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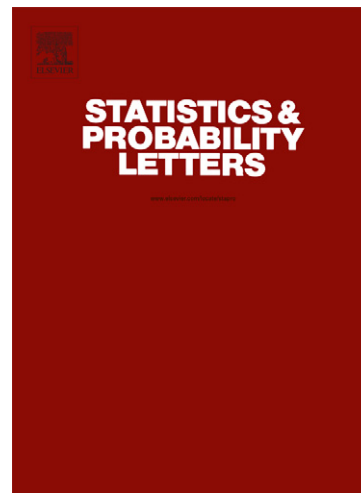
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The Optional Sampling Theorem for Submartingales in the Sequentially Planned Context

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Abstract

The optional sampling theorem is considered in the sequentially planned context. We prove the optional sampling theorem for direct successors and for sampling plans with a finite number of stages. Also, the theorem is studied in the general case under a uniform integrability condition; we obtain it for submartingales with a last element, and for submartingales that verify a bounded condition based on uniform integrability.

Keywords: Sampling plan, sequentially planned, optional sampling.

Ams: 62L10, 60G40.

1 Introduction

The Sequentially Planned Decision Procedures were introduced by Schmitz (1993); the underlying idea is to take sequential sub-samples whose random size will be determined (planned) according to the information gathered up to that time.

In this paper we study the Optional Sampling Theorem (OST) in the sequentially planned context; i.e. the submartingales are indexed by a set A of sampling sequences, and a partial order represented by \preceq is defined on A . Thus, let $a = (a_1, \dots, a_j) \in A$, $b = (b_1, \dots, b_m) \in A$ finite sampling sequences; $a \preceq b$ means that $a_1 = b_1, \dots, a_j = b_j$ with $j \leq m$, and each $a_i, b_k \in M \subset \mathbb{N}$. For details, see Schmitz (1993), Fenoy and Ibarrola (2003).

The optional sampling theorem for stopping times is already studied for submartingales indexed by different sets. For example, Doob (1953), Bochner (1955) studied the OST in the general context of a directed index set. Krickeberg (1956), Helms (1958) and Chow, Robbins, and Siegmund (1971) have obtained results for linearly ordered index sets. Washburn and Willsky (1981) proved it for *uniformly bounded submartingales* indexed by partially ordered sets for reachable pairs of stopping times. They also proved that any admissible sampling plan is reachable from the stopping time 0, but the results obtained are for stopping times (i.e. in a classical sequential context).

In the sequentially planned context we use sampling plans instead of stopping times. For $(\mathcal{X}, \mathcal{B}, P_\theta)$ a probability space and $(\mathcal{B}_a)_{a \in A}$ an isotonic family of sub- σ -fields of \mathcal{B} , let τ be a sampling plan if it verifies

$$\{\tau \preceq a\} \in \mathcal{B}_a, \quad \forall a \in A \tag{1}$$

$$\{\tau \succeq ah\} \in \mathcal{B}_a, \quad \forall a \in A, \quad h \in M. \tag{2}$$

The main difference between a stopping time and a sampling plan is given by condition (2): the size of the next sub-sample is determined sequentially and based on the information accumulated. In general, we focus our attention on admissible sampling plans, i.e. sampling plans which stop a.s.

In this article, we study the OST for any submartingale, and we obtain it for sampling plans with a fixed number of stages, not necessarily direct successors. Next, we study the OST for any sampling plans and we obtain it for submartingales uniformly integrable.

The structure of this paper is as follows:

In section 2, we obtain the OST for sampling plans that they are direct successors. This result allows us to prove the OST for any pair of sampling plans, τ , σ , and such that $\sigma \preceq \tau$ with a fixed number of stages. These results are obtained without any extra-conditions about the submartingales.

