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Change of angles in tent spaces

Pascal Auscher

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Abstract

We prove sharp bounds for the equivalence of norms in tent spaces with respect to changes of angles. Some applications are given.

Let $B(x, t)$ denote an open ball centered at $x \in \mathbb{R}^n$ with radius $t > 0$. Define for a locally square integrable function $g(t, y)$, $(t, y) \in \mathbb{R}_+^{n+1}$, for $\alpha > 0$ and $x \in \mathbb{R}^n$,

$$A^{(\alpha)}g(x) := \left(\int_{\mathbb{R}_+^{n+1}} \frac{1_{B(x, \alpha t)}(y)}{t^n} |g(t, y)|^2 \frac{dy dt}{t} \right)^{\frac{1}{2}}, \quad (0.1)$$

and for $0 < p < \infty$, say that $g \in T_\alpha^{p,2}$ if

$$\|g\|_{T_\alpha^{p,2}} := \|A^{(\alpha)}g\|_p < \infty.$$

This space was introduced in [4]. As sets the spaces for a given p are the same and the norms (or quasi-norms if $p < 1$) are equivalent: whenever $\alpha, \beta > 0$, one has

$$\|A^{(\alpha)}g\|_p \sim \|A^{(\beta)}g\|_p. \quad (0.2)$$

For $p = \infty$, the limiting space is defined with a Carleson measure condition. Let $T_\alpha B$ be the tent with aperture α above the open ball $B = B(x, r)$, *i.e.*, the set of (t, y) such that $0 < t < r/\alpha$ and $y \in B(x, r - \alpha t)$. We define $\|g\|_{T_\alpha^{\infty,2}}$ as the infimum of $C \geq 0$ such that for all ball B ,

$$\int_{T_\alpha B} |g(t, y)|^2 \frac{dy dt}{t} \leq C^2 \frac{|B|}{\alpha^n}. \quad (0.3)$$

Again, the spaces $T_\alpha^{\infty,2}$ are the same, the norms are equivalent and the isometry property holds.

To explain the choice of the normalisation in (0.3), we remark that for $p = \infty$ included, $g(t, y) \mapsto h(t, y) := \alpha^{n/2}g(t/\alpha, y)$ is an isometry between $T_1^{p,2}$ and $T_\alpha^{p,2}$ equipped with their respective (quasi-)norms.

It is shown in [4] that the spaces $T_1^{p,2}$, $0 < p \leq \infty$, interpolate by the complex method and the real method. The same results hold for $T_\alpha^{p,2}$ for fixed α , with constants (*i.e.*, the constants in the equivalence of norms between $T_\alpha^{p,2}$ and the interpolated space to which it is equal) independent of α by using the isometry property.

Motivated by an intensive usage of tent spaces in the development of new Hardy spaces associated to operators with Gaffney-Davies estimates first made in [2], [7], and also by the study of maximal regularity on tent spaces towards applications for parabolic PDE's ([3], and some more work in progress), it became interesting to know the sharp dependence of the bounds in (0.2) with respect to α, β . The L^2 bound is immediate by Fubini's theorem: $\|g\|_{T_\alpha^{2,2}} = (\alpha/\beta)^{n/2} \|g\|_{T_\beta^{2,2}}$. For $p \neq 2$, the argument in [4], originating from [6], does not give optimal dependence. The inequality

$$\|g\|_{T_\alpha^{p,2}} \leq C(1 + \log \alpha) \alpha^{n/\tau} \|g\|_{T_1^{p,2}},$$

for $1 < p < \infty$ and $\alpha \geq 1$, where $\tau = \min(p, 2)$ and C depends only on n and p , is proved in [8] as a special case of a Banach space valued result, and, moreover, the polynomial growth $\alpha^{n/\tau}$ is shown

to be optimal for such an inequality to hold. The restriction $p > 1$ occurs in this argument because the UMD property is required and a maximal inequality is used. Note that even the L^2 bounds is not immediate in a Banach (non Hilbert) space valued context. In discussion with T. Hytönen, we convinced ourselves that the logarithmic factor is not produced by this argument in the scalar case when $p \geq 2$. Still, this argument is quite involved and elimination of the logarithm in the $p < 2$ situation was unclear.

