bases of the biorthogonal MRA $(V_j^1, \tilde{V}_j^1)$. This point is classical, although different kinds of wavelets may be designed [1,9,11,21,17,4]. For $j \geq j_{min}$, the biorthogonal wavelet spaces associated to $V_j^1$ and $\tilde{V}_j^1$ are in all cases defined by :

$$W_j^1 = V_{j+1}^1 \cap (\tilde{V}_j^1)^\perp \quad \text{and} \quad \tilde{W}_j^1 = \tilde{V}_{j+1}^1 \cap (V_j^1)^\perp$$

The wavelet space $W_j^1$ has the following structure :

$$W_j^1 = W_j^{1,\flat} \oplus W_j^{1,int} \oplus W_j^{1,\sharp}$$

where

$$\begin{cases} W_j^{1,\flat} &= span\{\Psi_{j,\ell}^{1,\flat}(x) = 2^{j/2}\Psi_\ell^{1,\flat}(2^j x)\ ;\ \ell = 0,\ p_\flat - 1\} \\ W_j^{1,int} &= span\{\psi_{j,k}^1 = 2^{j/2}\psi^1(2^j x - k)\ ;\ k = p_\flat,\ 2^j - p_\sharp - 1\} \\ W_j^{1,\sharp} &= span\{\Psi_{j,\ell}^{1,\sharp}(1-x) = 2^{j/2}\Psi_\ell^{1,\sharp}(2^j(1-x))\ ;\ \ell = 0,\ p_\sharp - 1\} \end{cases} \tag{2.20}$$

$p_\flat$ and $p_\sharp$ introduced above are suitable integers to ensure that the support of each *interior wavelet* $\psi_{j,k}^1(x)$ of $W_j^{1,int}$ is included into $[\frac{\delta_\flat}{2^{j+1}}, 1 - \frac{\delta_\sharp}{2^{j+1}}]$. Recall that the support of $\psi^1$ (wavelet on $\mathbb{R}$ introduced in proposition 2.1) is $[\frac{n_{min} - \tilde{n}_{max}+1}{2}, \frac{n_{max} - \tilde{n}_{min}+1}{2}]$, then we deduce :

$$p_\flat = \lfloor \frac{\tilde{n}_{max} + k_\flat - 1}{2} \rfloor \quad \text{and} \quad p_\sharp = \lfloor \frac{k_\sharp - \tilde{n}_{min} + 1}{2} \rfloor$$

with $k_\flat = \delta_\flat - n_{min}$ and $k_\sharp = \delta_\sharp + n_{max}$. This construction of *edge wavelets* $\Psi_{j,\ell}^{1,\flat}$ and $\Psi_{j,\ell}^{1,\sharp}$ is borrowed from the work of Grivet-Talocia and Tabacco [17]. The biorthogonal spaces $\tilde{W}_j^1$ are constructed with the same structure, finally the wavelet bases of the two spaces must to be biorthogonalized identically as the scaling functions. The resulting wavelet bases are denoted by $\{\psi_{j,k}^1\}_{k=1,2^j}$ and $\{\tilde{\psi}_{j,k}^1\}_{k=1,2^j}$ without distinction.

Such construction leads to fast wavelet transforms, since the scaling functions and wavelets satisfy both a two-scale relations. Indeed, there exist sparse matrices $H_j^1$, $\tilde{H}_j^1$, $G_j^1$ and $\tilde{G}_j^1$ such that :

$$\varphi_{j,k}^1 = \sum_n (H_j^1)_{k,n}\varphi_{j+1,n}^1 \qquad \text{and} \qquad \psi_{j,k}^1 = \sum_n (G_j^1)_{k,n}\varphi_{j+1,n}^1$$

$$\tilde{\varphi}_{j,k}^1 = \sum_n (\tilde{H}_j^1)_{k,n}\tilde{\varphi}_{j+1,n}^1 \qquad \text{and} \qquad \tilde{\psi}_{j,k}^1 = \sum_n (\tilde{G}_j^1)_{k,n}\tilde{\varphi}_{j+1,n}^1$$

The main objective of this section is now to construct biorthogonal wavelet bases of $W_j^0$ and $\tilde{W}_j^0$, that will be linked to $\psi_{j,k}^1$ and $\tilde{\psi}_{j,k}^1$ by differentiation/integration. A first result in this direction is given by the following proposition :

**Proposition 2.5.**
*Let* $(V_j^1, \tilde{V}_j^1)$ *and* $(V_j^0, \tilde{V}_j^0)$ *be two biorthogonal multiresolution analyses*

*satisfying proposition 2.3.*
*The wavelet spaces* $W_j^0$ *and* $\tilde{W}_j^0$ *defined by :*

$$W_j^0 = V_{j+1}^0 \cap (\tilde{V}_j^0)^\perp \quad \text{and} \quad \tilde{W}_j^0 = \tilde{V}_{j+1}^0 \cap (V_j^0)^\perp$$

*are linked to the biorthogonal wavelet spaces associated to* $(V_j^1, \tilde{V}_j^1)$ *by differentiation and integration :*

$$W_j^0 = \frac{d}{dx} W_j^1 \quad \text{and} \quad \tilde{W}_j^0 = \int_0^x \tilde{W}_j^1 \tag{2.21}$$

**Proof.**
- First we prove the relation : $\frac{d}{dx} W_j^1 = W_j^0$. Since $V_{j+1}^1 = V_j^1 \oplus W_j^1$, we obtain :

$$\frac{d}{dx} V_{j+1}^1 = \frac{d}{dx} V_j^1 + \frac{d}{dx} W_j^1$$

By proposition 2.3, we have :

$$V_{j+1}^0 = \frac{d}{dx} V_{j+1}^1 \quad \text{and} \quad V_j^0 = \frac{d}{dx} V_j^1$$

Moreover $\frac{d}{dx} W_j^1 \subset (\tilde{V}_j^0)^\perp$. Indeed, let $w_j^1 \in W_j^1$. Proposition 2.4 $(i)$ leads to :

$$\mathcal{P}_j^0(\frac{d}{dx} w_j^1) = \frac{d}{dx} \mathcal{P}_j^1(w_j^1) = 0$$

thus $\frac{d}{dx} w_j^1 \in (\tilde{V}_j^0)^\perp$. Then $\frac{d}{dx} W_j^1 = W_j^0$.

- We now prove the relation $\tilde{W}_j^0 = \int_0^x \tilde{W}_j^1$.

On one hand we consider $\tilde{w}_j^0 \in \tilde{W}_j^0$. As $\tilde{w}_j^0 \in \tilde{V}_{j+1}^0$, there exists $\tilde{w}_j^1 \in \tilde{V}_{j+1}^1$ such that $\frac{d}{dx} \tilde{w}_j^0 = \tilde{w}_j^1$. Using proposition 2.4 $(ii)$ $(\tilde{w}_j^0 \in H_0^1(0,1))$ we get :

$$\tilde{\mathcal{P}}_j^1(\tilde{w}_j^1) = \tilde{\mathcal{P}}_j^1(\frac{d}{dx} \tilde{w}_j^0) = \frac{d}{dx} \tilde{\mathcal{P}}_j^0(\tilde{w}_j^0) = 0$$

This implies $\tilde{w}_j^1 \in \tilde{V}_{j+1}^1 \cap (V_j^1)^\perp = \tilde{W}_j^1$, thus $\frac{d}{dx} \tilde{W}_j^0 \subset \tilde{W}_j^1$. By integration, since $\tilde{W}_j^0 \subset H_0^1(0,1)$, we obtain $\tilde{W}_j^0 \subset \int_0^x \tilde{W}_j^1$.

On the other hand, let $\tilde{w}_j^1 \in \tilde{W}_j^1$. First we prove that $\int_0^x \tilde{w}_j^1 \in \tilde{V}_{j+1}^0$. Indeed, $\tilde{w}_j^1 \in \tilde{V}_{j+1}^1$ then $\int_0^x \tilde{w}_j^1 \in \int_0^x \tilde{V}_{j+1}^1$. Moreover, $\tilde{W}_j^1$ being a wavelet space (orthogonal to the constants), then $\int_0^1 \tilde{w}_j^1 = 0$, which implies $\int_0^x \tilde{w}_j^1 \in H_0^1(0,1)$.
This leads to the result : $\int_0^x \tilde{w}_j^1 \in \int_0^x \tilde{V}_{j+1}^1 \cap H_0^1(0,1) = \tilde{V}_{j+1}^0$.

Now to prove $\int_0^x \tilde{w}_j^1 \in (V_j^0)^\perp$ we use again proposition 2.4 (*ii*) to obtain :

$$\tilde{\mathcal{P}}_j^1(\tilde{w}_j^1) = 0 \Rightarrow \tilde{\mathcal{P}}_j^1(\frac{d}{dx}\int_0^x \tilde{w}_j^1) = \frac{d}{dx}\tilde{\mathcal{P}}_j^0(\int_0^x \tilde{w}_j^1) = 0$$
$$\Rightarrow \tilde{\mathcal{P}}_j^0(\int_0^x \tilde{w}_j^1) = 0$$

since $\tilde{\mathcal{P}}_j^0(\int_0^x \tilde{w}_j^1) \in \tilde{V}_j^0 \subset H_0^1(0,1)$. Then $\int_0^x \tilde{w}_j^1 \in (V_j^0)^\perp$ and finally $\int_0^x \tilde{W}_j^1 \subset \tilde{V}_{j+1}^0 \cap (V_j^0)^\perp = \tilde{W}_j^0$.
The proof is then complete. □

This proposition implies that the wavelet bases of $W_j^0$ and $\tilde{W}_j^0$ can be readily constructed by differentiating and integrating the wavelets of $W_j^1$ and $\tilde{W}_j^1$, as it was suggested in [18,24] :

**Definition 2.7.**
Let $\{\psi_{j,k}^1\}_{k=1,2^j}$ and $\{\tilde{\psi}_{j,k}^1\}_{k=1,2^j}$ be two biorthogonal wavelet bases of $W_j^1$ and $\tilde{W}_j^1$ respectively. The wavelets of $W_j^0$ and $\tilde{W}_j^0$ are defined by :

$$\psi_{j,k}^0 = 2^{-j}(\psi_{j,k}^1)' \quad \text{and} \quad \tilde{\psi}_{j,k}^0 = -2^j \int_0^x \tilde{\psi}_{j,k}^1$$

Interior wavelets $\psi_{j,k}^0(x) = 2^{j/2}\psi^0(2^j x - k)$ in this definition correspond to the classical wavelets arising from previous constructions [9,11,21,17,4], $\psi^0$ being a wavelet on $\mathbb{R}$ associated to the scaling function $\varphi^0$ as in proposition 2.1. On the other hand, in standard constructions, the edge wavelets do not verify the relations :

$$\frac{d}{dx}\Psi_{j,k}^{1,\flat} = 2^j \Psi_{j,k}^{0,\flat} \quad \text{or} \quad \tilde{\Psi}_{j,k}^{0,\flat} = -2^j \int_0^x \tilde{\Psi}_{j,k}^{1,\flat} \tag{2.22}$$

The following proposition guarantees (2.22) and the new edge wavelets provided by definition 2.7 preserve fast algorithms since they satisfy two-scale equations.