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The OST is studied under the uniform integrability condition in section 3; we have obtained it for submartingales with a last element. And we prove a version of the OST for submartingales that verifies the following condition based on uniform integrability:

$$E(|X_{bk} - X_b|/\mathcal{B}_b) \leq c < +\infty, \text{ on } \{\tau \geq b\}.$$

We do not discuss some concepts in detail since the background of this note can be found in the reference Fenoy and Ibarrola (2003).

2 The Optional Sampling Theorem

For a sampling plan τ we define $\mathcal{B}_\tau = \{C \in \mathcal{B} : \forall a \in A, C \cap \{\tau = a\} \in \mathcal{B}_a\}$. \mathcal{B}_τ is a σ -field, and for any two pairs of sampling plans, such that $\sigma \preceq \tau$, $\mathcal{B}_\sigma \subset \mathcal{B}_\tau$. Let τ, σ be two sampling plans; we say they are direct successors if $\tau \succeq \sigma$; and if $\tau \succ \sigma$, τ must only have one stage more than σ .

The theorem below shows that the OST holds for direct successors.

Theorem 2.1 *Let $\{X_a, \mathcal{B}_a\}_{a \in A}$ be a submartingale. Let $\{\tau_n\}_{n \in \mathbb{N}}$ be an admissible sampling plan sequence, such that $\forall n \in \mathbb{N}$ $\tau_n \preceq \tau_{n+1}$ are direct successors and $E(|X_{\tau_n}|) < +\infty$. Then $\{X_{\tau_n}, \mathcal{B}_{\tau_n}\}_{n \in \mathbb{N}}$ is a submartingale.*

Proof: $\{\mathcal{B}_{\tau_n}\}_{n \in \mathbb{N}}$ is a filtration since $\tau_n \preceq \tau_{n+1}$ (see Fenoy and Ibarrola (2003)). For $B \in \mathcal{B}_{\tau_n}$, it holds that

$$\int_B X_{\tau_{n+1}} dP \geq \int_B X_{\tau_n} dP \quad (3)$$

since $B = \bigcup_{a \in A} B \cap \{\tau_n = a\} = \bigcup_{a \in A} D_a$, $\forall a \in A$, $D_a \in \mathcal{B}_a$ and $\{\tau_n = a\} \subset \{\tau_{n+1} \succeq a\}$.

$$\begin{aligned} \int_{D_a} X_{\tau_{n+1}} dP &= \int_{B \cap \{\tau_n = a\}} X_{\tau_{n+1}} dP = \int_{D_a \cap \{\tau_{n+1} \succeq a\}} X_{\tau_{n+1}} dP \\ &= \int_{D_a \cap \{\tau_{n+1} = a\}} X_a dP + \sum_{k \in \tilde{M}_a} \int_{D_a \cap \{\tau_{n+1} = ak\}} E(X_{ak}/\mathcal{B}_a) dP \\ &\geq \int_{D_a \cap \{\tau_{n+1} = a\}} X_a dP + \sum_{k \in \tilde{M}_a} \int_{D_a \cap \{\tau_{n+1} = ak\}} X_a dP \\ &= \int_{D_a \cap \{\tau_{n+1} \succeq a\}} X_a dP = \int_{D_a} X_a dP = \int_{D_a} X_{\tau_n} dP \end{aligned} \quad (4)$$

(4) follows from $\{\tau_n = a\} \cap \{\tau_{n+1} = ak\} = \{\tau_n = a\} \cap \{\tau_{n+1} \succeq ak\} \in \mathcal{B}_a$, since τ_n and τ_{n+1} are direct successors.

