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GIRSANOV THEORY UNDER A FINITE ENTROPY CONDITION

CHRISTIAN LÉONARD

ABSTRACT. This paper is about Girsanov's theory. It (almost) doesn't contain new results but it is based on a simplified new approach which takes advantage of the (weak) extra requirement that some relative entropy is finite. Under this assumption, we present and prove all the standard results pertaining to the absolute continuity of two continuous-time processes on \mathbb{R}^d with or without jumps. We have tried to give as much as possible a self-contained presentation.

The main advantage of the finite entropy strategy is that it allows us to replace martingale representation results by the simpler Riesz representations of the dual of a Hilbert space (in the continuous case) or of an Orlicz function space (in the jump case).

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1. INTRODUCTION

This paper is about Girsanov's theory. It (almost) doesn't contain new results but it is based on a simplified new approach which takes advantage of the (weak) extra requirement that some relative entropy is finite. Under this assumption, we present and prove all the standard results pertaining to the absolute continuity of two continuous-time processes on \mathbb{R}^d with or without jumps.

This article intends to look like lecture notes and we have tried to give as much as possible a self-contained presentation of Girsanov's theory. The author hopes that it could be useful for students and also to readers already acquainted with stochastic calculus.

The main advantage of the finite entropy strategy is that it allows us to replace martingale representation results by the simpler Riesz representations of the dual of a Hilbert space (in the continuous case) or of an Orlicz function space (in the jump case). The gain is especially interesting in the jump case where martingale representation results are not easy, see [Jac75]. Another feature of this simplified approach is that very few about exponential martingales is needed.

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Girsanov's theory studies the relation between a reference process R and another process P which is assumed to be absolutely continuous with respect to R . In particular, it is known that if R is the law of an \mathbb{R}^d -valued semimartingale, then P is also the law of a semimartingale. In its wide meaning, this theory also provides us with a formula for the Radon-Nikodým density $\frac{dP}{dR}$.

In this article, we assume that the probability measure P has its relative entropy with respect to R :

$$H(P|R) := \begin{cases} E_P \log \left(\frac{dP}{dR} \right) \in [0, \infty] & \text{if } P \ll R \\ +\infty & \text{otherwise,} \end{cases}$$

which is finite, i.e.

$$H(P|R) = E_R \left[\frac{dP}{dR} \log \left(\frac{dP}{dR} \right) \right] < \infty. \quad (1)$$

In comparison, requiring $P \ll R$ only amounts to assume that

$$E_R \left(\frac{dP}{dR} \right) < \infty \quad (2)$$

since P has a finite mass. We are going to take advantage of the only difference between (1) and (2) which is the stronger integrability property carried by the extra term $\log \frac{dP}{dR}$.

A key argument of this approach is the variational representation of the relative entropy which is stated at Proposition 3.1. Some versions of this result are well-known and widely used. We decided to give a (usually unknown) complete picture of this very useful variational representation together with a complete elementary proof. We think that this complete picture is interesting in its own right.

A clear exposition of the general Girsanov's theorems, with no explicit expression of $\frac{dP}{dR}$ in terms of the characteristics of the processes, is given in P. Protter's textbook [Pro04]. The most complete results about Girsanov's theory for \mathbb{R}^d -valued processes, including explicit formulas for $\frac{dP}{dR}$, are available in J. Jacod's textbook [Jac79]. An alternate presentation of this realm of results is also given in the later book by J. Jacod and A. Shiryaev [JS87]. A good standard reference in the continuous case is D. Revuz and M. Yor's textbook [RY99] about continuous martingales.

Next Section 2 is devoted to the statement of the main results. At Section 3, we state and prove the above mentioned variational representation of the relative entropy. At Sections 4 and 5, we present the proofs of Theorems 2.1 and 2.3 which correspond to the continuous case. At Section 6, we give the proofs of Theorems 2.6 and 2.9 which correspond to the jump case.

2. STATEMENT OF THE RESULTS

We distinguish the cases where the sample paths are continuous and where they exhibit jumps.

Continuous processes in \mathbb{R}^d . The paths which we consider are built on the time interval $[0, 1]$. An \mathbb{R}^d -valued continuous stochastic process is a random variable taking its values in the set

$$\Omega = C([0, 1], \mathbb{R}^d)$$

of all continuous paths from $[0, 1]$ to \mathbb{R}^d . The canonical process $(X_t)_{t \in [0, 1]}$ is defined by

$$X_t(\omega) = \omega_t, \quad t \in [0, 1], \omega = (\omega_s)_{s \in [0, 1]} \in \Omega.$$

In other words, $X = (X_t)_{t \in [0, 1]}$ is the identity on Ω and $X_t : \Omega \rightarrow \mathbb{R}^d$ is the t -th projection. The set Ω is endowed with the σ -field $\sigma(X_t; t \in [0, 1])$ which is generated by the canonical

projections. We also consider the canonical filtration $(\sigma(X_{[0,t]}); t \in [0, 1])$ where for each t , $X_{[0,t]} := (X_s)_{s \in [0,t]}$.

Let us give ourself a reference probability measure R on Ω such that X admits the R -semimartingale decomposition

$$X = X_0 + B^R + M^R, \quad R\text{-a.s.} \quad (3)$$

This means that B^R is an adapted process with bounded variation sample paths R -a.s. and M^R is a local R -martingale, i.e. there exists an increasing sequence of stopping times $(\tau_k)_{k \geq 1}$ which converges to infinity R -a.s. and such that for each $k \geq 1$, the stopped process $t \mapsto M_{t \wedge \tau_k}^R$ is a uniformly integrable R -martingale.

As a typical example, one may think of the solution to the SDE (if it exists)

$$X_t = X_0 + \int_{[0,t]} b_s(X_{[0,s]}) ds + \int_{[0,t]} \sigma_s(X_{[0,s]}) dW_s, \quad 0 \leq t \leq 1 \quad (4)$$

where W is a Wiener process on \mathbb{R}^d and $b : [0, 1] \times \Omega \rightarrow \mathbb{R}^d$ and $\sigma : [0, 1] \times \Omega \rightarrow \mathbf{M}_{d \times d}$ are locally bounded. In this situation, a natural localizing sequence $(\tau_k)_{k \geq 1}$ is the sequence of exit times from the Euclidean balls of radius k , $B_t^R = \int_0^t b_s(X_{[0,s]}) ds$ has absolutely continuous sample paths R -a.s. and $M_t^R = \int_0^t \sigma_s(X_{[0,s]}) dW_s$ has the quadratic variation

$$[M^R, M^R]_t = \int_0^t a_s ds \quad R\text{-a.s.} \quad (5)$$

where $a_t := \sigma_t \sigma_t^*(X_{[0,t]})$ takes its values in the set \mathbf{S}_+ of all positive semi-definite $d \times d$ matrices.

More generally, we assume that the quadratic variation of M^R is a process which is R -a.s. equal to a random element of the set $\mathcal{M}_{\mathbf{S}_+}^{\text{na}}([0, 1])$ of all bounded measures on $[0, 1]$ with no atoms and taking their values in \mathbf{S}_+ :

$$[M^R, M^R](dt) = A(dt) \in \mathcal{M}_{\mathbf{S}_+}^{\text{na}}([0, 1]), \quad R\text{-a.s.} \quad (6)$$

and also that

$$t \in [0, 1] \mapsto [M^R, M^R]_t := A([0, t]) = A(t, X_{[0,t]}; [0, t]) \in \mathbf{S}_+, \quad R\text{-a.s.}$$

is an adapted process. The quadratic variation given at (6) might have an atomless singular part in addition to its absolutely continuous component $a_t dt$. This notation is concise: $A(dt)$ is random and for any \mathbb{R}^d -valued processes α, β , $\alpha_t \cdot A(dt) \beta_t$ is the infinitesimal element of a measure on $[0, 1]$. In particular, $t \mapsto \int_{[0,t]} A(ds) \beta_s \in \mathbb{R}^d$, $t \mapsto \int_{[0,t]} \alpha_s \cdot A(ds) \beta_s \in \mathbb{R}$ and the process $t \mapsto \int_{[0,t]} \beta_s \cdot A(ds) \beta_s \in \mathbb{R}$ is increasing.

Summing up, R is a solution to the *martingale problem* $\text{MP}(B^R, A)$. This means that the canonical process X is the sum (3) of a bounded variation adapted process B^R and a local R -martingale M^R whose quadratic variation is specified by A and (6). We write

$$R \in \text{MP}(B^R, A)$$

for short.

Theorem 2.1 (Girsanov's theorem). *Let R and P be as above, satisfying in particular the finite entropy condition (1). Then, P is the law of a semimartingale. More precisely, there exists an \mathbb{R}^d -valued adapted process β satisfying*

$$E_P \int_{[0,1]} \beta_t \cdot A(dt) \beta_t < \infty \quad (7)$$

and such that, defining

$$\widehat{B}_t := \int_{[0,t]} A(ds) \beta_s, \quad 0 \leq t \leq 1, \quad (8)$$

one obtains

$$X = X_0 + B^R + \widehat{B} + M^P, \quad P\text{-a.s.}$$

where M^P is a local P -martingale such that $[M^P, M^P] = [M^R, M^R]$, P -a.s.

In other words, $P \in \text{MP}(B^R + \widehat{B}, A)$.

Remarks 2.2.

- (a) The process β only needs to be defined P -a.s. (and not R -a.s.) for the statement of Theorem 2.1 to be meaningful. In fact, its proof only gives the “construction” of a process β , P -almost everywhere.
- (b) The process \widehat{B} is well defined. Indeed, by Cauchy-Schwarz inequality, for any \mathbb{R}^d -valued process ξ ,

$$\int_{[0,1]} |\xi_t \cdot A(dt) \beta_t| \leq \left(\int_{[0,1]} \beta_t \cdot A(dt) \beta_t \right)^{1/2} \left(\int_{[0,1]} \xi_t \cdot A(dt) \xi_t \right)^{1/2} \in [0, \infty], \quad P\text{-a.s.}$$

Looking at $A(\omega)$ with ω fixed as a matrix of measures, we see that $\sup\{\int_{[0,1]} \xi_t \cdot A(dt) \xi_t; \xi : |\xi_t| = 1, \forall t\}$ is bounded above by the sum of the total variations of the entries of A . Consequently, this supremum is finite P -a.s. On the other hand, as $E_P \int_{[0,1]} \beta_t \cdot A(dt) \beta_t < \infty$, we see that a fortiori $\int_{[0,1]} \beta_t \cdot A(dt) \beta_t < \infty$, P -a.s. It follows that $\int_{[0,1]} |A(dt) \beta_t| < \infty$, P -a.s. which means that \widehat{B} is well defined.

- (c) When the quadratic variation is given by (5), one retrieves the standard representation

$$\widehat{B}_t = \int_{[0,t]} a_s \beta_s ds.$$

It is then known that under the minimal assumption (2), Theorem 2.1 still holds true with

$$\int_{[0,1]} \beta_t \cdot a_t \beta_t dt < \infty, \quad P\text{-a.s.}$$

instead of $E_P \int_{[0,1]} \beta_t \cdot a_t \beta_t dt < \infty$ under the assumption (1), see for instance [JS87, Chp. III].

For any probability Q on Ω , let us denote $Q_0 = X_{0\#}Q$ the law of the initial position X_0 under Q .

Definition (Condition (U)). One says that $R \in \text{MP}(B^R, A)$ satisfies the *uniqueness condition* (U) if for any probability measure R' on Ω such that the initial laws $R'_0 = R_0$ are equal, $R' \ll R$ and $R' \in \text{MP}(B^R, A)$, we have $R = R'$.

It is known [Jac75] that if the SDE (4) admits a unique solution, for instance if the coefficients b and σ are locally Lipschitz, then its law R satisfies (U).

Theorem 2.3 (The density dP/dR). *Let R and P be as above, satisfying in particular the finite entropy condition (1). Keeping the notation of Theorem 2.1, we have*

$$H(P_0|R_0) + \frac{1}{2} E_P \int_{[0,1]} \beta_t \cdot A(dt) \beta_t \leq H(P|R).$$

If in addition it is assumed that R satisfies the uniqueness condition (U), then

$$H(P_0|R_0) + \frac{1}{2}E_P \int_{[0,1]} \beta_t \cdot A(dt)\beta_t = H(P|R)$$

and

$$\begin{aligned} \frac{dP}{dR} &= \mathbf{1}_{\{\frac{dP}{dR} > 0\}} \frac{dP_0}{dR_0}(X_0) \exp \left(\int_{[0,1]} \beta_t \cdot dM_t^R - \frac{1}{2} \int_{[0,1]} \beta_t \cdot A(dt)\beta_t \right) \\ &= \mathbf{1}_{\{\frac{dP}{dR} > 0\}} \frac{dP_0}{dR_0}(X_0) \exp \left(\int_{[0,1]} \beta_t \cdot (dX_t - dB_t^R) - \frac{1}{2} \int_{[0,1]} \beta_t \cdot A(dt)\beta_t \right). \end{aligned}$$

Recall that (7) implies that $\int_{[0,1]} \beta_t \cdot A(dt)\beta_t < \infty$, P -a.s. It follows that, although the process β is defined only P -a.s., the stochastic integral $\int_{[0,1]} \beta_t \cdot dM_t^R$ is meaningful on $\{\frac{dP}{dR} > 0\}$.

Processes with jumps in \mathbb{R}^d . The law of a process with jumps is a probability measure P on the canonical space

$$\Omega = D([0, 1], \mathbb{R}^d)$$

of all left limited and right continuous (càdlàg) paths, endowed with its canonical filtration. We denote $X = (X_t)_{t \in [0,1]}$ the canonical process,

$$\Delta X_t = X_t - X_{t-}$$

the jump at time t and $\mathbb{R}_*^d := \mathbb{R}^d \setminus \{0\}$ the set of all effective jumps.

A Lévy kernel is a random σ -finite positive measure

$$\bar{L}_\omega(dt dq) = \rho(dt) L_\omega(t, dq), \quad \omega \in \Omega$$

on $[0, 1] \times \mathbb{R}_*^d$ where ρ is assumed to be a σ -finite positive *atomless* measure on $[0, 1]$. As a definition, any Lévy kernel is assumed to be predictable, i.e. $L_\omega(t, dq) = L(X_{[0,t)}(\omega); t, dq)$ for all $t \in [0, 1]$.

Let B be a bounded variation *continuous* adapted process.

