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# Weak Solutions to the Continuous Coagulation Equation with Multiple Fragmentation

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**Abstract** The existence of weak solutions to the continuous coagulation equation with multiple fragmentation is shown for a class of unbounded coagulation and fragmentation kernels, the fragmentation kernel having possibly a singularity at the origin. This result extends previous ones where either boundedness of the coagulation kernel or no singularity at the origin for the fragmentation kernel were assumed.

**Keywords:** Coagulation; Multiple Fragmentation; Unbounded kernels; Existence; Weak compactness

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## 1 Introduction

The continuous coagulation and multiple fragmentation equation describes the evolution of the number density  $f = f(x, t)$  of particles of volume  $x \geq 0$  at time  $t \geq 0$  and reads

$$\begin{aligned} \frac{\partial f(x, t)}{\partial t} = & \frac{1}{2} \int_0^x K(x-y, y) f(x-y, t) f(y, t) dy - \int_0^\infty K(x, y) f(x, t) f(y, t) dy \\ & + \int_x^\infty b(x, y) S(y) f(y, t) dy - S(x) f(x, t), \end{aligned} \quad (1)$$

with

$$f(x, 0) = f_0(x) \geq 0. \quad (2)$$

The first two terms on the right-hand side of (1) accounts for the formation and disappearance of particles as a result of coagulation events and the coagulation kernel  $K(x, y)$  represents the

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rate at which particles of volume  $x$  coalesce with particles of volume  $y$ . The remaining two terms on the right-hand side of (1) describes the variation of the number density resulting from fragmentation events which might produce more than two daughter particles, and the breakage function  $b(x, y)$  is the probability density function for the formation of particles of volume  $x$  from the particles of volume  $y$ . Note that it is non-zero only for  $x < y$ . The selection function  $S(x)$  describes the rate at which particles of volume  $x$  are selected to fragment. The selection function  $S$  and breakage function  $b$  are defined in terms of the multiple-fragmentation kernel  $\Gamma$  by the identities

$$S(x) = \int_0^x \frac{y}{x} \Gamma(x, y) dy, \quad b(x, y) = \Gamma(y, x)/S(y). \quad (3)$$

The breakage function is assumed here to have the following properties

$$\int_0^y b(x, y) dx = N < \infty, \quad \text{for all } y > 0, \quad b(x, y) = 0 \quad \text{for } x > y, \quad (4)$$

and

$$\int_0^y xb(x, y) dx = y \quad \text{for all } y > 0. \quad (5)$$

The parameter  $N$  represents the number of fragments obtained from the breakage of particles of volume  $y$  and is assumed herein to be finite and independent of  $y$ . This is however inessential for the forthcoming analysis, see Remark 2.3 below. As for the condition (5), it states that the total volume of the fragments resulting from the splitting of a particle of volume  $y$  equals  $y$  and thus guarantees that the total volume of the system remains conserved during fragmentation events.

The existence of solutions to coagulation-fragmentation equations has already been the subject of several papers which however are mostly devoted to the case of binary fragmentation, that is, when the fragmentation kernel  $\Gamma$  satisfies the additional symmetry property  $\Gamma(x + y, y) = \Gamma(x + y, x)$  for all  $(x, y) \in ]0, \infty[^2$ , see the survey [7] and the references therein. The coagulation-fragmentation equation with multiple fragmentation has received much less attention over the years though it is already considered in the pioneering work [10], where the existence and uniqueness of solutions to (1)-(2) are established for bounded coagulation and fragmentation kernels  $K$  and  $\Gamma$ . A similar result was obtained later on in [9] by a different approach. The boundedness of  $\Gamma$  was subsequently relaxed in [4] where it is only assumed that  $S$  grows at most linearly, but still for a bounded coagulation kernel. Handling simultaneously unbounded coagulation and fragmentation kernels turns out to be more delicate and, to our knowledge, is only considered in [5] for coagulation kernels  $K$  of the form  $K(x, y) = r(x)r(y)$  with no growth restriction on  $r$  and a moderate growth assumption on  $\Gamma$  (depending on  $r$ ) and in [3] for coagulation kernels satisfying  $K(x, y) \leq \phi(x)\phi(y)$  for some sublinear function  $\phi$  and a moderate growth assumption on  $\Gamma$  (see also [6] for the existence of solutions for the corresponding discrete model). Still, the fragmentation kernel  $\Gamma$  is required to be bounded near the origin in [3, 5] which thus excludes kernels frequently encountered in the literature such as  $\Gamma(y, x) = (\alpha + 2) x^\alpha y^{\gamma - (\alpha + 1)}$  with  $\alpha > -2$  and  $\gamma \in \mathbb{R}$  [8].

The purpose of this note is to fill (at least partially) this gap and establish the existence of weak solutions to (1) for simultaneously unbounded coagulation and fragmentation kernels  $K$  and  $\Gamma$ , the latter being possibly unbounded for small and large volumes. More precisely, we

make the following hypotheses on the coagulation kernel  $K$ , multiple-fragmentation kernel  $\Gamma$ , and selection rate  $S$ .

**Hypotheses 1.1.** (H1)  $K$  is a non-negative measurable function on  $]0, \infty[ \times ]0, \infty[$  and is symmetric, i.e.  $K(x, y) = K(y, x)$  for all  $x, y \in ]0, \infty[$ ,

(H2)  $K(x, y) \leq \phi(x)\phi(y)$  for all  $x, y \in ]0, \infty[$  where  $\phi(x) \leq k_1(1+x)^\mu$  for some  $0 \leq \mu < 1$  and constant  $k_1 > 0$ .