Remark 1 *In Theorem 2.1, the condition $E(|X_{\tau_n}|) < +\infty$ will hold if the number of observations of τ_n is at most $k_n < +\infty$, $\forall n \in \mathbb{N}$, particularly, if A is a finite set.*

For any two sampling plans τ, σ , such that $\sigma \preceq \tau$, we introduce the following functions, $k \in \mathbb{N}$:

$$\tau_1(x) = \sigma(x)I_{\{\tau=\sigma\}}(x) + \sum_{a_1 \in M} (\sigma(x), a_1)I_{\{\tau \succeq (\sigma, a_1)\}}(x) \quad (5)$$

$$\tau_2(x) = \tau_1(x)I_{\{\tau=\tau_1\}}(x) + \sum_{a_1 \in M} (\tau_1(x), a_1)I_{\{\tau \succeq (\tau_1, a_1)\}}(x) \quad (6)$$

\vdots

$$\tau_k(x) = \tau_{k-1}(x)I_{\{\tau=\tau_{k-1}\}}(x) + \sum_{a_1 \in M} (\tau_{k-1}(x), a_1)I_{\{\tau \succeq (\tau_{k-1}, a_1)\}}(x) \quad (7)$$

\vdots

Let $\tau_0 = \sigma$.

Proposition 2.1 Let τ and σ be sampling plans, such that $\tau \succeq \sigma$; then τ_k , $k \in \mathbb{N}$ defined in (7), is an admissible sampling plan and $\tau_k \preceq \tau_{k+1}$ are direct successors. Also, if τ is admissible $\forall x$, there exists $k \in \mathbb{N}$, such that $\tau(x) = \tau_k(x)$. Thus

$$\tau(x) = \lim_{k \rightarrow \infty} \tau_k(x), \text{ a.s.}$$

Proof: For $j = 1$, $a \in A$, $h \in M$

$$\begin{aligned} \{\tau_1 \preceq a\} &= \{\tau_1 \preceq a\} \cap \{\sigma \preceq a\} \\ &= \bigcup_{i=1}^{h(a)-1} [\{\tau_1 \preceq a\} \cap \{\sigma = a^i\}] \cup \{\tau_1 \preceq a\} \cap \{\sigma = a\} \\ &= \bigcup_{i=1}^{h(a)-1} \{\sigma = a^i\} \cup \{\tau = a\} \cap \{\sigma = a\} \in \mathcal{B}_a \end{aligned}$$

$$\begin{aligned} \{\tau_1 \succeq ah\} &= \{\tau_1 \succeq ah\} \cap \{\sigma \succeq ah\} \cup \{\tau_1 \succeq ah\} \cap \{\sigma \not\succeq ah\} \\ &= \{\sigma \succeq ah\} \cup \{\sigma = a\} \cap \{\tau \succeq ah\} \in \mathcal{B}_a \end{aligned}$$

Thus, we obtain that τ_1 is an admissible sampling plan. Now suppose that the result is true for τ_{j-1} , it is easy to see that it is true for j , following a similar scheme. \square

Theorem 2.2 Let $\{X_a, \mathcal{B}_a\}_{a \in A}$ be a submartingale. Let τ, σ be admissible sampling plans such that, $\tau \succeq \sigma$. Suppose that $\exists k \in \mathbb{N}$, such that, $\tau(x) = \tau_k(x)$, $\forall x \in \mathcal{X}$ (for τ_k defined in (7)), and $\forall j = 0, \dots, k$, $E(|X_{\tau_j}|) < +\infty$, then

$$E(X_\tau / \mathcal{B}_\sigma) \geq X_\sigma. \quad (8)$$

Proof: First we see that condition (8) holds for $k = 1$. Since τ_1 defined in (5) is a direct successor of σ , we can apply Theorem 2.1, and we obtain $E(X_{\tau_1} / \mathcal{B}_\sigma) \geq X_\sigma$. On the other hand, for any $j = 1, \dots, k$, since τ_{j-1}, τ_j are direct successors, we obtain

$$E(X_{\tau_j} / \mathcal{B}_{\tau_{j-1}}) \geq X_{\tau_{j-1}}. \quad (9)$$

Finally, suppose that $E(X_{\tau_j} / \mathcal{B}_\sigma) \geq X_\sigma$ is true for $k - 1$; we prove that is true for k . By induction hypothesis and equation (9), it follows that

$$E(X_\tau / \mathcal{B}_\sigma) = E[E(X_{\tau_k} / \mathcal{B}_{\tau_{k-1}}) / \mathcal{B}_\sigma] \geq E(X_{\tau_{k-1}} / \mathcal{B}_\sigma) \geq X_\sigma. \quad \square$$

3 Uniform Integrability

Now we prove the OST under conditions based on the concepts related with uniform integrability. We introduce some necessary concepts about sampling plans.

