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# Progressive enlargement of filtrations and Backward SDEs with jumps

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## Abstract

This work deals with backward stochastic differential equation (BSDE) with random marked jumps, and their applications to default risk. We show that these BSDEs are linked with Brownian BSDEs through the decomposition of processes with respect to the progressive enlargement of filtrations. We show that the equations have solutions if the associated Brownian BSDEs have solutions. We also provide a uniqueness theorem for BSDEs with jumps by giving a comparison theorem based on the comparison for Brownian BSDEs. We give in particular some results for quadratic BSDEs. As applications, we study the pricing and the hedging of a European option in a complete market with a single jump, and the utility maximization problem in an incomplete market with a finite number of jumps.

**Keywords:** Backward SDE, quadratic BSDE, multiple random marked times, progressive enlargement of filtrations, decomposition in the reference filtration, exponential utility.

**MSC classification (2000):** 60G57, 60H20, 60J75, 91G10, 91G80, 93E30.

## 1 Introduction

In recent years, credit risk has come out to be one of most fundamental financial risk. The most extensively studied form of credit risk is the default risk. Many people, such as Bielecki, Jarrow, Jeanblanc, Pham, Rutkowski ([3, 4, 14, 15, 18, 26]) and many others, have worked on this subject. In several papers (see for example Ankirchner *et al.* [1], Bielecki and Jeanblanc [5], Lim and Quenez [21] and Peng and Xu [25]), related to this topic, backward stochastic differential equations (BSDEs) with jumps have appeared. Unfortunately, the results relative to these latter BSDEs are far from being as numerous as for Brownian BSDEs. In particular, there is not any general result on the existence and the uniqueness

of solution to quadratic BSDEs, except Ankirchner *et al.* [1], in which the assumptions on the driver are strong. In this paper, we study BSDEs with random marked jumps and apply the obtained results to mathematical finance where these jumps can be interpreted as default times. We give a general existence and uniqueness result for the solutions to these BSDEs, in particular we enlarge the result given by [1] for quadratic BSDEs.

A standard approach of credit risk modeling is based on the powerful technique of filtration enlargement, by making the distinction between the filtration  $\mathbb{F}$  generated by the Brownian motion, and its smallest extension  $\mathbb{G}$  that turns default times into  $\mathbb{G}$ -stopping times. This kind of filtration enlargement has been referred to as progressive enlargement of filtrations. This field is a traditional subject in probability theory initiated by fundamental works of the French school in the 80s, see e.g. Jeulin [16], Jeulin and Yor [17], and Jacod [13]. For an overview of applications of progressive enlargement of filtrations on credit risk, we refer to the books of Duffie and Singleton [9], of Bielecki and Rutkowski [3], or the lectures notes of Bielecki *et al.* [4].

The purpose of this paper is to combine results on Brownian BSDEs and results on progressive enlargement of filtrations in view of providing existence and uniqueness of solutions to BSDEs with random marked jumps. We consider a progressive enlargement with multiple random times and associated marks. These marks can represent for example the name of the firm which defaults or the jump sizes of asset values. Throughout the sequel, we make the classical assumption in the enlargement of filtration which ensures the stability of the class of semimartingale. This assumption, usually called **(H')** hypothesis, means that any  $\mathbb{F}$ -semimartingale remains a  $\mathbb{G}$ -semimartingale. This assumption is a fundamental property both in probability and finance where it is closely related to the absence of arbitrage.

Our approach consists in using the recent results of Pham [26] on the decomposition of predictable processes with respect to the progressive enlargement of filtrations to decompose a BSDE with random marked jumps into a sequence of Brownian BSDEs. By combining the solutions of Brownian BSDEs, we obtain a solution to the BSDE with random marked times. This method allows to get a general existence theorem. In particular, we get an existence result for quadratic BSDEs which is more general than the result of Ankirchner *et al* [1]. This decomposition approach also allows to obtain a uniqueness theorem under the stronger assumption **(H)** i.e. any  $\mathbb{F}$ -martingale remains a  $\mathbb{G}$ -martingale. We first set a general comparison theorem for BSDEs with jumps based on comparison theorems for Brownian BSDEs. Using this theorem, we prove, in particular, the uniqueness for quadratic BSDEs with a concave generator in  $z$ .

We illustrate our methodology with two financial applications in default risk management: the pricing and the hedging of a European option in a complete market, and the problem of utility maximization in an incomplete market. A similar problem (without marks) has recently been considered in Ankirchner *et al.* [1] and Lim and Quenez [21].

The paper is organized as follows. The next section presents the general framework of progressive enlargement of filtrations with successive random times and marks, and states the decomposition result for  $\mathbb{G}$ -predictable and  $\mathbb{G}$ -progressively measurable processes. In Section 3, we use this decomposition to make a link between Brownian BSDEs and BSDEs

with random marked jumps. This allows to give a general existence result. We then give two examples: quadratic BSDEs with marked jumps for the first one, and linear BSDEs arising in the pricing and hedging problem of a European option in a complete market with a single jump for the second one. In Section 4, we give a general comparison theorem for BSDEs and we use this result to give a uniqueness theorem for quadratic BSDEs. Finally, in Section 5, we apply our existence and uniqueness results to solve the exponential utility maximization problem in an incomplete market with a finite number of marked jumps.

## 2 Progressive enlargement of filtrations with successive random times and marks

We fix a probability space  $(\Omega, \mathcal{G}, \mathbb{P})$ , and we start with a reference filtration  $\mathbb{F} = (\mathcal{F}_t)_{t \geq 0}$  satisfying the usual conditions<sup>1</sup> and generated by a Brownian motion  $W$ . Throughout the sequel, we consider a finite sequence  $(\tau_k, \zeta_k)_{1 \leq k \leq n}$ , where

- $(\tau_k)_{1 \leq k \leq n}$  is a nondecreasing sequence of random times (i.e. nonnegative  $\mathcal{G}$ -random variables),
- $(\zeta_k)_{1 \leq k \leq n}$  is a sequence of random marks valued in some Borel subset  $E$  of  $\mathbb{R}^m$ .