Definition 2.4 (Lévy kernel and martingale problem). *We say that a probability measure P on Ω solves the martingale problem $\text{MP}(B, \bar{L})$ if the integrability assumption*

$$E_P \int_{[0,1] \times \mathbb{R}_*^d} (|q|^2 \wedge 1) \bar{L}(dt dq) < \infty \tag{9}$$

holds and for any function f in $\mathcal{C}_b^2(\mathbb{R}^d)$, the process

$$\begin{aligned} f(\tilde{X}_t) - f(\tilde{X}_0) - \int_{(0,t] \times \mathbb{R}_*^d} [f(\tilde{X}_{s-} + q) - f(\tilde{X}_{s-}) - \nabla f(\tilde{X}_{s-}) \cdot q] \mathbf{1}_{\{|q| \leq 1\}} \bar{L}(ds dq) \\ - \int_{(0,t] \times \mathbb{R}_*^d} [f(\tilde{X}_{s-} + q) - f(\tilde{X}_{s-})] \mathbf{1}_{\{|q| > 1\}} \bar{L}(ds dq) \end{aligned}$$

is a local P -martingale, where $\tilde{X} := X - B$. We write this

$$P \in \text{MP}(B, \bar{L})$$

for short. In this case, we also say that P admits the Lévy kernel \bar{L} and we denote this property

$$P \in \text{LK}(\bar{L})$$

for short.

If $P \in \text{MP}(B, \bar{L})$, the canonical process is decomposed as

$$X = X_0 + B + (\mathbf{1}_{\{|q|>1\}}q) \odot \mu^X + (\mathbf{1}_{\{|q|\leq 1\}}q) \odot \tilde{\mu}^L, \quad P\text{-a.s.} \quad (10)$$

where

$$\mu^X := \sum_{t \in [0,1]; \Delta X_t \neq 0} \delta_{(t, \Delta X_t)}$$

is the canonical jump measure, $\varphi(q) \odot \mu^X = \int_{[0,1] \times \mathbb{R}_*^d} \varphi d\mu^X = \sum_{t \in [0,1]; \Delta X_t \neq 0} \varphi(\Delta X_t)$ and $\varphi(q) \odot \tilde{\mu}^L$ is the P -stochastic integral with respect to the compensated sum of jumps

$$\tilde{\mu}_\omega^L(dtdq) := \mu_\omega^X(dtdq) - \bar{L}_\omega(dtdq).$$

Definition 2.5 (Class $\mathcal{H}_{p,r}(P, \bar{L})$). *Let P be a probability measure on Ω and \bar{L} a Lévy kernel such that $P \in \text{LK}(\bar{L})$. We say that a predictable integrand $h_\omega(t, q)$ is in the class $\mathcal{H}_{p,r}(P, \bar{L})$ if $E_P \int_{[0,1] \times \mathbb{R}_*^d} \mathbf{1}_{\{|q|\leq 1\}} |h_t(q)|^p \bar{L}(dtdq) < \infty$ and $E_P \int_{[0,1] \times \mathbb{R}_*^d} \mathbf{1}_{\{|q|>1\}} |h_t(q)|^r \bar{L}(dtdq) < \infty$.*

We denote $\mathcal{H}_{p,p}(P, \bar{L}) = \mathcal{H}_p(P, \bar{L})$.

We take our reference law R such that

$$R \in \text{MP}(B^R, \bar{L})$$

for some adapted continuous bounded variation process B^R . The integrability assumption (9) means that the integrand $|q|$ is in $\mathcal{H}_{2,0}(R, \bar{L})$. This will be always assumed in the future. We introduce the function

$$\theta(a) = \log \mathbb{E} e^{a(N-1)} = e^a - a - 1, \quad a \in \mathbb{R}.$$

where N is a Poisson(1) random variable. Its convex conjugate is

$$\theta^*(b) = \begin{cases} (b+1) \log(b+1) - 1 & \text{if } b > -1 \\ 1 & \text{if } b = -1 \\ \infty & \text{otherwise} \end{cases}, \quad b \in \mathbb{R}$$

Note that θ and θ^* are respectively equivalent to $a^2/2$ and $b^2/2$ near zero.

Theorem 2.6 (Girsanov's theorem. The jump case). *Let R and P be as above: $R \in \text{MP}(B^R, \bar{L})$ and $H(P|R) < \infty$. Then, there exists a unique predictable nonnegative process $\ell : \Omega \times [0, 1] \times \mathbb{R}_*^d \rightarrow [0, \infty)$ satisfying*

$$E_P \int_{[0,1] \times \mathbb{R}_*^d} \theta^*(|\ell - 1|) d\bar{L} < \infty, \quad (11)$$

such that $P \in \text{MP}(B^R + \hat{B}^\ell, \ell \bar{L})$ where

$$\hat{B}_t^\ell = \int_{[0,t] \times \mathbb{R}_*^d} \mathbf{1}_{\{|q|\leq 1\}} (\ell_s(q) - 1) q \bar{L}(dsdq), \quad t \in [0, 1]$$

is well-defined P -a.s.

It will appear that, in several respects, $\log \ell$ is analogous to β in Theorem 2.1. Again, ℓ only needs to be defined P -a.s. and not R -a.s. for the statement of Theorem 2.6 to be meaningful. And indeed, its proof will only provide a P -a.s.-construction of ℓ .

Corollary 2.7. *Suppose that in addition to the assumptions of Theorem 2.6, there exist some $a_o, b_o, c_o > 0$ such that*

$$E_R \exp \left(a_o \int_{[0,1] \times \mathbb{R}_*^d} \mathbf{1}_{\{|q|>c_o\}} e^{b_o|q|} \bar{L}(dtdq) \right) < \infty. \quad (12)$$

It follows immediately that $\mathbf{1}_{\{|q|>c_0\}}|q|$ is $R \otimes \bar{L}$ -integrable so that the stochastic integral $q \odot \tilde{\mu}^L$ is well-defined R -a.s. and we are allowed to rewrite (10) as

$$X = X_0 + B + q \odot \tilde{\mu}^L, \quad R\text{-a.s.},$$

for some adapted continuous bounded variation process B .

Then, there exists a unique predictable nonnegative process $\ell : \Omega \times [0, 1] \times \mathbb{R}_*^d \rightarrow [0, \infty)$ satisfying (11) such that

$$X = X_0 + B + \bar{B}^\ell + q \odot \tilde{\mu}^{\ell L}, \quad P\text{-a.s.},$$

where

$$\bar{B}_t^\ell = \int_{[0,t] \times \mathbb{R}_*^d} (\ell_s(q) - 1) q \bar{L}(ds dq), \quad t \in [0, 1]$$

is well-defined P -a.s. and the P -stochastic integral $q \odot \tilde{\mu}^{\ell L}$ with respect to the Lévy kernel $\ell \bar{L}$ is a local P -martingale.

Remarks 2.8.

- (a) The energy estimate (11) is equivalent to: $\mathbf{1}_{\{0 \leq \ell \leq 2\}}(\ell - 1)^2$ and $\mathbf{1}_{\{\ell \geq 2\}}\ell \log \ell$ are integrable with respect to $P \otimes \bar{L}$.
- (b) Together with (11), (12) implies that the integral for \bar{B}^ℓ is well-defined since

$$E_P \int_{[0,1] \times \mathbb{R}_*^d} (\ell_t(q) - 1) |q| \bar{L}(dt dq) < \infty. \quad (13)$$

In the present context of processes with jumps, the uniqueness condition (U) becomes:

Definition (Condition (U)). One says that $R \in \text{MP}(B^R, \bar{L})$ satisfies the *uniqueness condition* (U) if for any probability measure R' on Ω such that the initial laws $R'_0 = R_0$ are equal, $R' \ll R$ and $R' \in \text{MP}(B^R, \bar{L})$, we have $R = R'$.

Theorem 2.9 (The density dP/dR). Suppose that R and P verify $R \in \text{MP}(B, \bar{L})$ and $H(P|R) < \infty$. With ℓ given at Theorem 2.6, we have

$$H(P_0|R_0) + E_P \int_{[0,1] \times \mathbb{R}_*^d} (\ell \log \ell - \ell + 1) d\bar{L} \leq H(P|R)$$

with the convention $0 \log 0 - 0 + 1 = 1$.

If in addition it is assumed that R satisfies the uniqueness condition (U), then

$$H(P_0|R_0) + E_P \int_{[0,1] \times \mathbb{R}_*^d} (\ell \log \ell - \ell + 1) d\bar{L} = H(P|R)$$

and

$$\frac{dP}{dR} = \mathbf{1}_{\{\frac{dP}{dR} > 0\}} \frac{dP_0}{dR_0}(X_0) \widetilde{\exp} \left(\log \ell \odot \tilde{\mu}_1^L - \int_{[0,1] \times \mathbb{R}_*^d} \theta(\log \ell) d\bar{L} \right). \quad (14)$$

In formula (14), $\widetilde{\exp}$ indicates a shorthand for the rigorous following expression

$$\left\{ \begin{array}{l} \frac{dP}{dR} = \frac{dP_0}{dR_0}(X_0) Z^+ Z^- \quad \text{with} \\ Z^+ = \mathbf{1}_{\{\frac{dP}{dR} > 0\}} \exp \left([\mathbf{1}_{\{\ell \geq 1/2\}} \log \ell] \odot \tilde{\mu}_1^L - \int_{\{\ell \geq 1/2\}} (\ell - \log \ell - 1) d\bar{L} \right) \\ Z^- = \mathbf{1}_{\{\frac{dP}{dR} > 0, \tau^- = \infty\}} \exp \left(- \int_{\{0 \leq \ell < 1/2\}} [\ell - 1] d\bar{L} \right) \prod_{0 \leq t \leq 1; 0 < \ell(t, \Delta X_t) < 1/2} \ell(t, \Delta X_t) \end{array} \right. \quad (15)$$

where

$$\tau^- := \sup_{n \geq 1} \inf \{t \in [0, 1]; \ell(t, \Delta X_t) \leq 1/n\} \in [0, 1] \cup \{\infty\},$$

with the convention that $\inf \emptyset = \infty$.

Note that although ℓ is only defined P -a.s., Z^+ , Z^- and τ^- are meaningful thanks to the prefactors $\mathbf{1}_{\{\frac{dP}{dR} > 0\}}$.

Remarks 2.10.

- (a) Because of (11), the integral $\int_{\{\ell \geq 1/2\}} (\ell - \log \ell - 1) d\bar{L}$ inside Z^+ is finite P -a.s.
- (b) Similarly, the product $\prod_{0 \leq t \leq 1; 0 < \ell(t, \Delta X_t) < 1/2} \ell(t, \Delta X_t)$ doesn't vanish P -a.s. because it is proved at Lemma 6.3 that $P(\tau^- = \infty) = 1$.
- (c) Note that this product is well-defined in $[0, 1]$ since it contains P -a.s. at most countably many terms in $(0, 1/2]$. But, if it contains infinitely many such terms, it vanishes. Therefore, it contains P -a.s. finitely many terms.
- (d) Since $\inf \{t \in [0, 1]; \ell(t, \Delta X_t) = 0\} \geq \tau^-$, if $\ell(t, \Delta X_t) = 0$ for some $t \in [0, 1]$, then $\frac{dP}{dR} = 0$. Therefore, $\ell > 0$, P -a.s.
- (e) If $\mathbf{1}_{\{\ell \geq 1/2\}} \log \ell$ is $R \otimes \bar{L}$ -integrable, an alternate expression of $\frac{dP}{dR}$ is
$$\frac{dP}{dR} = \mathbf{1}_{\{\frac{dP}{dR} > 0, \tau^- = \infty\}} \frac{dP_0}{dR_0}(X_0) \exp \left(- \int_{[0, 1] \times \mathbb{R}_*^d} (\ell - 1) d\bar{L} \right) \prod_{0 \leq t \leq 1} \ell(t, \Delta X_t).$$
- (f) If $\mathbf{1}_{\{0 \leq \ell < 1/2\}} \log \ell$ is not $R \otimes \bar{L}$ -integrable, then $\log \ell \odot \tilde{\mu}_1^L$ is undefined and (14) with \exp instead of $\widetilde{\exp}$ is meaningless and must be replaced by (15).
- (g) On the other hand, if $\ell > 0$, R -a.s. and $E_R \int_{[0, 1] \times \mathbb{R}_*^d} \theta(\log \ell) d\bar{L} < \infty$, then (14) gives the rigorous expression for $\frac{dP}{dR}$ with \exp instead of $\widetilde{\exp}$.

For more details about the relationship between (14) and (15), see the discussion below Proposition A.1 at the Appendix.

3. VARIATIONAL REPRESENTATIONS OF THE RELATIVE ENTROPY

Theorems 2.1 and 2.6's proofs rely on some variational representation of the relative entropy which is stated and proved below.

Proposition 3.1 (Variational representations of the relative entropy). *Let R be a probability measure on some space Ω .*

(1) *For any signed bounded measure P on Ω , we have*

$$\begin{aligned} & \sup \left\{ \int u dP - \log \int e^u dR; u \text{ bounded measurable} \right\} \\ &= \sup \left\{ \int u dP - \log \int e^u dR; u \in L^\infty(P) \right\} \\ &= \begin{cases} H(P|R) \in [0, \infty], & \text{if } P \text{ is a probability measure and } P \ll R \\ \infty, & \text{otherwise.} \end{cases} \end{aligned}$$

(2) *For any probability measure P on Ω such that $P \ll R$,*

$$H(P|R) = \sup \left\{ \int u dP - \log \int e^u dR; u : \int e^u dR < \infty, \int u_- dP < \infty \right\} \in [0, \infty]$$

where u is measurable, $u_- = (-u) \vee 0$ and $\int u dP \in (-\infty, \infty]$ is well-defined for all u such that $\int u_- dP < \infty$.

- (3) If in addition it is known that the probability measure P satisfies $H(P|R) < \infty$, then any measurable function u such that $\int e^u dR < \infty$ verifies $u \in L^1(P)$ and we have

$$H(P|R) = \sup \left\{ \int u dP - \log \int e^u dR; u : \int e^u dR < \infty \right\}. \quad (16)$$

In this formula the supremum is taken over all measurable functions $u : \Omega \rightarrow [-\infty, \infty)$, possibly taking the value $-\infty$ with the convention $e^{-\infty} = 0$. On the other hand, the supremum is attained at $u^* = \mathbf{1}_{\{dP/dR>0\}} \log(dP/dR) - \mathbf{1}_{\{dP/dR=0\}} \infty$, corresponding to $e^{u^*} = dP/dR$. If R is not a Dirac measure, this supremum is uniquely attained.