For $a = (a_1, \dots, a_j) \in A$, let $h(a) = j$ be the number of sub-samples of the sample sequence a , and let $g(a) = \sum_{i=1}^j a_j$ be the total number of observations of the sample sequence a .

Definition 3.1 Let τ be a sampling plan; let us define ${}_k\tau$, $\forall k \in \mathbb{N}$, as follows:

$${}_k\tau = \begin{cases} \tau & h(\tau) \leq k \\ \tau^k = (\tau_1, \dots, \tau_k) & h(\tau) > k. \end{cases}$$

Remark 2 1. ${}_k\tau$, $\forall k \in \mathbb{N}$, is an admissible sampling plan with almost k -stages and ${}_k\tau \xrightarrow[k \rightarrow \infty]{} \tau$ a.s.

2. ${}_{k+1}\tau$ and ${}_k\tau$ are direct successors, $\forall k \in \mathbb{N}$.

3. $\mathcal{B}_{{}_k\tau} \subset \mathcal{B}_{{}_{k+1}\tau}$, $\forall k \in \mathbb{N}$.

4. $\mathcal{B}_{k\tau} \subset \mathcal{B}_\tau, \forall k \in \mathbb{N}$.

Definition 3.2 Let τ be a sampling plan, let us define ${}^k\tau, \forall k \in \mathbb{N}$, as follows:

$${}^k\tau = \begin{cases} \tau & g(\tau) \leq k \\ (\tau_1, \dots, \tau_j, b) & g(\tau_1, \dots, \tau_j) < k, \quad g(\tau_1, \dots, \tau_{j+1}) > k, \quad g(\tau_1, \dots, \tau_j, b) = k \end{cases}$$

$b \in M$.

Remark 3 ${}^k\tau, \forall k \in \mathbb{N}$, is an admissible sampling plan with almost k observations, and ${}^k\tau \xrightarrow[k \rightarrow \infty]{} \tau$ a.s.

Definition 3.3 A sequence $\{X_a, \mathcal{B}_a\}_{a \in A}$ is uniformly integrable if

$$\int_{\{|X_a| \geq c\}} |X_a| dP \xrightarrow[c \rightarrow \infty]{} 0$$

uniformly in $a \in A$.

Definition 3.4 Let $\{X_a, \mathcal{B}_a\}_{a \in A}$ be a submartingale; we say that it has a "last element" if a random variable X_∞ exists, such that $X_a \leq E(X_\infty/\mathcal{B}_a), \forall a \in A$.

Theorem 3.1 Let $\{X_a, \mathcal{B}_a\}_{a \in A}$ be a submartingale with a last element. Let τ and σ be sampling plans. Then

$$E(X_\tau/\mathcal{B}_\sigma) \geq X_\sigma \text{ c.s., } \forall \tau \succeq \sigma.$$

Proof:

For $a \in A$, let $Z_a = E(X_\infty/\mathcal{B}_a)$ and $W_a = X_a - E(X_\infty/\mathcal{B}_a)$, thus, $X_a = Z_a + W_a$.

