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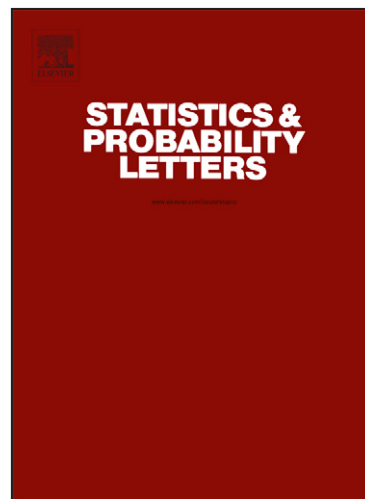
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THE PAIR CORRELATION FUNCTION OF SPATIAL HAWKES PROCESSES

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Abstract

Spatial Hawkes processes can be considered as spatial versions of classical Hawkes processes. We derive the pair correlation function of stationary spatial Hawkes processes and discuss the connection to the Bartlett spectrum and other summary statistics. Particularly, results for Gaussian fertility rates and the extension to spatial Hawkes processes with random fertility rates are discussed.

Keywords: Bartlett spectrum; Hawkes process; pair correlation function; spatial point processes; summary statistics; unpredictable marks.

AMS 2000 Subject Classification: Primary 60G55

Secondary 62M30

1. Introduction

Classical Hawkes processes [5, 6, 7, 8, 9] and their extensions to marked Hawkes processes [2, 5, 10, 11, 16] play a fundamental role in the theory of point processes and its applications. This paper considers a spatial Hawkes process defined as a superposition $X = \cup_{n=0}^{\infty} G_n$ of spatial point processes $G_n \subset \mathbb{R}^d$ ($d \geq 1$) defined by the following branching structure. The points in G_0 are called *immigrants*, and we assume that G_0 is stationary with intensity $\mu_0 \in (0, \infty)$ (where “stationarity” and “intensity” are defined in Section 2.1). For $n = 0, 1, \dots$, the $(n + 1)$ -th generation G_{n+1} given the

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previous generations G_0, \dots, G_n is a Poisson process on \mathbb{R}^d with intensity function

$$\lambda_{n+1}(\xi) = \sum_{\eta \in G_n} \gamma(\xi - \eta) \quad (1)$$

where γ is a non-negative locally integrable function called the *fertility rate*. We can view G_{n+1} as a superposition $\cup_{\eta \in G_n} \Phi(\eta)$, where conditional on G_n the $\Phi(\eta)$ are independent Poisson processes (and for $n \geq 1$ they are independent of G_0, \dots, G_{n-1}), and $\Phi(\eta)$ has intensity function $\gamma(\xi - \eta)$, $\xi \in \mathbb{R}^d$. We refer to the points in G_{n+1} as *offspring*, to $\Phi(\eta)$ as an *offspring process*, and to the point process given by an immigrant point and all its associated offspring generated after some steps as a *cluster*. Note that a classical Hawkes process on the line is the special case where $d = 1$, G_0 is a Poisson process, and $\gamma(\xi) = 0$ for $\xi < 0$.

To the best of our knowledge, spatial Hawkes processes have so far been studied very little in the literature. Brémaud, Massoulié, and Ridolfi [1] consider the extension of a spatial Hawkes process to the case of a random fertility rate specified by a so-called unpredictable mark (for details, see Section 3.3), and they obtain the Bartlett spectrum assuming the existence of the Bartlett spectrum of the immigrant process. As explained later our results easily extend to this case of a random fertility rate, but for ease of presentation we have chosen to concentrate on the deterministic case.

This paper takes another route than [1]: In Section 2, we derive the pair correlation function of a spatial Hawkes process, whereby the Bartlett spectrum can be obtained. Section 3.1 discusses the importance of our results in spatial statistics, Section 3.2 considers the case of a Gaussian fertility rate, and Section 3.3 deals with the extension of a spatial Hawkes process to the case of a random fertility rate specified by an unpredictable mark.

2. First- and second order characteristics

Throughout this paper we assume that the mean number of points in an offspring process is strictly less than one, i.e.

$$\nu \equiv \int \gamma(\eta) \, d\eta < 1 \quad (2)$$

and to avoid the trivial case where we have no offspring, we also assume that $\nu > 0$. Recall that a point process $Y \subset \mathbb{R}^d$ is stationary if the distribution of Y is invariant

under translations in \mathbb{R}^d , and its intensity is then given by the mean number of points of Y per unit volume (the intensity may be infinite). From (1) we easily obtain the following proposition, which shows that (2) is equivalent to assume that X has finite intensity.