**Proposition 2.6.**
*Let $\{\psi_{j,k}^1\}_{k=1,2^j}$ and $\{\tilde{\psi}_{j,k}^1\}_{k=1,2^j}$ be two biorthogonal wavelet bases of $W_j^1$ and $\tilde{W}_j^1$ associated respectively to filters $G_j^1$ and $\tilde{G}_j^1$ :*

$$\psi_{j,k}^1 = \sum_n (G_j^1)_{k,n}\ \varphi_{j+1,n}^1 \quad \textit{and} \quad \tilde{\psi}_{j,k}^1 = \sum_n (\tilde{G}_j^1)_{k,n}\ \tilde{\varphi}_{j+1,n}^1$$

*Then the following propositions hold :*
*(i) The system $\{\psi_{j,k}^0 = 2^{-j}(\psi_{j,k}^1)'\}_{k=1,2^j}$ and $\{\tilde{\psi}_{j,k}^0 = -2^j \int_0^x \tilde{\psi}_{j,k}^1\}_{k=1,2^j}$*

*form biorthogonal wavelet bases of* $W_j^0$ *and* $\tilde{W}_j^0$ *respectively.*
(*ii*) *There exist sparse matrices* $G_j^0$ *and* $\tilde{G}_j^0$ *defined by :*

$$G_j^0 = 2^{-j} G_j^1 L_{j+1}^1 \quad \text{and} \quad \tilde{G}_j^0 = -2^j \tilde{G}_j^1 L_{j+1}^{0T} \tag{2.23}$$

*such that the wavelets* $\psi_{j,k}^0$ *and* $\tilde{\psi}_{j,k}^0$ *satisfy :*

$$\psi_{j,k}^0 = \sum_n (G_j^0)_{k,n} \varphi_{j+1,n}^0 \quad \text{and} \quad \tilde{\psi}_{j,k}^0 = \sum_n (\tilde{G}_j^0)_{k,n} \tilde{\varphi}_{j+1,n}^0$$

**Proof.**
(*i*) This point follows directly from proposition 2.5.
(*ii*) The filters are derived from the definition of wavelets. Indeed :

$$\begin{aligned} 2^j \psi_{j,k}^0 &= \sum_n (G_j^1)_{k,n} (\varphi_{j+1,n}^1)' = \sum_{n,m} (G_j^1)_{k,n} (L_{j+1}^1)_{n,m} \varphi_{j+1,m}^0 \\ &= \sum_m [G_j^1 L_{j+1}^1]_{k,m} \varphi_{j+1,m}^0 \end{aligned}$$

since $\psi_{j,k}^0 = \sum_m (G_j^0)_{k,m} \varphi_{j+1,m}^0$, we obtain the expression of $G_j^0$.
Similarly, for $\tilde{\psi}_{j,k}^0$ we obtain :

$$\begin{aligned} \frac{d}{dx} \tilde{\psi}_{j,k}^0 &= \sum_m (\tilde{G}_j^0)_{k,m} (\tilde{\varphi}_{j+1,m}^0)' = \sum_{m,n} (\tilde{G}_j^0)_{k,m} (\tilde{L}_{j+1}^0)_{m,n} \tilde{\varphi}_{j+1,n}^1 \\ &= \sum_n [\tilde{G}_j^0 \tilde{L}_{j+1}^0]_{k,n} \tilde{\varphi}_{j+1,n}^1 \end{aligned}$$

since $\frac{d}{dx} \tilde{\psi}_{j,k}^0 = -2^j \tilde{\psi}_{j,k}^1 = -2^j \sum_n (\tilde{G}_j^1)_{k,n} \tilde{\varphi}_{j+1,n}^1$ this provides the expression of $\tilde{G}_j^0$. □

**Remark 2.4.**
The above construction of wavelets $\psi_{j,k}^0$ and $\tilde{\psi}_{j,k}^0$ has two main interests : their filters are directly accessible from those of $\psi_{j,k}^1$ and $\tilde{\psi}_{j,k}^1$, and there is no need for biorthogonalization as for classical constructions.

**Example 2.1.**
Figure 1 shows the plot of edge scaling functions and wavelets at 0 in $(V_j^1, \tilde{V}_j^1)$. The generators $(\varphi^1, \tilde{\varphi}^1)$ used are biorthogonal B-Splines with $r = \tilde{r} = 3$. Then we have : $n_{min} = -1$, $n_{max} = 2$, $\tilde{n}_{min} = -3$ and $\tilde{n}_{max} = 4$. The "free" integer parameters are chosen as $\delta_\flat = \delta_\sharp = 2$ and $\tilde{\delta}_\flat = \tilde{\delta}_\sharp = 0$. Figure 2 plots the corresponding edge scaling functions and wavelets of $(V_j^0, \tilde{V}_j^0)$ at 0. Figure 3 draws the corresponding non zero elements of filters $G_j^0$ and $\tilde{G}_j^0$ and Figure 4 these of matrices $L_j^0$ and $L_j^1$ respectively, for $j = 6$.

### 2.3. *Biorthogonal MRA of $\mathcal{H}_{div}(\Omega)$*

Let $\Omega$ be the square $[0,1]^2$. The aim of the present section is to provide a divergence-free MRA and wavelet bases of the space $\mathcal{H}_{div}(\Omega)$ [16] :

$$\mathcal{H}_{div}(\Omega) = \{\mathbf{u} \in (L^2(\Omega))^2 : \ \mathbf{div}(\mathbf{u}) = 0 \text{ and } \mathbf{u} \cdot \vec{\nu} = 0\}$$

Since this space is equal to :

$$\mathcal{H}_{div}(\Omega) = \{\mathbf{u} = \mathbf{curl}\ \chi\ ;\ \chi \in H_0^1(\Omega)\}$$

our construction consists in taking the curl of a regular MRA of the two-dimensional scalar space $H_0^1(\Omega)$.

Such MRA of $H_0^1(\Omega)$ is usually defined as tensor-product of one-dimensional MRA of $H_0^1(0,1)$. We now consider a regular one-dimensional MRA satisfying homogeneous boundary conditions :

$$V_j^D = V_j^1 \cap H_0^1(0,1)$$

as constructed in section 2.2, and which takes the form :

$$V_j^D = span\{\Phi_{j,\ell}^{1,\flat}\ ;\ \ell = 1, r-1\} \oplus V_j^{1,int} \oplus span\{\Phi_{j,\ell}^{1,\sharp}\ ;\ \ell = 1, r-1\}$$

To simplify, we denote by $\varphi_{j,k}^D$ the scaling functions of $V_j^D$ :

$$V_j^D = span\{\varphi_{j,k}^D\ ;\ k = 0,\ 2^j - k_\sharp - k_\flat + 2r - 2\}$$

and $\psi_{j,k}^D$ the corresponding wavelets. With these notations, the divergence-free scaling function spaces are defined below.

**Definition 2.8.**
For $j \geq j_{min}$, the *divergence-free scaling function spaces* $\mathbf{V}_j^{div}$ are defined by :

$$\mathbf{V}_j^{div} = \mathbf{curl}(V_j^D \otimes V_j^D) = span\{\Phi_{j,\mathbf{k}}^{div}\} \tag{2.24}$$

where the *divergence-free scaling functions* are given by [a] :

$$\Phi_{j,\mathbf{k}}^{div} := \mathbf{curl}[\varphi_{j,k_1}^D \otimes \varphi_{j,k_2}^D],\ j \geq j_{min} \tag{2.25}$$

The spaces $\mathbf{V}_j^{div}$ defined above constitute an increasing sequence of subspaces of $(L^2(\Omega))^2$ :

$$\mathbf{V}_j^{div} \subset \mathbf{V}_{j+1}^{div}$$

a. The **curl** of a 2D scalar function $\Psi$ is defined by $\mathbf{curl}\ \Psi = (\frac{\partial \Psi}{\partial y}, -\frac{\partial \Psi}{\partial x})$.

of dimension :

$$\dim(\mathbf{V}_j^{div}) = \dim(V_j^D)^2 = (2^j - k_\sharp - k_\flat + 2r - 1)^2 \\ = \left(2^j - (n_{max} - n_{min}) - (\delta_\flat + \delta_\sharp) + 2r - 1\right)^2$$

We will also consider a more standard multiresolution analysis $\vec{\mathbf{V}}_j$ of $(L^2(\Omega))^2$ defined as :

$$\vec{\mathbf{V}}_j = (V_j^1 \otimes V_j^0) \times (V_j^0 \otimes V_j^1) \tag{2.26}$$

$V_j^0$ being the spaces defined in section 2.2.2. By proposition 2.3, this discrete space $\vec{\mathbf{V}}_j$ preserves the divergence-free condition, as stated by Jouini-Lemarié-Rieusset[18] :

$$\mathbf{u} \in (L^2(\Omega))^2, \quad \mathbf{div}(\mathbf{u}) = 0 \;\Rightarrow\; \mathbf{div}[\vec{\mathbf{P}}_j(\mathbf{u})] = 0 \tag{2.27}$$

where $\vec{\mathbf{P}}_j$ is the biorthogonal projector on $\vec{\mathbf{V}}_j$ :

$$\vec{\mathbf{P}}_j = (\mathcal{P}_j^1 \otimes \mathcal{P}_j^0, \mathcal{P}_j^0 \otimes \mathcal{P}_j^1) \tag{2.28}$$

In the same way, we now introduce anisotropic divergence-free wavelets and wavelet spaces :

**Definition 2.9.**
The *anisotropic divergence-free wavelets and wavelet spaces* are given by :

$$\Psi_{\mathbf{j},\mathbf{k}}^{div,1} := \mathbf{curl}[\varphi_{j_{min},k_1}^D \otimes \psi_{j_2,k_2}^D] \quad \text{and} \quad \mathbf{W}_{\mathbf{j}}^{div,1} = span\{\Psi_{\mathbf{j},\mathbf{k}}^{div,1}\},\ j_2 \geq j_{min}$$

$$\Psi_{\mathbf{j},\mathbf{k}}^{div,2} := \mathbf{curl}[\psi_{j_1,k_1}^D \otimes \varphi_{j_{min},k_2}^D] \quad \text{and} \quad \mathbf{W}_{\mathbf{j}}^{div,2} = span\{\Psi_{\mathbf{j},\mathbf{k}}^{div,2}\},\ j_1 \geq j_{min}$$

$$\Psi_{\mathbf{j},\mathbf{k}}^{div,3} := \mathbf{curl}[\psi_{j_1,k_1}^D \otimes \psi_{j_2,k_2}^D] \quad \text{and} \quad \mathbf{W}_{\mathbf{j}}^{div,3} = span\{\Psi_{\mathbf{j},\mathbf{k}}^{div,3}\},\ j_1, j_2 \geq j_{min}$$

The following proposition proves that $(\mathbf{V}_j^{div})_{j \geq j_{min}}$ is a multiresolution analysis of $\mathcal{H}_{div}(\Omega)$.

**Proposition 2.7.**
*The divergence-free scaling function spaces* $\mathbf{V}_j^{div}$ *and wavelet spaces* $\mathbf{W}_{\mathbf{j}}^{div,\varepsilon}$ *for* $\varepsilon = 1, 2, 3$, *satisfy :*
*(i)* $\mathbf{V}_{j_{min}}^{div} \subset \cdots \subset \mathbf{V}_j^{div} \subset \mathbf{V}_{j+1}^{div} \subset \cdots \subset \mathcal{H}_{div}(\Omega)$ *and* $\overline{\cup \mathbf{V}_j^{div}} = \mathcal{H}_{div}(\Omega)$.
*(ii)* $\mathbf{V}_j^{div} = \mathbf{V}_{j_{min}}^{div} \bigoplus_{j_{min} \leq j_1, j_2 \leq j-1} (\oplus_{\varepsilon=1,2,3} \mathbf{W}_{\mathbf{j}}^{div,\varepsilon})$.
*(iii) For all* $\mathbf{j}$ *and* $\varepsilon = 1, 2, 3$, $\{\Psi_{\mathbf{j},\mathbf{k}}^{div,\varepsilon}\}$ *is a Riesz basis of* $\mathbf{W}_{\mathbf{j}}^{div,\varepsilon}$.

