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TRUNCATED TOEPLITZ OPERATORS AND BOUNDARY VALUES IN NEARLY INVARIANT SUBSPACES

ANDREAS HARTMANN & WILLIAM T. ROSS

ABSTRACT. We consider truncated Toeplitz operator on nearly invariant subspaces of the Hardy space H^2. Of some importance in this context is the boundary behavior of the functions in these spaces which we will discuss in some detail.

1. INTRODUCTION

If H^2 is the classical Hardy space [Dur70], we say a (closed) subspace $M \subset H^2$ is *nearly invariant* when

$$f \in M, f(0) = 0 \Rightarrow \frac{f}{z} \in M.$$

These subspaces have been completely characterized in [Hit88, Sar88] and continued to be studied in [AR96, HSS04, MP05, KN06, AK08, CCP10]. In this paper, we will examine certain properties of nearly invariant subspaces. We aim to accomplish three things.

First, in Theorem 3.1 we observe that much of what we already know about truncated Toeplitz operators on model spaces [Sar07] can be transferred, via a unitary operator to be introduced below, *mutatis mutandis* to truncated operators on nearly invariant subspaces.

Secondly, in Theorem 4.5 we will show that every function in a nearly invariant subspace M has a finite non-tangential limit at $|\zeta| = 1$ if and only if (i) g has a finite non-tangential limit at ζ and (ii) the reproducing kernel functions for M are uniformly norm bounded in Stolz regions with vertex at ζ. This parallels a result by Ahern and Clark in model spaces.

Third, in order to better understand the self-adjoint rank-one truncated Toeplitz operators on nearly invariant subspaces, we will discuss the non-tangential limits of functions in these spaces. It turns out (Theorem 5.2) there is a kind of dichotomy: If every function in a nearly invariant subspace M has a boundary limit at a fixed point $|\zeta| = 1$ then either ζ is a point where *every* function in $\frac{1}{g}M$ (where g is the extremal function for M - see definition below) has a finite non-tangential limit or *every* function in M has non-tangential limit 0 at ζ.

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A word concerning numbering in this paper: in each section, we have numbered theorems, propositions, lemmas, corollaries **and** equations consecutively.

2. Preliminaries

If H^2 is the Hardy space of the open unit disk $\mathbb{D}$ (with the usual norm $\|\cdot\|$) and I is inner, let $K_I = H^2 \ominus IH^2$ be the well-studied model space [Nik86]. Note that H^2, as well as K_I, are regarded, via non-tangential boundary values on $\mathbb{T} := \partial\mathbb{D}$, as subspaces of $L^2 := L^2(\mathbb{T}, d\theta/2\pi)$ [Dur70]. It is well known that H^2 is a reproducing kernel Hilbert space with kernel

$$k_\lambda(z) := \frac{1}{1-\overline{\lambda}z},$$

as is K_I with kernel

$$k_\lambda^I(z) := \frac{1-\overline{I(\lambda)}I(z)}{1-\overline{\lambda}z}.$$

Note that k_λ^I is bounded and that finite linear combinations of them form a dense subset of K_I. Also note that if P_I is the orthogonal projection of L^2 onto K_I, then

$$(P_I f)(\lambda) = \langle f, k_\lambda^I\rangle, \quad \lambda \in \mathbb{D}.$$

Note that for every $\lambda \in \mathbb{D}$ this formula extends to $f \in L^1 = L^1(\mathbb{T}, d\theta/2\pi)$ so that we can define P_I for L^1-functions (which in general does not give an H^1-function).

By a theorem of Ahern and Clark [AC70], every function $f \in K_I$ has a finite non-tangential limit at a boundary point $\zeta \in \mathbb{T}$, i.e.,

$$f(\zeta) := \angle\lim_{\lambda\to\zeta} f(\lambda)$$

exists for every $f \in K_I$, if and only if $\zeta \in \mathrm{ADC}(I)$. Here $\zeta \in \mathrm{ADC}(I)$ means that I has a finite angular derivative in the sense of Carathéodory, meaning

$$\angle\lim_{\lambda\to\zeta} I(\lambda) = \eta \in \mathbb{T}$$

and

$$\angle\lim_{\lambda\to\zeta} I'(\lambda) \quad \text{exists.}$$

Moreover, whenever $\zeta \in \mathrm{ADC}(I)$, the linear functional $f \mapsto f(\zeta)$ is continuous on K_I giving us a kernel function k_ζ^I for K_I at the boundary point ζ. That is to say