(H3)  $\Gamma$  is a non-negative measurable function on  $]0, \infty[ \times ]0, \infty[$  such that  $\Gamma(x, y) = 0$  if  $0 < x < y$ . Defining  $S$  and  $b$  by (3), we assume that  $b$  satisfies (5) and there are  $\theta \in [0, 1[$  and two non-negative functions  $k : ]0, \infty[ \rightarrow [0, \infty[$  and  $\omega : ]0, \infty[^2 \rightarrow [0, \infty[$  such that, for each  $R \geq 1$ :

(H4) we have  $\Gamma(y, x) \leq k(R) y^\theta$  for  $y > R$  and  $x \in ]0, R[$ ,

(H5) for  $y \in ]0, R[$  and any measurable subset  $E$  of  $]0, R[$ , we have

$$\int_0^y \mathbb{1}_E(x) \Gamma(y, x) dx \leq \omega(R, |E|), \quad y \in ]0, R[,$$

where  $|E|$  denotes the Lebesgue measure of  $E$ ,  $\mathbb{1}_E$  is the indicator function of  $E$  given by

$$\mathbb{1}_E(x) := \begin{cases} 1 & \text{if } x \in E, \\ 0 & \text{if } x \notin E, \end{cases}$$

and we assume in addition that

$$\lim_{\delta \rightarrow 0} \omega(R, \delta) = 0,$$

(H6)  $S \in L^\infty]0, R[$ .

We next introduce the functional setting which will be used in this paper: define the Banach space  $X$  with norm  $\|\cdot\|$  by

$$X = \{f \in L^1(0, \infty) : \|f\| < \infty\} \quad \text{where} \quad \|f\| = \int_0^\infty (1+x)|f(x)|dx,$$

together with its positive cone

$$X^+ = \{f \in X : f \geq 0 \text{ a.e.}\}.$$

For further use, we also define the norms

$$\|f\|_x = \int_0^\infty x|f(x)|dx \quad \text{and} \quad \|f\|_1 = \int_0^\infty |f(x)|dx, \quad f \in X.$$

The main result of this note is the following existence result:

**Theorem 1.2.** *Suppose that (H1)–(H6) hold and assume that  $f_0 \in X^+$ . Then (1)-(2) has a weak solution  $f$  on  $]0, \infty[$  in the sense of Definition 1.3 below. Furthermore,  $\|f(t)\|_x \leq \|f_0\|_x$  for all  $t \geq 0$ .*

Before giving some examples of coagulation and fragmentation kernels satisfying (H1)–(H6), we recall the definition of a weak solution to (1)-(2) [12].

**Definition 1.3.** *Let  $T \in ]0, \infty[$ . A solution  $f$  of (1)-(2) is a non-negative function  $f : [0, T[ \rightarrow X^+$  such that, for a.e.  $x \in ]0, \infty[$  and all  $t \in [0, T[$ ,*

(i)  $s \mapsto f(x, s)$  is continuous on  $[0, T[$ ,

(ii) the following integrals are finite

$$\int_0^t \int_0^\infty K(x, y) f(y, s) dy ds < \infty \quad \text{and} \quad \int_0^t \int_x^\infty b(x, y) S(y) f(y, s) dy ds < \infty,$$

(iii) the function  $f$  satisfies the following weak formulation of (1)-(2)

$$f(x, t) = f_0(x) + \int_0^t \left\{ \frac{1}{2} \int_0^x K(x-y, y) f(x-y, s) f(y, s) dy - \int_0^\infty K(x, y) f(x, s) f(y, s) dy + \int_x^\infty b(x, y) S(y) f(y, s) dy - S(x) f(x, s) \right\} ds.$$

Coming back to (H1)–(H6), it is clear that coagulation kernels satisfying  $K(x, y) \leq x^\mu y^\nu + x^\nu y^\mu$  for some  $\mu \in [0, 1[$  and  $\nu \in [0, 1[$  which are usually used in the mathematical literature satisfy (H1)–(H2), see also [3] for more complex choices. Let us now turn to fragmentation kernels which also fit in the classes considered in Hypotheses 1.1.

Clearly, if we assume that

$$\Gamma \in L^\infty(]0, \infty[ \times ]0, \infty[)$$

as in [3, 9], (H4) and (H5) are satisfied with  $k = \|\Gamma\|_{L^\infty}$ ,  $\theta = 0$ , and  $\omega(R, \delta) = \|\Gamma\|_{L^\infty} \delta$ . Now let us take

$$S(y) = y^\gamma \quad \text{and} \quad b(x, y) = \frac{\alpha + 2}{y} \left( \frac{x}{y} \right)^\alpha \quad \text{for } 0 < x < y, \quad (6)$$

where  $\gamma > 0$  and  $\alpha \geq 0$ , see [8, 11]. Then

$$\Gamma(y, x) = (\alpha + 2) x^\alpha y^{\gamma - (\alpha + 1)} \quad \text{for } 0 < x < y.$$