Here we give the sharp lower and upper bounds for (0.2) in the scalar case by a very simple argument. Define $\underline{h}(p, \alpha) = \min(\alpha^{-n/2}, \alpha^{-n/p})$, $\bar{h}(p, \alpha) = \max(\alpha^{-n/2}, \alpha^{-n/p})$. Note that $\underline{h}(p, \alpha) = \alpha^{-n/p}$ if $(\alpha - 1)(p - 2) \geq 0$ and $\underline{h}(p, \alpha) = \alpha^{-n/2}$ if $(\alpha - 1)(p - 2) \leq 0$, and inversely for $\bar{h}(p, \alpha)$.

Theorem 0.1. *Let $0 < p \leq \infty$ and $\alpha, \beta > 0$. There exist constants $C, C' > 0$ depending on n, p only, such that for any locally square integrable function g ,*

$$C\underline{h}(p, \alpha/\beta)\|g\|_{T_\alpha^{p,2}} \leq \|g\|_{T_\beta^{p,2}} \leq C'\bar{h}(p, \alpha/\beta)\|g\|_{T_\alpha^{p,2}} .$$

Moreover, the dependence in α/β is best possible in the sense that this growth is attained.

In particular, for $\alpha > 1$, $\tau = \min(2, p)$ and $\sigma = \max(2, p)$, one has

$$\|g\|_{T_\alpha^{p,2}} \leq C\alpha^{n/\tau}\|g\|_{T_1^{p,2}} , \quad (0.4)$$

$$\|g\|_{T_1^{p,2}} \leq C\alpha^{-n/\sigma}\|g\|_{T_\alpha^{p,2}} . \quad (0.5)$$

The second one improves the obvious bound $\|g\|_{T_1^{p,2}} \leq \|g\|_{T_\alpha^{p,2}}$. By symmetry using the relation $\underline{h}(p, \alpha)^{-1} = \bar{h}(p, \alpha^{-1})$ and scale invariance, all cases reduce to (0.4) and (0.5) with $\alpha > 1$.

Corollary 0.2. *Let $0 < p < \infty$. There is a constant $0 < C < \infty$ depending on n, p only such that for any locally square integrable g , if $Vg(x) = (\int_0^\infty |g(t, x)|^2 \frac{dt}{t})^{\frac{1}{2}}$,*

$$\|Vg\|_p \leq C\|g\|_{T_1^{p,2}}, \quad \text{if } p \leq 2, \quad \|Vg\|_p \geq C\|g\|_{T_1^{p,2}}, \quad \text{if } p \geq 2.$$

The corollary was proved by a different method in [1] when $p < 2$ and the $p > 2$ case dates back to [9]. The opposite inequalities are false if $p \neq 2$. Starting from Theorem 0.1, the proof is a mere application of Lebesgue differentiation theorem when $\alpha \rightarrow 0$ for $\alpha^{-n/2}A^{(\alpha)}g(x)$ converges to $Vg(x)$ assuming g smooth with compact support. This assumption is easily removed.

Corollary 0.3. *If $0 < p < \infty$ and $\lambda > \max(2/p, 1)$, for any locally square integrable g ,*

$$\left\| \left(\int_{\mathbb{R}_+^{n+1}} \left(\frac{t}{|x-y|+t} \right)^{n\lambda} |g(t, y)|^2 \frac{dydt}{t^{n+1}} \right)^{\frac{1}{2}} \right\|_p \leq C(n, p, \lambda) \|A^{(1)}g\|_p .$$