1. Z_a , is a uniformly integrable martingale. Let τ be a sampling plan; we consider the admissible sampling plans that we have defined in Definition 3.1, that is, ${}^k\tau, \forall k \in \mathbb{N}$. By the optional projection theorem (see Fenoy and Ibarrola (2003)), $Z_{k\tau} = E(X_\infty/\mathcal{B}_{k\tau}), Z_\tau = E(X_\infty/\mathcal{B}_\tau)$; and $Z_\tau, Z_{k\tau}$ are integrable random variables, $\forall k \in \mathbb{N}$:

$$\begin{aligned} E(|Z_\tau|) &= E(|E(X_\infty/\mathcal{B}_\tau)|) \leq E(E(|X_\infty|/\mathcal{B}_\tau)) = E(|X_\infty|) < +\infty \\ E(|Z_{k\tau}|) &= E(|E(X_\infty/\mathcal{B}_{k\tau})|) \leq E(E(|X_\infty|/\mathcal{B}_{k\tau})) = E(|X_\infty|) < +\infty \end{aligned}$$

Hence, $\{Z_{k\tau}, \mathcal{B}_{k\tau}\}_{k \in \mathbb{N}}$ is a convergence uniformly integrable martingale, and its limit is Z_τ . $\sigma \preceq \tau$, then $({}_k\sigma) \preceq ({}_k\tau), \forall k \in \mathbb{N}$, and $h({}_k\tau) \leq k$; also, $E(|X_{k\tau}|) < +\infty$. Thus, we can apply Theorem 2.2, obtaining

$$E(Z_{k\tau}/\mathcal{B}_{k\sigma}) \geq Z_{k\sigma} \text{ a.s. } \forall k \in \mathbb{N}.$$

Let us write $Z_\tau = Z_{k\tau} + (Z_\tau - Z_{k\tau})I_{\{\tau \neq k\tau\}}$. For $B \in \mathcal{B}_\sigma$

$$\begin{aligned} \int_B Z_\tau dP &= \int_{B \cap (h(\sigma) \leq k)} Z_{k\tau} dP + \int_{B \cap (h(\sigma) \leq k)} (Z_\tau - Z_{k\tau}) I_{\{\tau \neq k\tau\}} dP \\ &\quad + \int_{B \cap (h(\sigma) > k)} Z_{k\tau} dP + \int_{B \cap (h(\sigma) > k)} (Z_\tau - Z_{k\tau}) I_{\{\tau \neq k\tau\}} dP. \end{aligned} \tag{10}$$

Since $\forall k \in \mathbb{N}, B \cap (h(\sigma) \leq k) \in \mathcal{B}_{k\sigma}$ and

$$\int_{B \cap (h(\sigma) \leq k)} Z_{k\tau} dP \geq \int_{B \cap (h(\sigma) \leq k)} Z_{k\sigma} dP, \tag{11}$$

equation (10) becomes

$$\begin{aligned} \int_B Z_\tau dP &\geq \int_{B \cap (h(\sigma) \leq k)} Z_\sigma dP + \int_{B \cap (h(\sigma) \leq k)} (Z_\tau - Z_{k\tau}) I_{\{\tau \neq k\tau\}} dP \\ &\quad + \int_{B \cap (h(\sigma) > k)} Z_\tau I_{\{\tau \neq k\tau\}} dP. \end{aligned} \tag{12}$$

Now, we consider $\int_{B \cap (h(\sigma) \leq k)} Z_\sigma dP$. Since Z_σ is an integrable r.v., and σ is an admissible sampling plan, i.e. $I_{\{h(\sigma) \leq k\}} \xrightarrow[k \rightarrow \infty]{} 1$ a.s., we obtain

$$Z_\sigma I_{\{h(\sigma) \leq k\}} \xrightarrow[k \rightarrow \infty]{\text{a.s.}} Z_\sigma,$$

and by the dominated convergence theorem

$$\int_{B \cap (h(\sigma) \leq k)} Z_\sigma dP \xrightarrow[n \rightarrow \infty]{} \int_B Z_\sigma dP. \quad (13)$$