Let us first check (H5). Given  $R > 0$ ,  $y \in ]0, R[$ , and a measurable subset  $E$  of  $]0, R[$ , we deduce

from Hölder's inequality that

$$\begin{aligned}
\int_0^y \mathbb{1}_E(x) \Gamma(y, x) dx &= (\alpha + 2) y^{\gamma - (\alpha + 1)} \int_0^y \mathbb{1}_E(x) x^\alpha dx \\
&\leq (\alpha + 2) y^{\gamma - (\alpha + 1)} |E|^{\frac{\gamma}{\gamma + 1}} \left( \int_0^y x^{\alpha(\gamma + 1)} dx \right)^{\frac{1}{\gamma + 1}} \\
&\leq (\alpha + 2) |E|^{\frac{\gamma}{\gamma + 1}} (1 + \alpha(\gamma + 1))^{-\frac{1}{\gamma + 1}} y^{\alpha + \frac{1}{\gamma + 1} + \gamma - (\alpha + 1)} \\
&\leq C(\alpha, \gamma) y^{\frac{\gamma^2}{\gamma + 1}} |E|^{\frac{\gamma}{\gamma + 1}} \\
&\leq C(\alpha, \gamma) R^{\frac{\gamma^2}{\gamma + 1}} |E|^{\frac{\gamma}{\gamma + 1}}.
\end{aligned}$$

This shows that (H5) is fulfilled with  $\omega(R, \delta) = C(\alpha, \gamma) R^{\frac{\gamma^2}{\gamma + 1}} \delta^{\frac{\gamma}{\gamma + 1}}$ . As for (H4), for  $0 < x < R < y$ , we write

$$\Gamma(y, x) \leq (\alpha + 2) R^\alpha y^{\gamma - (\alpha + 1)} \leq \begin{cases} (\alpha + 2) R^{\gamma - 1} & \text{if } \gamma \leq \alpha + 1, \\ (\alpha + 2) R^\alpha y^{\gamma - (\alpha + 1)} & \text{if } \gamma > \alpha + 1, \end{cases}$$

and (H4) is satisfied provided  $\gamma < 2 + \alpha$  with  $k(R) = (\alpha + 2) R^{\gamma - 1}$  and  $\theta = 0$  if  $\gamma \in ]0, \alpha + 1[$  and  $k(R) = (\alpha + 2) R^\alpha$  and  $\theta = \gamma - (\alpha + 1) \in [0, 1[$  if  $\gamma \in ]\alpha + 1, \alpha + 2[$ . Therefore, Theorem 1.2 provides the existence of weak solutions to (1)-(2) for unbounded coagulation kernels  $K$  satisfying (H1)-(H2) and multiple fragmentation kernels  $\Gamma$  given by (6) with  $\alpha \geq 0$  and  $\gamma \in ]0, \alpha + 2[$ . Let us however mention that some fragmentation kernels which are bounded at the origin and considered in [3, 5] need not satisfy (H4)-(H5).

**Remark 1.4.** *While the requirement  $\gamma < \alpha + 2$  restricting the growth of  $\Gamma$  might be only of a technical nature, the constraint  $\gamma > 0$  might be more difficult to remove. Indeed, it is well-known that there is an instantaneous loss of matter in the fragmentation equation when  $S(x) = x^\gamma$  and  $\gamma < 0$  produced by the rapid formation of a large amount of particles with volume zero (dust), a phenomenon referred to as disintegration or shattering [8]. The case  $\gamma = 0$  thus appears as a borderline case.*

Let us finally outline the proof of Theorem 1.2. Since the pioneering work [12], it has been realized that  $L^1$ -weak compactness techniques are a suitable way to tackle the problem of existence for coagulation-fragmentation equations with unbounded kernels. This is thus the approach we use hereafter, the main novelty being the proof of the estimates needed to guarantee the expected weak compactness in  $L^1$ . These estimates are derived in Section 2.2 on a sequence of unique global solutions to truncated versions of (1)-(2) constructed in Section 2.1. After establishing weak equicontinuity with respect to time in Section 2.3, we extract a weakly convergent subsequence in  $L^1$  and finally show that the limit function obtained from the weakly convergent subsequence is actually a solution to (1)-(2) in Sections 2.4 and 2.5.

## 2 Existence

### 2.1 Approximating equations

In order to prove the existence of solutions to (1-2), we take the limit of a sequence of approximating equations obtained by replacing the kernel  $K$  and selection rate  $S$  by their “cut-off”

analogues  $K_n$  and  $S_n$  [12], where

$$K_n(x, y) := \begin{cases} K(x, y) & \text{if } x + y < n, \\ 0 & \text{if } x + y \geq n, \end{cases} \quad S_n(x) := \begin{cases} S(x) & \text{if } 0 < x < n, \\ 0 & \text{if } x \geq n, \end{cases}$$

for  $n \geq 1$ . Owing to the boundedness of  $K_n$  and  $S_n$  for each  $n \geq 1$ , we may argue as in [12, Theorem 3.1] or [13] to show that the approximating equation

$$\begin{aligned} \frac{\partial f^n(x, t)}{\partial t} = & \frac{1}{2} \int_0^x K_n(x - y, y) f^n(x - y, t) f^n(y, t) dy - \int_0^{n-x} K_n(x, y) f^n(x, t) f^n(y, t) dy \\ & + \int_x^n b(x, y) S_n(y) f^n(y, t) dy - S_n(x) f^n(x, t), \end{aligned} \quad (7)$$

with initial condition

$$f_0^n(x) := \begin{cases} f_0(x) & \text{if } 0 < x < n, \\ 0 & \text{if } x \geq n. \end{cases} \quad (8)$$

has a unique non-negative solution  $f^n \in C^1([0, \infty[; L^1]0, n])$  such that  $f^n(t) \in X^+$  for all  $t \geq 0$ . In addition, the total volume remains conserved for all  $t \in [0, \infty[$ , i.e.

$$\int_0^n x f^n(x, t) dx = \int_0^n x f_0^n(x) dx. \quad (9)$$

From now on, we extend  $f^n$  by zero to  $]0, \infty[ \times ]0, \infty[$ , i.e. we set  $f^n(x, t) = 0$  for  $x > n$  and  $t \geq 0$ . Observe that we then have the identity  $S_n f^n = S f^n$ .