Proposition 1. *Each G_n is stationary with intensity $\rho_n = \mu_0 \nu^n$, and X is stationary with intensity*

$$\rho = \mu_0 / (1 - \nu). \quad (3)$$

We now find the pair correlation function $g(\xi, \eta)$ for a Hawkes process. Loosely speaking, $\rho^2 g(\xi, \eta) d\xi d\eta$ is the probability for observing a pair of points from X occurring jointly in each of two infinitesimally small balls with centres ξ, η and volumes $d\xi, d\eta$. For further details, see Appendix A and [13, 17].

We first need the following terminology and notation. Consider any Lebesgue integrable functions f and h defined on \mathbb{R}^d . Let $f * h$ denote convolution, i.e. the Lebesgue integrable function $f * h(\xi) = \int f(\xi - \eta)h(\eta) d\eta$, $\xi \in \mathbb{R}^d$. Define \tilde{f} by $\tilde{f}(\xi) = f(-\xi)$. Let f^{*n} denote convolution of f with itself $n \geq 1$ times, and set $f^{*0} = \delta$, where δ denotes the Dirac delta function on \mathbb{R}^d : $\delta(\xi) = \infty$ if $\xi = 0$, $\delta(\xi) = 0$ if $\xi \in \mathbb{R}^d \setminus \{0\}$, and for any Lebesgue integrable or constant function f , $f(\xi) = \int \delta(\xi - \eta)f(\eta) d\eta$. Accordingly we set $\delta * f = f * \delta = f$ and $\delta * \delta = \delta$. The *normalized fertility rate* is the density ϕ (with respect to the Lebesgue measure) of an offspring (in the first generation) generated by a point at 0: $\phi(\xi) = \gamma(\xi)/\nu$, $\xi \in \mathbb{R}^d$. Furthermore, let χ denote the mixture density of the densities ϕ^{*n} with geometric weights $(1 - \nu)\nu^n$,

$$\chi(\xi) = (1 - \nu) \sum_{n=0}^{\infty} \nu^n \phi^{*n}(\xi), \quad \xi \in \mathbb{R}^d,$$

where in the trivial case $\nu = 0$ we set $\chi = \delta$. Finally, we abuse notation and write e.g. $g_0(\xi, \eta) = g_0(\xi - \eta)$ (which simply means that $g_0(\xi, \eta)$ depends only on (ξ, η) through $\xi - \eta$) for two different functions, however, it will always be clear from the context which function is used.

In the sequel we assume that G_0 has a translation invariant pair correlation function g_0 , i.e. $g_0(\xi, \eta) = g_0(\xi - \eta)$ for all $\xi, \eta \in \mathbb{R}^d$. The following main result is proved in Appendix A.

Theorem 1. *We have that*

$$g(\xi, \eta) = g(\kappa) = g(-\kappa), \quad g(\kappa) = \chi * \tilde{\chi} * \left[g_0 + \frac{1}{\mu_0(1-\nu)} \delta \right] (\kappa) \quad (4)$$

whenever $\kappa = \xi - \eta \neq 0$.

Since $\{(\xi, \xi) : \xi \in \mathbb{R}^d\}$ is a nullset with respect to Lebesgue measure, we define arbitrary the value of $g(0)$. The term g_0 in (4) corresponds to the case where ξ and η are not in the same cluster, while the other term $\delta/(\mu_0(1-\nu))$ corresponds to the case where ξ and η are in the same cluster. From (4) we obtain immediately the following result.

Corollary 1. *If $g_0 = 1$ then*

$$g(\kappa) = 1 + \frac{1}{\mu_0(1-\nu)} \chi * \tilde{\chi}(\kappa) \quad (5)$$

for all $\kappa \in \mathbb{R}^d \setminus \{0\}$.

Recall that the pair correlation function for a Poisson process is equal to one. By (5), $g > 1$, which is in agreement with the usual interpretation that this indicates aggregation of the points in X , cf. [13, 17].

We now consider the Bartlett spectrum of spatial Hawkes processes, which is derived in [1]. An alternative and more elementary way is to exploit its close connection to g . The result is stated below; we refer to our technical report [12] for details.