Proof. Let us first **prove (1)**. Denote

$$\kappa := \sup \left\{ \int u dP - \log \int e^u dR; u \text{ bounded measurable} \right\}$$

and

$$\kappa' := \sup \left\{ \int u dP - \log \int e^u dR; u \in L^\infty(P) \right\}.$$

Let us show that when P is not positive, i.e. $P_- \neq 0$, then $\kappa = \kappa' = \infty$. Let A be a measurable subset such that $P_+(A) = 0$ and $P_-(A) > 0$. Then, choosing $u_a = -a\mathbf{1}_A$ with $a > 0$, we see that $\kappa \geq \lim_{a \rightarrow \infty} (\int u_a dP - \log \int e^{u_a} dR) = \lim_{a \rightarrow \infty} (aP_-(A) - \log[1 + (e^{-a} - 1)R(A)]) = +\infty$. Similarly for κ' .

Now, suppose that P is a positive measure such that $P(\Omega) \neq 1$. Let us show that $\kappa = \kappa' = \infty$. Considering the constant functions $u \equiv a \in \mathbb{R}$, we see that $\int a dP - \log \int e^a dR = a(P(\Omega) - 1)$. Letting $|a|$ tend to infinity, we obtain $\kappa \geq \sup_a \{a(P(\Omega) - 1)\} = \infty$. And similarly for κ' .

Let us show that, if the probability measure P is not absolutely continuous with respect to R , then $\kappa = \kappa' = \infty$. For such a P , there is a measurable set A such that $P(A) > 0$ and $R(A) = 0$. Considering the functions $u = a\mathbf{1}_A$, we see that $\kappa \geq \sup_a \{aP(A) - 0\} = \infty$, and similarly for κ' .

From now on, P is a probability measure such that $P \ll R$.

Let us have a look at the first equality of assertion (1). Since the bounded functions are in $L^\infty(P)$, it is clear that $\kappa \leq \kappa'$. On the other hand, we also have $\kappa' \leq \kappa$. Indeed, one can write any u in $L^\infty(P)$ as $u = \mathbf{1}_{\{dP/dR>0\}}v + \mathbf{1}_{\{dP/dR=0\}}w$ where v is bounded and w is unspecified. But,

$$\begin{aligned} & \int u dP - \log \int e^u dR \\ &= \int v dP - \log \left(\int \mathbf{1}_{\{dP/dR>0\}} e^v dR + \int \mathbf{1}_{\{dP/dR=0\}} e^w dR \right) \\ &\leq \int v dP - \log \int \mathbf{1}_{\{dP/dR>0\}} e^v dR \\ &= \lim_{n \rightarrow \infty} \left(\int u_n dP - \log \int e^{u_n} dR \right) \end{aligned}$$

with $u_n := \mathbf{1}_{\{dP/dR>0\}}v - n\mathbf{1}_{\{dP/dR=0\}}$. As the functions u_n are bounded, we see that $\kappa' \leq \kappa$.

To prove (1), it remains to show that

$$\kappa = H(P|R). \quad (17)$$

We begin **proving (2)**. The identity (17) will appear as a step. The remainder of the proof relies on Fenchel's inequality for the convex $\theta(a) := a \log a - a + 1$ and on its equality case. This inequality is

$$ab \leq (a \log a - a + 1) + (e^b - 1) = \theta(a) + (e^b - 1) \quad (18)$$

for all $a \in [0, \infty)$, $b \in [-\infty, \infty)$, with the conventions $0 \log 0 = 0$, $e^{-\infty} = 0$ and $-\infty \times 0 = 0$ which are legitimated by limiting procedures. The equality is realized if and only if $a = e^b$. We denote $Z := \frac{dP}{dR}$ for a simpler notation. Taking $a = Z(\omega)$, $b = u(\omega)$ and integrating with respect to R leads us to

$$\int u dP \leq \int \theta(Z) dR + \int (e^u - 1) dR = H(P|R) + \int (e^u - 1) dR,$$

whose terms are meaningful provided that they are understood in $(-\infty, \infty]$, as soon as $\int u_- dP < \infty$. Formally, the equality case corresponds to $Z = e^u$. By the monotone convergence theorem, it can be approximated by the sequence $u_n = \log(Z \vee e^{-n})$, as n tends to infinity. This gives us

$$H(P|R) = \sup \left\{ \int u dP - \int (e^u - 1) dR; u : \int e^u dR < \infty, \inf u > -\infty \right\}, \quad (19)$$

which in turn implies that

$$H(P|R) = \sup \left\{ \int u dP - \int (e^u - 1) dR; u : \int e^u dR < \infty, \int u_- dP < \infty \right\}, \quad (20)$$

since the integral $\int \log Z dP = \int \theta(Z) dR \in [0, \infty]$ is well-defined.

Now, let us take advantage of the unit mass of P :

$$\int (u + b) dP - \int (e^{(u+b)} - 1) dR = \int u dP - e^b \int e^u dR + b + 1, \quad \forall b \in \mathbb{R}.$$

Thanks to the elementary identity $\log a = \inf_{b \in \mathbb{R}} \{ae^b - b - 1\}$, we obtain

$$\sup_{b \in \mathbb{R}} \left\{ \int (u + b) dP - \int (e^{(u+b)} - 1) dR \right\} = \int u dP - \log \int e^u dR.$$

Hence,

$$\begin{aligned} & \sup \left\{ \int u dP - \int (e^u - 1) dR; u : \int e^u dR < \infty, \int u_- dP < \infty \right\} \\ &= \sup \left\{ \int (u + b) dP - \int (e^{(u+b)} - 1) dR; b \in \mathbb{R}, u : \int e^u dR < \infty, \int u_- dP < \infty \right\} \\ &= \sup \left\{ \int u dP - \log \int e^u dR; u : \int e^u dR < \infty, \int u_- dP < \infty \right\}, \end{aligned}$$

With (20), this proves assertion (2).

But a similar reasoning, starting from (19) instead of (20), leads us to the similar following conclusion

$$H(P|R) = \sup \left\{ \int u dP - \log \int e^u dR; u : \int e^u dR < \infty, \inf u > -\infty \right\}.$$

Considering the functions $u \wedge n$ with $\inf u > -\infty$ and letting n tend to infinity, this leads us to (17) and proves assertion (1).

Let us prove (3). Suppose that $H(P|R) < \infty$. With the inequality (18), we obtain $|u|Z = |uZ| \leq \theta(Z) + e^u$. Therefore, if $\int e^u dR < \infty$, then

$$E_P|u| = E_R(|u|Z) \leq E_R\theta(Z) + E_R e^u = H(P|R) + E_R e^u < \infty.$$

This means that u is P -integrable and shows (16).

We check directly the equality case. The uniqueness of its realization comes from the *strict* concavity of the function $u \mapsto \int u dP - \log \int e^u dR$. One shows the strict convexity of $u \mapsto \log \int e^u dR$ by means of Hölder's inequality. But it is also possible to come back to the representation (20) which, with the same reasoning as above, leads us to

$$H(P|R) = \sup \left\{ \int u dP - \int (e^u - 1) dR; u : \int e^u dR < \infty \right\}.$$

Then, one directly reads the strict convexity of $u \mapsto \int (e^u - 1) dR$. \square

4. PROOF OF THEOREM 2.1

For the proof of Theorem 2.1 we need to exhibit a large enough family of exponential supermartingales.

Lemma 4.1 (Exponential supermartingales). *Let M be a local martingale, then*

$$Z_t^M = \exp \left(M_t - \frac{1}{2} [M, M]_t \right), \quad 0 \leq t \leq 1,$$

is also a local martingale and a supermartingale. In particular, $0 \leq E_R Z_1^M \leq 1$.

Proof. Recall Itô's formula

$$df(Y_t) = f'(Y_t) dY_t + \frac{1}{2} f''(Y_t) d[Y, Y]_t$$

which is valid for any \mathcal{C}^2 function f and any continuous semimartingale Y . Applying it to $Y_t = M_t - \frac{1}{2} [M, M]_t$ and $f(y) = e^y$, we obtain

$$dZ_t^M = Z_t^M \left(dM_t - \frac{1}{2} d[M, M]_t + \frac{1}{2} d[M, M]_t \right) = Z_t^M dM_t$$

which proves that Z^M is a local martingale. Since $Z^M \geq 0$, Fatou's lemma applied to the localized sequence $Z_{t \wedge \tau_k}^M$ as k tends to infinity tells us that Z^M is a R -supermartingale, with $(\tau_k)_{k \geq 1}$ an increasing sequence of stopping times which tends almost surely to infinity and localizes the local martingale M . In particular, $E(Z_1^M) \leq E(Z_0^M) = 1$. \square

The standard notation for the supermartingale of Lemma 4.1 is

$$\mathcal{E}(M) := \exp \left(M - \frac{1}{2} [M, M] \right).$$

We are now ready for the proof of Theorem 2.1.

Proof of Theorem 2.1. We start with some useful notation. Let Q be a probability measure on Ω ; later we shall take $Q = R$ or $Q = P$. For any measurable function g on $[0, 1] \times \Omega$, let us denote

$$(g, g)_A(\omega) := \int_{[0, 1]} g_t(\omega) \cdot A_t(\omega; dt) g_t(\omega) \in [0, \infty]$$

and introduce the function space

$$\mathcal{G}(Q) := \{g : [0, 1] \times \Omega \rightarrow \mathbb{R}^d; g \text{ measurable, } E_Q(g, g)_A < \infty\}$$

endowed with the seminorm $\|g\|_{\mathcal{G}(Q)} := (E_Q(g, g)_A)^{1/2}$. Identifying the functions with their equivalence classes when factorizing by the kernel of this seminorm, turns $\mathcal{G}(Q)$ into a Hilbert space. These equivalence classes are called $\mathcal{G}(Q)$ -classes and with some abuse, we say that two elements of the same class are equal $\mathcal{G}(Q)$ -almost everywhere. The relevant space of integrands for the stochastic integral is

$$\mathcal{H}^Q := \{h \in \mathcal{G}(Q); h \text{ adapted}\}.$$

Identity (3) says that $M^R = X - X_0 - B^R$ is a local R -martingale. For all $h \in \mathcal{H}^R$, let us denote the stochastic integral

$$h \cdot M_t^R := \int_0^t h_s dM_s^R, \quad t \in [0, 1].$$

By Lemma 4.1, $0 < E_R Z_1^{h \cdot M^R} \leq 1$ for all $h \in \mathcal{H}^R$ and because of (16), for any probability measure P such that $H(P|R) < \infty$, we have

$$E_P \left(h \cdot M_1^R - \frac{1}{2} [h \cdot M^R, h \cdot M^R]_1 \right) \leq H(P|R), \quad \forall h \in \mathcal{H}^R. \quad (21)$$

Note that, as $P \ll R$, $h \cdot M_1^R$ and $[h \cdot M^R, h \cdot M^R]_1$ which are defined R -a.s., are a fortiori defined P -a.s. With (6) and (21), we see that

$$E_P(h \cdot M^R) \leq H(P|R) + \frac{1}{2} E_P(h, h)_A, \quad \forall h \in \mathcal{G}(P) \cap \mathcal{H}^R. \quad (22)$$

The notation $\mathcal{G}(P) \cap \mathcal{H}^R$ is a little bit improper. Indeed, $\mathcal{G}(P)$ is a set of equivalence classes with respect to the equality $\mathcal{G}(P)$ -a.e., while \mathcal{H}^R is a set of $\mathcal{G}(R)$ -classes. But since $P \ll R$, keeping in mind that any $\mathcal{G}(P)$ -class is the union of some $\mathcal{G}(R)$ -classes, one can interpret $\mathcal{G}(P) \cap \mathcal{G}(R)$ as a set of $\mathcal{G}(P)$ -classes. It is also clear that $\mathcal{G}(P) \cap \mathcal{H}^R = \mathcal{H}^P \cap \mathcal{H}^R$ which is a set of $\mathcal{G}(P)$ -classes. Considering $-h$ in (22), we obtain for all $\lambda > 0$,

$$\left| E_P \left(\frac{h}{\lambda} \cdot M^R \right) \right| \leq H(P|R) + \frac{1}{2\lambda^2} E_P(h, h)_A, \quad \forall h \in \mathcal{H}^P \cap \mathcal{H}^R.$$

Let

$$S := \left\{ h : [0, 1] \times \Omega \rightarrow \mathbb{R}^d; h = \sum_{i=1}^k h_i \mathbf{1}_{[S_i, T_i]} \right\}$$

denote the set of all *simple* adapted processes h where k is finite and for all i , $h_i \in \mathbb{R}^d$ and $S_i \leq T_i$ are stopping times. As $S \subset \mathcal{H}^P \cap \mathcal{H}^R$, taking $\lambda = \|h\|_{\mathcal{G}(P)}$ in previous inequality, we obtain the keystone of the proof:

$$|E_P(h \cdot M^R)| \leq [H(P|R) + 1/2] \|h\|_{\mathcal{G}(P)}, \quad \forall h \in S.$$

This estimate still holds when $\|h\|_{\mathcal{H}(P)} = 0$. Indeed, for all real α , by (22) we see that $\alpha E_P(h \cdot M^R) \leq H(P|R) + \alpha^2/2 E_P(h, h)_A = H(P|R)$. Letting $|\alpha|$ tend to infinity, it follows that $E_P(h \cdot M^R) = 0$.