Next, we need to establish suitable estimates in order to apply the *Dunford-Pettis Theorem* [2, Theorem 4.21.2] and then the equicontinuity of the sequence  $(f^n)_{n \in \mathbb{N}}$  in time to use the *Arzelà-Ascoli Theorem* [1, Appendix A8.5]. This is the aim of the next two sections.

## 2.2 Weak compactness

**Lemma 2.1.** *Assume that (H1)–(H6) hold and fix  $T > 0$ . Then we have:*

(i) *There is  $L(T) > 0$  (depending on  $T$ ) such that*

$$\int_0^\infty (1 + x) f^n(x, t) dx \leq L(T) \quad \text{for } n \geq 1 \quad \text{and all } t \in [0, T],$$

(ii) *For any  $\varepsilon > 0$  there exists  $R_\varepsilon > 0$  such that for all  $t \in [0, T]$*

$$\sup_{n \geq 1} \left\{ \int_{R_\varepsilon}^\infty f^n(x, t) dx \right\} \leq \varepsilon,$$

(iii) *given  $\varepsilon > 0$  there exists  $\delta_\varepsilon > 0$  such that, for every measurable set  $E$  of  $]0, \infty[$  with  $|E| \leq \delta_\varepsilon$ ,  $n \geq 1$ , and  $t \in [0, T]$ ,*

$$\int_E f^n(x, t) dx < \varepsilon.$$

*Proof.* (i) Let  $n \geq 1$  and  $t \in [0, T]$ . Integrating (7) with respect to  $x$  over  $]0, 1[$  and using Fubini's Theorem, we have

$$\begin{aligned} \frac{d}{dt} \int_0^1 f^n(x, t) dx &= -\frac{1}{2} \int_0^1 \int_0^{1-x} K_n(x, y) f^n(x, t) f^n(y, t) dy dx \\ &\quad - \int_0^1 \int_{1-x}^{n-x} K_n(x, y) f^n(x, t) f^n(y, t) dy dx \\ &\quad + \int_0^1 \int_x^n b(x, y) S(y) f^n(y, t) dy dx - \int_0^1 S(x) f^n(x, t) dx. \end{aligned}$$

Since  $K_n$ ,  $f^n$ , and  $S$  are non-negative and  $\Gamma$  satisfies (3), we have

$$\begin{aligned} \frac{d}{dt} \int_0^1 f^n(x, t) dx &\leq \int_0^1 \int_x^n b(x, y) S(y) f^n(y, t) dy dx \\ &= \int_0^1 \int_x^1 \Gamma(y, x) f^n(y, t) dy dx + \int_0^1 \int_1^n \Gamma(y, x) f^n(y, t) dy dx, \end{aligned}$$

Using Fubini's Theorem and (H5) (with  $R = 1$  and  $E = ]0, 1[$ ) in the first term of the right-hand side and (H4) (with  $R = 1$ ) in the second one, we obtain

$$\begin{aligned} \frac{d}{dt} \int_0^1 f^n(x, t) dx &\leq \int_0^1 f^n(y, t) \int_0^y \Gamma(y, x) dx dy + k(1) \int_0^1 \int_1^n y f^n(y, t) dy dx \\ &\leq \omega(1, 1) \int_0^1 f^n(x, t) dx + k(1) \|f^n(t)\|_x. \end{aligned} \tag{10}$$

Recalling that  $\|f^n(t)\|_x = \|f^n(0)\|_x \leq \|f_0\|$  for  $t \geq 0$  by (9), we readily deduce from (10) that

$$\frac{d}{dt} \int_0^1 f^n(x, t) dx \leq \omega(1, 1) \int_0^1 f^n(y, t) dy + k(1) \|f_0\|.$$

Integrating with respect to time, we end up with

$$\int_0^1 f^n(x, t) dx \leq \|f_0\| \left( 1 + \frac{k(1)}{\omega(1, 1)} \right) \exp(\omega(1, 1)t), \quad t \in [0, T].$$

Using (9) again we may estimate

$$\begin{aligned} \int_0^\infty (1+x) f^n(x, t) dx &= \int_0^1 f^n(x, t) dx + \int_1^n f^n(x, t) dx + \int_0^n x f^n(x, t) dx \\ &\leq \int_0^1 f^n(x, t) dx + \int_1^n x f^n(x, t) dx + \|f_0\| \\ &\leq \|f_0\| \left[ \left( 1 + \frac{k(1)}{\omega(1, 1)} \right) \exp(\omega(1, 1)T) + 2 \right] =: L(T). \end{aligned}$$