The left hand side equals the grand square function of Stein when $g(t, y) = t\nabla u(t, y)$, u being the harmonic extension of a suitable distribution u_0 on \mathbb{R}^n . Hence, for all $0 < p < \infty$ and $\lambda > \max(2/p, 1)$, it is dominated in L^p by $\|A^{(1)}g\|_p$ which is the L^p norm of the area functional of Lusin defined from u_0 . However, it is known from Stein-Weiss' theory that $A^{(1)}g \in L^p(\mathbb{R}^n)$ if and only if $\frac{n-1}{n} < p < \infty$ and u_0 belongs to the Hardy space $H^p(\mathbb{R}^n)$ (See [9]). This gives a simple proof of Theorem 2, Chap. IV in [9]. The lower exponent $\frac{n-1}{n}$ is only due to the choice of the extension. Using an extension by convolution with $t^{-n}\varphi(x/t)$ with $\varphi \in C_0^\infty(\mathbb{R}^n)$ having all vanishing moments but the one of order 0, $\frac{n-1}{n}$ becomes 0 by the results in [6]. At $\lambda = 2/p$ and $p < 2$, a weak type inequality is plausible, the Lorentz norm $L^{p,\infty}$ replacing the Lebesgue norm L^p in the left hand side. It would give a simple proof of the weak type (p, p) result of Fefferman [5] for Stein's grand square function. We leave this open.

The proof of the corollary is easy by splitting the upper half space according to $|x-y|/t$ compared to powers 2^k , $k \geq 0$, and one obtains

$$\left(\int_{\mathbb{R}_+^{n+1}} \left(\frac{t}{|x-y|+t} \right)^{n\lambda} |g(t, y)|^2 \frac{dydt}{t^{n+1}} \right)^{\frac{1}{2}} \leq C(n, \lambda) \sum_{k \geq 0} 2^{-kn\lambda/2} A^{(2^{k+1})}g(x) .$$

It remains to use $\|A^{(\alpha)}g\|_p \leq C\alpha^{n/\tau}\|A^{(1)}g\|_p$ for $\alpha = 2^{k+1}$ in appropriate arguments for $p \geq 1$ or $p \leq 1$.

The proof of Theorem 0.1 is an easy matter using in part atomic theory for tent spaces, again proved in [4]. Recall that for $0 < p \leq 1$, the tent space $T_1^{p,2}$ has an atomic decomposition: A $T_1^{p,2}$ atom is a function $a(t, x)$ supported in a tent T_1B with the estimate

$$\int_{T_1B} |a(t, x)|^2 \frac{dxdt}{t} \leq |B|^{-\left(\frac{2}{p}-1\right)}.$$

There is a constant $C = C(n, p) > 0$ such that $\|a\|_{T_1^{p,2}} \leq C$. Any $T_1^{p,2}$ function g can be represented as a series $g = \sum \lambda_j a_j$ where a_j is a $T_1^{p,2}$ atom and $\sum |\lambda_j|^p \sim \|g\|_{T_1^{p,2}}^p$. By the isometry property, a $T_\alpha^{p,2}$ atom is a function $A(u, x)$ supported a tent $T_\alpha B$ with the estimate

$$\int_{T_\alpha B} |A(u, x)|^2 \frac{dxdu}{u} \leq \alpha^{-n} |B|^{-\left(\frac{2}{p}-1\right)}$$

and the decomposition theorem holds in $T_\alpha^{p,2}$.

Proof of Theorem 0.1. Recall that we may assume $\alpha > \beta = 1$ and it is enough to prove (0.4) and (0.5). Fix $p = \infty$ first. Let $\|g\|_{T_1^{\infty,2}} = 1$ and B be a ball. As $T_\alpha B \subset T_1 B$,

$$\int_{T_\alpha B} |g(t, x)|^2 \frac{dxdt}{t} \leq \int_{T_1 B} |g(t, x)|^2 \frac{dxdt}{t} \leq |B| = \alpha^n \frac{|B|}{\alpha^n}.$$

Hence, $\|g\|_{T_\alpha^{\infty,2}} \leq \alpha^{n/2}$. This shows $\|g\|_{T_\alpha^{\infty,2}} \leq \alpha^{n/2} \|g\|_{T_1^{\infty,2}}$ for all g .