We denote by  $\mu$  the random measure associated to the sequence  $(\tau_k, \zeta_k)_{1 \leq k \leq n}$  :

$$\mu([0, t] \times B) = \sum_{k=1}^n \mathbb{1}_{\{\tau_k \leq t, \zeta_k \in B\}}, \quad t \geq 0, \quad B \in \mathcal{B}(E).$$

For each  $k = 1, \dots, n$ , we consider  $\mathbb{D}^k = (\mathcal{D}_t^k)_{t \geq 0}$  the smallest right-continuous filtration for which  $\tau_k$  is a stopping time and  $\zeta_k$  is  $\mathcal{D}_{\tau_k}^k$ -measurable.  $\mathbb{D}^k$  is then given by  $\mathcal{D}_t^k = \tilde{\mathcal{D}}_{t+}^k$ , where  $\tilde{\mathcal{D}}_t^k = \sigma(\mathbb{1}_{\tau_k \leq s}, s \leq t) \vee \sigma(\zeta_k \mathbb{1}_{\tau_k \leq s}, s \leq t)$ . The global information is then defined by the progressive enlargement  $\mathbb{G} = (\mathcal{G}_t)_{t \geq 0}$  of the initial filtration  $\mathbb{F}$  where  $\mathbb{G} := \mathbb{F} \vee \mathbb{D}^1 \vee \dots \vee \mathbb{D}^n$ . The filtration  $\mathbb{G} = (\mathcal{G}_t)_{t \geq 0}$  is the smallest filtration containing  $\mathbb{F}$ , and such that for each  $k = 1, \dots, n$ ,  $\tau_k$  is a  $\mathbb{G}$ -stopping time, and  $\zeta_k$  is  $\mathcal{G}_{\tau_k}$ -measurable. We denote by  $\Delta_k$  the set where the random  $k$ -tuple  $(\tau_1, \dots, \tau_k)$  takes its values in  $\{\tau_n < \infty\}$ :

$$\Delta_k := \{(\theta_1, \dots, \theta_k) \in (\mathbb{R}_+)^k : \theta_1 \leq \dots \leq \theta_k\}, \quad 1 \leq k \leq n.$$

We introduce some notations used throughout the paper:

- $\mathcal{P}(\mathbb{F})$  (resp.  $\mathcal{P}(\mathbb{G})$ ) is the  $\sigma$ -algebra of  $\mathbb{F}$  (resp.  $\mathbb{G}$ )-predictable measurable subsets of  $\Omega \times \mathbb{R}_+$ , i.e. the  $\sigma$ -algebra generated by the left-continuous  $\mathbb{F}$  (resp.  $\mathbb{G}$ )-adapted processes.
- $\mathcal{PM}(\mathbb{F})$  (resp.  $\mathcal{PM}(\mathbb{G})$ ) is the  $\sigma$ -algebra of  $\mathbb{F}$  (resp.  $\mathbb{G}$ )-progressively measurable subsets of  $\Omega \times \mathbb{R}_+$ .
- For  $\theta = (\theta_1, \dots, \theta_n) \in \Delta_n$  and  $e = (e_1, \dots, e_n) \in E^n$ , we denote by

$$\theta_{(k)} = (\theta_1, \dots, \theta_k) \quad \text{and} \quad e_{(k)} = (e_1, \dots, e_k), \quad 1 \leq k \leq n.$$

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<sup>1</sup> $\mathcal{F}_0$  contains the  $\mathbb{P}$ -null sets and  $\mathbb{F}$  is right continuous:  $\mathcal{F}_t = \mathcal{F}_{t+} := \bigcap_{s>t} \mathcal{F}_s$ .

The following result provides the basic decomposition of predictable and progressive measurable processes with respect to this progressive enlargement of filtrations.

**Lemma 2.1.** – Any  $\mathcal{P}(\mathbb{G})$ -measurable process  $X = (X_t)_{t \geq 0}$  is represented as

$$X_t = X_t^0 \mathbb{1}_{t \leq \tau_1} + \sum_{k=1}^{n-1} X_t^k(\tau_{(k)}, \zeta_{(k)}) \mathbb{1}_{\tau_k < t \leq \tau_{k+1}} + X_t^n(\tau_{(n)}, \zeta_{(n)}) \mathbb{1}_{\tau_n < t}, \quad (2.1)$$

for all  $t \geq 0$ , where  $X^0$  is  $\mathcal{P}(\mathbb{F})$ -measurable, and  $X^k$  is  $\mathcal{P}(\mathbb{F}) \otimes \mathcal{B}(\Delta_k) \otimes \mathcal{B}(E^k)$ -measurable, for  $k = 1, \dots, n$ .

– Any càd-làg  $\mathcal{PM}(\mathbb{G})$ -measurable process  $X = (X_t)_{t \geq 0}$  is represented as

$$X_t = X_t^0 \mathbb{1}_{t < \tau_1} + \sum_{k=1}^{n-1} X_t^k(\tau_{(k)}, \zeta_{(k)}) \mathbb{1}_{\tau_k \leq t < \tau_{k+1}} + X_t^n(\tau_{(n)}, \zeta_{(n)}) \mathbb{1}_{\tau_n \leq t}, \quad (2.2)$$

for all  $t \geq 0$ , where  $X^0$  is  $\mathcal{PM}(\mathbb{F})$ -measurable, and  $X^k$  is  $\mathcal{PM}(\mathbb{F}) \otimes \mathcal{B}(\Delta_k) \otimes \mathcal{B}(E^k)$ -measurable, for  $k = 1, \dots, n$ .

The proof of the first point is given in Pham [26] and is therefore omitted. The proof of the second point is based on similar arguments. Hence, we postpone it to the appendix.

Throughout the sequel, we will use the convention  $\tau_0 = 0$ ,  $\tau_{n+1} = +\infty$ ,  $\theta_0 = 0$  and  $\theta_{n+1} = +\infty$  for any  $\theta \in \Delta_n$ , and  $X^0(\theta_{(0)}, e_{(0)}) = X^0$  to simplify the notation.