(ii) For  $\varepsilon > 0$ , set  $R_\varepsilon := \|f_0\|/\varepsilon$ . Then, by (9), for each  $n \geq 1$  and for all  $t \in [0, T]$  we have

$$\int_{R_\varepsilon}^\infty f^n(x, t) dx \leq \frac{1}{R_\varepsilon} \int_{R_\varepsilon}^\infty x f^n(x, t) dx \leq \frac{\|f_0\|}{R_\varepsilon} < \varepsilon.$$

(iii) Fix  $R > 0$ . For  $n \geq 1$ ,  $\delta \in (0, 1)$ , and  $t \in [0, T]$ , we define

$$p^n(\delta, t) = \sup \left\{ \int_0^R \mathbf{1}_E(x) f^n(x, t) dx : E \subset ]0, R[ \text{ and } |E| \leq \delta \right\}.$$

Consider a measurable subset  $E \subset ]0, R[$  with  $|E| \leq \delta$ . For  $n \geq 1$  and  $t \in [0, T]$ , it follows from the non-negativity of  $f^n$ , (3) and (7)-(8) that

$$\frac{d}{dt} \int_0^R \mathbf{1}_E(x) f^n(x, t) dx \leq \frac{1}{2} I_1^n(t) + I_2^n(t) + I_3^n(t), \quad (11)$$

where

$$\begin{aligned} I_1^n(t) &:= \int_0^R \mathbf{1}_E(x) \int_0^x K_n(x-y, y) f^n(x-y, t) f^n(y, t) dy dx, \\ I_2^n(t) &:= \int_0^R \mathbf{1}_E(x) \int_x^R \Gamma(y, x) f^n(y, t) dy dx, \\ I_3^n(t) &:= \int_0^R \mathbf{1}_E(x) \int_R^\infty \Gamma(y, x) f^n(y, t) dy dx. \end{aligned}$$

First, applying Fubini's Theorem to  $I_1^n(t)$  gives

$$\begin{aligned} I_1^n(t) &= \int_0^R f^n(y, t) \int_y^R \mathbf{1}_E(x) K_n(y, x-y) f^n(x-y, t) dx dy \\ &= \int_0^R f^n(y, t) \int_0^{R-y} \mathbf{1}_E(x+y) K_n(y, x) f^n(x, t) dx dy. \end{aligned}$$

Setting  $-y + E := \{z > 0 : z = -y + x \text{ for some } x \in E\}$ , it follows from (H2) and the above identity that

$$I_1^n(t) \leq k_1^2 (1+R)^\mu \int_0^R (1+y)^\mu f^n(y, t) \int_0^R f^n(x, t) \mathbf{1}_{-y+E \cap ]0, R-y[}(x) dx dy.$$

Since  $-y + E \cap ]0, R-y[ \subset ]0, R[$  and  $|-y + E \cap ]0, R-y[| \leq |-y + E| = |E| \leq \delta$ , we infer from the definition of  $p^n(\delta, t)$  and Lemma 2.1 (i) that

$$I_1^n(t) \leq k_1^2 (1+R)^\mu \left( \int_0^R (1+y)^\mu f^n(y, t) dy \right) p^n(\delta, t) \leq k_1^2 L(T) (1+R)^\mu p^n(\delta, t).$$

Next, applying Fubini's Theorem to  $I_2^n(t)$  and using (H5) and Lemma 2.1 (i) give

$$I_2^n(t) = \int_0^R f^n(y, t) \int_0^y \mathbf{1}_E(x) \Gamma(y, x) dx dy \leq \omega(R, |E|) \int_0^R f^n(y, t) dy \leq L(T) \omega(R, |E|).$$

Finally, owing to (H4) and (9), we have

$$\begin{aligned} I_3^n(t) &\leq k(R) \int_0^R \int_R^\infty \mathbf{1}_E(x) y^\theta f^n(y, t) dy dx \leq k(R) R^{\theta-1} |E| \int_R^\infty y f^n(y, t) dy \\ &\leq k(R) R^{\theta-1} \|f_0\| |E| \leq k(R) R^{\theta-1} \|f_0\| \delta. \end{aligned}$$

Collecting the estimates on  $I_j^n(t)$ ,  $1 \leq j \leq 3$ , we infer from (11) that there is  $C_1(R, T) > 0$  such that

$$\frac{d}{dt} \int_0^R \mathbf{1}_E(x) f^n(x, t) dx \leq C_1(R, T) (p^n(\delta, t) + \omega(R, \delta) + \delta).$$

Integrating with respect to time and taking the supremum over all  $E$  such that  $E \subset ]0, R[$  with  $|E| \leq \delta$  give

$$p^n(\delta, t) \leq p^n(\delta, 0) + TC_1(R, T)[\omega(R, \delta) + \delta] + C_1(R, T) \int_0^t p^n(\delta, s) ds, \quad t \in [0, T].$$

By Gronwall's inequality (see e.g. [14, p. 310]), we obtain

$$p^n(\delta, t) \leq [p^n(\delta, 0) + TC_1(R, T)(\omega(R, \delta) + \delta)] \exp \{C_1(R, T)t\}, \quad t \in [0, T]. \quad (12)$$

Now, since  $f^n(x, 0) \leq f_0(x)$  for  $x > 0$ , the absolute continuity of the integral guarantees that  $\sup_n \{p^n(\delta, 0)\} \rightarrow 0$  as  $\delta \rightarrow 0$  which implies, together with (H5) and (12) that

$$\lim_{\delta \rightarrow 0} \sup_{n \geq 1, t \in [0, T]} \{p^n(\delta, t)\} = 0.$$

Lemma 2.1 (iii) is then a straightforward consequence of this property and Lemma 2.1 (i).  $\square$

Lemma 2.1 and the *Dunford-Pettis Theorem* imply that, for each  $t \in [0, T]$ , the sequence of functions  $(f^n(t))_{n \geq 1}$  lies in a weakly relatively compact set of  $L^1]0, \infty[$  which does not depend on  $t \in [0, T]$ .