Under the assumption that $H(P|R)$ is finite, this means that $h \mapsto h \cdot M^R$ is continuous on S with respect to the Hilbert topology of \mathcal{H}^P . As S is dense in \mathcal{H}^P , this linear form extends uniquely as a continuous linear form on \mathcal{H}^P . It also appears that this extension is again a stochastic integral *with respect to P* . We still denote this extension by $h \cdot M^R$. As $h \mapsto h \cdot M^R$ is a continuous linear form on \mathcal{H}^P , we know by the Riesz representation theorem that there exists a unique $\beta \in \mathcal{H}^P$ such that

$$E_P(h \cdot M^R) = E_P(\beta, h)_A, \quad \forall h \in \mathcal{H}^P.$$

In other words,

$$E_P \int_{[0,1]} h_t dM_t^P = 0, \quad \forall h \in \mathcal{H}^P$$

where

$$M_t^P := M_t^R - \int_{[0,t]} A(ds) \beta_s = X_t - X_0 - B_t^R - \widehat{B}_t,$$

which means that M^P is a local P -martingale. \square

5. PROOF OF THEOREM 2.3

It relies on a transfer result which is stated below at Lemma 5.1. But we first need to introduce its framework and some additional notation. Let P be a probability measure on Ω such that $[X, X] = A$, P -a.s. and

$$X = X_0 + B + M^P, \quad P\text{-a.s.},$$

where B is a bounded variation process and M^P is a local P -martingale. Let γ be an adapted process such that $\int_{[0,1]} \gamma_t \cdot A(dt) \gamma_t < \infty$, P -a.s. We define

$$Z_t = \exp \left(\int_{[0,t]} \gamma_s dM_s^P - \frac{1}{2} \int_{[0,t]} \gamma_s \cdot A(ds) \gamma_s \right), \quad 0 \leq t \leq 1$$

and for all $k \geq 1$,

$$\sigma^k := \inf \left\{ t \in [0, 1]; \int_{[0,t]} \gamma_s \cdot A(ds) \gamma_s \geq k \right\} \in [0, 1] \cup \{\infty\},$$

with the convention $\inf \emptyset = \infty$.

We use the standard notation $Y_t^\tau = Y_{\tau \wedge t}$ for the process Y stopped at a stopping time τ . For all k , $P^k := X^{\sigma_k}_\# P$ is the push-forward of P with respect to the stopping procedure X^{σ_k} . Note that P^k and P match on the σ -field which is generated by $X_{[0, \sigma_k]}$.

Lemma 5.1. *Let P and γ as above. Then, for all $k \geq 1$, Z^{σ_k} is a genuine P -martingale and the measure*

$$Q^k := Z_1^{\sigma_k} P^k$$

is a probability measure on Ω which satisfies $Q^k \in \text{MP}(\widehat{B}^{\sigma_k}, A^{\sigma_k})$ where $\widehat{B}_t^{\sigma_k} = \int_{[0, t \wedge \sigma_k]} A(ds) \gamma_s$ and M^k is a local Q^k -martingale.

Proof. Let us first show that Z^{σ_k} is a P^k -martingale¹. The local martingale Z^{σ_k} is of the form $Z^{\sigma_k} = \mathcal{E}(N) := \exp(N - \frac{1}{2}[N, N])$ with N a local P^k -martingale such that $[N, N] \leq k$, P^k -a.s. For all $p \geq 0$, since $\mathcal{E}(N)^p = \exp(pN - \frac{p}{2}[N, N])$ and $\mathcal{E}(pN) = \exp(pN - \frac{p^2}{2}[N, N]) \geq e^{pN} e^{-kp^2/2}$, we obtain

$$\mathcal{E}(N)^p \leq e^{pN} \leq e^{kp^2/2} \mathcal{E}(pN).$$

As a nonnegative local martingale, $\mathcal{E}(pN)$ is a supermartingale. We deduce from this that $E_{P^k} \mathcal{E}(pN) \leq 1$ and

$$E_{P^k} \mathcal{E}(N)^p \leq e^{kp^2/2} E_{P^k} \mathcal{E}(pN) \leq e^{kp^2/2} < \infty.$$

Choosing $p > 1$, it follows that $\mathcal{E}(N)$ is uniformly integrable. In particular, this implies that

$$E_{P^k} \mathcal{E}(N)_1 = E_{P^k} \mathcal{E}(N)_0 = 1$$

¹It is a direct consequence of Novikov's criterion, but we prefer presenting an elementary proof which will be a guideline for a similar result with jump processes.

and proves that Q^k is a probability measure.

Suppose now that the supermartingale $\mathcal{E}(N)$ is not a martingale. This implies that there exists $0 \leq t < 1$ such that on a subset with positive measure, $E_{P^k}(\mathcal{E}(N)_1 \mid X_{[0,t]}) < \mathcal{E}(N)_t$. Integrating, we get $1 = E_{P^k}\mathcal{E}(N)_1 < E_{P^k}\mathcal{E}(N)_t$, which contradicts $E_{P^k}\mathcal{E}(N)_s \leq E_{P^k}\mathcal{E}(N)_0 = 1, \forall s$: a consequence of the supermartingale property of $\mathcal{E}(N)$. Therefore, $\mathcal{E}(N)$ is a genuine P^k -martingale.

Let us fix $k \geq 1$ and show that Q^k is a solution to $\text{MP}(\widehat{B}^{\sigma_k}, A^{\sigma_k})$. First of all, as it is assumed that $[X, X] = A$, P -a.s., we obtain $[X, X] = A^{\sigma_k}$, P^k -a.s. With $Q^k \ll P^k$, this implies that $[X, X] = A^{\sigma_k}$, Q^k -a.s.

Now, we check

$$X = X_0 + B^{\sigma_k} + \widehat{B}^{\sigma_k} + M^k \quad (23)$$

where M^k is a Q^k -martingale. Let τ be a stopping time and denote $F_t = \xi \cdot X_t^\tau$ with $\xi \in \mathbb{R}^d$. The martingale Z^{σ_k} is the stochastic exponential $\mathcal{E}(N)$ of $N_t = \int_{[0,t]} \mathbf{1}_{[0,\sigma_k]}(s) \gamma_s \cdot dM_s^P$. Hence, denoting $Z = Z^{\sigma_k}$, we have $dZ_t = Z_t \mathbf{1}_{[0,\sigma_k]}(t) \gamma_t \cdot dM_t^P$, $dF_t = \mathbf{1}_{[0,\tau]}(t) \xi \cdot (dB_t + dM_t^P)$ and $d[Z, F]_t = Z_t \mathbf{1}_{[0,\tau \wedge \sigma_k]}(t) \xi \cdot A(dt) \gamma_t$, P^k -a.s. Consequently,

$$\begin{aligned} E_{Q^k}[\xi \cdot (X_\tau - X_0)] &\stackrel{(a)}{=} E_{P^k}[Z_\tau F_\tau - Z_0 F_0] \\ &\stackrel{(b)}{=} E_{P^k} \left[\int_{[0,\tau]} (F_t dZ_t + Z_t dF_t + d[Z, F]_t) \right] \\ &= E_{P^k} \left[\int_{[0,\tau]} F_t dZ_t + \int_{[0,\tau]} Z_t \xi \cdot (dB_t + dM_t^P) + \int_{[0,\tau]} Z_t \xi \cdot A(dt) \gamma_t \right] \\ &\stackrel{(c)}{=} E_{P^k} \left[\int_{[0,\tau]} Z_t \xi \cdot dB_t + \int_{[0,\tau]} Z_t \xi \cdot A(dt) \gamma_t \right] \\ &\stackrel{(d)}{=} E_{Q^k} \left[\xi \cdot \int_{[0,\tau]} (dB_t + A(dt) \gamma_t) \right]. \end{aligned}$$

In order that all the above terms are meaningful, we choose τ such that it localizes F , B , M^P and $\xi \cdot A\gamma$. This is possible, taking for any $n \geq 1$, $\tau \leq \tau_n = \min(\tau_n^F, \tau_n^B, \tau_n^M, \tau_n^\gamma)$ where $\tau_n^F = \inf\{t \in [0, 1]; |X_t| \geq n\}$, $\tau_n^B = \inf\{t \in [0, 1]; \int_{[0,t]} |dB_s| \geq n\}$, $\tau_n^\gamma = \inf\{t \in [0, 1]; \int_{[0,t]} \gamma_s \cdot A(ds) \gamma_s \geq n\}$, and τ_n^M is a localizing sequence of the local martingale M^P . We have

$$\lim_{n \rightarrow \infty} \tau_n = \infty, \quad P^k\text{-a.s.} \quad (24)$$

We used the definition of Q^k and the martingale property of Z at (a) and (d), (b) is Itô's formula and (c) relies on the martingale property of Z and $(M^P)^\tau$. Finally, taking $\tau = \varsigma \wedge \tau_n$, we see that for any stopping time ς , any $n \geq 1$ and any $\xi \in \mathbb{R}^d$

$$E_{Q^k}[\xi \cdot (X_\varsigma^{\tau_n} - X_0^{\tau_n})] = E_{Q^k} \left[\xi \cdot \int_{[0, \varsigma \wedge \tau_n]} (dB_t + A(dt) \gamma_t) \right].$$

Taking (24) into account, this means that $X - X_0 - B - \widehat{B}$ is a local Q^k -martingale. We conclude remarking that for any process Y , we have $Y = Y^{\sigma_k}$, Q^k -a.s. This leads us to (23). \square

Let us denote $P^\tau = X^\tau \# P$ the law under P of the process X^τ which is stopped at the stopping time τ .

Lemma 5.2. *If R fulfills the condition (U), then for any stopping time τ , R^τ also fulfills it.*

Proof. Let us fix the stopping time τ . Our assumption on R implies that

$$X = X_0 + B + M, \quad R^\tau\text{-a.s.}$$

where $M = M^R$ is a local R -martingale and we denote $B = B^R$. Let $Q \ll R^\tau$ be given such that $Q_0 = R_0$ and

$$X = X_0 + B + M^Q, \quad Q\text{-a.s.}$$

where M^Q is a local Q -martingale. We wish to show that $Q = R^\tau$. The disintegration

$$R = R_{[0,\tau]} \otimes R(\cdot \mid X_{[0,\tau]})$$

means that for any bounded measurable function F on Ω , denoting $F = F(X) = F(X_{[0,\tau]}, X_{(\tau,1]})$,

$$E_R(F) = \int_{\Omega} E_R[F(\eta, X_{(\tau,1]}) \mid X_{[0,\tau]} = \eta] R_{[0,\tau]}(d\eta).$$

Similarly, we introduce the probability measure

$$R' := Q_{[0,\tau]} \otimes R(\cdot \mid X_{[0,\tau]}).$$

To complete the proof, it is enough to show that R' satisfies

$$X = X_0 + B + M', \quad R'\text{-a.s.} \quad (25)$$

with M' a local R' -martingale. Indeed, the condition (U) tells us that $R' = R$, which implies that $R'^\tau = R^\tau$. But $R'^\tau = Q$, hence $Q = R^\tau$.

Let us show (25). Let $\xi \in \mathbb{R}^d$ and a stopping time σ be given. We denote $(\tau_n)_{n \geq 1}$ a localizing sequence of $M = M^R$ and $B = B^R$. Then,

$$\begin{aligned} & E_{R'}[\xi \cdot (X_\sigma^{\tau_n} - X_0^{\tau_n})] \\ &= E_{R'}[\mathbf{1}_{\{\tau \leq \sigma\}} \xi \cdot (X_\sigma^{\tau_n} - X_\tau^{\tau_n})] + E_Q[\xi \cdot (X_\sigma^{\tau_n} - X_0^{\tau_n})] \\ &= \int_{\Omega} E_R[\mathbf{1}_{\{\tau \leq \sigma\}} \xi \cdot (X_\sigma^{\tau_n} - X_\tau^{\tau_n}) \mid X_{[0,\tau]} = \eta] Q(d\eta) + E_Q[\xi \cdot (X_\sigma^{\tau_n} - X_0^{\tau_n})] \\ &= \int_{\Omega} E_R[\mathbf{1}_{\{\tau \leq \sigma\}} \xi \cdot (B_\sigma^{\tau_n} - B_\tau^{\tau_n}) \mid X_{[0,\tau]} = \eta] Q(d\eta) + E_Q[\xi \cdot (B_\sigma^{\tau_n} - B_0^{\tau_n})] \\ &= E_{R'}[\xi \cdot (B_\sigma^{\tau_n} - B_0^{\tau_n})] \end{aligned}$$

This means that (25) is satisfied (with the localizing sequence $(\tau_n)_{n \geq 1}$) and completes the proof of the lemma. \square

For all $k \geq 1$, we consider the stopping time

$$\tau_k = \inf \left\{ t \in [0, 1]; \int_{[0,t]} \beta_s \cdot A(ds) \beta_s \geq k \right\} \in [0, 1] \cup \{\infty\}$$

where β is the process which is associated with P in Theorem 2.1 and as a convention $\inf \emptyset = \infty$. We are going to use this stopping time R -a.s. Since β is only defined P -a.s., we assume for the moment that P and R are equivalent measures: $P \sim R$.

Lemma 5.3. *Assume that $P \sim R$ and suppose that R satisfies the condition (U). Then, for all $k \geq 1$, on the stochastic interval $\llbracket 0, \tau_k \wedge 1 \rrbracket$ we have, R -almost everywhere*

$$\mathbf{1}_{\llbracket 0, \tau_k \wedge 1 \rrbracket} \frac{dP}{dR} = \mathbf{1}_{\llbracket 0, \tau_k \wedge 1 \rrbracket} \frac{dP_0}{dR_0}(X_0) \exp \left(\int_{[0, \tau_k \wedge 1]} \beta_t \cdot dM_t^R - \frac{1}{2} \int_{[0, \tau_k \wedge 1]} \beta_t \cdot A(dt) \beta_t \right). \quad (26)$$

Proof. By conditionning with respect to X_0 , we see that we can assume without loss of generality, that $R_0 := (X_0)_\# R = (X_0)_\# P =: P_0$, i.e. $\frac{dP_0}{dR_0}(X_0) = 1$. Let $k \geq 1$. Denote $R^k = R^{\tau_k}$, $P^k = P^{\tau_k}$. Applying Lemma 5.1 with $\gamma = -\beta$ and remarking that $\widehat{B}_{-\beta} = -\widehat{B}_\beta$, we see that

$$Q^k := \mathcal{E}(-\beta \cdot M^P)_{\tau_k \wedge 1} P^k \in \text{MP}(\mathbf{1}_{[0, \tau_k]}[(B^R + \widehat{B}_\beta) + \widehat{B}_{-\beta}], \mathbf{1}_{[0, \tau_k]} A) = \text{MP}(\mathbf{1}_{[0, \tau_k]} B^R, \mathbf{1}_{[0, \tau_k]} A).$$

But, it is known with Lemma 5.2 that R^k satisfies the condition (U). Therefore,

$$Q^k = R^k. \quad (27)$$

Applying twice Lemma 5.1, we observe on the one hand that

$$\widetilde{P}^k := \mathcal{E}(\beta \cdot M^R)_{\tau_k \wedge 1} R^k \in \text{MP}(\mathbf{1}_{[0, \tau_k]}(B^R + \widehat{B}_\beta), \mathbf{1}_{[0, \tau_k]} A), \quad (28)$$

and on the other hand that

$$\widetilde{Q}^k := \mathcal{E}(-\beta \cdot M^P)_{\tau_k \wedge 1} \widetilde{P}^k \in \text{MP}(\mathbf{1}_{[0, \tau_k]}[(B^R + \widehat{B}_\beta) - \widehat{B}_\beta], \mathbf{1}_{[0, \tau_k]} A) = \text{MP}(\mathbf{1}_{[0, \tau_k]} B^R, \mathbf{1}_{[0, \tau_k]} A).$$

As for the proof of (27), the condition (U) which is satisfied by R^k leads us to $\widetilde{Q}^k = R^k$. Therefore, we see with (27) that $Q^k = \widetilde{Q}^k$, i.e. $\mathcal{E}(-\beta \cdot M^P)_{\tau_k \wedge 1} P^k = \mathcal{E}(-\beta \cdot M^P)_{\tau_k \wedge 1} \widetilde{P}^k$. And since $\mathcal{E}(-\beta \cdot M^P)_{\tau_k \wedge 1} > 0$, we obtain $P^k = \widetilde{P}^k$ which is (26). \square

We are ready to complete the proof of Theorem 2.3.