Then each vector function $\mathbf{u}$ of $\mathcal{H}_{div}(\Omega)$ has a unique decomposition into the basis $\{\Phi^{div}_{j_{min},\mathbf{k}}, \Psi^{div,\varepsilon}_{\mathbf{j},\mathbf{k}}\}_{j_1,j_2\geq j_{min},\ \varepsilon=1,2,3}$ :

$$\mathbf{u} = \sum_k c^{div}_{j_{min},\mathbf{k}} \Phi^{div}_{j_{min},\mathbf{k}} + \sum_{\mathbf{j},\mathbf{k}} \sum_{\varepsilon=1,2,3} d^{div,\varepsilon}_{\mathbf{j},k} \Psi^{div,\varepsilon}_{\mathbf{j},\mathbf{k}}$$

with the norm-equivalence :

$$\|\mathbf{u}\|^2_{L^2} \sim \sum_{\mathbf{k}} |c^{div}_{j_{min},\mathbf{k}}|^2 + \sum_{\mathbf{j},\mathbf{k}} \sum_{\varepsilon=1,2,3} |d^{div,\varepsilon}_{\mathbf{j},\mathbf{k}}|^2$$

**Proof.**
$(i)$ Let $\vec{\mathbf{V}}_j$ be the spaces defined in (2.26). Since the spaces $\mathcal{H}_{div}(\Omega)\cap\vec{\mathbf{V}}_j$ provide a multiresolution analysis of $\mathcal{H}_{div}(\Omega)$[18], point $(i)$ is reduced to prove that : $\mathbf{V}^{div}_j = \mathcal{H}_{div}(\Omega)\cap\vec{\mathbf{V}}_j$.
According to proposition 2.2, we have $\mathbf{V}^{div}_j \subset \vec{\mathbf{V}}_j$ and $\mathbf{V}^{div}_j \subset \mathcal{H}_{div}(\Omega)$ by construction.
Conversely, let $\mathbf{u}\in\mathcal{H}_{div}(\Omega)\cap\vec{\mathbf{V}}_j$, and $\vec{\mathbf{P}}_j$ be the biorthogonal projector on $\vec{\mathbf{V}}_j$ defined in (2.28). We are going to prove that $\mathbf{u}\in\mathbf{V}^{div}_j$. On one hand, as $\mathbf{u}\in\vec{\mathbf{V}}_j$ we have $\mathbf{u}=\vec{\mathbf{P}}_j(\mathbf{u})$, on the other hand due to $\mathbf{u}\in\mathcal{H}_{div}(\Omega)$ we have $\mathbf{u}=\mathbf{curl}\ (\chi)$ with $\chi\in H^1_0(\Omega)$, and thus :

$$\mathbf{u} = \vec{\mathbf{P}}_j[\mathbf{curl}\ (\chi)]$$

Since the spaces $(V^D_j\otimes V^D_j)_{j\geq j_{min}}$ form a MRA of $H^1_0(\Omega)$, we can decompose $\chi$ as :

$$\chi = \mathbf{P}^D_j(\chi) + \sum_{j_1,j_2\geq j} \left(\mathbf{Q}^D_1(\chi) + \mathbf{Q}^D_2(\chi) + \mathbf{Q}^D_3(\chi)\right)$$

where

$$\mathbf{P}^D_j(\chi) = \sum_{\mathbf{k}} c_{\mathbf{k}}\ \varphi^D_{j,k_1}\otimes\varphi^D_{j,k_2},\quad \mathbf{Q}^D_2(\chi) = \sum_{j_1\geq j,\mathbf{k}} d^2_{j_1,\mathbf{k}}\ \psi^D_{j_1,k_1}\otimes\varphi^D_{j,k_2}$$

$$\mathbf{Q}^D_1(\chi) = \sum_{j_2\geq j,\mathbf{k}} d^1_{j_2,\mathbf{k}}\ \varphi^D_{j,k_1}\otimes\psi^D_{j_2,k_2},\quad \mathbf{Q}^D_3(\chi) = \sum_{j_1,j_2\geq j,\mathbf{k}} d^3_{\mathbf{j},\mathbf{k}}\ \psi^D_{j_1,k_1}\otimes\psi^D_{j_2,k_2}$$

are the biorthogonal projectors on respectively $V^D_j\otimes V^D_j$, $W^D_{j_1}\otimes V^D_j$, $V^D_j\otimes W^D_{j_2}$ and $W^D_{j_1}\otimes W^D_{j_2}$. Proposition 2.2 implies that :

$$\mathbf{curl}\ [\varphi^D_{j,k_1}\otimes\psi^D_{j_2,k_2}]\in(V^D_j\otimes W^0_{j_2})\times(V^0_j\otimes W^D_{j_2})$$

hence :

$$\vec{\mathbf{P}}_j(\mathbf{curl}\ [\varphi^D_{j,k_1}\otimes\psi^D_{j_2,k_2}]) = 0$$

and same for $\vec{\mathbf{P}}_j(\mathbf{curl}\ [\psi^D_{j_1,k_1}\otimes\varphi^D_{j,k_2}])$ and $\vec{\mathbf{P}}_j(\mathbf{curl}\ [\psi^D_{j_1,k_1}\otimes\psi^D_{j_2,k_2}])$. This leads to :

$$\vec{\mathbf{P}}_j(\mathbf{curl}\ (\chi)) = \vec{\mathbf{P}}_j(\mathbf{curl}\ [\mathbf{P}^D_j(\chi)]) = \mathbf{curl}\ [\mathbf{P}^D_j(\chi)]$$

By construction we have $\mathbf{curl}\ [\mathbf{P}^D_j(\chi)] \in \mathbf{V}^{div}_j$, which implies $\mathbf{u} \in \mathbf{V}^{div}_j$ and then completes the proof : $\mathbf{V}^{div}_j = \mathcal{H}_{div}(\Omega) \cap \vec{\mathbf{V}}_j$.
(*ii*) The spaces $V^D_j$ form a multiresolution analysis of $H^1_0(0,1)$, and we can write :

$$V^D_j \otimes V^D_j = (V^D_{j_{min}} \bigoplus_{j_1=j_{min}}^{j-1} W^D_{j_1}) \otimes (V^D_{j_{min}} \bigoplus_{j_2=j_{min}}^{j-1} W^D_{j_2})$$

By definition of $\mathbf{V}^{div}_j$, we obtain :

$$\mathbf{V}^{div}_j = \mathbf{curl}\left[(V^D_{j_{min}}\otimes V^D_{j_{min}}) \bigoplus_{j_{min}\le j_1,j_2\le j-1} [(V^D_{j_{min}}\otimes W^D_{j_2})\oplus(W^D_{j_1}\otimes V^D_{j_{min}})\oplus(W^D_{j_1}\otimes W^D_{j_2})]\right]$$

which is exactly $\mathbf{V}^{div}_j = \mathbf{V}^{div}_{j_{min}} \bigoplus \left[\bigoplus_{j_{min}\le j_1,j_2\le j-1}\left(\oplus_{\varepsilon=1,2,3}\mathbf{W}^{div,\varepsilon}_{\mathbf{j}}\right)\right]$.
(*iii*) Following [20,8], this point is a consequence of the proposition 2.8 below. □

We now introduce the biorthogonal divergence-free scaling functions and wavelets. Let :

$$\tilde{\Phi}^{div}_{j,\mathbf{k}} := \begin{vmatrix} \tilde{\varphi}^D_{j,k_1}\otimes\tilde{\gamma}_{j,k_2} \\ -\tilde{\gamma}_{j,k_1}\otimes\tilde{\varphi}^D_{j,k_2}\end{vmatrix}, \quad \tilde{\Psi}^{div,1}_{\mathbf{j},\mathbf{k}} := \begin{vmatrix} 2^{j_2}\tilde{\varphi}^D_{j_{min},k_1}\otimes\tilde{\psi}^0_{j_2,k_2} \\ -\tilde{\gamma}_{j_{min},k_1}\otimes\tilde{\psi}^D_{j_2,k_2}\end{vmatrix} \tag{2.29}$$

$$\tilde{\Psi}^{div,2}_{\mathbf{j},\mathbf{k}} := \begin{vmatrix} \tilde{\psi}^D_{j_1,k_1}\otimes\tilde{\gamma}_{j_{min},k_2} \\ -2^{j_1}\tilde{\psi}^0_{j_1,k_1}\otimes\tilde{\varphi}^D_{j_{min},k_2}\end{vmatrix}, \quad \tilde{\Psi}^{div,3}_{\mathbf{j},\mathbf{k}} := \begin{vmatrix} 2^{j_2}\tilde{\psi}^D_{j_1,k_1}\otimes\tilde{\psi}^0_{j_2,k_2} \\ -2^{j_1}\tilde{\psi}^0_{j_1,k_1}\otimes\tilde{\psi}^D_{j_2,k_2}\end{vmatrix} \tag{2.30}$$

where : $\tilde{\gamma}_{j,k} = -\int_0^x \tilde{\varphi}^D_{j,k}$. Remark that $\tilde{\gamma}_{j,k}(0) = \tilde{\gamma}_{j,k}(1)$ since $\tilde{\varphi}^D_{j,k} \in H^1_0(0,1)$.

**Proposition 2.8.**
*For a fixed $j \ge j_{min}$, the normalized families* $\left\{\frac{1}{\sqrt{2}}\Phi^{div}_{j,\mathbf{k}}, \frac{1}{\sqrt{4^{j_2}+1}}\Psi^{div,1}_{\mathbf{j},\mathbf{k}}, \frac{1}{\sqrt{4^{j_1}+1}}\Psi^{div,2}_{\mathbf{j},\mathbf{k}}, \frac{1}{\sqrt{4^{j_1}+4^{j_2}}}\Psi^{div,3}_{\mathbf{j},\mathbf{k}}\ ;\ j_1,j_2\ge j\ ,\ \mathbf{k}\right\}$ *and* $\left\{\frac{1}{\sqrt{2}}\tilde{\Phi}^{div}_{j,\mathbf{k}}, \frac{1}{\sqrt{4^{j_2}+1}}\tilde{\Psi}^{div,1}_{\mathbf{j},\mathbf{k}}, \frac{1}{\sqrt{4^{j_1}+1}}\tilde{\Psi}^{div,2}_{\mathbf{j},\mathbf{k}}, \frac{1}{\sqrt{4^{j_1}+4^{j_2}}}\tilde{\Psi}^{div,3}_{\mathbf{j},\mathbf{k}}\ ;\ j_1,j_2\ge j\ ,\ \mathbf{k}\right\}$ *are biorthogonal in $(L^2(\Omega))^2$.*

**Remark 2.5.**
Unlike the classical definitions of MRAs, the ($L^2$-normalized)

divergence-free scaling functions $(\Phi^{div}_{j,\mathbf{k}})_{\mathbf{k}}$ don't form a Riesz basis of the space $\mathbf{V}^{div}_j$, since they do not verify :

$$\|\sum_{\mathbf{k}} c^{div}_{j,\mathbf{k}}\ \Phi^{div}_{j,\mathbf{k}}\|^2_{(L^2(\Omega))^2} \sim \sum_{\mathbf{k}} |c^{div}_{j,\mathbf{k}}|^2$$

for all $(c^{div}_{j,\mathbf{k}}) \in \ell^2$ and independently of $j$. A counterexample is given by :

$$\mathbf{u} = \sum_{k_\flat \le k_1,k_2 \le 2^j - k_\sharp} \Phi^{div}_{j,k_1,k_2},$$

which satisfy : $\sum_{\mathbf{k}} |c^{div}_{j,\mathbf{k}}|^2 = (2^j - k_\sharp - k_\flat + 1)^2 \sim 2^{2j}$.
On the other hand :

$$\mathbf{u} = \sum_{k_\flat \le k_1,k_2 \le 2^j - k_\sharp} \frac{2^{-j}}{\sqrt{2}} \begin{vmatrix} \varphi^D_{j,k_1} \otimes (\varphi^D_{j,k_2})' \\ -(\varphi^D_{j,k_1})' \otimes \varphi^D_{j,k_2} \end{vmatrix}$$

and

$$\sum_{k_\flat \le k_2 \le 2^j - k_\sharp} (\varphi^D_{j,k_2})' = \sum_{k_\flat \le k_2 \le 2^j - k_\sharp} 2^j(\varphi^0_{j,k_2} - \varphi^0_{j,k_2+1}) = 2^j(\varphi^0_{j,k_\flat} - \varphi^0_{j,2^j-k_\sharp+1})$$

Let $h(x) = \sum_{k_\flat \le k \le 2^j - k_\sharp} \varphi^D_{j,k}(x) \sim 2^{j/2}\chi_{[\frac{\delta_\flat}{2^j}, 1-\frac{\delta_\sharp}{2^j}]}$ , $\quad \forall x \in ]0,1[$
then

$$\|\mathbf{u}\|^2_{(L^2(\Omega))^2} = 2^{-2j}\left(\int_0^1 h^2\right)\left(\int_0^1 2^{2j}(\varphi^0_{j,k_\flat} - \varphi^0_{j,2^j-k_\sharp+1})^2\right) \sim 2^j$$

**Example 2.2.**
Figure 5 and Figure 6 show the vector representation of divergence-free edge scaling functions **curl** $[\Phi^{1,\flat}_1 \otimes \Phi^{1,\flat}_1]$, **curl** $[\Phi^{1,\flat}_2 \otimes \Phi^{1,\flat}_2]$ and wavelets **curl** $[\Psi^{1,\flat}_1 \otimes \Psi^{1,\flat}_1]$, **curl** $[\Psi^{1,\flat}_2 \otimes \Psi^{1,\flat}_2]$, constructed from biorthogonal B-Spline generators $(\varphi^1, \tilde{\varphi}^1)$ with $r = \tilde{r} = 3$ used in example 2.1.