**Remark 2.1.** In the case where the studied process  $X$  depends on another parameter  $x$  evolving in a Borelian subset  $\mathcal{X}$  of  $\mathbb{R}^p$ , and if  $X$  is  $\mathcal{P}(\mathbb{G}) \otimes \mathcal{B}(\mathcal{X})$  (resp.  $\mathcal{PM}(\mathbb{G}) \otimes \mathcal{B}(\mathcal{X})$ )-measurable, then, decomposition (2.1) (resp. (2.2)) is still true but where  $X^k$  is  $\mathcal{P}(\mathbb{F}) \otimes \mathcal{B}(\Delta_k) \otimes \mathcal{B}(E^k) \otimes \mathcal{B}(\mathcal{X})$  (resp.  $\mathcal{PM}(\mathbb{F}) \otimes \mathcal{B}(\Delta_k) \otimes \mathcal{B}(E^k) \otimes \mathcal{B}(\mathcal{X})$ )-measurable. Indeed, it is obvious for the processes generating  $\mathcal{P}(\mathbb{G}) \otimes \mathcal{B}(\mathcal{X})$  (resp.  $\mathcal{PM}(\mathbb{G}) \otimes \mathcal{B}(\mathcal{X})$ ) of the form  $X_t(\omega, x) = L_t(\omega)R(x)$ ,  $(t, \omega, x) \in \mathbb{R}_+ \times \Omega \times \mathcal{X}$ , where  $L$  is  $\mathcal{P}(\mathbb{G})$  (resp.  $\mathcal{PM}(\mathbb{G})$ )-measurable and  $R$  is  $\mathcal{B}(\mathcal{X})$ -measurable. Then, the result is extended to any  $\mathcal{P}(\mathbb{G}) \otimes \mathcal{B}(\mathcal{X})$  (resp.  $\mathcal{PM}(\mathbb{G}) \otimes \mathcal{B}(\mathcal{X})$ )-measurable process by the monotone class theorem.

We shall make, throughout the sequel, the standing assumption of the semimartingale invariance property, also called **(H')**-hypothesis, i.e. any  $\mathbb{F}$ -semimartingale remains a  $\mathbb{G}$ -semimartingale. This result is related in finance to no-arbitrage conditions, and is thus also a desirable property from an economical viewpoint. We also notice that under **(H')** the stochastic integral w.r.t. the  $\mathbb{F}$ -Brownian motion  $W$  is well defined in  $\mathbb{G}$ .

We now introduce a density assumption on the random times and their associated marks by assuming that the distribution of  $(\tau_1, \dots, \tau_n, \zeta_1, \dots, \zeta_n)$  is absolutely continuous with respect to the Lebesgue measure  $d\theta de$  on  $\mathcal{B}(\Delta_n) \otimes \mathcal{B}(E^n)$ . More precisely, we assume that there exists a positive  $\mathcal{P}(\mathbb{F}) \otimes \mathcal{B}(\Delta_n) \otimes \mathcal{B}(E^n)$ -measurable map  $\gamma$  such that

$$(HD) \quad \mathbb{P}[(\tau_1, \dots, \tau_n, \zeta_1, \dots, \zeta_n) \in d\theta de | \mathcal{F}_t] = \gamma_t(\theta_1, \dots, \theta_n, e_1, \dots, e_n) d\theta_1 \dots d\theta_n de_1 \dots de_n.$$

We then introduce some notation. Define the (deterministic) process  $\gamma^0$  by

$$\gamma_t^0 = \mathbb{P}[\tau_1 > t] = \int_{\Delta_n \times E^n} \mathbb{1}_{\theta_1 > t} \gamma_t(\theta, e) d\theta de ,$$

and the map  $\gamma^k$  an  $\mathcal{P}(\mathbb{F}) \otimes \mathcal{B}(\Delta_k) \otimes \mathcal{B}(E^k)$ -measurable process,  $k = 1, \dots, n-1$ , by

$$\begin{aligned} & \gamma_t^k(\theta_1, \dots, \theta_k, e_1, \dots, e_k) \\ = & \int_{\Delta_{n-k} \times E^{n-k}} \mathbb{1}_{\theta_{k+1} > t} \gamma_t(\theta_1, \dots, \theta_n, e_1, \dots, e_n) d\theta_{k+1} \dots d\theta_n de_{k+1} \dots de_n . \end{aligned}$$

We shall use the natural convention  $\gamma^n = \gamma$ . We obtain that under (HD), the random measure admits a compensator absolutely continuous w.r.t. the Lebesgue measure. The intensity  $\lambda$  is given by the following proposition.

**Proposition 2.1.** *The random measure admits a compensator for the filtration  $\mathbb{G}$  given by  $\lambda_t(e)dedt$ , where the intensity  $\lambda$  is defined by*

$$\lambda_t(e) = \sum_{k=1}^n \lambda_t^k(e, \tau_{(k-1)}, \zeta_{(k-1)}) \mathbb{1}_{\tau_{k-1} < t \leq \tau_k} , \quad (2.3)$$

with

$$\lambda_t^k(e, \theta_{(k-1)}, e_{(k-1)}) = \frac{\gamma_t^k(\theta_{(k-1)}, t, e_{(k-1)}, e)}{\gamma_t^{k-1}(\theta_{(k-1)}, e_{(k-1)})} , \quad (\theta_{(k-1)}, t, e_{(k-1)}, e) \in \Delta_{k-1} \times \mathbb{R}_+ \times E^k .$$

The proof of Proposition 2.1 is based on similar arguments to those of [11]. We therefore postpone it to the appendix.