First, recall the notion of Fourier transform of a tempered distribution: For a Borel measure m on \mathbb{R}^d , let $\langle m, \psi \rangle = \int \psi(\xi) m(d\xi)$, $\psi \in \mathcal{S}$, where \mathcal{S} is the set of the rapid decreasing functions, see e.g. [5]. By definition, the Fourier transform of the tempered distribution $\langle m, \cdot \rangle$ is the tempered distribution $\langle \hat{m}, \cdot \rangle$ such that $\langle \hat{m}, \psi \rangle = \langle m, \hat{\psi} \rangle$, $\psi \in \mathcal{S}$, where $\hat{\psi}(\omega) = \int \exp(i \omega \cdot \xi) \psi(\xi) d\xi$, $\omega \in \mathbb{R}^d$, is the usual Fourier transform; here \cdot is the usual inner product on \mathbb{R}^d . Second, recall that $c(\kappa) = \rho^2(g(\kappa) - 1) + \rho\delta(\kappa)$ is the reduced covariance function of X , and denote by $C(d\xi) = c(\xi) d\xi$ the reduced covariance measure of X (see e.g. [5]). The Bartlett spectrum of X is the Borel measure \hat{C} on \mathbb{R}^d defined by the tempered distribution $\langle \hat{C}, \cdot \rangle$, see e.g. [5]. Third, there exists a locally finite Borel measure α_0 on \mathbb{R}^d such that $\int g_0(\xi) \hat{\psi}(\xi) d\xi = \int \psi(\xi) \alpha_0(d\xi)$, $\psi \in \mathcal{S}$, and the Bartlett spectrum of G_0 is

$$\hat{C}_0(d\xi) = \mu_0^2 [\alpha_0(d\xi) - (2\pi)^d \delta(\xi) d\xi] + \mu_0 d\xi \quad (6)$$

see [5, 12] for details. In many applications, g_0 is of the form $g_0 = 1 + f_0$, where f_0 is Lebesgue integrable, in which case \hat{C}_0 is absolutely continuous with respect to Lebesgue measure, with density

$$\hat{c}_0(\xi) = \mu_0 \left(\mu_0 \hat{f}_0(\xi) + 1 \right). \quad (7)$$

We refer to \hat{c}_0 as the spectral density of G_0 . Finally, let $|z|$ denote the modulus of a complex number z .

Corollary 2. *We have that*

$$\hat{C}(d\xi) = \frac{1}{|1 - \hat{\gamma}(\xi)|^2} \left[\hat{C}_0(d\xi) + (\mu_0 \nu / (1 - \nu)) d\xi \right] \quad (8)$$

where \hat{C}_0 is given by (6) (and $|z|$ denotes the modulus of a complex number z).

When G_0 has spectral density (7), the spatial Hawkes process has spectral density

$$\hat{c}(\xi) = \frac{\mu_0}{|1 - \hat{\gamma}(\xi)|^2} \left[\mu_0 \hat{f}_0(\xi) + \frac{1}{1 - \nu} \right] \quad (9)$$

(that is, $\hat{C}(B) = \int_B \hat{c}(\xi) d\xi$ for Borel sets $B \subseteq \mathbb{R}^d$).

3. Examples and discussion

3.1. Statistical applications

This section briefly discusses the potential statistical applications of our results in Section 2.

Spatial Hawkes processes may provide natural models for e.g. a population of reproducing individuals or the development of an epidemic. However, to the best of our knowledge, spatial Hawkes processes have yet not been used to model a real data set. A natural and interesting application could be the weed plant dataset previously modelled by a log Gaussian Cox process or a shot noise Cox process in [3, 4, 13].

The pair correlation function and the closely related K -function

$$K(r) = \int_{\|\kappa\| \leq r} g(\kappa) d\kappa, \quad r > 0$$

are frequently used in spatial statistics, not only as characteristics of the second order properties of a spatial point process but also for parameter estimation (so-called

minimum contrast estimation), see [13, 14, 17]. On the other hand, the use of the Bartlett spectrum has played a minor role in spatial statistics [15], possibly because g is easier to interpret. However, in light of the much simpler expression (9) compared to (4), using the spectral density as a second order characteristic for spatial Hawkes processes seems appealing.

The empty space function F , the nearest-neighbour distribution function G , and the related J function, which are all widely used summary statistics (see [13] and the references therein), seem intractable for spatial Hawkes processes. If G_0 is Poisson, then by [17], since X is a Poisson cluster process, $J \leq 1$.

3.2. Gaussian fertility rates

A particular tractable case of g occurs if we consider a two-dimensional radially symmetric Gaussian fertility rate, i.e. when ϕ is the density of $N_2(0, \sigma^2 I)$ where $\sigma^2 > 0$ is the variance. Then $\phi^{*n}(\xi) = \phi^{*n}(r)$ depends only on $r = \|\xi\|$ and is the density of $N_2(0, n\sigma^2 I)$, $n \geq 1$, and a straightforward calculation shows that

$$\chi * \tilde{\chi}(r) = (1 - \nu)^2 \sum_{n=0}^{\infty} (n+1) \nu^n \phi^{*n}(r). \quad (10)$$