Proof of Theorem 2.3. Derivation of $\frac{dP}{dR}$. Provided that R satisfies the condition (U), when $P \sim R$ we obtain the announced formula

$$\frac{dP}{dR} = \frac{dP_0}{dR_0}(X_0) \exp \left(\int_{[0,1]} \beta_t \cdot dM_t^R - \frac{1}{2} \int_{[0,1]} \beta_t \cdot A(dt) \beta_t \right), \quad (29)$$

letting k tend to infinity in (26), remarking that $\tau := \lim_{k \rightarrow \infty} \tau_k = \inf\{t \in [0, 1]; \int_{[0,t]} \beta_s \cdot A(ds) \beta_s = \infty\}$ and that (7) implies

$$\tau = \infty, \quad P\text{-a.s.} \quad (30)$$

and, since $P \sim R$, we also have $\tau = \infty$, R -a.s. Indeed, since $\tau(\omega) = \infty$, there is some $k_o \geq 1$ such that $\tau_{k_o}(\omega) = \infty$ and applying Lemma 5.3 with $k = k_o : \frac{dP}{dR}(\omega) = \frac{dP_0}{dR_0}(\omega_0) \exp \left(\int_{[0,1]} \beta_t \cdot dM_t^R - \frac{1}{2} \int_{[0,1]} \beta_t \cdot A(dt) \beta_t \right) (\omega) > 0$.

Now, we consider the general case when P might not be equivalent to R . The main idea is to approximate P by a sequence $(P_n)_{n \geq 1}$ such that $P_n \sim R$ for all $n \geq 1$, and to rely on our previous intermediate results. We consider

$$P_n := \left(1 - \frac{1}{n}\right) P + \frac{1}{n} R, \quad n \geq 1.$$

Clearly, $P_n \sim R$ and by convexity $H(P_n|R) \leq (1 - \frac{1}{n})H(P|R) + \frac{1}{n}H(R|R) \leq H(P|R) < \infty$. More precisely, the function $x \in [0, 1] \mapsto H(xP + (1-x)R|R) \in [0, \infty]$ is a finitely valued convex continuous and increasing. It follows that $\lim_{n \rightarrow \infty} H(P_n|R) = H(P|R)$.

It is clear that $\lim_{n \rightarrow \infty} P_n = P$ in total variation norm. Let us prove that the stronger convergence

$$\lim_{n \rightarrow \infty} H(P|P_n) = 0 \quad (31)$$

also holds. It is easy to check that $\mathbf{1}_{\{\frac{dP}{dR} \geq 1\}} dP/dP_n$ and $\mathbf{1}_{\{\frac{dP}{dR} \leq 1\}} dP/dP_n$ are respectively decreasing and increasing sequences of functions. It follows by monotone convergence

that

$$\begin{aligned} \lim_{n \rightarrow \infty} H(P|P_n) &= \lim_{n \rightarrow \infty} \int \log(dP/dP_n) dP \\ &= \lim_{n \rightarrow \infty} \int_{\{\frac{dP}{dR} \geq 1\}} \log(dP/dP_n) dP + \lim_{n \rightarrow \infty} \int_{\{\frac{dP}{dR} < 1\}} \log(dP/dP_n) dP = 0. \end{aligned}$$

By Theorem 2.1, there exist two vector fields β^n and β which are respectively defined R -a.s. and P -a.s. such that $E_{P_n} \int_{[0,1]} \beta_t^n \cdot A(dt) \beta_t^n < \infty$, $E_P \int_{[0,1]} \beta_t \cdot A(dt) \beta_t < \infty$ and

$$dX_t = dB_t^R + A(dt) \beta_t^n + dM_t^{P_n}, \quad R\text{-a.s.}; \quad dX_t = dB_t^R + A(dt) \beta_t + dM_t^P, \quad P\text{-a.s.}$$

where M^{P_n} and M^P are respectively a local P_n -martingale and a local P -martingale. Therefore,

$$dM_t^{P_n} = dM_t^P + A(dt)(\beta_t - \beta_t^n), \quad P\text{-a.s.} \quad (32)$$

Extending β arbitrarily by $\beta = 0$ on the P -null set where it is unspecified, we know that

$$\exp \left(\int_{[0,t]} (\beta_s - \beta_s^n) \cdot dM_s^{P_n} - \frac{1}{2} \int_{[0,t]} (\beta_s - \beta_s^n) \cdot A(ds) (\beta_s - \beta_s^n) \right)$$

is a P^n -supermartingale. It follows with Proposition 3.1, (32) and a standard monotone convergence argument that

$$\begin{aligned} H(P|P_n) &\geq E_P \left(\int_{[0,1]} (\beta_s - \beta_s^n) \cdot dM_s^{P_n} - \frac{1}{2} \int_{[0,1]} (\beta_s - \beta_s^n) \cdot A(ds) (\beta_s - \beta_s^n) \right) \\ &= \frac{1}{2} E_P \int_{[0,1]} (\beta_s - \beta_s^n) \cdot A(ds) (\beta_s - \beta_s^n). \end{aligned}$$

With (31), this shows the key estimate

$$\lim_{n \rightarrow \infty} E_P \int_{[0,1]} (\beta_s - \beta_s^n) \cdot A(ds) (\beta_s - \beta_s^n) = 0. \quad (33)$$

Since $H(P_n|R) < \infty$ and $P_n \sim R$, under the condition (U) we can invoke (29) to write

$$\frac{dP_n}{dR} = \frac{dP_{n,0}}{dR_0}(X_0) \exp \left(\int_{[0,1]} \beta_t^n \cdot dM_t^R - \frac{1}{2} \int_{[0,1]} \beta_t^n \cdot A(dt) \beta_t^n \right).$$

As $\lim_{n \rightarrow \infty} P_n = P$ in total variation norm, up to the extraction of a R -a.s.-convergent subsequence we have $\lim_{n \rightarrow \infty} dP_n/dR = dP/dR$ and $\lim_{n \rightarrow \infty} dP_{n,0}/dR_0 = dP_0/dR_0$. On the other hand, (33) implies that P -a.s., $\lim_{n \rightarrow \infty} \frac{1}{2} \int_{[0,1]} \beta_t^n \cdot A(dt) \beta_t^n = \frac{1}{2} \int_{[0,1]} \beta_t \cdot A(dt) \beta_t$. It follows that

$$\frac{dP}{dR} = \mathbf{1}_{\{\frac{dP}{dR} > 0\}} \frac{dP_0}{dR_0}(X_0) \exp \left(\int_{[0,1]} \beta_t \cdot dM_t^R - \frac{1}{2} \int_{[0,1]} \beta_t \cdot A(dt) \beta_t \right).$$

where (33) also implies that the limit of the stochastic integrals

$$\lim_{n \rightarrow \infty} \int_{[0,1]} \beta_t^n \cdot dM_t^R = \int_{[0,1]} \beta_t \cdot dM_t^R, \quad P\text{-a.s.}$$

exists P -a.s. □

It remains to compute $H(P|R)$.

End of the proof of Theorem 2.3. Computation of $H(P|R)$. Let us first compute $H(P|R)$ when R satisfies (U). Remark that in the proof of Lemma 5.3, for all $k \geq 1$ the local \tilde{P}^k -martingale $N^k = M^R - \widehat{B}$ which is behind (28) is a genuine martingale. It is a consequence of the first statement of Lemma 5.1. As $\tilde{P}^k = P^k$, N^k is a genuine P^k -martingale. This still holds when $P \sim R$ fails. Indeed, this hypothesis has only been invoked to insure that τ_k is well-defined R -a.s. But in the present situation, τ_k only needs to be defined P -a.s. With (26), we have

$$\begin{aligned}
H(P^k|R^k) &= E_{P^k} \log \frac{dP^k}{dR^k} \\
&\stackrel{(26)}{=} E_P \left(\log \frac{dP_0}{dR_0}(X_0) \right) + E_{P^k} \left(\int_{[0,1]} \beta_t \cdot dM_t^R - \frac{1}{2} \int_{[0,1]} \beta_t \cdot A(dt) \beta_t \right) \\
&\stackrel{(28)}{=} H(P_0|R_0) + E_{P^k} \left(\int_{[0,1]} \beta_t \cdot (dN_t^k + d\widehat{B}_t) - \frac{1}{2} \int_{[0,1]} \beta_t \cdot A(dt) \beta_t \right) \\
&\stackrel{(8)}{=} H(P_0|R_0) + \frac{1}{2} E_{P^k} \left(\int_{[0,1]} \beta_t \cdot A(dt) \beta_t \right) + E_{P^k} \left(\int_{[0,1]} \beta_t \cdot dN_t^k \right) \\
&= H(P_0|R_0) + \frac{1}{2} E_P \left(\int_{[0, \tau^k \wedge 1]} \beta_t \cdot A(dt) \beta_t \right)
\end{aligned}$$

where the last equality comes from the P^k -martingale property of N^k . It remains to let k tend to infinity to see that

$$H(P|R) = H(P_0|R_0) + \frac{1}{2} E_P \left(\int_{[0,1]} \beta_t \cdot A(dt) \beta_t \right).$$

Indeed, because of (30) and since the sequence $(\tau_k)_{k \geq 1}$ is increasing, we obtain by monotone convergence that

$$\lim_{k \rightarrow \infty} E_P \left(\int_{[0, \tau^k \wedge 1]} \beta_t \cdot A(dt) \beta_t \right) = \frac{1}{2} E_P \left(\int_{[0,1]} \beta_t \cdot A(dt) \beta_t \right).$$

As regards the left hand side of the equality, with Proposition 3.1-(1) and (30), we see that

$$\begin{aligned}
H(P|R) &= \sup \{ E_P u(X) - \log E_R e^{u(X)}; u \in L^\infty(P) \} \\
&= \sup_k \sup \{ E_P u(X^{\tau^k}) - \log E_R e^{u(X^{\tau^k})}; u \in L^\infty(P) \} \\
&= \lim_{k \rightarrow \infty} H(P^k|R^k).
\end{aligned}$$

It remains to check that, without the condition (U), we have

$$H(P|R) \geq H(P_0|R_0) + \frac{1}{2} E_P \left(\int_{[0,1]} \beta_t \cdot A(dt) \beta_t \right). \quad (34)$$

Let us extend β by $\beta = 0$ on the P -null set where it is unspecified and define

$$\tilde{u}(X) := \log \frac{dP_0}{dR_0}(X_0) + \int_{[0, \tau^k \wedge 1]} \beta_t \cdot dM_t^R - \frac{1}{2} \int_{[0, \tau^k \wedge 1]} \beta_t \cdot A(dt) \beta_t.$$

Choosing $\tilde{u}(X)$ at inequality $\stackrel{(i)}{\geq}$ below, thanks to an already used supermartingale argument, we obtain the inequality $\stackrel{(ii)}{\geq}$ below and

$$\begin{aligned}
H(P^k|R^k) &\stackrel{(16)}{=} \sup \left\{ \int u dP^k - \log \int e^u dR^k; u : \int e^u dR^k < \infty \right\} \\
&\stackrel{(i)}{\geq} \int \tilde{u} dP^k - \log \int e^{\tilde{u}} dR^k \\
&\stackrel{(ii)}{\geq} \int \tilde{u} dP^k \\
&\stackrel{(iii)}{=} H(P_0|R_0) + E_{P^k} \left(\int_{[0, \tau^k \wedge 1]} \beta_t \cdot d\widehat{B}_t - \frac{1}{2} \int_{[0, \tau^k \wedge 1]} \beta_t \cdot A(dt) \beta_t \right) \\
&\stackrel{(8)}{=} H(P_0|R_0) + \frac{1}{2} E_{P^k} \int_{[0, \tau^k \wedge 1]} \beta_t \cdot A(dt) \beta_t.
\end{aligned}$$

Equality (iii) is a consequence of

$$\tilde{u}(X) = \log \frac{dP_0}{dR_0}(X_0) + \int_{[0, \tau^k \wedge 1]} \beta_t \cdot (dM_t^P + d\widehat{B}_t) - \frac{1}{2} \int_{[0, \tau^k \wedge 1]} \beta_t \cdot A(dt) \beta_t, \quad P^k\text{-a.s.}$$

which comes from Theorem 2.1. It remains to let k tend to infinity, to obtain as above with (30) that (34) holds true. This completes the proof of the theorem. \square

6. PROOFS OF THEOREMS 2.6 AND 2.9

We begin recalling Itô's formula. Let P be the law of a semimartingale

$$dX_t = b_t \rho(dt) + dM_t^P$$

with M^P a local P -martingale such that $M^P = q \odot \tilde{\mu}^K$, P -a.s. That is $P \in \text{LK}(\overline{K})$ for some Lévy kernel \overline{K} . For any f in $\mathcal{C}^2(\mathbb{R}^d)$ which satisfies:

(*) *When localizing with an increasing sequence $(\tau_k)_{k \geq 1}$ of stopping times tending P -almost surely to infinity, for each $k \geq 1$ the truncated process $\mathbf{1}_{\{q > 1\}} \mathbf{1}_{\{t \leq \tau_k\}} [f(X_{t-} + q) - f(X_{t-})]$ is a $\mathcal{H}_1(P, \overline{K})$ integrand,*

Itô's formula is

$$\begin{aligned}
df(X_t) &= \left[\int_{\mathbb{R}_*^d} [f(X_{t-} + q) - f(X_{t-}) - \nabla f(X_{t-}) \cdot q] K_t(dq) \right] \rho(dt) \\
&\quad + \nabla f(X_{t-}) \cdot b_t \rho(dt) + dM_t, \quad P\text{-a.s.}
\end{aligned} \tag{35}$$

where M is a local P -martingale. This identity would fail if ρ was not assumed to be atomless.