$$f(\zeta) = \langle f, k_\zeta^I\rangle \quad \forall f \in K_I.$$

Sarason [Sar07] began a study, taken up by others, of the *truncated Toeplitz operators* on K_I. These operators are defined as follows: For $\varphi \in L^2$, define the truncated Toeplitz operator A_φ densely on the bounded $f \in K_I$ by

$$A_\varphi f = P_I(\varphi f).$$

Let $\mathcal{T}_I$ denote the A_φ which extend to be bounded operators on K_I. Certainly when φ is bounded, then $A_\varphi \in \mathcal{T}_I$, but there are unbounded φ which yield $A_\varphi \in \mathcal{T}_I$. Moreover, there are bounded truncated Toeplitz operators which can not be represented by a bounded symbol [BCF+10]. Much

is known about these operators (see the Sarason paper [Sar07] for a detailed discussion) but we list a few interesting facts below:

(1) $\mathcal{T}_I$ is a weakly closed linear subspace of operators on K_I.
(2) $A_\varphi \equiv 0$ if and only if $\varphi \in IH^2 + \overline{IH^2}$.
(3) The operator $Cf := \overline{zfI}$ (considered as boundary functions on $\mathbb{T}$) defines an isometric, anti-linear, involution on K_I for which $CA_\varphi C = A_\varphi^* = A_{\overline{\varphi}}$. The operator C is called a conjugation. This makes $\mathcal{T}_I$ a collection of complex symmetric operators [GP06, GP07].
(4) A bounded operator A on K_I belongs to $\mathcal{T}_I$ if and only if there are functions $\varphi_1, \varphi_2 \in K_I$ so that
$$A - A_z A A_z^* = (\varphi_1 \otimes k_0^I) + (k_0^I \otimes \varphi_2).$$
In the above, we use the notation $f \otimes g$ for the rank-one operator
$$(f \otimes g)(h) := \langle h, g\rangle f.$$
(5) An operator A on K_I is a rank-one truncated Toeplitz operator on if and only if it can be written as constant multiple of one of the following three types
$$k_\lambda^I \otimes Ck_\lambda^I, \quad Ck_\lambda^I \otimes k_\lambda^I, \quad k_\zeta^I \otimes k_\zeta^I$$
where $\lambda \in \mathbb{D}$, $\zeta \in \mathrm{ADC}(I)$. The two first ones are the non-selfadjoint and the last ones are the selfadjoint truncated Toeplitz operators.
(6) Sedlock [Sed] (see also [GRW]) showed that every maximal algebra in $\mathcal{T}_I$ can be written as the commutant of a generalization of the Clark unitary operator.
(7) In [CGRW10] they show, for two inner functions I_1 and I_2, that $\mathcal{T}_{I_1}$ is spatially isomorphic to $\mathcal{T}_{I_2}$ if and only if either $I_1 = \psi \circ I_2 \circ \phi$ or $I_1 = \psi \circ \overline{I_2(\overline{z})} \circ \phi$ for some disk automorphisms ϕ, ψ.

As we have already mentioned, a *nearly invariant* subspace M is a closed subspace of H^2 such that
$$f \in M, f(0) = 0 \Rightarrow \frac{f}{z} \in M.$$
A result of Hitt [Hit88] says that if g (called the extremal function for M) is the unique solution to the extremal problem

$$\sup\{\mathrm{Re} g(0) : g \in M, \|g\| = 1\}, \tag{2.1}$$

then there is an inner function I so that
$$M = gK_I$$
and moreover, the map
$$U_g : K_I \to M, \quad U_g f = gf,$$
is isometric.

Note that necessarily we have the condition $I(0) = 0$ since $g \in M = gK_I$ and so $1 \in K_I$. If one were to choose any g and any inner I, then gK_I is, in general, not even a subspace of H^2.

Sarason shows in [Sar88] that when $I(0) = 0$, every isometric multiplier on K_I takes the form

$$g = \frac{a}{1 - Ib}, \tag{2.2}$$

where a and b are in the unit ball of H^∞ with $|a|^2 + |b|^2 = 1$ a.e. on $\mathbb{T}$. As a consequence, gK_I is a (closed) nearly invariant subspace of H^2 with extremal function g as in (2.1).

Remark 2.3. From now on, whenever we speak of a nearly invariant subspace $M = gK_I$, we will always assume that $I(0) = 0$ and that g is extremal for M (as in (2.1)) and of the form in (2.2). We will say that gK_I is a nearly invariant subspace with extremal function g and suitable inner function I with $I(0) = 0$.

Our first step towards defining a truncated Toeplitz operator on the nearly invariant subspace $M = gK_I$, as Sarason did for K_I, is to understand P_M, the orthogonal projection of L^2 onto M.