Throughout the sequel, we add an assumption on the intensity  $\lambda$ :

$$(HBI) \quad \text{The process } \left( \int_E \lambda_t(e) de \right)_{t \geq 0} \text{ is bounded on } [0, \infty) .$$

We now consider one dimensional BSDEs driven by  $W$  and the random measure  $\mu$ . To define solutions, we need to introduce the following spaces, where  $a, b \in \mathbb{R}_+$  with  $a \leq b$ , and  $T < \infty$  is the terminal time:

- $\mathcal{S}_{\mathbb{G}}^\infty[a, b]$  (resp.  $\mathcal{S}_{\mathbb{F}}^\infty[a, b]$ ) is the set of  $\mathcal{P}\mathcal{M}(\mathbb{G})$  (resp.  $\mathcal{P}\mathcal{M}(\mathbb{F})$ )-measurable processes  $(Y_t)_{t \in [a, b]}$  essentially bounded:

$$\|Y\|_{\mathcal{S}^\infty[a, b]} := \operatorname{ess\,sup}_{t \in [a, b]} |Y_t| < \infty .$$

- $L_{\mathbb{G}}^2[a, b]$  (resp.  $L_{\mathbb{F}}^2[a, b]$ ) is the set of  $\mathcal{P}(\mathbb{G})$  (resp.  $\mathcal{P}(\mathbb{F})$ )-measurable processes  $(Z_t)_{t \in [a, b]}$  such that

$$\|Z\|_{L^2[a, b]} := \left( \mathbb{E} \left[ \int_a^b Z_t^2 dt \right] \right)^{\frac{1}{2}} < \infty .$$

–  $L^2(\mu)$  is the set of  $\mathcal{P}(\mathbb{G}) \otimes \mathcal{B}(E)$ -measurable processes  $U$  such that

$$\|U\|_{L^2(\mu)} := \left( \mathbb{E} \left[ \int_0^T \int_E |U_s(e)|^2 \mu(de, ds) \right] \right)^{\frac{1}{2}} < \infty.$$

We then consider BSDEs of the form: find a triple  $(Y, Z, U) \in \mathcal{S}_{\mathbb{G}}^\infty[0, T] \times L_{\mathbb{G}}^2[0, T] \times L^2(\mu)$  such that<sup>2</sup>

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s, U_s) ds - \int_t^T Z_s dW_s - \int_t^T \int_E U_s(e) \mu(de, ds), \quad 0 \leq t \leq T, \quad (2.4)$$

where  $\xi$  is a  $\mathcal{G}_T$ -measurable random variable and  $f$  is a  $\mathcal{P}(\mathbb{G}) \otimes \mathcal{B}(\mathbb{R}) \otimes \mathcal{B}(\mathbb{R}^d) \otimes \mathcal{B}(\mathbb{R}^E)$ -measurable map.

### 3 Existence of a solution

In this section, we use the decompositions given by Lemma 2.1 to solve BSDEs with a finite number of jumps. We use a similar approach to Ankirchner *et al.* [1]: one can explicitly construct a solution by combining solutions of an associated recursive system of Brownian BSDEs. But contrary to them, we suppose that there exist  $n$  random times and  $n$  random marks. Our assumptions on the driver are also weaker. We first give a general existence theorem which links the studied BSDEs with jumps with a system of recursive Brownian BSDEs. We then illustrate our general result with concrete examples.

#### 3.1 The existence theorem

To prove the existence of a solution to BSDE (2.4), we introduce the decomposition of the coefficients  $\xi$  and  $f$  as given in Lemma 2.1.

Considering the càd-làg  $\mathcal{PM}(\mathbb{G})$ -measurable process  $(\xi \mathbb{1}_{t=T})_{0 \leq t \leq T}$ , we get from Lemma 2.1 the following decomposition

$$\xi = \sum_{k=0}^n \xi^k(\tau_{(k)}, \zeta_{(k)}) \mathbb{1}_{\tau_k \leq T < \tau_{k+1}}, \quad (3.1)$$

where  $\xi^0$  is  $\mathcal{F}_T$ -measurable and  $\xi^k$  is  $\mathcal{F}_T \otimes \mathcal{B}(\Delta_k) \otimes \mathcal{B}(E^k)$ -measurable for each  $k = 1, \dots, n$ . Then, using Remark 2.1, we get the following decomposition for  $f$

$$f(t, y, z, u) = \sum_{k=0}^n f^k(t, y, z, u, \tau_{(k)}, \zeta_{(k)}) \mathbb{1}_{\tau_k \leq t < \tau_{k+1}}, \quad (3.2)$$

where  $f^0$  is  $\mathcal{P}(\mathbb{F}) \otimes \mathcal{B}(\mathbb{R}) \otimes \mathcal{B}(\mathbb{R}^d) \otimes \mathcal{B}(\mathbb{R}^E)$ -measurable and  $f^k$  is  $\mathcal{P}(\mathbb{F}) \otimes \mathcal{B}(\mathbb{R}) \otimes \mathcal{B}(\mathbb{R}^d) \otimes \mathcal{B}(\mathbb{R}^E) \otimes \mathcal{B}(\Delta_k) \otimes \mathcal{B}(E^k)$ -measurable for each  $k = 1, \dots, n$ .

In the following theorem, we show how BSDEs driven by  $W$  and  $\mu$  are related to a recursive system of Brownian BSDEs involving the coefficients  $\xi^k$  and  $f^k$ ,  $k = 0, \dots, n$ .

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<sup>2</sup>The symbol  $\int_s^t$  stands for the integral on the interval  $(s, t]$  for all  $s, t \in \mathbb{R}_+$ .