## 2.4. *Biorthogonal MRA of* $\mathcal{H}^{\perp}_{div}(\Omega)$

In this section, the curl-free function space $\mathcal{H}_{curl}(\Omega)$ that we consider is the following :

$$\mathcal{H}_{curl}(\Omega) = \{\mathbf{u} = \nabla\ q :\ q \in H^1_0(\Omega)\}$$

This space is a proper subspace of $\mathcal{H}^{\perp}_{div}(\Omega)$ :

$\mathcal{H}^{\perp}_{div}(\Omega) = \mathcal{H}_{curl}(\Omega) \oplus \mathcal{H}^{\Delta}(\Omega)$, with $\mathcal{H}^{\Delta}(\Omega) = \{\nabla\ q :\ q \in H^1(\Omega)$ and $\Delta\ q = 0\}$

To construct a multiresolution analysis of $\mathcal{H}_{curl}(\Omega)$, it suffices to consider the gradient of a MRA of $H_0^1(\Omega)$. With the same notations as in previous section 2.3, curl-free scaling functions spaces are defined by :

**Definition 2.10.**
For $j \geq j_{min}$, a *curl-free scaling function space* $\mathbf{V}_j^{\nabla}$ is defined by :

$$\mathbf{V}_j^{\nabla} = \nabla(V_j^D \otimes V_j^D) = span\{\Phi_{j,\mathbf{k}}^{\nabla}\} \tag{2.31}$$

where the *curl-free scaling functions* are given by :

$$\Phi_{j,\mathbf{k}}^{\nabla} := \nabla[\varphi_{j,k_1}^D \otimes \varphi_{j,k_2}^D],\ j \geq j_{min} \tag{2.32}$$

The spaces $(\mathbf{V}_j^{\nabla})$ form an increasing sequence of subspaces of $(L^2(\Omega))^2$, of dimension : $\dim(\mathbf{V}_j^{\nabla}) = \dim(V_j^D \otimes V_j^D) = (2^j - k_\sharp - k_\flat + 2r - 1)^2$.

Let $\vec{\mathbf{V}}_j^*$ be the standard multiresolution analysis of $(L^2(\Omega))^2$ defined by :

$$\vec{\mathbf{V}}_j^* = (V_j^0 \otimes V_j^1) \times (V_j^1 \otimes V_j^0) \tag{2.33}$$

By proposition 2.3, the spaces $\mathbf{V}_j^{\nabla}$ are contained in $\vec{\mathbf{V}}_j^*$. We now define the corresponding irrotational wavelets.

**Definition 2.11.**
The *anisotropic curl-free wavelets* and wavelet spaces are defined by :

$$\Psi_{\mathbf{j},\mathbf{k}}^{\nabla,1} := \nabla[\varphi_{j_{min},k_1}^D \otimes \psi_{j_2,k_2}^D] \quad \text{and} \quad \mathbf{W}_{\mathbf{j}}^{\nabla,1} = span\{\Psi_{\mathbf{j},\mathbf{k}}^{\nabla,1}\},\ j_2 \geq j_{min}$$

$$\Psi_{\mathbf{j},\mathbf{k}}^{\nabla,2} := \nabla[\psi_{j_1,k_1}^D \otimes \varphi_{j_{min},k_2}^D] \quad \text{and} \quad \mathbf{W}_{\mathbf{j}}^{\nabla,2} = span\{\Psi_{\mathbf{j},\mathbf{k}}^{\nabla,2}\},\ j_1 \geq j_{min}$$

$$\Psi_{\mathbf{j},\mathbf{k}}^{\nabla,3} := \nabla[\psi_{j_1,k_1}^D \otimes \psi_{j_2,k_2}^D] \quad \text{and} \quad \mathbf{W}_{\mathbf{j}}^{\nabla,3} = span\{\Psi_{\mathbf{j},\mathbf{k}}^{\nabla,3}\},\ j_1, j_2 \geq j_{min}$$

The following proposition holds :

**Proposition 2.9.**
*The spaces* $\mathbf{V}_j^{\nabla}$ *and* $\mathbf{W}_{\mathbf{j}}^{\nabla,\varepsilon}$ *for* $\epsilon = 1,2,3$ *verify :*
*(i)* $\mathbf{V}_{j_{min}}^{\nabla} \subset \cdots \subset \mathbf{V}_j^{\nabla} \subset \mathbf{V}_{j+1}^{\nabla} \subset \cdots \subset \mathcal{H}_{curl}(\Omega)$ *and* $\overline{\cup \mathbf{V}_j^{\nabla}} = \mathcal{H}_{curl}(\Omega)$
*(ii)* $\mathbf{V}_j^{\nabla} = \mathbf{V}_{j_{min}}^{\nabla} \bigoplus_{j_{min} \leq j_1, j_2 \leq j-1} (\oplus_{\varepsilon=1,2,3} \mathbf{W}_{\mathbf{j}}^{\nabla,\varepsilon})$
*(iii) For all* $\mathbf{j}$ *and* $\varepsilon = 1,2,3$, $\{\Psi_{\mathbf{j},\mathbf{k}}^{\nabla,\varepsilon}\}$ *is a Riesz basis of* $\mathbf{W}_{\mathbf{j}}^{\nabla,\varepsilon}$.

The proof uses the same arguments as in proposition 2.7. In addition, these construction of curl-free scaling functions and wavelets can readily be extended to higher dimensions, for more details see [22,23].

**Example 2.3.**
Figure 7 and Figure 8 show the vector representation of curl-free edge scaling functions $\nabla[\Phi_1^{1,\flat} \otimes \Phi_1^{1,\flat}]$, $\nabla[\Phi_2^{1,\flat} \otimes \Phi_2^{1,\flat}]$ and wavelets $\nabla[\Psi_1^{1,\flat} \otimes \Psi_1^{1,\flat}]$, $\nabla[\Psi_2^{1,\flat} \otimes \Psi_2^{1,\flat}]$, constructed from biorthogonal B-Spline generators $(\varphi^1, \tilde{\varphi}^1)$ with $r = \tilde{r} = 3$ used in example 2.1.

## 3. Fast divergence-free wavelet transform

We describe in this section the practical computation of divergence-free scaling function and wavelet coefficients of a vector field $\mathbf{u} \in \mathbf{V}_j^{div}$. We use the same notations as in previous sections. The starting point is the decomposition of $\mathbf{u}$ in the MRA of $(L^2(\Omega))^2$ provided by $\vec{\mathbf{V}}_j$ :

$$\vec{\mathbf{V}}_j = (V_j^1 \otimes V_j^0) \times (V_j^0 \otimes V_j^1)$$

On the scaling functions basis of $\vec{\mathbf{V}}_j$, the vector field $\mathbf{u} = (\mathbf{u}_1, \mathbf{u}_2)$ can be written as :

$$\mathbf{u}_1 = \sum_{\mathbf{k}} c_{j,\mathbf{k}}^1 \ \varphi_{j,k_1}^1 \otimes \varphi_{j,k_2}^0 \quad \text{and} \quad \mathbf{u}_2 = \sum_{\mathbf{k}} c_{j,\mathbf{k}}^2 \ \varphi_{j,k_1}^0 \otimes \varphi_{j,k_2}^1 \tag{3.1}$$

The computation of divergence-free coefficients will use the change of bases between $(\frac{d}{dx}\varphi_{j,k}^1)$ and $(\varphi_{j,k}^0)$ introduced in definition 2.6. The following proposition explicits the form of the change of basis matrices (before the biorthogonalization process, or when it does not operate on these functions, which will be the case in our numerical tests).

**Proposition 3.1.**
*Let $\Delta_j$ be the dimension of $V_j^1$ $(j \geq j_{min})$ and consider the matrices $L_j^0$ and $L_j^1$ defined in definition 2.6, renormalized by :*

$$L_j^1 = 2^j L^1 \qquad \text{and} \qquad L_j^0 = 2^{-j} L^0$$

*Then the only non zero elements of matrices $L^1$ and $L^0$ are given below.*
*(i) $L^1$ is a rectangular matrix of size $\Delta_j \times (\Delta_j - 1)$ whose non zero elements are :*

$$L_{k,k-1}^1 = 1, \quad L_{k,k}^1 = -1, \quad r+1 \leq k \leq \Delta_j - r$$

*and for $2 \leq k \leq r$ :*

$$L_{1,r}^1 = -1, \ L_{k,k-1}^1 = 1, \ L_{k,r}^1 = -\tilde{p}_{k-1}^1(k_\flat - 1)$$
$$L_{\Delta_j,\Delta_j-r}^1 = 1, \ L_{\Delta_j-k+1,\Delta_j-k+1}^1 = -1, \ L_{\Delta_j-k+1,\Delta_j-r}^1 = \tilde{p}_{k-1}^1(2^j - k_\sharp + 1)$$

*(ii) $L^0$ is a rectangular matrix of size $(\Delta_j - 1) \times \Delta_j$ whose no zero elements are :*

$$L^0_{k,m} = -1, \quad L^0_{k,\Delta_j} = -1, \quad k+1 \leq m \leq \Delta_j - r \text{ and } r \leq k \leq \Delta_j - r$$

*and for $2 \leq k \leq r$ :*

$$L^0_{r,1} = 1,\ L^0_{k-1,k} = -1,\ L^0_{k-1,1} = \tilde{p}^1_{k-1}(k_\flat - 1)$$
$$L^0_{\Delta_j - r,\Delta_j} = -1,\ L^0_{\Delta_j - k+1,\Delta_j - k+1} = 1,\ L^0_{\Delta_j - k+1,\Delta_j} = -\tilde{p}^1_{k-1}(2^j - k_\sharp + 1)$$

**Proof.**
*(i)* The expression of $L^1$ is straightforward using proposition 2.2. Indeed, the differentiation relation on interior scaling functions (proposition 2.2 *(i)*) leads to $L^1_{k,k-1} = 1, \quad L^1_{k,k} = -1, \quad r+1 \leq k \leq \Delta_j - r$.
In the same way, proposition 2.2 *(ii)* corresponds to the differentiation relation on edge scaling functions, for instance :

$$(\Phi^{1,\flat}_\ell)' = \Phi^{0,\flat}_{\ell-1} - \tilde{p}^1_\ell(k_\flat - 1)\varphi^0_{k_\flat}$$

rewrites with the new notations for the scaling functions :

$$(\varphi^1_{j,k})' = 2^j[\varphi^0_{j,k-1} - \tilde{p}^1_\ell(k_\flat - 1)\varphi^0_{j,r}], \quad 2 \leq k \leq r$$

leading to : $L^1_{k,k-1} = 1,\ L^1_{k,r} = -\tilde{p}^1_{k-1}(k_\flat - 1)$.
*(ii)* To obtain the relation on $L^0$, as it is still true by differentiation we get :

$$-\Phi^{0,\flat}_{j,\ell} = \sum_{k=0}^{r-1} L^0_{\ell,k}\ (\Phi^{1,\flat}_{j,k})' \quad \text{and} \quad -\Phi^{0,\sharp}_{j,\ell} = \sum_{k=0}^{r-1} L^0_{\Delta_j+\ell,\Delta_j+k}\ (\Phi^{1,\sharp}_{j,k})'$$

Then using the proposition 2.2 again, we have :

$$\Phi^{0,\flat}_{j,\ell} = 2^{-j}[(\Phi^{1,\flat}_{\ell+1})' - \tilde{p}^1_{\ell+1}(k_\flat - 1)\ (\Phi^{1,\flat}_{j,0})']$$

and

$$\Phi^{0,\sharp}_{j,\ell}(1-.) = 2^{-j}[(\Phi^{1,\sharp}_{\ell+1}(1-.))' + \tilde{p}^1_{\ell+1}(2^j - k_\sharp + 1)\ (\Phi^{1,\sharp}_{j,0}(1-.))']$$

since :

$$\varphi^0_{j,k_\flat} = -2^{-j}(\Phi^{1,\flat}_{j,0})' \text{ and } \varphi^0_{j,2^j-k_\sharp+1} = 2^{-j}(\Phi^{1,\sharp}_{j,0}(1-.))'$$

For interior scaling functions, proposition 2.2 leads to :

$$\varphi^0_{j,k} = 2^{-j}(\varphi^1_{j,k+1})' + \varphi^0_{j,k+1}, \quad r \leq k \leq \Delta_j - r$$

and recursively we get :

$$\varphi^0_{j,k} = 2^{-j}(\varphi^1_{j,k})' + \varphi^0_{j,k+1} = 2^{-j}[(\varphi^1_{j,k})' + \cdots + (\varphi^1_{j,2^j-k_{max}})'] + \varphi^0_{j,2^j-k_{max}+1}$$
$$= 2^{-j}[(\varphi^1_{j,k})' + \cdots + (\varphi^1_{j,2^j-k_{max}})' + (\Phi^{1,\sharp}_{j,0}(1-.))']$$

This completes the proof. □

Using matrices $L_j^0$ and $L_j^1$ we can rewrite the components of $\mathbf{u}$ as follows :

$$\mathbf{u}_1 = \sum_{\mathbf{k}} c^1_{j,\mathbf{k}}\ \varphi^1_{j,k_1} \otimes \varphi^0_{j,k_2} = -\sum_{\mathbf{k}} [C_j^1 L_j^0]_{\mathbf{k}}\ \varphi^1_{j,k_1} \otimes (\varphi^1_{j,k_2})' \tag{3.2}$$

and

$$\mathbf{u}_2 = \sum_{\mathbf{k}} c^2_{j,\mathbf{k}}\ \varphi^0_{j,k_1} \otimes \varphi^1_{j,k_2} = -\sum_{\mathbf{k}} [L_j^{0\ T} C_j^2]_{\mathbf{k}}\ (\varphi^1_{j,k_1})' \otimes \varphi^1_{j,k_2} \tag{3.3}$$

with $C_j^1 = [c^1_{j,\mathbf{k}}]$ and $C_j^2 = [c^2_{j,\mathbf{k}}]$. If $\mathbf{u} \in \mathbf{V}_j^{div}$, it can be uniquely written as :

$$\mathbf{u} = \sum_{\mathbf{k}} c^{div}_{j,\mathbf{k}} \Phi^{div}_{j,\mathbf{k}} \tag{3.4}$$

Therefore, we have the following proposition.