First, let $g_0 = 1$ and consider $(g(r) - 1)\mu_0$ which by (5) does not depend on the parameter μ_0 . By (5) and (10),

$$(g(r) - 1)\mu_0 = (1 - \nu) \sum_{n=1}^{\infty} (n+1) \nu^n \phi^{*n}(r), \quad r > 0,$$

which we can calculate by numerical methods using e.g. Maple. The left plot in Figure 1 shows $(g(r) - 1)\mu_0$. The effect of increasing σ^2 from 1 to 4 and ν from 0.5 to 0.9 is clearly visible. For comparison we have also shown $(g(r) - 1)\mu_0 = \phi^{*2}(r)$ for a Neyman-Scott (or modified Thomas) process when $(\sigma^2, \nu) = (1, 0.9)$ (i.e. when g is the pair correlation function for offspring of the first generation). This curve has to some extent a similar shape as for a spatial Hawkes process, though it is much below the curve for the spatial Hawkes process with $(\sigma^2, \nu) = (1, 0.9)$.

Next, the upper curve in Figure 1 is $(g(r) - 1)\mu_0$ when the immigrant pair correlation function is that of the Thomas process above, i.e. when $g_0 = 1 + \phi^{*2}/\mu_0$ and

$$(g(r) - 1)\mu_0 = (1 - \nu) \sum_{n=1}^{\infty} (n+1) \nu^n \left[\phi^{*n}(r) + (1 - \nu) \phi^{*(n+2)}(r) \right], \quad r > 0,$$

and where again $(\sigma^2, \nu) = (1, 0.9)$. This curve is only slightly above the corresponding curve where $g_0 = 1$.

The right plot in Figure 1 shows spectral densities (9) on a logarithmic scale when still $(\sigma^2, \nu) = (1, 0.9)$ and $\mu_0 = 1$ or $\mu_0 = 10$ and $g_0 = 1$ or $g_0 = 1 + \phi^{*2}/\mu_0$. For a low immigrant intensity, it is hard to distinguish between the two cases of spectral densities.

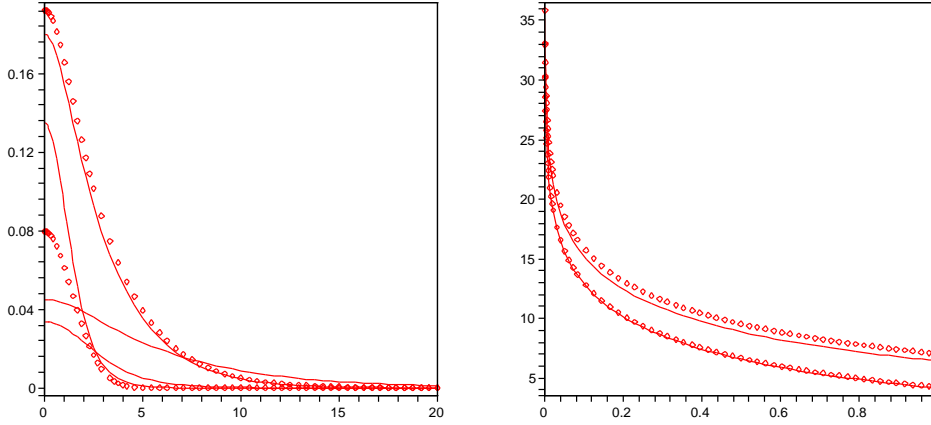


FIGURE 1: *Left*: Transformed pair correlation functions $(g(r) - 1)\mu_0$ with $d = 2$ and a radially symmetric Gaussian density ϕ with variance σ^2 . Full lines: for spatial Hawkes processes with $(\sigma^2, \nu) = (1, 0.9), (1, 0.5), (4, 0.9), (4, 0.5)$ (from top to bottom at $r = 0$) and $g_0 = 1$. Dotted lines: for a modified Thomas process (bottom) and for a spatial Hawkes process when g_0 is the pair correlation of the modified Thomas process (top), with $(\sigma^2, \nu) = (1, 0.9)$ in both cases. *Right*: Log spectral densities when $(\sigma^2, \nu) = (1, 0.9)$ and $\mu_0 = 1$ (bottom) or $\mu_0 = 10$ (top) and $g_0 = 1$ (full lines) or $g_0 = 1 + \phi^{*2}/\mu_0$ (dotted lines).