**Theorem 3.1.** Assume that for all  $(\theta, e) \in \Delta_n \times E^n$ , the Brownian BSDE

$$\begin{aligned} Y_t^n(\theta, e) &= \xi^n(\theta, e) + \int_t^T f^n\left(s, Y_s^n(\theta, e), Z_s^n(\theta, e), 0, \theta, e\right) ds \\ &\quad - \int_t^T Z_s^n(\theta, e) dW_s, \quad \theta_n \wedge T \leq t \leq T, \end{aligned} \quad (3.3)$$

admits a solution  $(Y^n(\theta, e), Z^n(\theta, e)) \in \mathcal{S}_{\mathbb{F}}^\infty[\theta_n \wedge T, T] \times L_{\mathbb{F}}^2[\theta_n \wedge T, T]$ , and that for each  $k = 0, \dots, n-1$ , the Brownian BSDE

$$\begin{aligned} Y_t^k(\theta_{(k)}, e_{(k)}) &= \xi^k(\theta_{(k)}, e_{(k)}) + \int_t^T f^k\left(s, Y_s^k(\theta_{(k)}, e_{(k)}), Z_s^k(\theta_{(k)}, e_{(k)}), \right. \\ &\quad \left. Y_s^{k+1}(\theta_{(k)}, s, e_{(k)}, \cdot) - Y_s^k(\theta_{(k)}, e_{(k)}), \theta_{(k)}, e_{(k)}\right) ds \\ &\quad - \int_t^T Z_s^k(\theta_{(k)}, e_{(k)}) dW_s, \quad \theta_k \wedge T \leq t \leq T, \end{aligned} \quad (3.4)$$

admits a solution  $(Y^k(\theta_{(k)}, e_{(k)}), Z^k(\theta_{(k)}, e_{(k)})) \in \mathcal{S}_{\mathbb{F}}^\infty[\theta_k \wedge T, T] \times L_{\mathbb{F}}^2[\theta_k \wedge T, T]$ . Assume moreover that each  $Y^k$  (resp.  $Z^k$ ) is  $\mathcal{P}\mathcal{M}(\mathbb{F}) \otimes \mathcal{B}(\Delta_k) \otimes \mathcal{B}(E^k)$ -measurable (resp.  $\mathcal{P}(\mathbb{F}) \otimes \mathcal{B}(\Delta_k) \otimes \mathcal{B}(E^k)$ -measurable).

If all these solutions satisfy

$$\sup_{(k, \theta, e)} \|Y^k(\theta_{(k)}, e_{(k)})\|_{\mathcal{S}^\infty[\theta_k \wedge T, T]} < \infty, \quad (3.5)$$

and

$$\mathbb{E} \left[ \int_{\Delta_n \times E^n} \left( \int_0^{\theta_1 \wedge T} |Z_s^0|^2 ds + \sum_{k=1}^n \int_{\theta_k \wedge T}^{\theta_{k+1} \wedge T} |Z_s^k(\theta_{(k)}, e_{(k)})|^2 ds \right) \gamma_T(\theta, e) d\theta de \right] < \infty,$$

then, BSDE (2.4) admits a solution  $(Y, Z, U) \in \mathcal{S}_{\mathbb{G}}^\infty[0, T] \times L_{\mathbb{G}}^2[0, T] \times L^2(\mu)$  given by

$$\left\{ \begin{aligned} Y_t &= Y_t^0 \mathbf{1}_{t < \tau_1} + \sum_{k=1}^n Y_t^k(\tau_{(k)}, \zeta_{(k)}) \mathbf{1}_{\tau_k \leq t < \tau_{k+1}}, \\ Z_t &= Z_t^0 \mathbf{1}_{t \leq \tau_1} + \sum_{k=1}^n Z_t^k(\tau_{(k)}, \zeta_{(k)}) \mathbf{1}_{\tau_k < t \leq \tau_{k+1}}, \\ U_t(\cdot) &= U_t^0(\cdot) \mathbf{1}_{t \leq \tau_1} + \sum_{k=1}^{n-1} U_t^k(\tau_{(k)}, \zeta_{(k)}, \cdot) \mathbf{1}_{\tau_k < t \leq \tau_{k+1}}, \end{aligned} \right. \quad (3.6)$$

with  $U_t^k(\tau_{(k)}, \zeta_{(k)}, \cdot) = Y_t^{k+1}(\tau_{(k)}, t, \zeta_{(k)}, \cdot) - Y_t^k(\tau_{(k)}, \zeta_{(k)})$  for each  $k = 0, \dots, n-1$ .

**Proof.** To alleviate notation, we shall often write  $\xi^k$  and  $f^k(t, y, z, u)$  instead of  $\xi^k(\theta_{(k)}, e_{(k)})$  and  $f^k(t, y, z, u, \theta_{(k)}, e_{(k)})$ , and  $Y_t^k(t, e)$  instead of  $Y_t^k(\theta_{(k-1)}, t, e_{(k-1)}, e)$ .

**Step 1:** We prove that for  $t \in [0, T]$ ,  $(Y, Z, U)$  defined by (3.6) satisfied the equation

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s, U_s) ds - \int_t^T Z_s dW_s - \int_t^T \int_E U_s(e) \mu(de, ds). \quad (3.7)$$



We make an induction on the number  $k$  of jumps in  $(t, T]$ .

• Suppose that  $k = 0$ . We distinguish two cases.

**Case 1:** there are  $n$  jumps before  $t$ . We then have  $\tau_n \leq t$  and from (3.6) we get  $Y_t = Y_t^n$ . Using BSDE (3.3), we can see that

$$Y_t^n = \xi^n + \int_t^T f^n(s, Y_s^n, Z_s^n, 0) ds - \int_t^T Z_s^n dW_s.$$

Since  $\tau_n \leq T$ , we have  $\xi^n = \xi$  from (3.1). In the same way, we have  $Y_s = Y_s^n$ ,  $Z_s = Z_s^n$  and  $U_s = 0$  for all  $s \in (t, T]$  from (3.6). Using (3.2), we also get  $f^n(s, Y_s^n, Z_s^n, 0) = f(s, Y_s, Z_s, U_s)$  for all  $s \in (t, T]$ . Hence, we have

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s, U_s) ds - \int_t^T Z_s dW_s - \int_t^T \int_E U_s(e) \mu(de, ds).$$

**Case 2:** there are  $i$  jumps before  $t$  with  $i < n$ . Since there is no jump after  $t$ , we have  $Y_s = Y_s^i$ ,  $Z_s = Z_s^i$ ,  $U_s^i(\cdot) = Y_s^{i+1}(s, \cdot) - Y_s^i$ ,  $\xi = \xi^i$  and  $f^i(s, Y_s^i, Z_s^i, U_s^i) = f(s, Y_s, Z_s, U_s)$  for all  $s \in (t, T]$ , and  $\int_t^T \int_E U_s(e) \mu(de, ds) = 0$ . Combining these equalities with (3.4), we get

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s, U_s) ds - \int_t^T Z_s dW_s - \int_t^T \int_E U_s(e) \mu(de, ds).$$

• Suppose equation (3.7) holds true when there are  $k$  jumps in  $(t, T]$ , and consider the case where there are  $k + 1$  jumps in  $(t, T]$ .