**Proposition 3.2.**
*The matrices of coefficients $C_j^1$ and $C_j^1$ are linked to the matrix of coefficients $C_j^{div} = [c^{div}_{j,\mathbf{k}}]$ by :*

$$2^{1/2} C_j^{div} = (L_j^0)^T C_j^2 - C_j^1 L_j^0 \tag{3.5}$$

*and conversely :*

$$C_j^1 = 2^{-1/2} C_j^{div} L_j^1 \qquad \textit{and} \qquad C_j^1 = -2^{-1/2} (L_j^1)^T C_j^{div} \tag{3.6}$$

**Proof.**
Following the form of divergence-free scaling functions (proposition 2.8, and using (3.2) and (3.3) we have :

$$\langle \mathbf{u} / \tilde{\Phi}^{div}_{j,\mathbf{k}} \rangle = 2^{-1/2} \left[ (L_j^0)^T C_j^2 - C_j^1 L_j^0 \right]_{\mathbf{k}}$$

which proves (3.5). The second relation (3.6) is proved using the change of basis provided by $L_j^1$. □

Now the objective is to compute the divergence-free wavelet coefficients of $\mathbf{u}$ :

$$\begin{aligned} \mathbf{u} = &\sum_{\mathbf{k}} c^{div}_{j,\mathbf{k}}\ \Phi^{div}_{j,\mathbf{k}} + \sum_{j_2 \geq j, \mathbf{k}} d^{div,1}_{\mathbf{j},\mathbf{k}}\ \Psi^{div,1}_{\mathbf{j},\mathbf{k}} \\ &+ \sum_{j_1 \geq j, \mathbf{k}} d^{div,2}_{\mathbf{j},\mathbf{k}}\ \Psi^{div,2}_{\mathbf{j},\mathbf{k}} + \sum_{j_1, j_2 \geq j, \mathbf{k}} d^{div,3}_{\mathbf{j},\mathbf{k}}\ \Psi^{div,3}_{\mathbf{j},\mathbf{k}} \end{aligned}$$

We start with the standard wavelet decomposition of $\mathbf{u} = (\mathbf{u}_1, \mathbf{u}_2)$ in $\vec{\mathbf{V}}_j$ :

$$\mathbf{u}_1 = \sum_{\mathbf{k}} c^1_{j,\mathbf{k}}\ \varphi^1_{j,k_1} \otimes \varphi^0_{j,k_2} + \sum_{j_2 \geq j,\mathbf{k}} d^{1,1}_{j_2,\mathbf{k}}\ \varphi^1_{j,k_1} \otimes \psi^0_{j_2,k_2}$$
$$+ \sum_{j_1 \geq j,\mathbf{k}} d^{1,2}_{j_1,\mathbf{k}}\ \psi^1_{j_1,k_1} \otimes \varphi^0_{j,k_2} + \sum_{j_1,j_2 \geq j,\mathbf{k}} d^{1,3}_{\mathbf{j},\mathbf{k}}\ \psi^1_{j_1,k_1} \otimes \psi^0_{j_2,k_2}$$

and

$$\mathbf{u}_2 = \sum_{\mathbf{k}} c^2_{j,\mathbf{k}}\ \varphi^0_{j,k_1} \otimes \varphi^1_{j,k_2} + \sum_{j_2 \geq j,\mathbf{k}} d^{2,1}_{j_2,\mathbf{k}}\ \varphi^0_{j,k_1} \otimes \psi^1_{j_2,k_2}$$
$$+ \sum_{j_1 \geq j,\mathbf{k}} d^{2,2}_{j_1,\mathbf{k}} \psi^0_{j_1,k_1} \otimes \varphi^1_{j,k_2} + \sum_{j_1,j_2 \geq j,\mathbf{k}} d^{2,3}_{\mathbf{j},\mathbf{k}}\ \psi^0_{j_1,k_1} \otimes \psi^1_{j_2,k_2}$$

This rewrites, using the matrices $L^0_j$ and $L^1_j$ :

$$\mathbf{u}_1 = -\sum_{\mathbf{k}} [C^1_j L^0_j]_{\mathbf{k}}\ \varphi^1_{j,k_1} \otimes (\varphi^1_{j,k_2})' + \sum_{j_2 \geq j,\mathbf{k}} d^{1,1}_{j_2,\mathbf{k}}\ \varphi^1_{j,k_1} \otimes \psi^0_{j_2,k_2}$$
$$- \sum_{j_1 \geq j,\mathbf{k}} [d^{1,2}_{j_1} L^0_j]_{\mathbf{k}}\ \psi^1_{j_1,k_1} \otimes (\varphi^1_{j,k_2})' + \sum_{j_1,j_2 \geq j,\mathbf{k}} d^{1,3}_{\mathbf{j},\mathbf{k}}\ \psi^1_{j_1,k_1} \otimes \psi^0_{j_2,k_2}$$

and

$$\mathbf{u}_2 = -\sum_{\mathbf{k}} [(L^0_j)^T C^2_j]_{\mathbf{k}}\ (\varphi^1_{j,k_1})' \otimes \varphi^1_{j,k_2} - \sum_{j_2 \geq j,\mathbf{k}} [(L^0_j)^T d^{2,1}_{j_2}]_{\mathbf{k}}\ (\varphi^1_{j,k_1})' \otimes \psi^1_{j_2,k_2}$$
$$+ \sum_{j_1 \geq j,\mathbf{k}} d^{2,2}_{j_1,\mathbf{k}} \psi^0_{j_1,k_1} \otimes \varphi^1_{j,k_2} + \sum_{j_1,j_2 \geq j,\mathbf{k}} d^{2,3}_{\mathbf{j},\mathbf{k}}\ \psi^0_{j_1,k_1} \otimes \psi^1_{j_2,k_2}$$

noting $d^{1,2}_{j_1} = [d^{1,2}_{j_1,\mathbf{k}}]$ and $d^{2,1}_{j_2} = [d^{2,1}_{j_2,\mathbf{k}}]$. We now prove the following proposition.

**Proposition 3.3.**
*For $\epsilon = 1, 2, 3$, the coefficients $d^{1,\epsilon}_{\mathbf{j}} = [d^{1,\epsilon}_{\mathbf{j},\mathbf{k}}]$ and $d^{2,\epsilon}_{\mathbf{j}} = [d^{2,\epsilon}_{\mathbf{j},\mathbf{k}}]$ are linked to the divergence-free wavelet coefficients $[d^{div,\epsilon}_{\mathbf{j},\mathbf{k}}]$ by :*

$$d^{div,1}_{\mathbf{j},\mathbf{k}} = \frac{1}{\sqrt{4^{j_2}+1}}[2^{j_2}(d^{1,1}_{\mathbf{j}}) - (L^0_j)^T(d^{2,1}_{\mathbf{j}})]_{\mathbf{j},\mathbf{k}} \tag{3.7}$$

$$d^{div,2}_{\mathbf{j},\mathbf{k}} = \frac{1}{\sqrt{4^{j_1}+1}}[(d^{1,2}_{\mathbf{j}})L^0_j - 2^{j_1}(d^{2,2}_{\mathbf{j}})]_{\mathbf{j},\mathbf{k}} \tag{3.8}$$

$$d^{div,3}_{\mathbf{j},\mathbf{k}} = \frac{1}{\sqrt{4^{j_1}+4^{j_2}}}[2^{j_2}(d^{1,3}_{\mathbf{j}}) - 2^{j_1}(d^{2,3}_{\mathbf{j}})]_{\mathbf{j},\mathbf{k}} \tag{3.9}$$

*Inversely we have :*

$$[d^{1,1}_{\mathbf{j},\mathbf{k}}] = \frac{2^{j_2}}{\sqrt{4^{j_2}+1}}[d^{div,1}_{\mathbf{j},\mathbf{k}}] \quad \textit{and} \quad [d^{2,1}_{\mathbf{j},\mathbf{k}}] = -\frac{1}{\sqrt{4^{j_2}+1}}(L^1_j)^T[d^{div,1}_{\mathbf{j},\mathbf{k}}] \quad (3.10)$$

$$[d^{1,2}_{\mathbf{j},\mathbf{k}}] = \frac{1}{\sqrt{4^{j_1}+1}}[d^{div,2}_{\mathbf{j},\mathbf{k}}]L^1_j \quad \textit{and} \quad [d^{2,1}_{\mathbf{j},\mathbf{k}}] = -\frac{2^{j_1}}{\sqrt{4^{j_1}+1}}[d^{div,2}_{\mathbf{j},\mathbf{k}}] \quad (3.11)$$

$$[d^{1,3}_{\mathbf{j},\mathbf{k}}] = \frac{2^{j_2}}{\sqrt{4^{j_1}+4^{j_2}}}[d^{div,3}_{\mathbf{j},\mathbf{k}}] \quad \textit{and} \quad [d^{2,3}_{\mathbf{j},\mathbf{k}}] = -\frac{2^{j_1}}{\sqrt{4^{j_1}+4^{j_2}}}[d^{div,3}_{\mathbf{j},\mathbf{k}}] \quad (3.12)$$

**Proof.**
Using proposition 2.8, the formula are obtained by considering the inner products :

$$\langle \mathbf{u}/\tilde{\Psi}^{div,1}_{\mathbf{j},\mathbf{k}}\rangle, \quad \langle \mathbf{u}/\tilde{\Psi}^{div,2}_{\mathbf{j},\mathbf{k}}\rangle \quad \text{and} \quad \langle \mathbf{u}/\tilde{\Psi}^{div,3}_{\mathbf{j},\mathbf{k}}\rangle$$

The reconstruction formula are still a consequence of proposition 2.2.□

**Example 3.1.**
We have considered a velocity field $\mathbf{u}$ of resolution $256 \times 256$, arising from a numerical simulation of lid driven cavity flow [22], and we have computed its divergence-free scaling function and wavelet coefficients using propositions 3.2 and 3.3, with the B-Spline generators of Figure 1. Figure 9 shows the vector field $\mathbf{u}$ (left) and corresponding coefficients. Divergence-free wavelet coefficients are localized near shear zones.

## 4. Applications

This section illustrates some practical uses of the divergence-free and curl-free wavelets constructed before. We first show on a numerical example their powerful properties of nonlinear approximation. Then, we consider two problems relevant for the numerical simulation of incompressible flows : the Helmholtz decomposition and the Stokes problem, with homogeneous boundary conditions.

### 4.1. *Nonlinear approximation*

Constructed by tensor-product, divergence-free and curl-free wavelet bases provide nonlinear approximation estimates, governed by the approximation orders of the one-dimensional spaces involved in their construction [7]. In this part, we investigate the convergence rate obtained from the $N$-best terms approximation.

We consider the vector field $\mathbf{u}$ introduced in example 3.1 and the same wavelets. On Figure 10, we plot the $\ell_2$ error on each component, provided by the nonlinear approximation on $\mathbf{u}$ onto the divergence-free wavelet basis. We obtain the same approximation order as that obtained with the standard multiresolution analysis $\vec{\mathbf{V}}_j$. It is consistent with the theoretical result proved in [22], since we recover a slope $\approx -3$, which corresponds to the lowest approximation order provided by $V_j^0$, equal to 2. Figure 11 highlights the classical boundary wavelet error phenomenon on the interval. It is well known that this error at the boundary does not affect the convergence of the multiscale projector on the whole domain [21].