Proof of Theorem 2.6. Based on Itô's formula, we start computing a large family of exponential local martingales. Recall that we denote

$$a \mapsto \theta(a) := e^a - a - 1 = \sum_{n \geq 2} a^n / n!, \quad a \in \mathbb{R}.$$

Lemma 6.1 (Exponential martingale). *Let $h : \Omega \times [0, 1] \times \mathbb{R}_*^d \rightarrow \mathbb{R}$ be a real valued predictable process which satisfies*

$$E_R \int_{[0, 1] \times \mathbb{R}_*^d} \theta[h_t(q)] \overline{L}(dtdq) < \infty. \tag{36}$$

Then, h and $e^h - 1$ belong to $\mathcal{H}_{1,2}(R, \bar{L})$. In particular, $h \odot \tilde{\mu}^L$ is a R -martingale. Moreover,

$$Z_t^h := \exp \left(h \odot \tilde{\mu}_t^L - \int_{(0,t] \times \mathbb{R}_*^d} \theta[h_s(q)] \bar{L}(ds dq) \right), \quad t \in [0, 1]$$

is a local R -martingale and a positive R -supermartingale which satisfies

$$dZ_t^h = Z_{t-}^h [(e^{h(q)} - 1) \odot d\tilde{\mu}_t^L].$$

Proof. The function θ is nonnegative, quadratic near zero, linear near $-\infty$ and it grows exponentially fast near $+\infty$. Therefore, (36) implies that h and $e^h - 1$ belong to $\mathcal{H}_{1,2}(R, \bar{L})$. In particular, $M^h := h \odot \tilde{\mu}^L$ is a R -martingale.

Let us denote $Y_t = M_t^h - \int_{(0,t]} \beta_s \rho(ds)$ where $\beta_t = \int_{\mathbb{R}_*^d} \theta[h_t(q)] L_t(dq)$. Remark that (36) implies that these integrals are almost everywhere well-defined. Applying (35) with $f(y) = e^y$ and $dY_t = -\beta_t \rho(dt) + dM_t^h$, we obtain

$$de^{Y_t} = e^{Y_{t-}} \left[-\beta_t + \int_{\mathbb{R}_*^d} \theta[h_t(q)] L_t(dq) \right] \rho(dt) + dM_t = dM_t$$

where M is a local martingale. We are allowed to do this because $(*)$ is satisfied. Indeed, with $f(y) = e^y$, $f(Y_{t-} + h_t(q)) - f(Y_{t-}) - f'(Y_{t-})h_t(q) = e^{Y_{t-}} \theta[h_t(q)]$ and if $Y_t^\sigma := Y_{t \wedge \sigma}$ is stopped at $\sigma := \inf\{t \in [0, 1]; Y_t \notin C\} \in [0, 1] \cup \{\infty\}$ for some compact subset C with the convention $\inf \emptyset = \infty$, we see with (36) and the fact that any path in Ω is bounded, that $\exp(Y_{t-}^\sigma) \theta[h_t(q)]$ is in $\mathcal{H}_1(R, \bar{L})$. Now, choosing the compact set C to be the ball of radius k and letting k tend to infinity, we obtain an increasing sequence of stopping times $(\sigma_k)_{k \geq 1}$ which tends almost surely to infinity. This proves that $Z^h := e^Y$ is a local martingale. We see that $dM_t = e^{Y_{t-}} d[(e^{h(q)} - 1) \odot \tilde{\mu}_t^L]$, keeping track of the martingale terms in the above differential formula:

$$\begin{aligned} de^{Y_t} &= e^{Y_{t-}} [\theta(\Delta Y_t) + dY_t] \\ &= e^{Y_{t-}} [\theta[h_t(q)] \odot d\tilde{\mu}_t^L + \left(\int_{\mathbb{R}_*^d} \theta[h_t(q)] L_t(dq) \right) \rho(dt) - \beta_t \rho(dt) + h(q) \odot d\tilde{\mu}_t^L] \\ &= e^{Y_{t-}} [\theta[h_t(q)] \odot d\tilde{\mu}_t^L + h_t(q) \odot d\tilde{\mu}_t^L] \\ &= e^{Y_{t-}} [(e^{h_t(q)} - 1) \odot d\tilde{\mu}_t^L]. \end{aligned}$$

By Fatou's lemma, any nonnegative local martingale is also a supermartingale. \square

Proof of Theorem 2.6. It follows the same line as the proof of Theorem 2.1. By Lemma 6.1, $0 < E_R Z_1^h \leq 1$ for all h satisfying the assumption (36). By (16), for any probability measure P such that $H(P|R) < \infty$, we have

$$E_P \left(h \odot \tilde{\mu}_1^L - \int_{[0,1] \times \mathbb{R}_*^d} \theta(h) d\bar{L} \right) \leq H(P|R).$$

As in the proof of Theorem 2.1, see that

$$|E_P(h \odot \tilde{\mu}_1^L)| \leq (H(P|R) + 1) \|h\|_\theta, \quad \forall h$$

where

$$\|h\|_\theta := \inf \left\{ a > 0; E_P \int_{[0,1] \times \mathbb{R}_*^d} \theta(h/a) d\bar{L} \leq 1 \right\} \in [0, \infty] \quad (37)$$

is the Luxemburg norm of the Orlicz space

$$L_\theta := \left\{ h : [0, 1] \times \mathbb{R}_*^d \times \Omega \rightarrow \mathbb{R}; \text{measurable s.t. } E_P \int_{[0,1] \times \mathbb{R}_*^d} \theta(b_o |h|) d\bar{L} < \infty, \text{ for some } b_o > 0 \right\}.$$

It differs from the corresponding *small* Orlicz space

$$S_\theta := \left\{ h : [0, 1] \times \mathbb{R}_*^d \times \Omega \rightarrow \mathbb{R}; \text{measurable s.t. } E_P \int_{[0,1] \times \mathbb{R}_*^d} \theta(b|h|) d\bar{L} < \infty, \forall b \geq 0 \right\}$$

because the function $\theta(|a|)$ grows exponentially fast.

We introduce the space \mathcal{B} of all the bounded processes such that $E_P \int_{[0,1] \times \mathbb{R}_*^d} |h| d\bar{L} < \infty$, and its subspace $\mathcal{H} \subset \mathcal{B}$ which consists of the processes in \mathcal{B} which are predictable. We have $\mathcal{B} \subset S_\theta$ and any h in \mathcal{H} satisfies (36), which is the hypothesis of Lemma 6.1. Hence, (37) holds for all $h \in \mathcal{H}$ and, as $H(P|R) < \infty$, it tells us that the linear mapping $h \mapsto E_P(h \odot \tilde{\mu}_1^L)$ is continuous on \mathcal{H} equipped with the norm $\|\cdot\|_\theta$. Since the convex conjugate of the Young function $\theta(|a|)$ is $\theta^*(|b|)$, the dual space of $(S_\theta, \|\cdot\|_\theta)^2$ (see [RR91]), is isomorphic to

$$L_{\theta^*} := \left\{ k : [0, 1] \times \mathbb{R}_*^d \times \Omega \rightarrow \mathbb{R}; \text{measurable s.t. } E_P \int_{[0,1] \times \mathbb{R}_*^d} \theta^*(|k|) d\bar{L} < \infty \right\}.$$

Therefore, there exists some $k \in L_{\theta^*}$ such that

$$E_P h \odot \tilde{\mu}_1^L = E_P \int_{[0,1] \times \mathbb{R}_*^d} k h d\bar{L}, \quad \forall h \in \mathcal{H}. \quad (38)$$

Let us introduce the predictable projection k^{pr} of k which is defined by $k_t^{\text{pr}} := E_P(k | X_{[0,t)})$, $t \in [0, 1]$. As the space \mathcal{B} is dense in S_θ ³, \mathcal{H} is dense in the subspace of all predictable processes in S_θ and it follows that any g and k in L_{θ^*} which both satisfy (38), share the same predictable projection: $g^{\text{pr}} = k^{\text{pr}}$. Consequently, there is a *unique* predictable process k in the space

$$\mathcal{K}(P) := \left\{ k : [0, 1] \times \mathbb{R}_*^d \times \Omega \rightarrow \mathbb{R}; \text{predictable s.t. } E_P \int_{[0,1] \times \mathbb{R}_*^d} \theta^*(|k|) d\bar{L} < \infty \right\}$$

which verifies (38).

As \mathcal{H} is included in $\mathcal{H}_1(P, \bar{L})$, we have for all $h \in \mathcal{H}$, $h \odot \tilde{\mu}^L - h \odot k \bar{L} = h \odot (\mu^X - \bar{L} - h \odot k \bar{L} = h \odot (\mu^X - \ell \bar{L})$ with $\ell := k + 1$. Consequently, (38) is equivalent to

$$E_P[h \odot (\mu^X - \ell \bar{L})] = 0, \quad \forall h \in \mathcal{H}, \quad (39)$$

which is the content of the theorem. It remains however to note that, being an expectation of the positive measure μ^X , $\ell \bar{L}$ is also a positive measure. Therefore, ℓ is nonnegative. This completes the proof of the theorem. \square

Proof of Corollary 2.7. It is mainly a remark based on Hölder's inequality in Orlicz spaces.

Proof of Corollary 2.7. We are under the exponential integrability assumption (12) and we denote $Z = \frac{dP}{dR}$. The finite entropy assumption (1) is equivalent to Z belongs to the Orlicz space $L_{\theta^*}(R)$, i.e. $\|Z\|_{\theta^*, R} < \infty$. Hölder's inequality in Orlicz spaces⁴ expressed with the Luxemburg norms (see (37)) gives us for any nonnnegative random variable U : $E_P(U) = E_R(ZU) \leq 2\|Z\|_{\theta^*, R}\|U\|_{\theta, R}$. This quantity is finite if $\|U\|_{\theta, R} < \infty$, and this is equivalent to $E_R(e^{a_o U}) < \infty$ for some $a_o > 0$. As a consequence, (12) implies that $E_P \int_{[0,1] \times \mathbb{R}_*^d} \mathbf{1}_{\{|q| \geq 1\}} e^{b_o |q|} \bar{L}(dtdq) < \infty$ for some b_o . But this is equivalent to: $\mathbf{1}_{\{|q| \geq 1\}} |q|$

²This doesn't hold with L_θ instead of S_θ .

³In general, it is not dense in L_θ .

⁴It is an easy consequence of Fenchel's inequality: $|ab| \leq \theta(|a|) + \theta^*(|b|)$, for all $a, b \in \mathbb{R}$.

belongs to the Orlicz space $L_\theta(P \otimes \bar{L})$. With (11) we see that $(\ell - 1)$ is in $L_{\theta^*}(P \otimes \bar{L})$ and by Hölder's inequality again, we obtain

$$E_P \int_{[0,1] \times \mathbb{R}_*^d} \mathbf{1}_{\{|q| \geq 1\}} |q| |\ell(t, q) - 1| \bar{L}(dtdq) < \infty.$$

The small jump part: $E_P \int_{[0,1] \times \mathbb{R}_*^d} \mathbf{1}_{\{|q| < 1\}} |q| |\ell(t, q) - 1| \bar{L}(dtdq) < \infty$, is a direct consequence of Hölder's inequality in L_2 . This proves (13).

We write symbolically

$$\tilde{\mu}^L = \mu - \bar{L} = \mu - \ell \bar{L} + (\ell - 1) \bar{L} = \hat{\mu} + (\ell - 1) \bar{L}.$$

Hence, $q \odot \tilde{\mu}^L = q \odot \hat{\mu} + \int (\ell - 1) q d\bar{L}$ provided that all these terms are well defined. But, we have assumed that $q \odot \tilde{\mu}^L$ is well-defined and we have just proved that $\int (\ell - 1) q d\bar{L}$ is well-defined. Therefore, the remaining term is also well-defined and the proof is complete. \square

Proof of Theorem 2.9. It is similar to the proof of Theorem 2.3. We begin with a transfer result in the spirit of Lemma 5.1. Let P be a probability measure on Ω such that

$$P \in \text{MP}(B, \bar{K})$$

where B is a continuous bounded variation adapted process and \bar{K} is some Lévy kernel

$$\bar{K}(dtdq) := \rho(dt)K(t; dq).$$

Let λ be a $[-\infty, \infty)$ -valued predictable process on $[0, 1] \times \mathbb{R}_*^d$ such that $\int_{\{\lambda \geq -1\}} \theta(\lambda) d\bar{K} < \infty$ and $\bar{K}(-\infty \leq \lambda < -1) < \infty$, P -a.s. We define for all $t \in [0, 1]$,

$$Z_t = \widetilde{\exp} \left(\lambda \odot \tilde{\mu}_t^K - \int_{[0,t] \times \mathbb{R}_*^d} \theta(\lambda) d\bar{K} \right) := Z_t^+ Z_t^-$$

with

$$\begin{cases} Z_t^+ &= \exp \left(\lambda^+ \odot \tilde{\mu}_t^K - \int_{(0,t] \times \mathbb{R}_*^d} \theta(\lambda^+) d\bar{K} \right) \\ Z_t^- &= \mathbf{1}_{\{t < \tau^\lambda\}} \exp \left(\sum_{0 \leq s \leq t} \lambda^-(s, \Delta X_s) - \int_{(0,t] \times \mathbb{R}_*^d} (e^{\lambda^-} - 1) d\bar{K} \right) \end{cases}$$

where

$$\lambda^+ = \mathbf{1}_{\{\lambda \geq -\alpha\}} \lambda, \quad \lambda^- = \mathbf{1}_{\{-\infty \leq \lambda < -\alpha\}} \lambda$$

with $\alpha > 0$, $e^{-\infty} = 0$ and $\tau^\lambda = \inf\{t \in [0, 1], \lambda(t, \Delta X_t) = -\infty\}$. Remark that, although Z^+ and Z^- both depend on the choice of α , their product $Z = Z^+ Z^-$ doesn't depend on $\alpha > 0$. For all $j, k \geq 1$, we define

$$\sigma_j^k := \inf \left\{ t \in [0, 1]; \int_{[0,t] \times \mathbb{R}_*^d} \theta(\lambda^+) d\bar{K} \geq k \text{ or } \lambda(t, \Delta X_t) \notin [-j, k] \right\} \in [0, 1] \cup \{\infty\}$$

and $P_j^k := X^{\sigma_j^k} \# P$.