Lemma 2.4. *Let $M = gK_I$ be a nearly invariant subspace with extremal function g and associated inner function I, $I(0) = 0$. Then*

$$P_M f = gP_I(\overline{g}f), \quad f \in M.$$

Proof. To show that the map $f \mapsto gP_I(\overline{g}f)$ is indeed P_M, we need to show that this map is the identity on M and vanishes on $L^2 \ominus M$.

Let $f = gh \in gK_I$ with $h \in K_I$. Then for every $\lambda \in \mathbb{D}$ we have

$$(P_I\overline{g}gh)(\lambda) = \langle \overline{g}gh, k_\lambda^I\rangle = \langle gh, gk_\lambda^I\rangle.$$

But, from our previous discussion, multiplication by the extremal function g is an isometry from K_I onto M and so

$$\langle gh, gk_\lambda^I\rangle = \langle h, k_\lambda^I\rangle = h(\lambda).$$

Thus

$$g(\lambda)(P_I\overline{g}f)(\lambda) = f(\lambda),$$

in other words, $f \mapsto gP_I(\overline{g}f)$ is the identity on M.

To finish the proof, we need to show that the map $f \mapsto gP_I(\overline{g}f)$ vanishes on $L^2 \ominus M$. Clearly, when $f \in \overline{H_0^2} = \{\overline{h} : h \in H^2, h(0) = 0\}$ then $P_I(\overline{g}f) = 0$. When $f \in H^2 \ominus M$ we have

$$(P_I\overline{g}f)(\lambda) = \langle \overline{g}f, k_\lambda^I\rangle = \langle f, gk_\lambda^I\rangle = 0$$

since $gk_\lambda^I \in M$ and $f \perp M$. ∎

Corollary 2.5. *The reproducing kernel for $M = gK_I$ is given by*

$$k_\lambda^M(z) = \overline{g(\lambda)}g(z)\frac{1 - \overline{I(\lambda)}I(z)}{1 - \overline{\lambda}z}.$$

Proof. Observe that $k_\lambda^M(z) = (P_M k_\lambda)(z) = g(z)(P_I\overline{g}k_\lambda)(z)$ and

$$(P_I\overline{g}k_\lambda)(z) = \langle \overline{g}k_\lambda, k_z^I\rangle = \langle k_\lambda, gk_z^I\rangle = \overline{(gk_z^I)(\lambda)} = \overline{g(\lambda)}k_\lambda^I(z).$$

∎

3. Truncated Toeplitz operators on nearly invariant subspaces

We are now able to introduce truncated Toeplitz operators on nearly invariant subspaces. Certainly whenever φ is a bounded function we can use Lemma 2.4 to see that the operator

$$A_\varphi^M f := P_M \varphi f = g P_I(\overline{g}\varphi f), \quad f \in M,$$

is well-defined and bounded.

The most general situation is when the symbol φ is a Lebesgue measurable function on $\mathbb{T}$ and $|g|^2\varphi \in L^2$. Then, for every bounded $h \in K_I$, the function $|g|^2\varphi h$ belongs to L^2 and so $P_I(|g|^2\varphi h) \in K_I$, and, by the isometric multiplier property of g in K_I, we get

$$P_M(\varphi h) = g P_I(|g|^2\varphi h) \in g K_I = M.$$

Note that by the isometric property of g, the set gK_I^∞, where K_I^∞ is the set of bounded functions in K_I, is dense in gK_I. Hence, in this situation, A_φ^M is densely defined.

Let $\mathcal{T}^M$ denote the densely defined A_φ^M, $|g|^2\varphi \in L^2$, which extend to be bounded on M and recall that $U_g : K_I \to M, U_g f = gf$, is unitary.

Theorem 3.1. *Let $M = gK_I$ be a nearly invariant subspace with extremal function g and suitable inner function I, $I(0) = 0$. Then for any Lebesgue measurable function φ on $\mathbb{T}$ with $|g|^2\varphi \in L^2$ we have*

$$U_g^* A_\varphi^M U_g = A_{|g|^2\varphi}.$$

Proof. Let $h \in K_I$, then

$$A_\varphi^M U_g h = A_\varphi^M g h = g P_I |g|^2 \varphi h = U_g A_{|g|^2\varphi} h.$$

■

The above shows that the map

$$A_\varphi^M \mapsto A_{|g|^2\varphi}$$

establishes a spatial isomorphism between $\mathcal{T}^M$ (the bounded truncated Toeplitz operators on M) and $\mathcal{T}_I$ (the bounded truncated Toeplitz operators on K_I) induced, via conjugation, by the unitary operator $U_g : K_I \to M = gK_I$, i.e., $\mathcal{T}^M = U_g \mathcal{T}_I U_g^*$.