### 4.2. *Helmholtz decomposition*

The Helmholtz decomposition of a vector field $\mathbf{u}$ of $(L^2(\Omega))^2$, is a unique decomposition of $\mathbf{u}$ of the form :

$$\mathbf{u} = \mathbf{curl}\ (\chi) + \nabla q \tag{4.1}$$

with $\chi \in H_0^1(\Omega)$ and $q \in H^1(\Omega)$. The objective in this section is to compute an approximation in $\mathbb{V}_j^{div}$ of the divergence-free part $u^{div} = \mathbf{curl}\ (\chi)$, using the divergence-free bases built in section 2.3. For simplicity, we use the scaling function basis $(\Phi_{j,\mathbf{k}}^{div})$ , but the wavelet one can be deduced using one-dimensional fast wavelet transform along each direction. $\mathbf{u}_j^{div}$ is searched as its decomposition :

$$\mathbf{u}_j^{div} = \sum_{\mathbf{k}} c_{j,\mathbf{k}}^{div}\ \Phi_{j,\mathbf{k}}^{div} \tag{4.2}$$

By orthogonality of the decomposition (4.1) in $(L^2(\Omega))^2$, one obtains :

$$\langle \mathbf{u}/\Phi_{j,\mathbf{k}}^{div}\rangle = \langle \mathbf{u}_j^{div}/\Phi_{j,\mathbf{k}}^{div}\rangle \quad \text{thus} \quad \mathbb{M}(c_{j,\mathbf{k}}^{div}) = (\langle \mathbf{u}/\Phi_{j,\mathbf{k}}^{div}\rangle) \tag{4.3}$$

where $\mathbb{M}$ the Gram matrix of the basis $\{\Phi_{j,\mathbf{k}}^{div}\}$. The computation of the coefficients $(c_{j,\mathbf{k}}^{div})$ is then reduced to the resolution of a linear system of matrix $\mathbb{M}$. This system can be easily inverted, since $\mathbb{M}$ is no more than the stiffness matrix of the Laplacian onto the scalar scaling function basis $\{\varphi_{j,k_1}^D \otimes \varphi_{j,k_2}^D\}$. Indeed :

$$\forall\ \psi, \phi \in H_0^1(\Omega); \quad \int_\Omega \mathbf{curl}(\psi) \cdot \mathbf{curl}(\phi) dx = \int_\Omega \nabla\psi \cdot \nabla\phi dx \tag{4.4}$$

More details on the implementation and resolution of (4.3) can be found in [23]. Figure 12 presents the $\ell_2$ error on $\mathbf{u}_j^{div}$ of the algorithm, according to the space resolution $j$, using wavelets of example 3.1. This test is performed with the following vector field $\mathbf{u}$, where $\mathbf{u}^{div}$ is known analytically :

$$\mathbf{u} = \mathbf{curl}\ [\sin(2\pi x)x^2(1-x)^2y^2(1-y)^2] + \nabla\ [\cos(2\pi x)x^2y^2]$$

The convergence rate $(-2.3)$ is in accordance with the slope provided by the approximation order $r = 3$ in $V_j^1$ and $r = 2$ in $V_j^0$.

### 4.3. *Stokes problem*

The Stokes problem is a simple test case for the simulation of incompressible flows. In the non-stationary case, and for a velocity $\mathbf{u}$ vanishing at the boundary, it is described by the following equations :

$$\begin{cases} \partial_t \mathbf{v} - \nu\Delta\mathbf{v} + \nabla\mathbf{p} = f \quad \text{in} \quad [0,T]\times\Omega \\ \nabla\cdot\mathbf{v} = 0 \quad \text{in} \quad \Omega \\ \mathbf{v} = 0 \quad \text{on} \quad \Gamma \\ \mathbf{v}(0,x) = \mathbf{v}_0 \end{cases} \tag{4.5}$$

where $\mathbf{p}$ is the corresponding pressure.

Urban was the first who uses *interior* divergence-free wavelets for the resolution of the stationary case[27]. His method uses a variational method[16] in $\mathcal{H}_{div}(\Omega)$ and thus requires the inversion of the stiffness matrix in the divergence-free wavelet basis.

We propose here to use the Helmholtz decomposition to simplify a classical method of resolution, called the Chorin projection method[6]. Our algorithm to compute $\mathbf{v}^n(x) \approx \mathbf{v}(x, n\delta t)$ is the following :

Starting with an initial value $\mathbf{v}^0 = \mathbf{v}(0,x)$, repeat for $1 \le n \le N$

**Step** 1 : Find $\mathbf{a}(x) \in \vec{\mathbf{V}}_j$ solution of

$$\frac{\mathbf{a}-\mathbf{v}^n}{\delta t} = \nu\Delta\frac{1}{2}(\mathbf{a}+\mathbf{v}^n) + \mathbb{P}(f), \quad x \in\ [0,1]^2 \tag{4.6}$$

$$\mathbf{a} = 0 \quad \text{on} \quad \Gamma \tag{4.7}$$

**Step** 2 Find $\mathbf{v}^{n+1} \in \mathbf{V}_j^{div}$ solution of

$$\mathbf{v}^{n+1} = \mathbb{P}(\mathbf{a}) \tag{4.8}$$

where $\mathbb{P}$ is the orthogonal projector from $(L^2(\Omega))^2$ onto $\mathcal{H}_{div}(\Omega)$ computed in practice by the Helmholtz decomposition described in section 4.2.

This method has the advantage of decoupling the resolution of the diffusion term and the incompressibility constraint. Moreover, for more general boundary conditions (for example $\mathbf{v} \cdot \vec{\nu} = 0$ on $\Gamma$), we can incorporate this boundary condition directly in the basis of $\mathcal{H}_{div}(\Omega)$ for the computation (4.8).

Figure 13 plots the $\ell_2$ error between $\mathbf{v}$ and its approximation $\mathbf{v}^n \in \vec{\mathbf{V}}_j$, at time $t = 10^{-2}$ for $\nu = 10^{-5}$ and $\delta t = 1/1000$. The exact solution $\mathbf{v}$ is taken from [19] :

$$\mathbf{v}(x, y, t) = \mathbf{curl}\ [1000x^2(1-x)^2y^2(1-y)^2] \qquad \nabla \mathbf{p} = x^2 + y^2 - \frac{2}{3} \quad (4.9)$$

We used the divergence-free wavelets of example 3.1 to compute the Step 2. The stationary solution is achieved at $t = 10^{-2}$. The convergence slope behaves as $s \approx -2.3$, which is in accordance with the approximation order provided by $\vec{\mathbf{V}}_j$.

## 5. Conclusion

In this article we have presented a practical construction of divergence-free and irrotational multiresolution analyses and wavelets on the square. Our construction, based on one-dimensional analyses on the interval allowing the reproduction of polynomials, respects the theoretical framework established by Jouini and Lemarié-Rieusset [18]. Moreover our construction can incorporate homogeneous boundary conditions in the basis functions, which allows the representation of more physical divergence-free vector functions spaces. This ability is not present, for instance, in the attempt addressed by Stevenson [24].
Associated fast wavelet transforms have been implemented satisfactorily, opening new prospects for the realistic simulation of incompressible flows. First realizations have been successfully presented in this article with the Helmholtz decomposition of a vector flow, or with the computation of a Stokes problem solution. Work on more complex problems are underway, such as the direct simulation of turbulence, and this will be the subject of a forthcoming paper.

## Bibliographie

1. L. Andersson, N. Hall, B. Jawerth, G. Peters, Wavelets on closed subsets of the real line, *Recents Advances in Wavelets Analysis (L.L. Schumaker and G. Webb eds)*, Academic Press (1993) 1–61.
2. G. Battle, P. Federbush, Divergence-free vector wavelets, *Michigan Math. Journ.* **40** (1993) 181–195.
3. G. Beylkin, On the representation of operator in bases of compactly supported wavelets, *SIAM J.Numer.Anal.* **6(6)** (1992) 1716–1740.
4. G. Chiavassa, J. Liandrat, On the Effective Construction of Compactly Supported Wavelets Satisfying Homogeneous Boundary Conditions on the Interval, *Appl. Comput. Harmon. Anal.* **4** (1997) 62–73.
5. Z. Cieselski, T. Figiel, Spline bases in classical function spaces on compact manifolds, *part II, Studia Math.* **LXXVCI** (1983) 95–136.
6. A.J. Chorin, Numerical simulation of the Navier-Stokes equation, *Math. Comp.* **22** (1968) 745–762.
7. A. Cohen, *Wavelet methods in numerical analysis*, Handbook of Numerical Analysis, vol. VII, P.G. Ciarlet and J.L. Lions eds., Elsevier, Amsterdam, 2000.
8. A. Cohen, *Numerical Analysis of Wavelet Methods*, (Elsevier, 2003).
9. A. Cohen, I. Daubechies, J.-C. Feauveau, Biorthogonal bases of compactly supported wavelets,*Comm. Pure Appli. Maths.* **45** (1992) 485–560.
10. A. Cohen, I. Daubechies, P. Vial, Wavelets on the Interval and Fast Wavelet Transforms, *Appl. Comput. Harmon. Anal.* **1** (1993) 54–81.
11. W. Dahmen, A. Kunoth, K. Urban, Biorthogonal Spline-wavelets on the interval-stability and moment conditions, *App. Comput. Harmon. Anal.* **6** (1999) 132–196.
12. W. Dahmen, K. Urban, J. Vorloeper, Adaptive Wavelet Methods-Basic Concepts and Applications to the Stokes Problem, *Wavelet Analysis–Twenty Years Developments, Ding-Xuan Zhou ed., World Scientific, New Jersey* (2002) 39–80.
13. E. Deriaz, V. Perrier, Orthogonal Helmholtz decomposition in arbitrary dimension using divergence-free and curl-free wavelets, *Appl. Comput. Harmon. Anal.* **26(2)** (2009) 249–269.
14. E. Deriaz, V. Perrier, Direct Numerical Simulation of Turbulence using divergence-free wavelets, *Multiscale Modeling & Simulation, SIAM* **7(3)** (2008) 1101–1129.
15. E. Deriaz, V. Perrier, Divergence-free and curl-free wavelets in 2D and 3D, application to turbulent flows, *J. of Turbulence* **7(3)** (2006) 1–37.
16. V. Girault, P.A. Raviart, *Finite element methods for Navier-Stokes equations* Springer-Verlag Berlin, 1986.
17. S. Grivet-Talocia, A. Tabacco, Wavelet on the interval with optimal localization, *Math. Models. Meth. Appl. Sci.* **10(3)** (2000) 441–462.
18. A. Jouini, P.G. Lemarié-Rieusset, Analyse multirésolution biorthogonale sur l'intervalle et applications, *Annales de l'I.H.P. Section C* **10** (1993) 453–476.
19. S. Krell, Stabilized DDFV schemes for Stokes problem, *Proceedings of algorithmy* (2009) 21–30.
20. P.G. Lemarié-Rieusset, Analyses multi-résolutions non orthogonales, commutation entre projecteurs et dérivation et ondelettes vecteurs à divergence nulle, *Revista Matemática Iberoamericana* **8(2)** (1992) 221–236.
21. P. Monasse, V. Perrier, Orthogonal Wavelet Bases Adapted For Partial Differential Equations With Boundary Conditions, *SIAM J.Math. Anal.* **29** (1998) 1040–1065.
22. K.-H. Souleymane, *Ondelettes pour la prise en compte de conditions aux limites en turbulence incompressible*, Phd thesis, Grenoble University, 2010.
23. K.-H. Souleymane, V. Perrier, Helmholtz-Hodge Decomposition on $[0,1]^d$ by

Divergence-free and Curl-free Wavelets, Grenoble university, preprint, submitted 2010.
24. R. Stevenson, Divergence-free wavelet bases on the hypercube, *Appl. Comput. Harmon. Anal.* (2010), doi :10.1016/j.acha.2010.01.007.
25. K. Urban, *Using divergence-free wavelets for the numerical solution of the Stokes problem*, AMLI'96 : Proceedings of the Conference on Algebraic Multilevel Iteration Methods with Applications, **2** : 261–277, University of Nijmegen, The Netherlands, 1996.
26. K. Urban, *Wavelet Bases in H(div) and H(curl)*, *Math. Comput.* **70** (2000) 739–766.
27. K. Urban, *Wavelets in Numerical Simulation*, (Springer Berlin, 2002).