Denote by  $i$  the number of jumps in  $[0, t]$ . Then, we have  $Z_s = Z_s^i$ ,  $U_s^i(\cdot) = Y_s^{i+1}(s, \cdot) - Y_s^i$  for all  $s \in (t, \tau_{i+1}]$ , and  $Y_s = Y_s^i$  and  $f(s, Y_s, Z_s, U_s) = f^i(s, Y_s^i, Z_s^i, U_s^i)$  for all  $s \in (t, \tau_{i+1}]$ . Using (3.2), we have

$$\begin{aligned} Y_t &= Y_{\tau_{i+1}}^i + \int_t^{\tau_{i+1}} f(s, Y_s, Z_s, U_s) ds - \int_t^{\tau_{i+1}} Z_s dW_s \\ &= Y_{\tau_{i+1}}^{i+1} + \int_t^{\tau_{i+1}} f(s, Y_s, Z_s, U_s) ds - \int_t^{\tau_{i+1}} Z_s dW_s \\ &\quad - \int_t^{\tau_{i+1}} \int_E U_s(e) \mu(de, ds). \end{aligned} \quad (3.8)$$

Using the induction assumption on  $(\tau_{i+1}, T]$ , we have

$$Y_{\tau_{i+1}}^{i+1} = \xi + \int_{\tau_{i+1}}^T f(s, Y_s, Z_s, U_s) ds - \int_{\tau_{i+1}}^T Z_s dW_s - \int_{\tau_{i+1}}^T \int_E U_s(e) \mu(de, ds). \quad (3.9)$$

Combining (3.8) and (3.9), we get (3.7).

**Step 2:** Notice that the process  $Y$  (resp.  $Z, U$ ) is  $\mathcal{PM}(\mathbb{G})$  (resp.  $\mathcal{P}(\mathbb{G})$ ,  $\mathcal{P}(\mathbb{G}) \otimes \mathcal{B}(E)$ )-measurable since each  $Y^k$  (resp.  $Z^k$ ) is  $\mathcal{PM}(\mathbb{F}) \otimes \mathcal{B}(\Delta_k) \otimes \mathcal{B}(E^k)$  (resp.  $\mathcal{P}(\mathbb{F}) \otimes \mathcal{B}(\Delta_k) \otimes \mathcal{B}(E^k)$ )-measurable.

**Step 3:** We now prove that the solution satisfies the integrability conditions. Suppose that the processes  $Y^k$ ,  $k = 0, \dots, n$ , satisfy (3.5). By definition of  $Y$ , we have

$$\begin{aligned} \operatorname{ess\,sup}_{t \in [0, T]} |Y_t| &\leq \operatorname{ess\,sup}_{(t, \theta, e) \in [0, T] \times \Delta_n \times E^n} \left| Y_t^0 \mathbb{1}_{t < \theta_1} + \sum_{k=1}^n Y_t^k(\theta^{(k)}, e^{(k)}) \mathbb{1}_{\theta_k \leq t < \theta_{k+1}} \right|, \\ &\leq \|Y^0\|_{\mathcal{S}^\infty[0, T]} + \sum_{k=1}^n \sup_{(\theta^{(k)}, e^{(k)}) \in \Delta_k \times E^k} \|Y^k(\theta^{(k)}, e^{(k)})\|_{\mathcal{S}^\infty[\theta_k \wedge T, T]}. \end{aligned}$$

Thus,  $Y \in \mathcal{S}_{\mathbb{G}}^{\infty}[0, T]$  since the processes  $Y^k$ ,  $k = 0, \dots, n$ , satisfy

$$\sup_{(k, \theta, e) \in \{0, \dots, n\} \times \Delta_n \times E^n} \|Y^k(\theta^{(k)}, e^{(k)})\|_{\mathcal{S}^{\infty}[\theta_k \wedge T, T]} < \infty.$$

In the same way, using (HD) and the tower property of conditional expectation, we get

$$\mathbb{E} \left[ \int_0^T |Z_s|^2 ds \right] = \mathbb{E} \left[ \int_{\Delta_n \times E^n} \left( \int_0^{\theta_1 \wedge T} |Z_s^0|^2 ds + \sum_{k=1}^n \int_{\theta_k \wedge T}^{\theta_{k+1} \wedge T} |Z_s^k(\theta^{(k)}, e^{(k)})|^2 ds \right) \gamma_T(\theta, e) d\theta de \right].$$

Thus,  $Z \in L_{\mathbb{G}}^2[0, T]$  since the processes  $Z^k$ ,  $k = 0, \dots, n$ , satisfy

$$\mathbb{E} \left[ \int_{\Delta_n \times E^n} \left( \int_0^{\theta_1 \wedge T} |Z_s^0|^2 ds + \sum_{k=1}^n \int_{\theta_k \wedge T}^{\theta_{k+1} \wedge T} |Z_s^k(\theta^{(k)}, e^{(k)})|^2 ds \right) \gamma_T(\theta, e) d\theta de \right] < \infty.$$