3.3. Unpredictable marks

The construction of the spatial Hawkes process in Section 1 immediately extends to the case of a random fertility rate specified by an unpredictable mark. More precisely, let the immigrants G_0 still be defined as in Section 1, and conditional on G_0 , let $(Z_\eta)_{\eta \in G_0}$ be independent and identically distributed (iid) “marks” (i.e. random variables) with a distribution Π which is independent of G_0 (the marks are

therefore said to be unpredictable). Further, for $n = 0, 1, \dots$, conditional on the first n generations G_0, \dots, G_n and their associated marks $(Z_\eta)_{\eta \in G_0}, \dots, (Z_\eta)_{\eta \in G_n}$, let G_{n+1} be a Poisson process with intensity function $\lambda_{n+1}(\xi) = \sum_{\eta \in G_n} \gamma(\xi - \eta, Z_\eta)$, where $\gamma(\xi, Z_\eta) \geq 0$ is locally integrable with respect to $\xi \in \mathbb{R}^d$. Furthermore, conditional on G_{n+1} , we associate iid marks $(Z_\eta)_{\eta \in G_{n+1}}$ with distribution Π independent of the “previous history” given by G_0, \dots, G_n, G_{n+1} and $(Z_\eta)_{\eta \in G_0}, \dots, (Z_\eta)_{\eta \in G_n}$ (the marks are therefore said to be unpredictable). We call $X = \cup_{n=0}^{\infty} G_n$ a spatial Hawkes process (this definition coincides with that one in Brémaud et al. [1]).

Note that all the marks are iid and independent of G_0 , but G_{n+1} ($n \geq 0$) may depend on the marks associated to G_n . Define $\bar{\gamma}(\cdot) = \mathbb{E}\gamma(\cdot, Z)$, where the expectation is with respect to a generic mark Z . Redefine $\nu = \int \bar{\gamma}(\eta) d\eta$ and $\phi(\cdot) = \bar{\gamma}(\cdot)/\nu$, where we again assume that $0 < \nu < 1$. Then, concerning the intensity, Proposition 1 is straightforwardly seen to hold. Define $F(\xi) = \int_{\mathbb{R}^d} \text{cov}(\gamma(\xi + x, Z), \gamma(x, Z)) dx$ (where the covariance is zero in the case of a deterministic fertility rate), and assume that

$$\mathbb{E} \left[\left(\int_{\mathbb{R}^d} \gamma(\xi, Z) d\xi \right)^2 \right] < \infty \quad (11)$$

which ensures that F is Lebesgue integrable. Exploiting the independence properties in the branching construction of the generations G_n and their associated marks, it can be verified (see Appendix B) that Theorem 1 extends to the following result.

Theorem 2. *The pair correlation function of X is*

$$g(\xi, \eta) = g(\kappa) = g(-\kappa), \quad g(\kappa) = \chi * \tilde{\chi} * \left[g_0 + \frac{1}{\mu_0(1-\nu)}(\delta + F) \right] (\kappa)$$

whenever $\kappa = \xi - \eta \neq 0$.

From Theorem 2 follows immediately the extension of Corollary 1. Concerning the Bartlett spectrum of X , using Theorem 2 and following the same lines as in the case where $\gamma(\cdot)$ is deterministic, Corollary 2 extends to

$$\hat{C}(d\xi) = \frac{1}{|1 - \mathbb{E}[\hat{\gamma}(\xi, Z)]|^2} \left[\hat{C}_0(d\xi) + \frac{\nu\mu_0}{1-\nu} d\xi + \frac{\mu_0}{1-\nu} \text{var}(\hat{\gamma}(\xi, Z)) d\xi \right] \quad (12)$$

where

$$\hat{\gamma}(\omega, Z) = \int_{\mathbb{R}^d} \exp(i\omega \cdot \xi) \gamma(\xi, Z) d\xi, \quad \omega \in \mathbb{R}^d.$$

Indeed (12) is in accordance with Theorem 3 in [1].

Finally, we notice that it is of course possible to extend the definition above of a spatial Hawkes process to the case with predictable marks, but then we do not see how to derive the pair correlation function and the Bartlett spectrum along similar lines as in the present paper. In fact we believe this to be a much harder problem.

Appendix A

For integers $m, n \geq 0$ and Borel sets $C \subseteq \mathbb{R}^d \times \mathbb{R}^d$, define the measure

$$\alpha_{m,n}(C) = \mathbb{E} \sum_{\xi \in G_m, \eta \in G_n: \xi \neq \eta} \mathbf{1}[(\xi, \eta) \in C]$$

where $\mathbf{1}[\cdot]$ is the indicator function. In the terminology of [13], $\alpha_{m,n}$ is the cross moment measure of (G_m, G_n) if $m \neq n$, and $\alpha_{n,n}$ is the second order factorial moment measure of G_n . Since the point processes G_0, G_1, \dots are almost surely pairwise disjoint, the second order factorial moment measure of $X = \cup_{n=0}^{\infty} G_n$ is given by