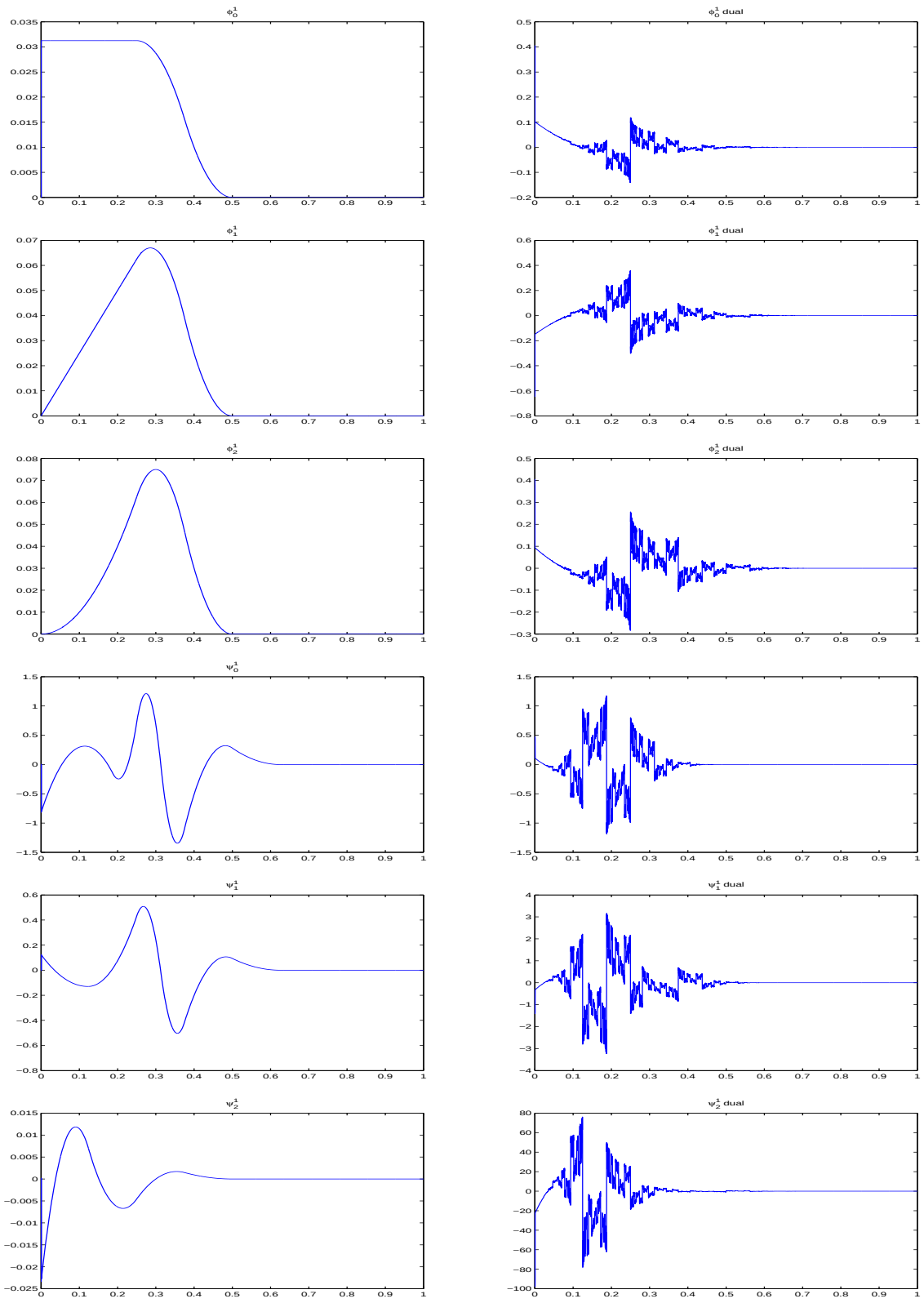


FIGURE 1. Left : scaling functions $\Phi_\ell^{1,\flat}$ (three first rows) and wavelets $\Psi_\ell^{1,\flat}$ (three last rows) and their biorthogonals $\tilde{\Phi}_\ell^{1,\flat}$, $\tilde{\Psi}_\ell^{1,\flat}$ (right) : B-Spline generators with $r = \tilde{r} = 3$.

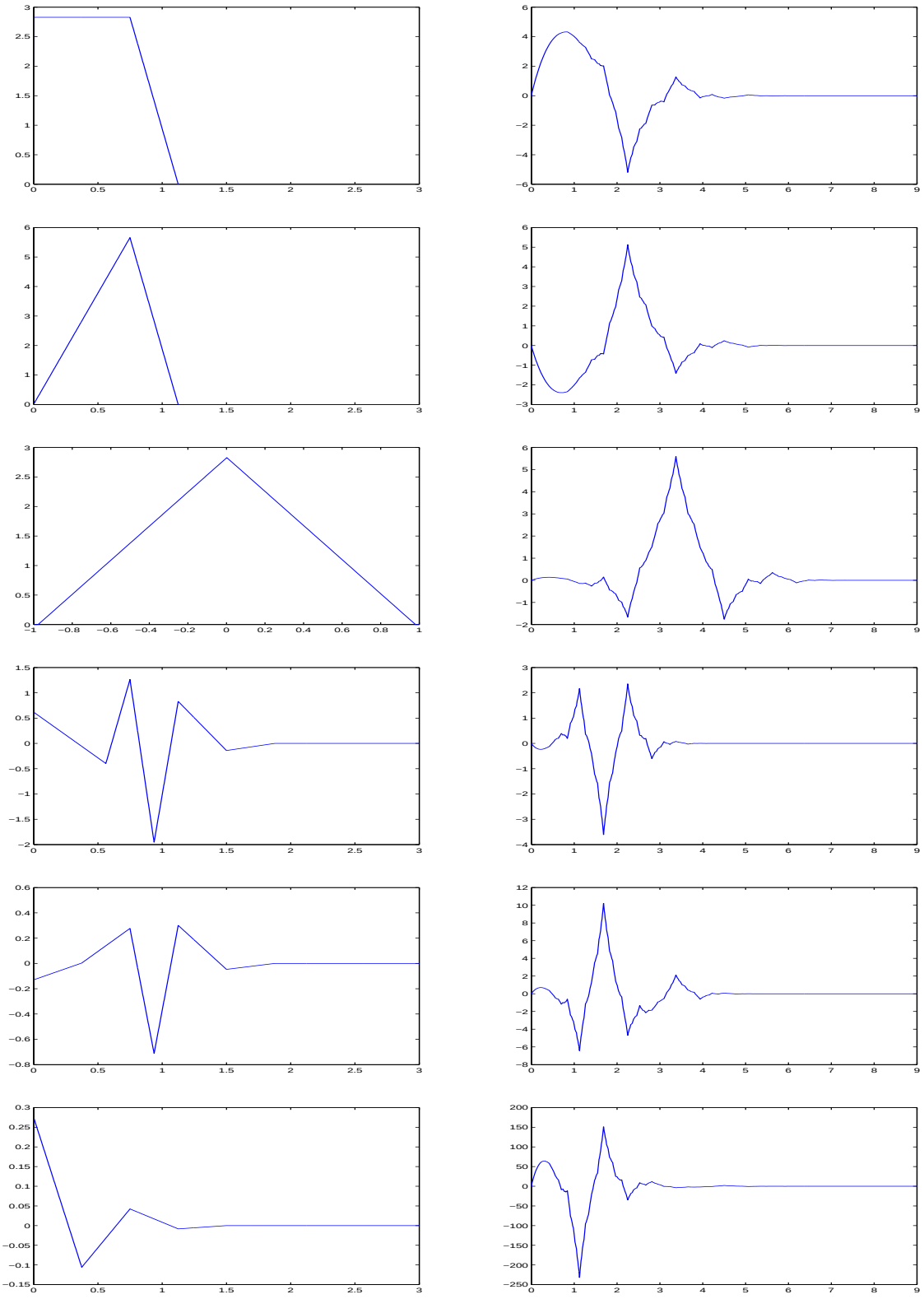

FIGURE 2. Left : scaling functions $\Phi_\ell^{0,\flat}$ (three first rows) and wavelets $\Psi_\ell^{0,\flat}$ (three last rows) and their biorthogonals $\tilde{\Phi}_\ell^{0,\flat}$, $\tilde{\Psi}_\ell^{0,\flat}$ (right), satisfying Dirichlet boundary condition.

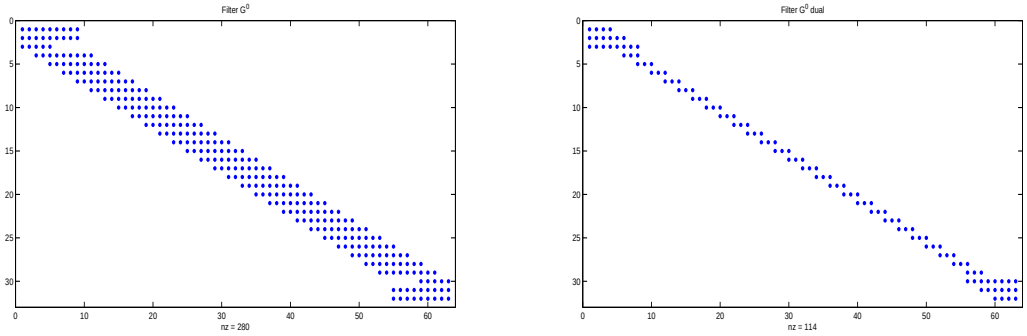


FIGURE 3. Wavelet filter $G_j^0$ (left) and conjugate wavelet filter $\tilde{G}_j^0$ (right), associated to functions of Figure 2 for $j = 6$.

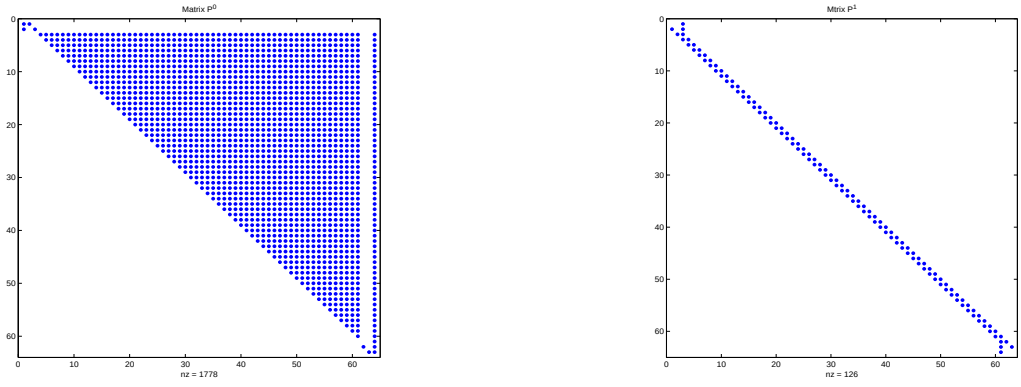


FIGURE 4. Matrices of change of bases $L_j^0$ (left) and $L_j^1$ (right), for $j = 6$.

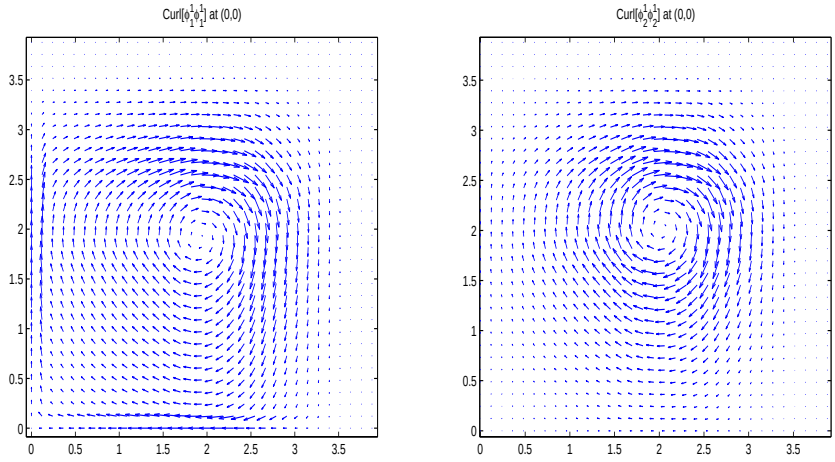


FIGURE 5. Vector field of divergence-free scaling functions $\mathbf{curl}[\Phi_1^{1,b} \otimes \Phi_1^{1,b}]$ and $\mathbf{curl}[\Phi_2^{1,b} \otimes \Phi_2^{1,b}]$, constructed from edge scaling functions : $\ell = 1,\ 2$.