### 2.3 Equicontinuity in time

Now we proceed to show the time equicontinuity of the sequence  $(f^n)_{n \in \mathbb{N}}$ . Though the coagulation terms can be handled as in [3, 5, 12], we sketch the proof below for the sake of completeness. Let  $T > 0$ ,  $\varepsilon > 0$ , and  $\phi \in L^\infty]0, \infty[$  and consider  $s, t \in [0, T]$  with  $t \geq s$ . Fix  $R > 1$  such that

$$\frac{2L(T)}{R} < \frac{\varepsilon}{2}, \quad (13)$$

the constant  $L(T)$  being defined in Lemma 2.1 (i). For each  $n$ , by Lemma 2.1 (i),

$$\int_R^\infty |f^n(x, t) - f^n(x, s)| dx \leq \frac{1}{R} \int_R^\infty x \{f^n(x, t) + f^n(x, s)\} dx \leq \frac{2L(T)}{R}. \quad (14)$$

By (7), (13), and (14), we get

$$\begin{aligned} & \left| \int_0^\infty \phi(x) \{f^n(x, t) - f^n(x, s)\} dx \right| \\ & \leq \left| \int_0^R \phi(x) \{f^n(x, t) - f^n(x, s)\} dx \right| + \int_R^\infty |\phi(x)| |f^n(x, t) - f^n(x, s)| dx \\ & \leq \|\phi\|_{L^\infty} \int_s^t \left[ \frac{1}{2} \int_0^R \int_0^x K_n(x-y, y) f^n(x-y, \tau) f^n(y, \tau) dy dx \right. \\ & \quad \left. + \int_0^R \int_0^{n-x} K_n(x, y) f^n(x, \tau) f^n(y, \tau) dy dx + \int_0^R \int_x^n b(x, y) S(y) f^n(y, \tau) dy dx \right. \\ & \quad \left. + \int_0^R S(x) f^n(x, \tau) dx \right] d\tau + \|\phi\|_{L^\infty} \frac{\varepsilon}{2}. \end{aligned} \quad (15)$$

By Fubini's Theorem, (H2), and Lemma 2.1 (i), the first term of the right-hand side of (15) may be estimated as follows:

$$\begin{aligned}
& \frac{1}{2} \int_0^R \int_0^x K_n(x-y, y) f^n(x-y, \tau) f^n(y, \tau) dy dx \\
&= \frac{1}{2} \int_0^R \int_y^R K_n(x-y, y) f^n(x-y, \tau) f^n(y, \tau) dx dy \\
&= \frac{1}{2} \int_0^R \int_0^{R-y} K_n(x, y) f^n(x, \tau) f^n(y, \tau) dx dy \\
&\leq \frac{k_1^2}{2} \int_0^R \int_0^{R-y} (1+x)^\mu (1+y)^\mu f^n(x, \tau) f^n(y, \tau) dy dx \\
&\leq \frac{k_1^2 L(T)^2}{2}.
\end{aligned}$$

Similarly, for the second term of the right-hand side of (15), it follows from (H2) that

$$\begin{aligned}
\int_0^R \int_0^{n-x} K_n(x, y) f^n(x, \tau) f^n(y, \tau) dy dx &\leq k_1^2 \int_0^R \int_0^{n-x} (1+x)^\mu (1+y)^\mu f^n(x, \tau) f^n(y, \tau) dy dx \\
&\leq k_1^2 L(T)^2.
\end{aligned}$$

For the third term of the right-hand side of (15), we use Fubini's Theorem, (H4), (H5), and Lemma 2.1 (i) to obtain

$$\begin{aligned}
& \int_0^R \int_x^n b(x, y) S(y) f^n(y, \tau) dy dx \\
&\leq \int_0^R \int_0^y \Gamma(y, x) f^n(y, \tau) dx dy + \int_0^R \int_R^\infty \Gamma(y, x) f^n(y, \tau) dy dx \\
&\leq \int_0^R f^n(y, \tau) \int_0^y \mathbb{1}_{]0, R[}(x) \Gamma(y, x) dx dy + k(R) \int_0^R \int_R^\infty y^\theta f^n(y, \tau) dy dx \\
&\leq \omega(R, R) \int_0^R f^n(y, \tau) dy + k(R) \int_0^R \int_R^\infty y f^n(y, \tau) dy dx \\
&\leq [\omega(R, R) + Rk(R)] L(T).
\end{aligned}$$