In view of the results in Sarason's paper [Sar07] one can use the above spatial isomorphism to prove the following facts:

(1) $\mathcal{T}_M$ is a weakly closed linear subspace of operators on M.
(2) $A_\varphi^M \equiv 0$ if and only if $|g|^2\varphi \in IH^2 + \overline{IH^2}$.
(3) Recalling the conjugation C mentioned earlier, define $C_g := U_g C U_g^*$. This defines a conjugation on M and $C_g A C_g = A^*$ for every $A \in \mathcal{T}^M$, and since $(A_\varphi^M)^* = A_{\overline{\varphi}}^M$ (as can be verified by direct inspection), we see that $\mathcal{T}^M$ is also a collection of complex symmetric operators (with respect to C_g).

(4) If $S_g := U_g A_z U_g^*$, then a bounded operator A on M belongs to $\mathcal{T}^M$ if and only if there are functions $\varphi_1, \varphi_2 \in M$ so that
$$A - S_g A S_g^* = (\varphi_1 \otimes k_0^M) + (k_0^M \otimes \varphi_2).$$

(5) Recall from our earlier discussion of [Sar07] that A is a rank-one truncated Toeplitz operator on K_I if and only if it can be written as constant multiple of one of the following three types (two of which are non-selfadjoint, and one of which is selfadjoint)
$$k_\lambda^I \otimes Ck_\lambda^I, \quad Ck_\lambda^I \otimes k_\lambda^I, \quad k_\zeta^I \otimes k_\zeta^I$$
where $\lambda \in \mathbb{D}$, $\zeta \in \mathrm{ADC}(I)$. It follows immediately that for $\lambda \in \mathbb{D}$,
$$U_g(k_\lambda^I \otimes Ck_\lambda^I)U_g^* = U_g k_\lambda^I \otimes U_g Ck_\lambda^I = gk_\lambda^I \otimes gCk_\lambda^I,$$
$$U_g(Ck_\lambda^I \otimes k_\lambda^I)U_g^* = U_g Ck_\lambda^I \otimes U_g k_\lambda^I = gCk_\lambda^I \otimes gk_\lambda^I,$$
are rank-one truncated Toeplitz operators on $M = gK_I$. What are the self-adjoint truncated Toeplitz operators on M? The result here is the following: Suppose that $\zeta \in \mathrm{ADC}(I)$, then
$$gk_\zeta^I \otimes gk_\zeta^I$$
is a self-adjoint rank-one truncated Toeplitz operator on M. Conversely, if A is a non-trivial rank-one self adjoint truncated Toeplitz operator on M, then there exists a $\zeta \in \mathrm{ADC}(I)$ so that $A = gk_\zeta^I \otimes gk_\zeta^I$.

(6) Recall from our earlier discussion that every maximal algebra in $\mathcal{T}_I$ can be written as the commutant of a generalization of the Clark unitary operator. Conjugating by U_g we get an analogous result for the maximal algebras of $\mathcal{T}^M$.

(7) Again from our earlier discussion, for two inner functions I_1 and I_2, that $\mathcal{T}_{I_1}$ is spatially isomorphic to $\mathcal{T}_{I_2}$ if and only if either $I_1 = \psi \circ I_2 \circ \phi$ or $I_1 = \psi \circ \overline{I_2(\overline{z})} \circ \phi$ for some disk automorphisms ϕ, ψ. Using Theorem 3.1 we get the exact same result for $\mathcal{T}^{M_1}, \mathcal{T}^{M_2}$. Notice how this result is independent of the corresponding extremal functions g_1 and g_2.

4. Existence of non-tangential limits

With trivial examples one can show that there is no point $\zeta \in \mathbb{T}$ such that every $f \in H^2$ has a non-tangential limit at ζ. However, there are model spaces K_I and $\zeta \in \mathbb{T}$ so that every $f \in K_I$ has a non-tangential limit at ζ. This situation was thoroughly discussed by Ahern and Clark [AC70] with the following theorem. For $\zeta \in \mathbb{T}$ and $\alpha > 1$ let
$$\Gamma_\alpha(\zeta) := \left\{ z \in \mathbb{D} : \frac{|z - \zeta|}{1 - |z|} < \alpha \right\}$$
be the standard Stolz domains with vertex at ζ.