$$\alpha^{(2)}(C) = \sum_{m,n \geq 0} \alpha_{m,n}(C). \quad (13)$$

If $\alpha^{(2)}$ is absolutely continuous with respect to Lebesgue measure on $\mathbb{R}^d \times \mathbb{R}^d$ with density $\rho^{(2)}$, then $\rho^{(2)}$ is called the second order product density, and the pair correlation function of X is given by $g(\xi, \eta) = \rho^{(2)}(\xi, \eta)/\rho^2$. By (13), if $\alpha_{m,n}$ is absolutely continuous with respect to Lebesgue measure on $\mathbb{R}^d \times \mathbb{R}^d$ with density $\rho_{m,n}$, we can take $\rho^{(2)} = \sum_{m,n} \rho_{m,n}$. As in Theorem 1 we assume that G_0 has pair correlation function $g_0(\xi, \eta) = g_0(\xi - \eta) = g_0(\eta - \xi)$, and write $\rho_0^{(2)}(\xi) = g_0(\xi)\mu_0^2$ for its second order product density. We show in Lemma 1 below that $\rho_{m,n}$ exists. In particular, $g_n = \rho_{n,n}/\rho_n^2$ is the pair correlation function of G_n , where $\rho_n = \mu_0 \nu^n$, cf. Proposition 1.

Lemma 1. *We have that $\alpha_{m,n}$ is absolutely continuous with respect to Lebesgue measure on $\mathbb{R}^d \times \mathbb{R}^d$, and for all $m, n \geq 0$ and $\xi \neq \eta$,*

$$\rho_{m,n}(\xi, \eta) = \rho_{m,n}(\xi - \eta) = \gamma^{*m} * \tilde{\gamma}^{*n} * \rho_0^{(2)}(\xi - \eta) + \sum_{k=0}^{\min\{m-1, n-1\}} \mu_0 \nu^k \gamma^{*(m-k)} * \tilde{\gamma}^{*(n-k)}(\xi - \eta) \quad (14)$$

with the convention $\sum_{k=0}^{-1} \dots = 0$.

Proof. The result is trivially true when $m = n = 0$.

Let $m < n$, and note that (G_m, G_n) is determined by the marked point process obtained by attaching to each point $\xi \in G_m$ a mark M_ξ given by the point process of all those points $\eta - \xi$ such that $\eta \in G_n$ and η is an offspring generated by ξ in $n - m$ steps. The point processes M_ξ , $\xi \in G_m$, are i.i.d. and independent of G_0, \dots, G_m . Furthermore, if $m > 0$, conditional on G_0, \dots, G_{m-1} , we have that G_m has intensity function λ_m . Hence by Campbell's theorem,

$$\mathbb{E} \left[\sum_{\xi \in G_m} f(\xi, M_\xi) \middle| G_0, \dots, G_{m-1} \right] = \mathbb{E} \left[\int \lambda_m(\xi) f(\xi, M_\xi) d\xi \middle| G_0, \dots, G_{m-1} \right] \quad (15)$$

for non-negative measurable functions f , where we set $\lambda_0 = \mu_0$ and condition on nothing if $m = 0$. Thereby, since with probability one the translated point processes $M_\xi + \xi$, $\xi \in G_m$, are pairwise disjoint and their union is equal to G_n ,

$$\alpha_{m,n}(C) = \mathbb{E} \int \lambda_m(\xi) \sum_{\eta \in G_n} \mathbf{1}[(\xi, \eta) \in C] d\xi = \mathbb{E} \int \lambda_n(\eta) \int \lambda_m(\xi) \mathbf{1}[(\xi, \eta) \in C] d\xi d\eta \quad (16)$$

where to obtain the first equality we consider $\mathbb{E}[\dots] = \mathbb{E}\mathbb{E}[\dots | G_0, \dots, G_{m-1}]$ and to obtain the second equality we consider $\mathbb{E}[\dots] = \mathbb{E}\mathbb{E}[\dots | G_0, \dots, G_{n-1}]$.

Similarly, for $m > n$, we obtain (16). For $m = n > 0$, since G_n conditional on G_0, \dots, G_{n-1} is a Poisson process with intensity function λ_n , Slivnyak-Mecke's theorem (see e.g. [13]) and considering again $\mathbb{E}[\dots] = \mathbb{E}\mathbb{E}[\dots | G_0, \dots, G_{n-1}]$ imply that

$$\alpha_{n,n}(C) = \mathbb{E} \int \lambda_n(\xi) \mathbb{E} \left[\sum_{\eta \in G_n} \mathbf{1}[(\xi, \eta) \in C] \middle| G_0, \dots, G_{n-1} \right] d\xi$$

and so by Campbell's theorem we obtain (16) with $m = n$.