Finally, the fourth term of the right-hand side of (15) is estimated with the help of (H6) and Lemma 2.1 (i) and we get

$$\int_0^R S(x) f^n(x, t) dx \leq \|S\|_{L^\infty]0, R[} L(T).$$

Collecting the above estimates and setting

$$C_2(R, T) = \frac{3k_1^2 L(T)^2}{2} + \{\omega(R, R) + Rk(R) + \|S\|_{L^\infty]0, R[}\} L(T)$$

the inequality (15) reduces to

$$\left| \int_0^\infty \phi(x) \{f^n(x, t) - f^n(x, s)\} dx \right| \leq C_2(R, T) \|\phi\|_{L^\infty} (t - s) + \|\phi\|_{L^\infty} \frac{\varepsilon}{2} < \|\phi\|_{L^\infty} \varepsilon, \quad (16)$$

whenever  $t - s < \delta$  for some suitably small  $\delta > 0$ . The estimate (16) implies the time equicontinuity of the family  $\{f^n(t), t \in [0, T]\}$  in  $L^1]0, \infty[$ . Thus, according to a refined version of the *Arzelà-Ascoli Theorem*, see [12, Theorem 2.1], we conclude that there exist a subsequence  $(f^{n_k})$  and a non-negative function  $f \in L^\infty(]0, T[; L^1]0, \infty[)$  such that

$$\lim_{n_k \rightarrow \infty} \sup_{t \in [0, T]} \left\{ \left| \int_0^\infty \{f^{n_k}(x, t) - f(x, t)\} \phi(x) dx \right| \right\} = 0, \quad (17)$$

for all  $T > 0$  and  $\phi \in L^\infty]0, \infty[$ . In particular, it follows from the non-negativity of  $f^n$  and  $f$ , (9), and (17) that, for  $t \geq 0$  and  $R > 0$ ,

$$\int_0^R x f(x, t) dx = \lim_{n_k \rightarrow \infty} \int_0^R x f^{n_k}(x, t) dx \leq \|f_0\|_x < \infty.$$

Letting  $R \rightarrow \infty$  implies that  $\|f(t)\|_x \leq \|f_0\|_x$  and thus  $f(t) \in X^+$ .

## 2.4 Passing to the limit

Now we have to show that the limit function  $f$  obtained in (17) is actually a weak solution to (1)-(2). To this end, we shall use weak continuity and convergence properties of some operators which define now: for  $g \in X^+$ ,  $n \geq 1$ , and  $x \in ]0, \infty[$ , we put

$$\begin{aligned} Q_1^n(g)(x) &= \frac{1}{2} \int_0^x K_n(x-y, y) g(x-y) g(y) dy, & Q_2^n(g)(x) &= \int_0^{n-x} K_n(x, y) g(x) g(y) dy, \\ Q_1(g)(x) &= \frac{1}{2} \int_0^x K(x-y, y) g(x-y) g(y) dy, & Q_2(g)(x) &= \int_0^\infty K(x, y) g(x) g(y) dy, \\ Q_3(g)(x) &= S(x) g(x), & Q_4(g)(x) &= \int_x^\infty b(x, y) S(y) g(y) dy, \end{aligned}$$

and  $Q^n = Q_1^n - Q_2^n - Q_3 + Q_4$ ,  $Q = Q_1 - Q_2 - Q_3 + Q_4$ .

We then have the following result:

**Lemma 2.2.** *Let  $(g^n)_{n \in \mathbb{N}}$  be a bounded sequence in  $X^+$ ,  $\|g^n\| \leq L$ , and  $g \in X^+$  such that  $g^n \rightharpoonup g$  in  $L^1]0, \infty[$  as  $n \rightarrow \infty$ . Then, for each  $R > 0$  and  $i \in \{1, \dots, 4\}$ , we have*

$$Q_i^n(g^n) \rightharpoonup Q_i(g) \text{ in } L^1]0, R[ \text{ as } n \rightarrow \infty. \quad (18)$$

*Proof.* The proof of (18) for  $i = 1, 2$  is the same as that in [3, 12] to which we refer. The case  $i = 3$  is obvious since  $\phi S$  belongs to  $L^\infty]0, R[$  by (H6) and (18) follows at once from the weak convergence of  $(g^n)$  in  $L^1]0, \infty[$ . For  $i = 4$ , we consider  $\phi \in L^\infty]0, R[$  and use (3) and Fubini's Theorem to compute, for  $r > R$ ,

$$\begin{aligned} \left| \int_0^R \phi(x) \{Q_4(g^n)(x) - Q_4(g)(x)\} dx \right| &= \left| \int_0^R \int_x^\infty \phi(x) S(y) b(x, y) \{g^n(y) - g(y)\} dy dx \right| \\ &\leq \left| \int_0^R \int_0^y \phi(x) S(y) b(x, y) \{g^n(y) - g(y)\} dx dy \right| \\ &\quad + \left| \int_R^\infty \int_0^R \phi(x) \Gamma(y, x) \{g^n(y) - g(y)\} dx dy \right|. \end{aligned}$$

This can be further written as

$$\left| \int_0^R \phi(x) \{Q_4(g^n)(x) - Q_4(g)(x)\} dx \right| = J_1^n + J_2^n(r) + J_3^n(r), \quad (19)$$

with

$$\begin{aligned} J_1^n &= \left| \int_0^R \{g^n(y) - g(y)\} \int_0^y \phi(x) S(y) b(x, y) dx dy \right| \\ J_2^n(r) &= \left| \int_R^r \{g^n(y) - g(y)\} \int_0^R \phi(x) \Gamma(y, x) dx dy \right| \\ J_3^n(r) &= \left| \int_r^\infty \{g^n(y) - g(y)\} \int_0^R \phi(x) \Gamma(y, x) dx dy \right|. \end{aligned}$$