Finally, we check that  $U \in L^2(\mu)$ . Using (HD), we have

$$\begin{aligned} \|U\|_{L^2(\mu)}^2 &= \sum_{k=1}^n \int_{\Delta_n \times E^n} \mathbb{E} \left[ |Y_{\theta_k}^k(\theta^{(k)}, e^{(k)}) - Y_{\theta_k}^{k-1}(\theta^{(k-1)}, e^{(k-1)})|^2 \gamma_T(\theta, e) \right] d\theta de \\ &\leq 2 \sum_{k=1}^n \left( \|Y^k(\theta^{(k)}, e^{(k)})\|_{\mathcal{S}^{\infty}[\theta_k \wedge T, T]}^2 + \|Y^{k-1}(\theta^{(k-1)}, e^{(k-1)})\|_{\mathcal{S}^{\infty}[\theta_{k-1} \wedge T, T]}^2 \right) \\ &< \infty. \end{aligned}$$

Hence,  $U \in L^2(\mu)$ . □

### 3.2 Application to quadratic BSDEs with jumps

We suppose that the random variable  $\xi$  and the generator  $f$  satisfy the following conditions:

(HEQ1) The random variable  $\xi$  is bounded: there exists a constant such that

$$|\xi| \leq C, \quad \mathbb{P} - a.s.$$

(HEQ2) The generator  $f$  is quadratic in  $z$ : there exists a constant  $C$  such that

$$|f(t, y, z, u)| \leq C \left( 1 + |y| + |z|^2 + \int_E |u(e)| \lambda_t(e) de \right),$$

for all  $(t, y, z, u) \in [0, T] \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^E$ .

**Proposition 3.1.** *Under (HBI), (HEQ1) and (HEQ2), BSDE (2.4) admits a solution in  $\mathcal{S}_{\mathbb{G}}^{\infty}[0, T] \times L_{\mathbb{G}}^2[0, T] \times L^2(\mu)$ .*

**Proof. Step 1.** Since  $\xi$  is a bounded random variable, we can choose  $\xi^k$  bounded for each  $k = 0, \dots, n$ . Indeed, let  $C$  be a positive constant such that  $|\xi| \leq C$ ,  $\mathbb{P} - a.s.$ , then, we have

$$\xi = \sum_{k=0}^n \tilde{\xi}^k(\tau_1, \dots, \tau_k, \zeta_1, \dots, \zeta_k) \mathbb{1}_{\tau_k \leq T < \tau_{k+1}},$$

with  $\tilde{\xi}^k(\tau_1, \dots, \tau_k, \zeta_1, \dots, \zeta_k) = (\xi^k(\tau_1, \dots, \tau_k, \zeta_1, \dots, \zeta_k) \wedge C) \vee (-C)$ , for each  $k = 1, \dots, n$ .

**Step 2.** Since  $f$  is quadratic in  $z$ , it is possible to choose the functions  $f^k$ ,  $k = 0, \dots, n$ , quadratic in  $z$ . Indeed, if  $C$  is a positive constant such that  $|f(t, y, z, u)| \leq C(1 + |y| + |z|^2 + \int_E |u(e)| \lambda_t(e) de)$ , for all  $(t, y, z, u) \in [0, T] \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^E$ ,  $\mathbb{P} - a.s.$  and  $f$  has the following decomposition

$$f(t, y, z, u) = \sum_{k=0}^n f^k(t, y, z, u, \tau_k, \zeta_k) \mathbb{1}_{\tau_k \leq t < \tau_{k+1}},$$

then,  $f$  satisfies the same decomposition with  $\tilde{f}^k$  instead of  $f^k$  where

$$\begin{aligned} \tilde{f}^k(t, y, z, u, \theta_{(k)}, e_{(k)}) &= f^k(t, y, z, u, \theta_{(k)}, e_{(k)}) \wedge \left( C \left( 1 + |y| + |z|^2 + \int_E |u(e)| \lambda_t(e) de \right) \right) \\ &\quad \vee \left( -C \left( 1 + |y| + |z|^2 + \int_E |u(e)| \lambda_t(e) de \right) \right), \end{aligned}$$

for all  $(t, y, z, u) \in [0, T] \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^E$  and  $(\theta, e) \in \Delta_n \times E^n$ .

**Step 3.** We now prove by a backward induction that there exists for each  $k = 0, \dots, n-1$  (resp.  $k = n$ ), a solution  $(Y^k, Z^k)$  to BSDE (3.4) (resp. (3.3)) s.t.  $Y^k$  (resp.  $Z^k$ ) is a  $\mathcal{P}\mathcal{M}(\mathbb{F}) \otimes \mathcal{B}(\Delta_k) \otimes \mathcal{B}(E^k)$  (resp.  $\mathcal{P}(\mathbb{F}) \otimes \mathcal{B}(\Delta_k) \otimes \mathcal{B}(E^k)$ )-measurable process and

$$\sup_{(\theta_{(k)}, e_{(k)}) \in \Delta_k \times E^k} \|Y^k(\theta_{(k)}, e_{(k)})\|_{\mathcal{S}^\infty[\theta_k \wedge T, T]} + \|Z^k(\theta_{(k)}, e_{(k)})\|_{L^2[\theta_k \wedge T, T]} < \infty.$$

• Choosing  $\xi^n(\theta_n, e_n)$  bounded as in Step 1, we get from Theorem 2.3 of Kobylanski [20] the existence of a solution  $(Y^n(\theta_n, e_n), Z^n(\theta_n, e_n))$  to BSDE (3.3).