Let $\|g\|_{T_\alpha^{\infty,2}} = 1$ and B be a ball. As $T_1 B \subset T_\alpha(\alpha B)$ where αB is the ball concentric with B dilated by α ,

$$\int_{T_1 B} |g(t, x)|^2 \frac{dxdt}{t} \leq \int_{T_\alpha(\alpha B)} |g(t, x)|^2 \frac{dxdt}{t} \leq \frac{|\alpha B|}{\alpha^n} = |B|.$$

Hence, $\|g\|_{T_1^{\infty,2}} \leq 1$ and $\|g\|_{T_1^{\infty,2}} \leq \|g\|_{T_\alpha^{\infty,2}}$ for all g .

Fix now $p \leq 1$. Let B be a ball and a be a $T_1^{p,2}$ atom supported in a tent $T_1 B$. As $T_1 B \subset T_\alpha(\alpha B)$, we have a is supported in $T_\alpha(\alpha B)$ and

$$\int_{T_\alpha(\alpha B)} |a(t, x)|^2 \frac{dxdt}{t} = \int_{T_1 B} |a(t, x)|^2 \frac{dxdt}{t} \leq |B|^{-\left(\frac{2}{p}-1\right)} = \alpha^{2n/p} (\alpha^{-n} |\alpha B|^{-\left(\frac{2}{p}-1\right)}).$$

Thus $\alpha^{-n/p} a$ is a $T_\alpha^{p,2}$ atom. An atomic decomposition of any element of $T_1^{p,2}$ is up to multiplication by $\alpha^{-n/p}$ an atomic decomposition in $T_\alpha^{p,2}$, proving $\|g\|_{T_1^{p,2}} \leq C(n, p) \alpha^{n/p} \|g\|_{T_\alpha^{p,2}}$.

Next, let B be a ball and a be a $T_\alpha^{p,2}$ atom supported in a tent $T_\alpha B$. As $T_\alpha B \subset T_1 B$, we have a is supported in $T_1 B$ and

$$\int_{T_1 B} |a(t, x)|^2 \frac{dxdt}{t} = \int_{T_\alpha B} |a(t, x)|^2 \frac{dxdt}{t} \leq \alpha^{-n} |B|^{-\left(\frac{2}{p}-1\right)}.$$

Thus $\alpha^{n/2} a$ is a $T_1^{p,2}$ atom. As above, we conclude that $\|g\|_{T_1^{p,2}} \leq C(n, p) \alpha^{-n/2} \|g\|_{T_\alpha^{p,2}}$.

For $1 < p < 2$ and $2 < p < \infty$, we conclude by interpolation with the $p = 2$ equality $\|g\|_{T_\alpha^{2,2}} = \alpha^{n/2} \|g\|_{T_1^{2,2}}$. We have shown (0.4) and (0.5).

The sharpness of the bounds is seen by saturating these inequalities. Fix $\alpha > 1$ large. Let B be the unit ball. Set $a_1(t, y) = 1_{T_1 B}(t, y) 1_{[1/2, 1]}(t)$. It is easy to see that $\|a_1\|_{T_1^{p,2}} \sim 1$. Now, we have that $A^{(\alpha)} a_1$ has support equal to $\overline{B}(0, \alpha)$, is bounded by a constant $c(n) > 0$ and equal to that constant on the ball $B(0, \frac{\alpha+1}{2})$. Thus $\|a_1\|_{T_\alpha^{p,2}} \sim \alpha^{n/p}$. This proves that (0.4) is optimal when $p \leq 2$ and (0.5) is optimal when $p \geq 2$. Next, let $a_2(t, y) = a_1(\alpha t, y)$. By scaling $\|a_2\|_{T_\alpha^{p,2}} = \alpha^{n/2} \|a_1\|_{T_1^{p,2}} \sim \alpha^{n/2}$. Now, $A^{(1)} a_2$ is supported in \overline{B} , bounded by a constant $c(n) > 0$ and equal to that constant on $B(0, \frac{\alpha-1}{2\alpha})$. Thus $\|a_2\|_{T_1^{p,2}} \sim 1$. This proves that (0.4) is optimal when $p \geq 2$ and (0.5) is optimal when $p \leq 2$. \square

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