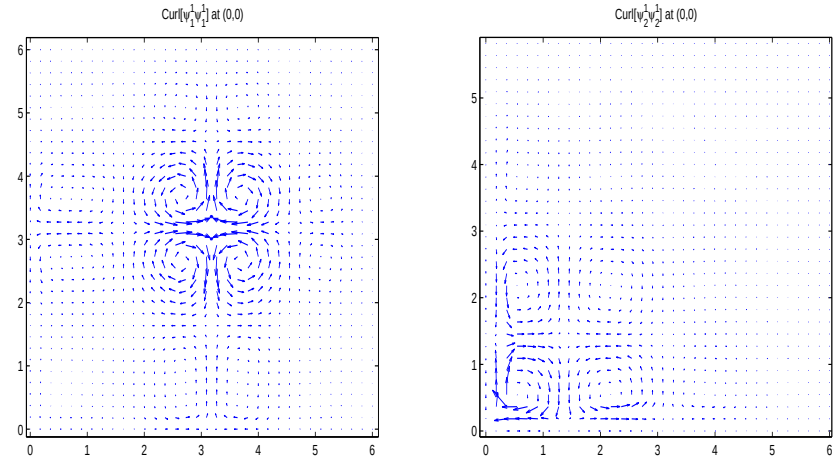


FIGURE 6. Vector field of divergence-free wavelets $\mathbf{curl}[\Psi_1^{1,b} \otimes \Psi_1^{1,b}]$ and $\mathbf{curl}[\Psi_2^{1,b} \otimes \Psi_2^{1,b}]$, constructed from edge wavelets : $\ell = 1,\ 2$.

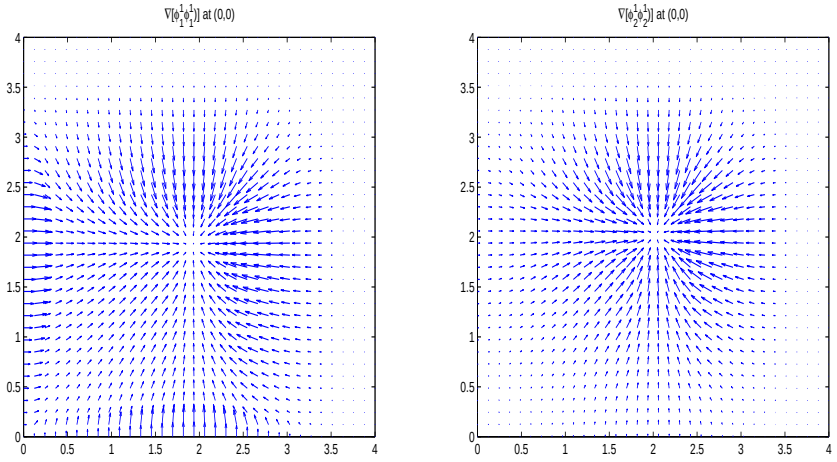


FIGURE 7. Vector field of curl-free scaling functions $\nabla[\Phi_1^{1,\flat} \otimes \Phi_1^{1,\flat}]$ and $\nabla[\Phi_2^{1,\flat} \otimes \Phi_2^{1,\flat}]$, constructed from edge scaling functions : $\ell = 1, 2$.

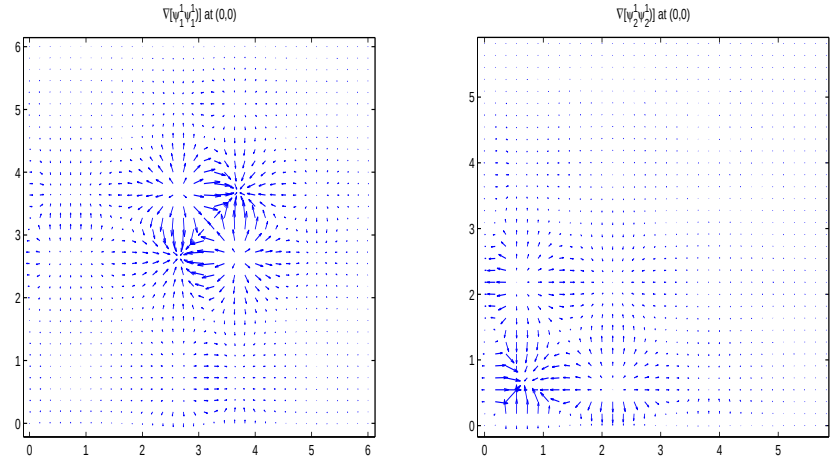


FIGURE 8. Vector field of curl-free wavelets $\nabla[\Psi_1^{1,\flat} \otimes \Psi_1^{1,\flat}]$ and $\nabla[\Psi_2^{1,\flat} \otimes \Psi_2^{1,\flat}]$, constructed from edge wavelets : $\ell = 1, 2$.

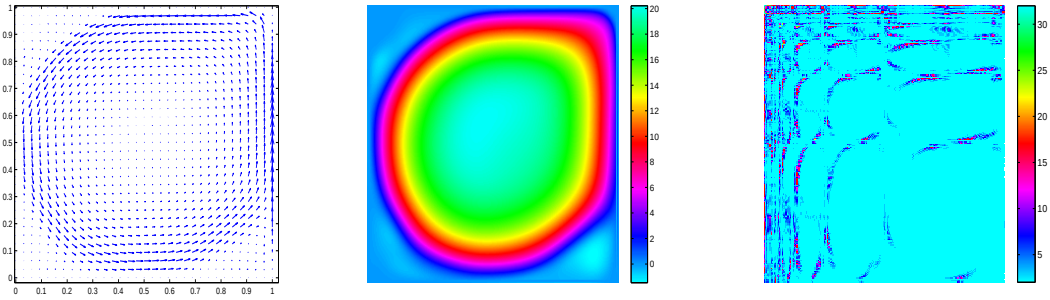

FIGURE 9. Example of vector field (left), its divergence-free scaling function coefficients (middle) and renormalized wavelet coefficients (right).

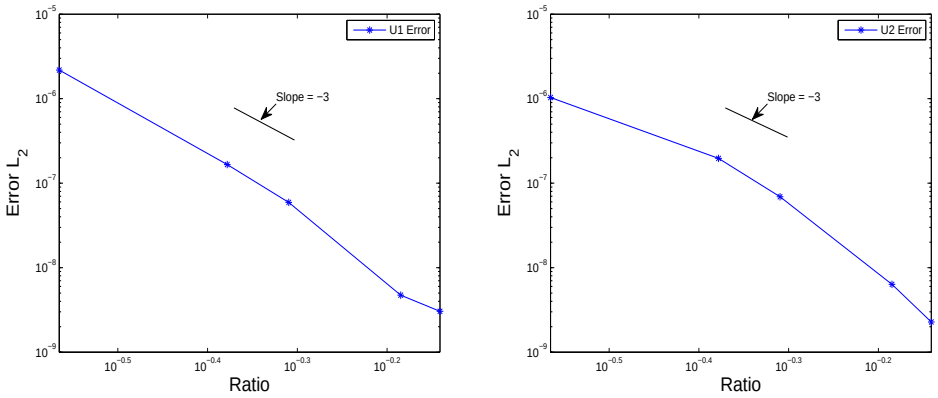


FIGURE 10. Error on the first component $\mathbf{u}_1$ (left) and on the second component $\mathbf{u}_2$ (right), provided by the nonlinear approximation of $\mathbf{u}$ in $\mathbb{V}_j^{div}$, versus the ratio of retained coefficients (log-log scale).

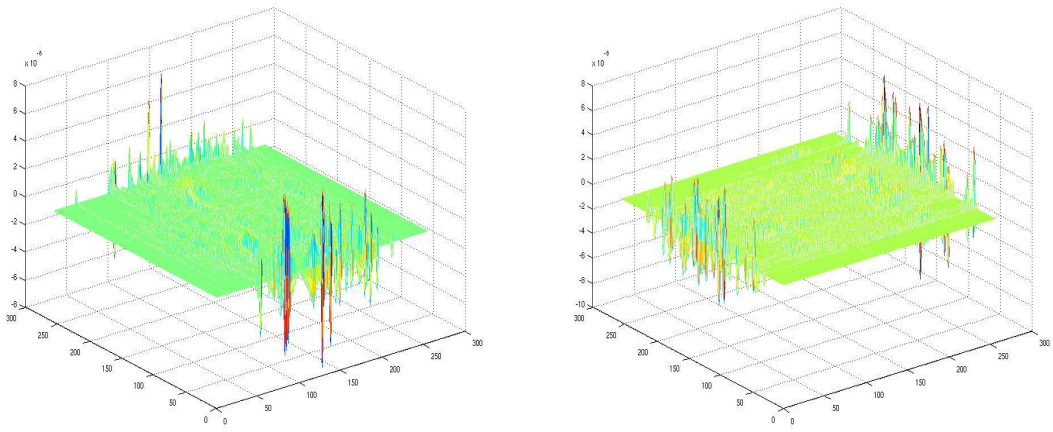

FIGURE 11. Error on $\mathbf{u}_1$ (left) and $\mathbf{u}_2$ (right) reconstructed from 32% of their divergence-free wavelet coefficients.

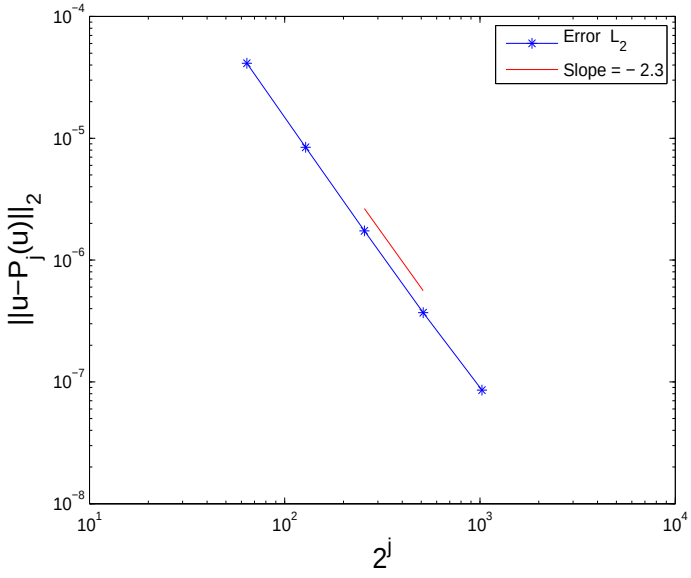


FIGURE 12. $l_2$-projection error of $\mathbf{u}$ onto $\mathbf{V}_j^{div}$, versus $2^j$ (log-log scale).

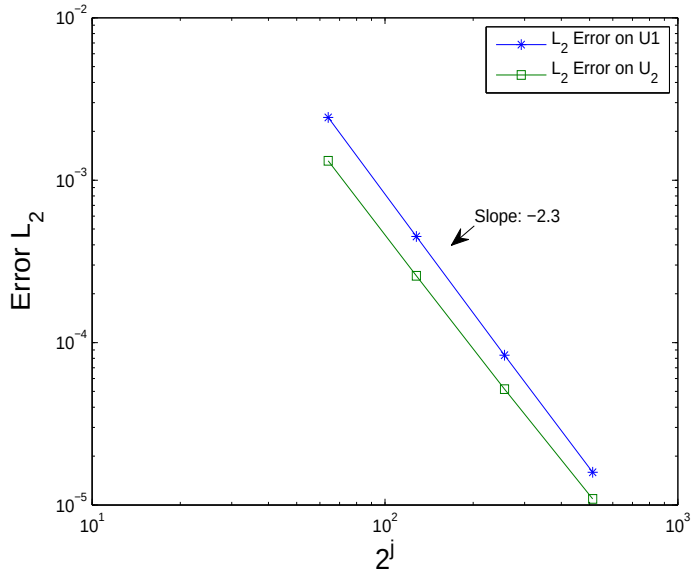


FIGURE 13. Relative $\ell_2$ errors on the components $\mathbf{u}_1$ and $\mathbf{u}_2$ of the numerical Stokes problem solution, according to the resolution $j$, at time $t = 10^{-2}$ for $\nu = 10^{-5}$.