Theorem 4.1 (Ahern-Clark)**.** *Let I be an inner function. Every function $f \in K_I$ has a non-tangential boundary limit at $\zeta \in \mathbb{T}$ if and only if for every fixed Stolz domain $\Gamma_\alpha(\zeta)$ at ζ, the family*
$$F_{\alpha,\zeta} := \left\{ k_\lambda^I : \lambda \in \Gamma_\alpha(\zeta) \right\}$$

is uniformly norm bounded, i.e.,

$$\sup\left\{\|k_\lambda^I\|^2 = \frac{1-|I(\lambda)|^2}{1-|\lambda|^2} : \lambda \in \Gamma_\alpha(\zeta)\right\} < \infty.$$

The original proof of this theorem involved a technical lemma using operator theory. We will give a new proof which avoids this technical lemma. This will allow us to consider the situation of nearly invariant subspaces where the operator theory lemma of Ahern-Clark no longer works. Also note that the uniform boundedness of the families $F_{\alpha,\zeta}$ is equivalent to the condition $\zeta \in \mathrm{ADC}(I)$.

Proof of Theorem 4.1. From the uniform boundedness principle it is clear that if every function in K_I has a non-tangential limit at $\zeta \in \mathbb{T}$, then the kernels k_λ^I are uniformly bounded in Stolz domains $\Gamma_\alpha(\zeta)$.

Let us consider the sufficiency part. If, for fixed α and ζ, the family $F_{\alpha,\zeta}$ is uniformly bounded, then there exists a sequence $\Lambda := \{\lambda_n\}_{n\geq 1} \subset \Gamma_\alpha(\zeta)$ such that $k_{\lambda_n}^I$ converges weakly to some $k_\zeta^{I,\Lambda} \in K_I$, i.e., for every function $f \in K_I$,

$$f(\lambda_n) = \langle f, k_{\lambda_n}^I\rangle \to \langle f, k_\zeta^{I,\Lambda}\rangle =: f_\zeta^\Lambda \quad \text{as } n \to \infty. \tag{4.2}$$

Observe that

$$(A_{\overline{z}} f)(\lambda) = \frac{f(\lambda) - f(0)}{\lambda}$$

and

$$(A_{\overline{z}}^N f)(\lambda) = \frac{f(\lambda) - \sum_{j=0}^{N-1} \widehat{f}(j)\lambda^j}{\lambda^N},$$

where $\widehat{f}(j)$ is the j-th Fourier coefficient of f. As a result we have

$$\|A_{\overline{z}}^N f\|^2 = \sum_{j=N}^{\infty} |\widehat{f}(j)|^2 \to 0 \quad \text{as } N \to \infty.$$

Apply (4.2) to the functions $A_{\overline{z}}^N f$ to get

$$\langle A_{\overline{z}}^N f, k_\lambda^I\rangle = (A_{\overline{z}}^N f)(\lambda) = \frac{f(\lambda) - \sum_{j=0}^{N-1} \widehat{f}(j)\lambda^j}{\lambda^N}.$$

Hence

$$\begin{aligned}\langle A_{\overline{z}}^N f, k_\zeta^{I,\Lambda}\rangle &= \lim_{n\to\infty} \langle A_{\overline{z}}^N f, k_{\lambda_n}^I\rangle = \lim_{n\to\infty} \frac{f(\lambda_n) - \sum_{j=0}^{N-1} \widehat{f}(j)\lambda_n^j}{\lambda_n^N} \\ &= \frac{f_\zeta^\Lambda - \sum_{j=0}^{N-1} \widehat{f}(j)\zeta^j}{\zeta^N}.\end{aligned}$$

Now using the fact that $\|A_{\overline{z}}^N f\| \to 0$ when $N \to \infty$, we obtain from $\langle A_{\overline{z}}^N f, k_\zeta^{I,\Lambda}\rangle \to 0$ that

$$\lim_{N\to\infty} \frac{f_\zeta^\Lambda - \sum_{j=0}^{N-1} \zeta^j \widehat{f}(j)}{\zeta^N} = 0,$$

and, since the denominator in the above limit is harmless, we get

$$\sum_{j=0}^{\infty} \widehat{f}(j)\zeta^j = f_\zeta^\Lambda.$$

This means that the Fourier series for f converges at ζ.

Now take any sequence $\Lambda := \{\lambda_n\}_{n\geq 1}$ converging non-tangentially to ζ. In view of the uniform boundedness of the family $F_{\alpha,\zeta}$, there exists a weakly convergent subsequence $\Lambda' := \{k_{\lambda_{n_l}}^I\}_{l\geq 1}$. Repeating the above proof, we obtain

$$\sum_{j=0}^{\infty} \zeta^j \widehat{f}(j) = f_\zeta^{\Lambda'}.$$

In other words, for every sequence $\{\lambda_n\}_{n\geq 1}$ converging non-tangentially to ζ there exists a subsequence $\{\lambda_{n_j}\}_{j\geq 1}$ such that $\{f(\lambda_{n_j})\}_{j\geq 1}$ converges to the same limit