Therefore, by (16) and Fubini's theorem, for all $m, n \geq 0$, $\alpha_{m,n}$ is absolutely continuous with respect to Lebesgue measure on $\mathbb{R}^d \times \mathbb{R}^d$, with density

$$\rho_{m,n}(\xi, \eta) = \mathbb{E}[\lambda_m(\xi)\lambda_n(\eta)]. \quad (17)$$

Consequently,

$$\rho_{m,n}(\xi, \eta) = \rho_{n,m}(\eta, \xi) \quad (18)$$

for all $\xi, \eta \in \mathbb{R}^d$ and $m, n \geq 0$.

In the remainder of this proof, let $\xi \neq \eta$.

We now prove by induction that (14) is satisfied when $m = n$. For $m = n = 0$, this is trivially satisfied. Assume that it is satisfied for a fixed $m = n \geq 0$. By (1) and (17),

$$\rho_{n+1,n+1}(\xi, \eta) = \mathbb{E} \sum_{x_1, x_2 \in G_n: x_1 \neq x_2} \gamma(\xi - x_1) \gamma(\eta - x_2) + \mathbb{E} \sum_{x \in G_n} \gamma(\xi - x) \gamma(\eta - x).$$

Applying the definition of $\rho_{n,n}$ in the first term and Campbell's theorem in the second term above,

$$\rho_{n+1,n+1}(\xi, \eta) = \int \int \gamma(\xi - x_1) \gamma(\eta - x_2) \rho_{n,n}(x_1, x_2) dx_1 dx_2 + \int \gamma(\xi - x) \gamma(\eta - x) \rho_n dx$$

which after a straightforward computation reduces to

$$\rho_{n+1,n+1}(\xi, \eta) = \gamma * \tilde{\gamma} * (\rho_n \delta + \rho_{n,n})(\xi - \eta).$$

Therefore, by the induction hypothesis,

$$\begin{aligned} \rho_{n+1,n+1}(\xi, \eta) &= \mu_0 \nu^n \gamma * \tilde{\gamma}(\xi - \eta) + \gamma^{*(n+1)} * \tilde{\gamma}^{*(n+1)} * \rho_0^{(2)}(\xi - \eta) \\ &+ \sum_{k=0}^{n-1} \mu_0 \nu^k \gamma^{*(n+1-k)} * \tilde{\gamma}^{*(n+1-k)}(\xi - \eta) \end{aligned}$$

whereby the induction proof is completed. Note that from this and (17) we obtain that

$$\rho_{n,n}(\xi, \eta) = \rho_{n,n}(\xi - \eta) = \rho_{n,n}(\eta - \xi). \quad (19)$$

Next, let $m < n$. Then

$$\begin{aligned} \rho_{m,n}(\xi, \eta) &= \mathbb{E}[\lambda_m(\xi) \lambda_n(\eta)] \\ &= \mathbb{E} \sum_{x_1 \in G_{n-1}} \gamma(\eta - x_1) \lambda_m(\xi) \\ &= \mathbb{E} \mathbb{E} \left[\sum_{x_1 \in G_{n-1}} \gamma(\eta - x_1) \lambda_m(\xi) \mid G_0, \dots, G_{n-2} \right] \\ &= \mathbb{E} \int \gamma(\eta - x_1) \lambda_m(\xi) \lambda_{n-1}(x_1) dx_1 \\ &= \int \gamma(\eta - x_1) \rho_{m,n-1}(\xi, x_1) dx_1 \end{aligned}$$

where we have used (17) in the first and last equalities, (1) in the second equality, and Campbell's theorem in the fourth equality. Iterating this calculation we obtain

$$\begin{aligned} \rho_{m,n}(\xi, \eta) &= \int \gamma(\eta - x_1) \int \gamma(x_1 - x_2) \cdots \int \gamma(x_{n-m-1} - x_{n-m}) \rho_{m,m}(\xi, x_{n-m}) \\ &\quad dx_{n-m} \cdots dx_2 dx_1 \\ &= \tilde{\gamma}^{*(n-m)} * \rho_{m,m}(\xi - \eta) \end{aligned}$$

using (19). Since $\rho_{m,m}$ satisfies (14), we obtain that $\rho_{m,n}$ satisfies (14).

Third, let $m > n$. Observe that if $h = f_1 * f_2$ then $\tilde{h} = \tilde{f}_1 * \tilde{f}_2$. Combining this with (14) (for the case so far verified), (18), and the fact that $\rho_0^{(2)}(\cdot)$ is symmetric, we obtain that

$$\begin{aligned}\rho_{m,n}(\xi, \eta) &= \gamma^{*n} * \tilde{\gamma}^{*m} * \rho_0^{(2)}(\eta - \xi) + \sum_{k=0}^{n-1} \mu_0 \nu^k \gamma^{*(n-k)} * \tilde{\gamma}^{*(m-k)}(\eta - \xi) \\ &= \gamma^{*m} * \tilde{\gamma}^{*n} * \rho_0^{(2)}(\xi - \eta) + \sum_{k=0}^{n-1} \mu_0 \nu^k \gamma^{*(m-k)} * \tilde{\gamma}^{*(n-k)}(\xi - \eta).\end{aligned}$$

Thereby (14) is verified for all $m, n \geq 0$.