We use (H6) and (4) to observe that, for  $y \in ]0, R[$ ,

$$\begin{aligned} \left| \int_0^y \phi(x) S(y) b(x, y) dx \right| &\leq \|S\|_{L^\infty]0, R[} \|\phi\|_{L^\infty]0, R[} \int_0^y b(x, y) dx \\ &\leq N \|S\|_{L^\infty]0, R[} \|\phi\|_{L^\infty]0, R[}. \end{aligned}$$

This shows that the function  $y \mapsto \int_0^y \phi(x) \Gamma(y, x) dx$  belongs to  $L^\infty]0, R[$ . Since  $g^n \rightharpoonup g$  in  $L^1]0, \infty[$  as  $n \rightarrow \infty$ , it thus follows that

$$\lim_{n \rightarrow \infty} J_1^n = 0. \quad (20)$$

We next infer from (H4) that, for  $y \in ]0, R[$ ,

$$\left| \int_0^R \phi(x) \Gamma(y, x) dx \right| \leq k(R) y^\theta \int_0^R \phi(x) dx \leq Rk(R) \|\phi\|_{L^\infty]0, R[} y^\theta. \quad (21)$$

On the one hand, (21) guarantees that the function  $y \mapsto \int_0^R \phi(x) \Gamma(y, x) dx$  belongs to  $L^\infty]R, r[$  and the weak convergence of  $(g^n)$  to  $g$  in  $L^1]0, \infty[$  entails that

$$\lim_{n \rightarrow \infty} J_2^n(r) = 0 \quad \text{for all } r > R. \quad (22)$$

On the other hand, we deduce from (21) and the boundedness of  $(g^n)$  and  $g$  in  $X^+$  that

$$\begin{aligned} \left| \int_r^\infty \{g^n(y) - g(y)\} \int_0^R \phi(x) \Gamma(y, x) dx dy \right| &\leq Rk(R) \|\phi\|_{L^\infty]0, R[} \int_r^\infty y^\theta \{g^n(y) + g(y)\} dy \\ &\leq \frac{Rk(R)(L + \|g\|)}{r^{1-\theta}} \|\phi\|_{L^\infty]0, R[} \end{aligned}$$

which is asymptotically small (as  $r \rightarrow \infty$ ) uniformly with respect to  $n$ . We thus conclude that

$$\lim_{r \rightarrow \infty} \sup_{n \geq 1} \{J_3^n(r)\} = 0. \quad (23)$$

Substituting (20) and (22) into (19), we obtain

$$\limsup_{n \rightarrow \infty} \left| \int_0^R \phi(x) \{Q_4(g^n)(x) - Q_4(g)(x)\} dx \right| \leq \sup_{n \geq 1} \{J_3^n(r)\}$$

for all  $r > R$ . Owing to (23), we may let  $r \rightarrow \infty$  and conclude that (18) holds true for  $i = 4$  thanks to the arbitrariness of  $\phi$  and the proof of Lemma 2.2 is complete.  $\square$

## 2.5 Existence

Now we are in a position to prove the main result.

*Proof of Theorem 1.2.* Fix  $R > 0$ ,  $T > 0$ , and consider  $t \in [0, T]$  and  $\phi \in L^\infty]0, R[$ . Owing to Lemma 2.2, we have for each  $s \in [0, t]$ ,

$$\int_0^R \phi(x) \{Q^{n_k}(f^{n_k}(s))(x) - Q(f(s))(x)\} dx \rightarrow 0 \text{ as } n_k \rightarrow \infty. \quad (24)$$

Arguing as in Section 2.3, it follows from (H2), (H4)–(H6), and Lemma 2.1 (i) that there is  $C_3(R, T) > 0$  such that, for  $n \geq 1$ , and  $s \in [0, t]$ , we have

$$\left| \int_0^R \phi(x) Q^n(f^n(s))(x) dx \right| \leq C_3(R, T) \|\phi\|_{L^\infty]0, R[}. \quad (25)$$

Since the right-hand side of (25) is in  $L^1]0, t[$ , it follows from (24), (25) and the dominated convergence theorem that

$$\left| \int_0^t \int_0^R \phi(x) \{Q^{n_k}(f^{n_k}(s))(x) - Q(f(s))(x)\} dx ds \right| \rightarrow 0 \text{ as } n_k \rightarrow \infty. \quad (26)$$

Since  $\phi$  is arbitrary in  $L^\infty]0, R[$ , Fubini's Theorem and (26) give

$$\int_0^t Q^{n_k}(f^{n_k}(s)) ds \rightarrow \int_0^t Q(f(s)) ds \text{ in } L^1]0, R[ \text{ as } n_k \rightarrow \infty. \quad (27)$$

It is then straightforward to pass to the limit as  $n_k \rightarrow \infty$  in (7)–(8) and conclude that  $f$  is a solution to (1)–(2) on  $[0, \infty[$  (since  $T$  is arbitrary). This completes the proof of Theorem 1.2.  $\square$

**Remark 2.3.** *It is worth pointing out that the assumption (4)  $\int_0^y b(x, y) dx = N$  is only used to prove (20) and it is clear from that proof that the assumption*

$$\sup_{y \in ]0, R[} \int_0^y b(x, y) dx < \infty \text{ for all } R > 0$$

*is sufficient. Thus, Theorem 1.2 is actually valid under this weaker assumption.*

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