$$l_\zeta := \sum_{j=0}^{\infty} \widehat{f}(j)\zeta^j.$$

It follows that f has a finite non-tangential limit at ζ. ■

We will now prove an analog of Theorem 4.1 for nearly invariant subspaces where the situation is a bit more complicated. Certainly a necessary condition for the existence of finite non-tangential limits for every function in gK_I at ζ is that g has a finite non-tangential limit at ζ (since indeed g is extremal and so $g \in gK_I$). As we will see with the next example, it is not possible to deduce the existence of the non-tangential limits for every function in gK_I merely from the uniform norm boundedness of the reproducing kernels in a Stolz angles.

Example 4.3. Given any inner function I, with $I(0) = 0$, we have already cited Sarason's result [Sar88] stating that every isometric multiplier on K_I is of the form

$$g = \frac{a}{1 - Ib}, \tag{4.4}$$

where a and b are functions in the ball of H^∞ with $|a|^2 + |b|^2 = 1$ a.e. on $\mathbb{T}$. Let $\Lambda_1 = \{1 - 4^{-n}\}_{n\geq 1}$ and $\Lambda_2 = \{1 - 2^{-n}\}_{n\geq 1}$, which are both interpolating sequences. Observe that Λ_2 consists of the sequence Λ_1 to which we have added in a certain way the (pseudohyperbolic) midpoints of two consecutive points of Λ_1. Let B_i be the Blaschke product whose zeros are Λ_i. In particular, B_1 vanishes on Λ_1 and is big on $\Lambda_2 \setminus \Lambda_1$, say $|B_1(\lambda)| \geq \delta > 0$ for $\lambda \in \Lambda_2 \setminus \Lambda_1$ (this comes from the fact that B_1 is an interpolating Blaschke product, and that on $\Lambda_2 \setminus \Lambda_1$ we are pseudohyperbolically far from the zeros of B_1).

Next, define

$$a = \frac{1}{2} + \frac{1}{4} B_1$$

which is a function oscillating on Λ_2 between the values $1/2$ (on Λ_1) and $|a(\lambda) - 1/2| \geq \delta/4$, $\lambda \in \Lambda_2 \setminus \Lambda_1$.

By construction we have $1/4 \leq |a| \leq 3/4$. In particular there is an outer function b_0 in the ball of H^∞ such that $|a|^2 + |b_0|^2 = 1$ a.e. on $\mathbb{T}$. Moreover $0 < \sqrt{7}/4 \leq |b_0| = \sqrt{1-|a|^2} \leq \sqrt{15}/4 < 1$ a.e. on $\mathbb{T}$ and this extends by the maximum/minimum principles to the disk (b_0 being outer). As a consequence, for every inner function J, the function $g = a/(1 - Jb_0)$ will be a bounded outer function (actually invertible). We will use this fact in particular for $J = IB_2$:

$$g = \frac{a}{1 - Jb_0} = \frac{a}{1 - I\underbrace{B_2 b_0}_{=b \text{ in } (4.4)}}.$$

Then

$$\left|g(\lambda) - \frac{1}{2}\right| = \left|a(\lambda) - \frac{1}{2}\right| \begin{cases} = 0 & \text{when } \lambda \in \Lambda_1 \\ \geq \frac{\delta}{4} & \text{when } \lambda \in \Lambda_2 \setminus \Lambda_1. \end{cases}$$

Hence g has no limit at $\zeta = 1$. Choose now an inner function I such that $\zeta = 1 \in \mathrm{ADC}(I)$. Then, the kernels $k_\lambda^M = \overline{g(\lambda)} g k_\lambda^I$ are uniformly bounded in a fixed Stolz region $\Gamma_\alpha(1)$ at $\zeta = 1$. Indeed g is bounded, the kernels k_λ^I are uniformly bounded in $\Gamma_\alpha(1)$ by $1 \in \mathrm{ADC}(I)$ (see Theorem 4.1), and $\|k_\lambda^M\| = |g(\lambda)| \|k_\lambda^I\|$.

Theorem 4.5. *Let $M = gK_I$ be a non-trivial nearly invariant subspace with extremal function g and corresponding inner function I. Then every function in M has a finite non-tangential limit at $\zeta \in \mathbb{T}$ if and only if the following two conditions are satisfied:*

(1) *g has a finite non-tangential limit at ζ.*
(2) *For every Stolz region $\Gamma_\alpha(\zeta)$ at ζ, the family $F_{\alpha,\zeta}^M := \{k_\lambda^M : \lambda \in \Gamma_\alpha(\zeta)\}$ is uniformly norm bounded, i.e.,*

$$\sup\left\{\|k_\lambda^M\|^2 = |g(\lambda)|^2 \frac{1 - I(\lambda)|^2}{1 - |\lambda|^2} : \lambda \in \Gamma_\alpha(\zeta)\right\} < \infty.$$

Proof. If every function in M has a finite non-tangential limit at ζ, then (1) holds since $g \in M$. Condition (2) holds by the uniform boundedness principle.