On the other hand,

$$\left| \int_{B \cap (h(\sigma) \leq k)} (Z_\tau - Z_{k\tau}) I_{\{\tau \neq k\tau\}} dP \right| \leq \int_{B \cap (h(\sigma) \leq k)} |(Z_\tau - Z_{k\tau})| dP \xrightarrow[k \rightarrow \infty]{} 0, \quad (14)$$

and

$$\int_{B \cap (h(\sigma) > k)} Z_\tau I_{\{\tau \neq k\tau\}} dP \xrightarrow[k \rightarrow \infty]{} 0. \quad (15)$$

Taking limits in equation (12), it follows that

$$E(Z_\tau / \mathcal{B}_\sigma) \geq Z_\sigma \text{ a.s.} \quad (16)$$

2. W_a is an integrable random variable, such that $W_a \leq 0, \forall a \in A$. For τ , we consider the sampling plan j_τ defined in 3.2. Then,

$$E(|W_{j_\tau}|) \leq \sum_{\substack{a \in A \\ g(a) \leq j}} E(|W_a|) < +\infty, \quad \forall a \in A$$

and by the Fatou's lemma we deduce that W_τ is integrable. Also, for $B \in \mathcal{B}_\sigma, k \in \mathbb{N}$, since $B \cap (h(\sigma) \leq k) \in \mathcal{B}_{k\sigma}$, and from Theorem 2.2, we have

$$\int_{B \cap (h(\sigma) \leq k)} W_\sigma dP = \int_{B \cap (h(\sigma) \leq k)} W_{k\sigma} dP \leq \int_{B \cap (h(\tau) \leq k)} W_{k\tau} dP = \int_{B \cap (h(\tau) \leq k)} W_\tau dP. \quad (17)$$

By the dominated convergence theorem, it follows that $\forall B \in \mathcal{B}_\sigma$,

$$\int_B W_\sigma dP \leq \int_B W_\tau dP. \quad (18)$$

From equations (16) and (18) the result follows. \square

Theorem 3.2 Let $\{X_b, \mathcal{B}_b\}_{b \in A}$ be a submartingale and τ an admissible sampling plan, such that, $\tau \succeq a$, for some $a \in A$. Let us suppose

1. $E(h(\tau)) < +\infty$.
2. $E(|X_{bk} - X_b| / \mathcal{B}_b) \leq c < +\infty$, in $\{\tau \succeq b\}$, with $b \succeq a$, and $k \in M$.

Then

$$E(X_\tau / \mathcal{B}_a) \geq X_a \text{ a.s.}$$

Proof:

For $b \in A, X_b = \sum_{i=1}^{h(b)} (X_{b^i} - X_{b^{i-1}})$, with $X_\emptyset = 0$. Thus,

$$|X_\tau| \leq \sum_{i=h(a)}^{h(\tau)} |X_{i\tau} - X_{i-1\tau}| + |X_{a^{h(a)-1}}| = \sum_{i=h(a)}^{h(\tau)} Z_{i\tau} + |X_{a^{h(a)-1}}| = Y_\tau + |X_{a^{h(a)-1}}| \quad (19)$$

and $\forall k \geq h(a)$, $|X_{k\tau}| \leq Y_\tau + |X_{a^{h(a)}-1}|$ is an integrable random variable, since $X_{a^{h(a)}-1}$ is integrable and $E(Y_\tau) \leq cE(h(\tau)) < +\infty$. For all $k \geq h(a)$, by Theorem 2.2, we have that

$$E(X_{k\tau}/\mathcal{B}_a) \geq X_a. \quad (20)$$

Also

$$\liminf_{k \rightarrow \infty} E((X_\tau - X_{k\tau})I_{\{\tau \neq k\tau\}}/\mathcal{B}_a) = 0. \quad (21)$$

Since

$$E(X_\tau/\mathcal{B}_a) = E(X_{k\tau}/\mathcal{B}_a) + E((X_\tau - X_{k\tau})I_{\{\tau \neq k\tau\}}/\mathcal{B}_a),$$

we can deduce from (20) and (21) that

$$E(X_\tau/\mathcal{B}_a) \geq X_a \text{ a.s.}$$

□

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