Proof of Theorem 1: By (14), $\rho^{(2)}(\xi, \eta) = \rho^{(2)}(\xi - \eta)$ whenever $\xi \neq \eta$, where for $\kappa \neq 0$,

$$\begin{aligned}\rho^{(2)}(\kappa) &= \sum_{m \geq 0, n \geq 0} \rho_{m,n}(\kappa) \\ &= \left(\sum_{m \geq 0} \gamma^{*m} \right) * \left(\sum_{n \geq 0} \tilde{\gamma}^{*n} \right) * \rho_0^{(2)}(\kappa) \\ &+ \sum_{k, m, n: m \geq k, n \geq k, k \geq 0} \mu_0 \nu^k \gamma^{*(m+1-k)} * \tilde{\gamma}^{*(n+1-k)}(\kappa) \\ &= \left(\sum_{m \geq 0} \gamma^{*m} \right) * \left(\sum_{n \geq 0} \tilde{\gamma}^{*n} \right) * \left(\rho_0^{(2)} + \rho\delta \right) (\kappa)\end{aligned}\tag{20}$$

using (3) and that $\delta(\kappa) = 0$ since $\kappa \neq 0$. By (3) and (20), $g(\xi, \eta) = g(\xi - \eta) = \rho^{(2)}(\xi - \eta)/\rho^2$ is easily seen to be given by the last expression in (4). Finally, it follows that $g(\cdot)$ is symmetric.

Appendix B

We sketch the proof of Theorem 2. First, (17) in the proof of Lemma 1 holds with $\gamma(\xi, Z_\xi)$ in place of $\gamma(\xi)$. Second, following the same lines as in the proof of Lemma 1 and using that the marks are unpredictable, for any $n \geq 0$,

$$\begin{aligned}\rho_{n+1, n+1}(\xi, \eta) &= \int \int \bar{\gamma}(\xi - x_1) \bar{\gamma}(\eta - x_2) \rho_{n,n}(x_1, x_2) dx_1 dx_2 + \int \bar{\gamma}(\xi - x) \bar{\gamma}(\eta - x) \rho_n dx \\ &+ \rho_n F(\xi - \eta)\end{aligned}$$

which after a straightforward computation reduces to

$$\rho_{n+1,n+1}(\xi, \eta) = \bar{\gamma} * \tilde{\gamma} * (\rho_n \delta + \rho_{n,n})(\xi - \eta) + \rho_n F(\xi - \eta).$$

Therefore, we obtain

$$\begin{aligned} \rho_{n+1,n+1}(\xi, \eta) &= \mu_0 \nu^n \bar{\gamma} * \tilde{\gamma}(\xi - \eta) + \bar{\gamma}^{*(n+1)} * \tilde{\gamma}^{*(n+1)} * \rho_0^{(2)}(\xi - \eta) \\ &+ \sum_{k=0}^{n-1} \mu_0 \nu^k \bar{\gamma}^{*(n+1-k)} * \tilde{\gamma}^{*(n+1-k)}(\xi - \eta) \\ &+ \mu_0 \nu^n F(\xi - \eta) + \sum_{k=0}^{n-1} \mu_0 \nu^k \bar{\gamma}^{*(n-k)} * \tilde{\gamma}^{*(n-k)} * F(\xi - \eta) \end{aligned}$$

Now, arguing as in the proof of Lemma 1, for $m, n \geq 0$,

$$\begin{aligned} \rho_{m,n}(\xi, \eta) = \rho_{m,n}(\xi - \eta) &= \bar{\gamma}^{*m} * \tilde{\gamma}^{*n} * \rho_0^{(2)}(\xi - \eta) + \sum_{k=0}^{\min\{m-1, n-1\}} \mu_0 \nu^k \bar{\gamma}^{*(m-k)} * \tilde{\gamma}^{*(n-k)}(\xi - \eta) \\ &+ \sum_{k=0}^{\min\{m-1, n-1\}} \mu_0 \nu^k \bar{\gamma}^{*(m-1-k)} * \tilde{\gamma}^{*(n-1-k)} * F(\xi - \eta) \end{aligned}$$

(with the convention $\sum_{k=0}^{-1} \dots = 0$). As in the proof of Theorem 1, Theorem 2 follows by summing over all $m, n \geq 0$ and dividing by ρ^2 .

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