Now suppose that conditions (1) and (2) hold.

Case 1: $\zeta \in \mathrm{ADC}(I)$. In this case, every function in M has a non-tangential boundary limit at ζ since every function in K_I has as well as g does.

Case 2: $\zeta \notin \mathrm{ADC}(I)$. From the Ahern-Clark theorem (Theorem 4.1), we know that $\zeta \in \mathrm{ADC}(I)$ if and only if

$$\|k_\lambda^I\|^2 = \frac{1 - |I(\lambda)|^2}{1 - |\lambda|^2}$$

is uniformly bounded in Stolz regions $\Gamma_\alpha(\zeta)$. Hence, if $\zeta \notin \mathrm{ADC}(I)$, then there is a sequence $\Lambda := \{\lambda_n\}_{n\geq 1}$ in $\Gamma_\alpha(\zeta)$ such that $\|k_{\lambda_n}^I\|^2 \to +\infty$. Recall that $k_\lambda^M = \overline{g(\lambda)} g k_\lambda^I$ so that $\|k_{\lambda_n}^M\| =$

$|g(\lambda_n)|\|k^I_{\lambda_n}\|$. In order that this last expression is uniformly bounded it is necessary that $g(\lambda_n) \to 0$ when $n \to \infty$.

Now, since $F^M_{\alpha,\zeta}$ is a uniformly bounded family, there exists a subsequence $\Lambda' := \{\lambda_{n_j}\}_{j\geq 1}$ so that $k^M_{\lambda_{n_j}}$ converges weakly to a function $k^{M,\Lambda'}_\zeta$.

Let us introduce the following operator

$$Q_N = U_g A^N_{\overline{z}} U^*_g$$

from $M = gK_I$ to itself. Let $f = gh \in gK_I = M$. Recall that $U^*_g f = h$. Then

$$\begin{aligned}(Q_N f)(z) &= \Big((U_g A^N_{\overline{z}} U^*_g) f\Big)(z) = \Big((U_g A^N_{\overline{z}} h\Big)(z) = g(z)\frac{h(z) - \sum_{m=0}^{N-1}\widehat{h}(m)z^m}{z^N} \\ &= \frac{f(z)}{z^N} - \frac{g(z)}{z^N}\sum_{m=0}^{N-1}\widehat{h}(m)z^m.\end{aligned}$$

Now since $Q_N f \in M$, we get, from the weak convergence of $k^M_{\lambda_{n_j}} \to k^{M,\Lambda'}_\zeta$ as $j \to \infty$,

$$\langle Q_N f, k^{M,\Lambda'}_\zeta\rangle = \lim_{j\to\infty}(Q_N f)(\lambda_{n_j}) = \lim_{j\to\infty}\left(\frac{f(\lambda_{n_j})}{\lambda^N_{n_j}} - \frac{g(\lambda_{n_j})}{\lambda^N_{n_j}}\sum_{m=0}^{N-1}\widehat{h}(m)\lambda^m_{n_j}\right). \tag{4.6}$$

The last sum appearing above is a harmless polynomial converging to some number when $j \to \infty$. Also $\lambda_{n_j} \to \zeta$ and $g(\lambda_{n_j}) \to 0$ when $j \to \infty$. Hence the limit in (4.6) is equal to

$$\langle Q_N f, k^{M,\Lambda'}_\zeta\rangle = \frac{f^{\Lambda'}_\zeta}{\zeta^N},$$

where

$$f^{\Lambda'}_\zeta = \langle f, k^{M,\Lambda'}_\zeta\rangle,$$

which exists by construction.

Since

$$\|Q_N f\| = \|U_g A^N_{\overline{z}} h\| = \|g A^N_{\overline{z}} h\| = \|A^N_{\overline{z}} h\| \to 0 \quad \text{as } N \to \infty,$$

we have

$$\lim_{N\to\infty}\frac{f^{\Lambda'}_\zeta}{\zeta^N} = \lim_{N\to\infty}\langle Q_N f, k^{M,\Lambda'}_\zeta\rangle = 0$$

which is possible precisely when $f^{\Lambda'}_\zeta = 0$.