Lemma 6.2. *Let P and λ be as above. Then, for all $j, k \geq 1$, $Z^{\sigma_j^k}$ is a genuine P -martingale and the measure*

$$Q_j^k := Z_1^{\sigma_j^k} P^k$$

is a probability measure on Ω which satisfies

$$Q_j^k \in \text{MP} \left(B^{\sigma_j^k} + \widehat{B}^{\sigma_j^k}, \mathbf{1}_{[0, \sigma_j^k]} e^{\lambda} \bar{K} \right)$$

where

$$\widehat{B}_t = \int_{[0,t] \times \mathbb{R}_*^d} \mathbf{1}_{\{|q| \leq 1\}} (e^\lambda - 1) q d\overline{K}, \quad t \in [0, 1]. \quad (40)$$

Note that \widehat{B}_t might not be well defined in the general case. Only the stopped processes $\widehat{B}_j^{\sigma_j^k}$ are asserted to be meaningful.

Proof. Let us fix $j, k \geq 1$. We have $Z^{\sigma_j^k} = \widehat{\exp}(\lambda_j^k \odot \tilde{\mu}^K - \int_{[0,1] \times \mathbb{R}_*^d} \theta(\lambda_j^k) d\overline{K})$ with $\lambda_j^k = \mathbf{1}_{[0, \sigma_j^k]} \lambda$ which is predictable since λ is predictable and $\mathbf{1}_{[0, \sigma_j^k]}$ is left continuous. We drop the subscripts and superscripts j, k and write $\lambda = \lambda_j^k$, $\lambda^+ = (\lambda_j^k)^+$, $\lambda^- = (\lambda_j^k)^-$, $Z^{\sigma_j^k} = Z$ for the remainder of the proof. By the definition of σ_j^k , we obtain with this simplified notation

$$\int_{[0,1] \times \mathbb{R}_*^d} \theta(\lambda^+) d\overline{K} \leq k, \quad -j \leq \lambda \leq k, \quad P_j^k\text{-a.s.} \quad (41)$$

Let us first prove that Z is a P_j^k -martingale. Since it is a local martingale, it is enough to show that

$$E_{P_j^k} Z_1^p < \infty, \quad \text{for some } p > 1.$$

Choosing $\alpha = j$ in the definition of $(Z^{\sigma_j^k})^+$ and $(Z^{\sigma_j^k})^-$, we see that $Z^{\sigma_j^k} = (Z^{\sigma_j^k})^+ = Z^+ = \mathcal{E}((e^{\lambda^+} - 1) \odot \tilde{\mu}^K)$. For all $p \geq 0$,

$$(Z^+)^p = \exp \left(p\lambda^+ \odot \tilde{\mu}^K - p \int_{[0,1] \times \mathbb{R}_*^d} \theta(\lambda^+) d\overline{K} \right) \leq \exp(p\lambda^+ \odot \tilde{\mu}^K)$$

and

$$\mathcal{E}((e^{p\lambda^+} - 1) \odot \tilde{\mu}^K) = \exp \left(p\lambda^+ \odot \tilde{\mu}^K - \int_{[0,1] \times \mathbb{R}_*^d} \theta(p\lambda^+) d\overline{K} \right) \geq e^{p\lambda^+ \odot \tilde{\mu}^K} / C(k, p)$$

for some finite deterministic constant $C(k, p) > 0$. To derive $C(k, p)$, we must take account of (41) and rely upon the inequality $\theta(pa) \leq c(k, p)\theta(a)$ which holds for all $a \in (-\infty, k]$ and some $0 < c(k, p) < \infty$. With this in hand, we obtain

$$(Z^+)^p \leq e^{p\lambda^+ \odot \tilde{\mu}^K} \leq C(k, p) \mathcal{E}((e^{p\lambda^+} - 1) \odot \tilde{\mu}^K).$$

We know with Lemma 6.1 that $\mathcal{E}((e^{p\lambda^+} - 1) \odot \tilde{\mu}^K)$ is a nonnegative local martingale. Therefore, it is a supermartingale. We deduce from this that $E_{P_j^k} \mathcal{E}((e^{p\lambda^+} - 1) \odot \tilde{\mu}^K) \leq 1$ and

$$E_{P_j^k} (Z^+)^p \leq C(k, p) E_{P_j^k} \mathcal{E}((e^{p\lambda^+} - 1) \odot \tilde{\mu}^K) \leq C(k, p) < \infty.$$

Choosing $p > 1$, it follows that $\mathcal{E}((e^{\lambda^+} - 1) \odot \tilde{\mu}^K)$ is uniformly integrable. We conclude as in Lemma 5.1's proof that $\mathcal{E}((e^\lambda - 1) \odot \tilde{\mu}^K)$ is a genuine P_j^k -martingale.

Now, let us show that

$$Q_j^k \in \text{LK} \left(\mathbf{1}_{[0, \sigma_j^k]} e^{\lambda \overline{K}} \right).$$

Let τ be a finitely valued stopping time and f a measurable function on $[0, 1] \times \mathbb{R}_*^d$ which will be specified later. We denote $F_t = \sum_{0 \leq s \leq t \wedge \tau} f(s, \Delta X_s)$ with the convention that $f(t, 0) = 0$ for all $t \in [0, 1]$. By Lemma 6.1, the martingale Z satisfies $dZ_t =$

$\mathbf{1}_{[0, \sigma_j^k]}(t) Z_{t-} [(e^\lambda - 1) \odot \tilde{\mu}^K]$. We have also $dF_t = \mathbf{1}_{[0, \tau]}(t) f(t, \Delta X_t)$ and $d[Z, F]_t = \mathbf{1}_{[0, \sigma_j^k \wedge \tau]}(t) Z_{t-} (e^{\lambda(\Delta X_t)} - 1) f(t, \Delta X_t)$, P_j^k -a.s. Consequently,

$$\begin{aligned}
& E_{Q_j^k} \sum_{0 \leq t \leq \tau} f(t, \Delta X_t) \\
&= E_{P_j^k} (Z_\tau F_\tau - Z_0 F_0) \\
&= E_{P_j^k} \int_{[0, \tau]} (F_t dZ_t + Z_t dF_t + d[Z, F]_t) \\
&= E_{P_j^k} \left[\int_{[0, \tau]} F_t dZ_t + \sum_{0 \leq t \leq \tau} Z_{t-} f(t, \Delta X_t) + \sum_{0 \leq t \leq \tau} Z_{t-} (e^{\lambda(t, \Delta X_t)} - 1) f(t, \Delta X_t) \right] \\
&= E_{P_j^k} \sum_{0 \leq t \leq \tau} Z_{t-} e^{\lambda(t, \Delta X_t)} f(t, \Delta X_t) \\
&= E_{P_j^k} \int_{[0, \tau] \times \mathbb{R}_*^d} Z_{t-} f(t, q) e^{\lambda(t, q)} \overline{K}(dt dq) \\
&= E_{Q_j^k} \int_{[0, \tau] \times \mathbb{R}_*^d} f(t, q) e^{\lambda(t, q)} \overline{K}(dt dq).
\end{aligned}$$

We are going to choose τ such that the above terms are meaningful. For each $n \geq 1$, consider $\tau_n := \inf\{t \in [0, 1]; \sum_{0 \leq s \leq t \wedge \tau} |f(s, \Delta X_s)| \geq n\}$ and take f in $L_1(P_j^k \otimes \overline{K})$ to obtain $\lim_{n \rightarrow \infty} \tau_n = \infty$, P_j^k -a.s. and a fortiori Q_j^k -a.s. It remains to take $\tau = \sigma \wedge \tau_n$ with any stopping time σ to see that the Lévy kernel of Q_j^k is $e^\lambda \overline{K} = e^{\lambda_j} \overline{K}$.

It remains to compute the drift term. Let us denote $X_t^* := \sum_{0 \leq s \leq t} \mathbf{1}_{\{|\Delta X_s| > 1\}} \Delta X_s$ the cumulated sum of large jumps of X , and $X^\Delta := X - X^*$ its complement. Let τ be a finitely valued stopping time and take $G_t = \xi \cdot X_{t \wedge \tau}^\Delta$ with $\xi \in \mathbb{R}^d$. We have $dG_t = \mathbf{1}_{[0, \tau]}(t) \xi \cdot (dB_t + (\mathbf{1}_{\{|q| \leq 1\}} q) \odot d\tilde{\mu}_t^K)$ and $d[Z, G]_t = \mathbf{1}_{[0, \sigma_j^k \wedge \tau]}(t) Z_{t-} (e^{\lambda(\Delta X_t)} - 1) \mathbf{1}_{\{|\Delta X_t| \leq 1\}} \xi \cdot \Delta X_t$, P_j^k -a.s. Therefore,

$$\begin{aligned}
& E_{Q_j^k} [\xi \cdot (X_\tau^\Delta - X_0^\Delta)] \\
&= E_{P_j^k} [Z_\tau G_\tau - Z_0 G_0] \\
&= E_{P_j^k} \left[\int_{[0, \tau]} (G_t dZ_t + Z_t dG_t + d[Z, G]_t) \right] \\
&= E_{P_j^k} \left[\int_{[0, \tau]} G_t dZ_t + \int_{[0, \tau]} Z_{t-} \xi \cdot (dB_t + (\mathbf{1}_{\{|q| \leq 1\}} q) \odot d\tilde{\mu}_t^K) \right. \\
&\quad \left. + \sum_{0 \leq t \leq \tau} Z_{t-} \mathbf{1}_{\{|\Delta X_t| \leq 1\}} (e^{\lambda(t, \Delta X_t)} - 1) \xi \cdot \Delta X_t \right] \\
&= E_{P_j^k} \left[\int_{[0, \tau]} Z_{t-} \xi \cdot dB_t + \sum_{0 \leq t \leq \tau} Z_{t-} \mathbf{1}_{\{|\Delta X_t| \leq 1\}} (e^{\lambda(t, \Delta X_t)} - 1) \xi \cdot \Delta X_t \right] \\
&= E_{P_j^k} \left[\int_{[0, \tau]} Z_{t-} \xi \cdot dB_t + \int_{[0, \tau]} Z_{t-} \left\{ \int_{\mathbb{R}_*^d} \mathbf{1}_{\{|q| \leq 1\}} (e^{\lambda(t, q)} - 1) \xi \cdot q K_t(dq) \right\} \rho(dt) \right] \\
&= E_{Q_j^k} \int_{[0, \tau]} \xi \cdot \left(dB_t + \left\{ \int_{\mathbb{R}_*^d} \mathbf{1}_{\{|q| \leq 1\}} (e^{\lambda(t, q)} - 1) q K_t(dq) \right\} \rho(dt) \right)
\end{aligned}$$

where we take $\tau = \tau_n := \inf\{t \in [0, 1]; |X_t| \geq n\}$ which tends to ∞ as n tends to infinity. This shows that the drift term of X under Q_j^k is $(B + \hat{B})^{\sigma_k}$ where \hat{B} is given at (40) and the stopped process \hat{B}^{σ_k} is well-defined. \square

As a first step, it is assumed that $P \sim R$ for the stopping times τ_j^k , τ_j and τ^- to be defined (below) R -a.s. and not only P -a.s.

Following the proofs of Lemmas 5.2 and 5.3, except for minor changes (but we skip the details), we arrive at analogous results:

- (i) If R fulfills the uniqueness condition (U), then for any stopping time τ , R^τ also fulfills (U).
- (ii) If $P \sim R$, then for any $j, k \geq 1$, we have

$$\mathbf{1}_{[0, \tau_j^k \wedge 1]} \frac{dP}{dR} = \mathbf{1}_{[0, \tau_j^k \wedge 1]} \frac{dP_0}{dR_0}(X_0) \exp \left((\mathbf{1}_{(0, \tau_j^k \wedge 1]} \log \ell) \odot \tilde{\mu}^L - \int_{(0, \tau_j^k \wedge 1] \times \mathbb{R}_*^d} \theta(\log \ell) d\bar{L} \right)$$

where

$$\tau_j^k := \inf \left\{ t \in [0, 1]; \int_{[0, t] \times \mathbb{R}_*^d} \mathbf{1}_{\{\ell > 1/2\}} \theta(\log \ell) d\bar{L} \geq k \text{ or } \log \ell(t, \Delta X_t) \notin [-j, k] \right\} \in [0, 1] \cup \{\infty\}.$$

For the proof of (ii), we use Lemma 6.1 where $\lambda = \log \ell$ plays the same role as β in Lemma 5.3, and we go backward with $-\lambda$ which corresponds to ℓ^{-1} .

We fix j , and let k tend to infinity to obtain with (11) that

$$\lim_{k \rightarrow \infty} \tau_j^k = \tau_j := \inf \{ t \in [0, 1]; \ell(t, \Delta X_t) < e^{-j} \} \in [0, 1] \cup \{\infty\}, \quad P\text{-a.s.}$$

and therefore R -a.s. also. More precisely, this increasing sequence is stationary after some time: there exists $K(\omega) < \infty$ such that $\tau_j^k(\omega) = \tau_j(\omega)$, for all $k \geq K(\omega)$. It follows that for all $j \geq 1$,

$$\mathbf{1}_{[0, \tau_j \wedge 1]} \frac{dP}{dR} = \mathbf{1}_{[0, \tau_j \wedge 1]} \frac{dP_0}{dR_0}(X_0) \exp \left((\mathbf{1}_{(0, \tau_j \wedge 1]} \log \ell) \odot \tilde{\mu}^L - \int_{(0, \tau_j \wedge 1] \times \mathbb{R}_*^d} \theta(\log \ell) d\bar{L} \right). \quad (42)$$

Lemma 6.3. *We do not assume that $P \sim R$ and we extend ℓ by $\ell = 1$ on the P -negligible subset where it is unspecified. Defining $\tau^- := \sup_{j \geq 1} \tau_j$, we have $P(\tau^- = \infty) = 1$.*

Proof. For all $j \geq 1$, we have $\tau^- \leq 1 \Rightarrow \sum_{t \leq 1} \mathbf{1}_{\{\ell(t, \Delta X_t) \leq e^{-j}\}} \geq 1$. Therefore,

$$\begin{aligned} P(\tau^- \leq 1) &\leq P \left(\sum_{t \leq 1} \mathbf{1}_{\{\ell(t, \Delta X_t) \leq e^{-j}\}} \geq 1 \right) \leq E_P \sum_{t \leq 1} \mathbf{1}_{\{\ell(t, \Delta X_t) \leq e^{-j}\}} \\ &\stackrel{\vee}{=} E_P \int_{[0, 1] \times \mathbb{R}_*^d} \mathbf{1}_{\{\ell \leq e^{-j}\}} \ell d\bar{L} \leq e^{-j} E_P \bar{L}(\ell \leq e^{-j}) \leq e^{-j} E_P \bar{L}(\ell \leq 1/2) \end{aligned}$$

where we used (39) at the marked equality. The result will follow letting j tend to infinity, provided that we show that $E_P \bar{L}(\ell \leq 1/2) < \infty$.