The above construction is independent of the choice of the sequence $\{\lambda_n\}_{n\geq 1}$ so that for every such sequence there is a subsequence $\{\lambda_{n_j}\}_{j\geq 1}$ such that $f(\lambda_{n_j}) \to 0$ when $j \to \infty$. As in the proof of Theorem 4.1 we conclude that $\angle\lim_{\lambda\to\zeta} f(\lambda)$ exists and is, in fact, equal to 0.

■

5. THE DICHOTOMY

For a nearly invariant subspace $M = gK_I$, let

$$N^M = \left\{ \zeta \in \mathbb{T} : \angle \lim_{\lambda \to \zeta} f(\lambda) = 0 \ \forall f \in M \right\}.$$

The sets $\mathrm{ADC}(I)$ and N^M are not necessarily disjoint. If the function g vanishes at a point $\zeta \in \mathrm{ADC}(I)$ then automatically every function $f \in gK_I$ vanishes at ζ so that $\zeta \in N^M$.

Another situation is the following. Observing that every function $f \in H^2$ satisfies the following well known growth condition

$$|f(z)| = O\left(\frac{1}{\sqrt{1-|z|}} \right), \tag{5.1}$$

we see that whenever $|g(z)| = o(\sqrt{|z-\zeta|})$ in a Stolz domain $\Gamma_\alpha(\zeta)$ based on ζ, every function gh tends non tangentially to 0 at ζ, and so $\zeta \in N^M$ independent of I. Notice that for growth in a Stolz domain $\Gamma_\alpha(\zeta)$ we can replace "big-Oh" by "little-oh" in (5.1), and then "little-oh" by "big-oh" in the growth of g in $\Gamma_\alpha(\zeta)$. The main theorem of this section is the following.

Theorem 5.2. *Let M be a non-trivial nearly invariant subspace with extremal function g and associated inner function I. Suppose that every function in gK_I has a finite non-tangential limit at $\zeta \in \mathbb{T}$. Then $\zeta \in \mathrm{ADC}(I)$ or $\zeta \in N^M$.*

Note again that the intersection $\mathrm{ADC}(I) \cap N^M$ can be non-empty.

Proof of Theorem 5.2. If every function in gK_I has a non-tangential limit at $\zeta \in \mathbb{T}$, then, in particular, g has a non-tangential limit at ζ. As a reminder, recall that g belongs to gK_I since $I(0) = 0$ (and thus the constants belong to K_I).

If

$$\angle \lim_{\lambda \to \zeta} g(\lambda) \neq 0,$$

then, since every function $f = gh \in gK_I$ has a finite non-tangential limit at ζ, we can divide by g to get $h = f/g \in K_I$ which has a non-tangential limit at ζ. So, every function in K_I has non-tangential limit and we conclude from [AC70] that $\zeta \in \mathrm{ADC}(I)$.

Now suppose that

$$\angle \lim_{\lambda \to \zeta} g(\lambda) = 0. \tag{5.3}$$

First note that if every function in gK_I has a non-tangential limit at $\zeta \in \mathbb{T}$, then, as already mentioned, there exists a reproducing kernel $k_\zeta^M \in M$ at ζ. Let now $\{\lambda_n\}_{n \geq 1}$ be any sequence with $\lambda_n \in \Gamma_\alpha(\zeta)$ tending to ζ. Then for $z \in \mathbb{D}$,

$$k_\zeta^M(z) = \langle k_\zeta^M, k_z^M \rangle = \overline{\langle k_z^M, k_\zeta^M \rangle} = \overline{k_z^M(\zeta)} = \lim_{n \to \infty} \overline{k_z^M(\lambda_n)} = \lim_{n \to \infty} \left(g(z)\overline{g(\lambda_n)} \frac{1 - \overline{I(\lambda_n)} I(z)}{1 - \overline{\lambda_n} z} \right)$$

Since $z \in \mathbb{D}$ is fixed, by (5.3), the above limit will be zero, which forces the kernel k_ζ^M to vanish identically. Thus for every $f \in M = gK_I$ we have

$$f(\zeta) = \lim_{n\to 0}\langle f, k_{\lambda_n}^M\rangle = \langle f, 0\rangle = 0.$$

■

As a consequence of this result, if $\zeta \in N^M \setminus \mathrm{ADC}(I)$, then the point evaluation operator at ζ is just the zero-functional. We state this observation as a separate result.

Corollary 5.4. *Let M be a non-trivial nearly invariant subspace with extremal function g and associated inner function I. On M, the only non-zero point evaluation functionals at a point $\zeta \in \mathbb{T}$ are those for which $\zeta \in \mathrm{ADC}(I)$ and g admits a non-tangential limit different from zero at ζ.*

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