But, we know with (11) that $E_P \int_{[0, 1] \times \mathbb{R}_*^d} \theta^*(|\ell - 1|) d\bar{L} < \infty$. Hence, $E_P \bar{L}(\ell \leq 1/2) \leq E_P \int_{[0, 1] \times \mathbb{R}_*^d} \theta^*(|\ell - 1|) d\bar{L} / \theta^*(1/2) < \infty$ and the proof is complete. \square

Lemma 6.4. *Assume $P \sim R$. Let R_j and P_j be the laws of the stopped process $X^{\tau_j \wedge 1}$ under R and P respectively. Then, under the condition (U) we have for all $j \geq 1$*

$$H(P_j|R_j) = H(P_0|R_0) + E_P \int_{(0, \tau_j \wedge 1] \times \mathbb{R}_*^d} (\ell \log \ell - \ell - 1) d\bar{L}.$$

Proof. We denote R_j^k and P_j^k the laws of the stopped process $X^{\tau_j^k \wedge 1}$ under R and P respectively. With the expression of $\frac{dP}{dR}$ on $\llbracket 0, \tau_j^k \wedge 1 \rrbracket$ we see that

$$\begin{aligned} & H(P_j^k|R_j^k) \\ &= H(P_0|R_0) + E_{P_j^k} \left((\mathbf{1}_{(0, \tau_j^k \wedge 1]} \log \ell) \odot \tilde{\mu}^L - \int_{(0, \tau_j^k \wedge 1] \times \mathbb{R}_*^d} \theta(\log \ell) d\bar{L} \right) \\ &= H(P_0|R_0) + E_{P_j^k} \left((\mathbf{1}_{(0, \tau_j^k \wedge 1]} \log \ell) \odot \tilde{\mu}^{\ell L} + \int_{(0, \tau_j^k \wedge 1] \times \mathbb{R}_*^d} [(\ell - 1) - \theta(\log \ell)] d\bar{L} \right) \\ &= H(P_0|R_0) + E_{P_j} \int_{(0, \tau_j^k \wedge 1] \times \mathbb{R}_*^d} (\ell \log \ell - \ell - 1) d\bar{L} \end{aligned}$$

where we invoke Lemma 6.2 at the last equality. We complete the proof letting k tend to infinity. \square

Conclusion of the proof of Theorem 2.9. When $P \sim R$, by Lemma 6.3, P -almost surely there exists j_o large enough such that for all $j \geq j_o$, $\tau_j = \infty$ and (42) tells us that

$$\frac{dP}{dR} = \frac{dP_0}{dR_0}(X_0) \exp \left((\log \ell) \odot \tilde{\mu}^L - \int_{[0, 1] \times \mathbb{R}_*^d} \theta(\log \ell) d\bar{L} \right)$$

and also that the product appearing in Z^- contains P -almost surely a finite number of terms which are all positive. Note that we do not use any limit result for stochastic or standard integrals; it is an immediate ω -by- ω result with a stationary sequence. This is the desired expression for $\frac{dP}{dR}$ when $P \sim R$.

Let us extend this result to the case when P might not be equivalent to R . We proceed exactly as in Theorem 2.3's proof and start from (31): $\lim_{n \rightarrow \infty} H(P|P_n) = 0$ where $P_n := (1 - 1/n)P + R/n$, $n \geq 1$. Let us write $\lambda = \log \ell$ and $\lambda^n = \log \ell^n$ which are well-defined P -a.s. Thanks to Theorem 2.6, we see that

$$\begin{aligned} H(P|P_n) &\geq E_P \left((\lambda - \lambda^n) \odot \tilde{\mu}^{\ell^n L} - \int_{[0, 1] \times \mathbb{R}_*^d} \theta(\lambda - \lambda^n) \ell^n d\bar{L} \right) \\ &= E_P \left((\lambda^n - \lambda) \odot \tilde{\mu}^{\ell L} + \int_{[0, 1] \times \mathbb{R}_*^d} [\ell/\ell^n \log(\ell/\ell^n) - \ell/\ell^n + 1] \ell^n d\bar{L} \right) \\ &= E_P \int_{[0, 1] \times \mathbb{R}_*^d} [\ell^n/\ell - \log(\ell^n/\ell) - 1] d\ell \bar{L} \\ &= E_P \int_{[0, 1] \times \mathbb{R}_*^d} \theta(\lambda^n - \lambda) d\ell \bar{L} \end{aligned}$$

which leads to the entropic estimate analogous to (33):

$$\lim_{n \rightarrow \infty} E_P \int_{[0, 1] \times \mathbb{R}_*^d} \theta(\lambda^n - \lambda) d\ell \bar{L} = 0. \quad (43)$$

Taking the difference between $\log(dP_n/dR) = \lambda^n \odot \tilde{\mu}^L - \int_{[0,1] \times \mathbb{R}_*^d} \theta(\lambda^n) d\bar{L}$ and the logarithm of the announced formula (14) for dP/dR on the set $\{\frac{dP}{dR} > 0\}$, we obtain

$$(\lambda^n - \lambda) \odot \tilde{\mu}^L - \int_{[0,1] \times \mathbb{R}_*^d} \theta(\lambda^n - \lambda) d\bar{L}, \quad P\text{-a.s.}$$

and the desired convergence follows from (43). Note that $\theta(a) = a^2/2 + o_{a \rightarrow 0}(a^2)$. This completes the proof of (14).

As in the proof of Theorem 2.3, we obtain the announced formula for $H(P|R)$ under the condition (U) with Lemmas 6.3 and 6.4, and the corresponding general inequality follows from choosing

$$\tilde{u}(X) := \log \frac{dP_0}{dR_0}(X_0) + (\mathbf{1}_{(0, \tau_j^k \wedge 1]} \log \ell) \odot \tilde{\mu}^L - \int_{(0, \tau_j^k \wedge 1] \times \mathbb{R}_*^d} \theta^*(\log \ell) d\bar{L}$$

in the variational representation formula (16), and then letting k and j tend to infinity. \square

APPENDIX A. AN EXPONENTIAL MARTINGALE WITH JUMPS

Next proposition is about exponential martingale with jumps. We didn't use it during the proofs of this paper. But we give it here for having a more complete picture of the Girsanov theory.

In this result, integrands h are considered which may attain the value $-\infty$. This is because with $h = \log \ell$, $h = -\infty$ corresponds to $\ell = 0$.

Proposition A.1 (Exponential martingale). *Let $h : \Omega \times [0, 1] \times \mathbb{R}_*^d \rightarrow [-\infty, \infty)$ be an extended real valued predictable process which may take the value $-\infty$ and satisfies*

$$E_R \int_{[0,1] \times \mathbb{R}_*^d} \mathbf{1}_{\{h_t(q) \geq -1\}} \theta[h_t(q)] \bar{L}(dtdq) < \infty, \quad (44)$$

$$E_R \int_{[0,1] \times \mathbb{R}_*^d} \mathbf{1}_{\{h_t(q) < -1\}} \bar{L}(dtdq) < \infty. \quad (45)$$

Let us introduce the stopping time

$$\tau^h := \inf\{t \in [0, 1]; h(\Delta X_t) = -\infty\} \in [0, 1] \cup \{\infty\}$$

and the convention $e^{-\infty} = 0$.

Then, $e^h - 1$ is in $\mathcal{H}_{1,2}(R, \bar{L})$ and

$$Z_t^h := \mathbf{1}_{\{t < \tau^h\}} \widetilde{\exp} \left(h \odot \tilde{\mu}_t^L - \int_{(0,t] \times \mathbb{R}_*^d} \theta[h_s(q)] \bar{L}(dsdq) \right), \quad t \in [0, 1] \quad (46)$$

is a local R -martingale and a nonnegative R -supermartingale which satisfies

$$dZ_t^h = \mathbf{1}_{\{t \leq \tau^h\}} Z_{t-}^h [(e^{h(q)} - 1) \odot d\tilde{\mu}_t^L]. \quad (47)$$

The standard notation is $Z^h := \mathcal{E}([e^h - 1] \odot \tilde{\mu}^L)$, the stochastic exponential of $[e^h - 1] \odot \tilde{\mu}^L$. Some details are necessary to make precise the sense of the inner stochastic integral $h \odot \tilde{\mu}_t^L$ in the expression of Z_t^h . We denote

$$\begin{aligned} h^+ &:= \mathbf{1}_{\{h \geq -1\}} h \in \mathbb{R} \\ h^- &:= \mathbf{1}_{\{h < -1\}} h \in [-\infty, 0]. \end{aligned}$$

Under the assumption (44), $h^+ \odot \tilde{\mu}^L$ is well defined as a stochastic integral. On the other hand, (45) implies that $h^-(t, \Delta X_t)$ has R -a.s. finitely many jumps. It follows that $\sum_{0 \leq s \leq t} h^-(s, \Delta X_s)$ is meaningful for all $t < \tau^h$. But the integral $\int_{(0,t] \times \mathbb{R}_*^d} h_s^-(q) \bar{L}(dsdq)$

might not be defined under (45) and $h^- \odot \tilde{\mu}_t^L = \sum_{0 \leq s \leq t} h^-(s, \Delta X_s) - \int_{(0,t] \times \mathbb{R}_*^d} h_s^-(q) \bar{L}(dsdq)$ is meaningless in this case. Nevertheless, the full expression in the exponential $\zeta(h) := h \odot \tilde{\mu}^L - \int \theta(h) d\bar{L}$ is defined as follows. We put $\zeta(h^-) := \sum_{0 \leq s \leq t} h^-(s, \Delta X_s) - \int_{(0,t] \times \mathbb{R}_*^d} [e^{h_s^-(q)} - 1] \bar{L}(dsdq)$ which is well defined under (45) and is obtained by cancelling the terms $\int_{(0,t] \times \mathbb{R}_*^d} h_s^-(q) \bar{L}(dsdq)$. As $\theta(0) = 0$, we have $\zeta(h) = \zeta(h^+ + h^-) = \zeta(h^+) + \zeta(h^-)$ and for all $t \in [0, 1]$,

$$\begin{cases} Z_t^h &= Z_t^{h^+} Z_t^{h^-} \quad \text{with} \\ Z_t^{h^+} &:= \exp \left(h^+ \odot \tilde{\mu}_t^L - \int_{(0,t] \times \mathbb{R}_*^d} \theta[h_s^+(q)] \bar{L}(dsdq) \right), \\ Z_t^{h^-} &:= \mathbf{1}_{\{t < \tau^h\}} \exp \left(\sum_{0 \leq s \leq t} h^-(s, \Delta X_s) - \int_{(0,t] \times \mathbb{R}_*^d} [e^{h_s^-(q)} - 1] \bar{L}(dsdq) \right). \end{cases} \quad (48)$$

This is what is meant by the concise expression (46).

Proof. Now, we consider the general case where h may attain the value $-\infty$ and (36) is weakened by (44) and (45). We use the decomposition (48) and write $Z^+ = Z^{h^+}$ and $Z^- = Z^{h^-}$ for short. Clearly, Z^+ and Z^- do not jump at the same times and $d[Z^+, Z^-] = \Delta Z^+ \Delta Z^- = 0$. Hence,

$$dZ_t = Z_t^+ dZ_t^- + Z_t^- dZ_t^+. \quad (49)$$

The h^+ -part enters the framework of Lemma 6.1 and we have

$$dZ_t^+ = Z_t^+ \left([e^{h^+} - 1] \odot \tilde{\mu}_t^L \right). \quad (50)$$

Let us look at the h^- -part. We need to compute dZ_t^- . For all $t < \tau^h$, put

$$Y_t^- = \sum_{0 \leq s \leq t} h^-(s, \Delta X_s) - \int_{(0,t] \times \mathbb{R}_*^d} [e^{h_s^-(q)} - 1] \bar{L}(dsdq).$$

Then, with the convention that $h^-(t, 0) = 0$, $dY_t^- = h^-(t, \Delta X_t) - \gamma_t \rho(dt)$ with $\gamma_t = \int_{\mathbb{R}_*^d} [e^{h_t^-(q)} - 1] L_t(dq)$, $\Delta Y_t^- = h^-(t, \Delta X_t)$ and with Itô's formula, we arrive at

$$\begin{aligned} de^{Y_t^-} &= e^{Y_t^-} \left([e^{\Delta Y_t^-} - 1] + dY_t^- - \Delta Y_t^- \right) = e^{Y_t^-} \left([e^{h^-(t, \Delta X_t)} - 1] - \gamma_t \rho(dt) \right) \\ &= e^{Y_t^-} \left([e^{h^-} - 1] \odot d\tilde{\mu}_t^L \right). \end{aligned}$$

It follows that

$$dZ_t^- = Z_t^- \left([e^{h^-} - 1] \odot d\tilde{\mu}_t^L \right), \quad t < \tau^h. \quad (51)$$

At $t = \tau^h$, by the definition (48) of Z^- , we have

$$dZ_{|t=\tau^h}^- = -Z_{(\tau^h)^-}^- = Z_{(\tau^h)^-}^- \times [e^{-\infty} - 1]$$

which is (51) at $t = \tau^h$ with the convention $e^{-\infty} = 0$. This provides us with

$$dZ_t^- = \mathbf{1}_{\{t \leq \tau^h\}} Z_t^- \left([e^{h^-} - 1] \odot \tilde{\mu}_t^L \right).$$

Together with (49) and (50), this proves (47) which implies that Z^h is a local R -martingale. By Fatou's lemma, any nonnegative local martingale is also a supermartingale. \square

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