We now check that we can choose  $Y^n$  (resp.  $Z^n$ ) as a  $\mathcal{P}\mathcal{M}(\mathbb{F}) \otimes \mathcal{B}(\Delta_n) \otimes \mathcal{B}(E^n)$  (resp.  $\mathcal{P}(\mathbb{F}) \otimes \mathcal{B}(\Delta_n) \otimes \mathcal{B}(E^n)$ )-measurable process. Indeed, we know (see [20]) that we can construct the solution  $(Y^n, Z^n)$  as limits of solutions to Lipschitz BSDEs. From Proposition C.1, we then get a  $\mathcal{P}(\mathbb{F}) \otimes \mathcal{B}(\Delta_n) \otimes \mathcal{B}(E^n)$ -measurable solution as limit of  $\mathcal{P}(\mathbb{F}) \otimes \mathcal{B}(\Delta_n) \otimes \mathcal{B}(E^n)$ -measurable processes. Hence,  $Y$  (resp.  $Z$ ) is a  $\mathcal{P}\mathcal{M}(\mathbb{F}) \otimes \mathcal{B}(\Delta_n) \otimes \mathcal{B}(E^n)$  (resp.  $\mathcal{P}(\mathbb{F}) \otimes \mathcal{B}(\Delta_n) \otimes \mathcal{B}(E^n)$ )-measurable process. Applying Proposition 2.1 of [20] to  $(Y^n, Z^n)$ , we get from (HEQ1) and (HEQ2)

$$\sup_{(\theta, e) \in \Delta_n \times E^n} \|Y^n(\theta_n, e_n)\|_{\mathcal{S}^\infty} + \|Z^n(\theta_n, e_n)\|_{L^2(W)} < \infty.$$

• Suppose that the result holds true for  $k+1$ : there exists  $(Y^{k+1}, Z^{k+1})$  such that

$$\begin{aligned} \sup_{(\theta_{(k+1)}, e_{(k+1)}) \in \Delta_{k+1} \times E^{k+1}} &\left\{ \|Y^{k+1}(\theta_{(k+1)}, e_{(k+1)})\|_{\mathcal{S}^\infty[\theta_{k+1} \wedge T, T]} \right. \\ &\quad \left. + \|Z^{k+1}(\theta_{(k+1)}, e_{(k+1)})\|_{L^2[\theta_{k+1} \wedge T, T]} \right\} < \infty. \end{aligned}$$

Then, using (HBI), there exists a constant  $C > 0$  such that

$$\left| f^k \left( s, y, z, Y_s^{k+1}(\theta_{(k)}, s, e_{(k)}, \cdot) - y, \theta_{(k)}, e_{(k)} \right) \right| \leq C(1 + |y| + |z|^2).$$

Choosing  $\xi^k(\theta_{(k)}, e_{(k)})$  bounded as in Step 1, we get from Theorem 3.1 of Kobylanski [20] the existence of a solution  $(Y^k(\theta_{(k)}, e_{(k)}), Z^k(\theta_{(k)}, e_{(k)}))$ .

As for  $k = n$ , we can choose  $Y^k$  (resp.  $Z^k$ ) as a  $\mathcal{P}\mathcal{M}(\mathbb{F}) \otimes \mathcal{B}(\Delta_k) \otimes \mathcal{B}(E^k)$  (resp.  $\mathcal{P}(\mathbb{F}) \otimes \mathcal{B}(\Delta_k) \otimes \mathcal{B}(E^k)$ )-measurable process.

Applying Proposition 2.1 of [20] to  $(Y^k(\theta_{(k)}, e_{(k)}), Z^k(\theta_{(k)}, e_{(k)}))$ , we get from (HEQ1) and (HEQ2)

$$\sup_{(\theta_{(k)}, e_{(k)}) \in \Delta_k \times E^k} \|Y^k(\theta_{(k)}, e_{(k)})\|_{\mathcal{S}^\infty[\theta_k \wedge T, T]} + \|Z^k(\theta_{(k)}, e_{(k)})\|_{L^2[\theta_k \wedge T, T]} < \infty .$$

**Step 4.** From Step 3, we can apply Theorem 3.1. We then get the existence of a solution to BSDE (2.4).  $\square$

### 3.3 Application to the pricing of a European option in a complete market with a jump

In this example, we assume that there is a single random time  $\tau$  representing the time occurrence of a shock in the prices on the market. We denote by  $N$  the associated pure jump process:

$$N_t = \mathbb{1}_{\tau \leq t}, \quad 0 \leq t \leq T .$$

We consider a financial market which consists of

- a non-risky asset  $S^0$ , whose strictly positive price process is defined by

$$dS_t^0 = r_t S_t^0 dt, \quad 0 \leq t \leq T, \quad S_0^0 = 1,$$

with  $r_t \geq 0$ , for all  $t \in [0, T]$ ,

- two risky assets with respective price processes  $S^1$  and  $S^2$  defined by

$$dS_t^1 = S_{t-}^1 (b_t dt + \sigma_t dW_t + \beta dN_t), \quad 0 \leq t \leq T, \quad S_0^1 = s_0^1,$$

and

$$dS_t^2 = S_t^2 (\bar{b}_t dt + \bar{\sigma}_t dW_t), \quad 0 \leq t \leq T, \quad S_0^2 = s_0^2,$$

with  $\sigma_t > 0$  and  $\bar{\sigma}_t > 0$  and  $\beta > -1$  (to ensure that the price process  $S^1$  always remains strictly positive).

We make the following assumption which ensures the existence of the processes  $S^0$ ,  $S^1$ , and  $S^2$ :

(HB) The coefficients  $r, b, \bar{b}, \sigma, \bar{\sigma}, \frac{1}{\sigma}$  and  $\frac{1}{\bar{\sigma}}$  are bounded: there exists a constant  $C$  s.t.

$$|r_t| + |b_t| + |\bar{b}_t| + |\sigma_t| + |\bar{\sigma}_t| + \left| \frac{1}{\sigma_t} \right| + \left| \frac{1}{\bar{\sigma}_t} \right| \leq C, \quad 0 \leq t \leq T, \quad \mathbb{P} - a.s.$$

We assume that the coefficients  $r, b, \bar{b}, \sigma$  and  $\bar{\sigma}$  have the following forms

$$\begin{cases} r_t = r^0 \mathbb{1}_{t < \tau} + r^1(\tau) \mathbb{1}_{t \geq \tau}, \\ b_t = b^0 \mathbb{1}_{t < \tau} + b^1(\tau) \mathbb{1}_{t \geq \tau}, \\ \bar{b}_t = \bar{b}^0 \mathbb{1}_{t < \tau} + \bar{b}^1(\tau) \mathbb{1}_{t \geq \tau}, \\ \sigma_t = \sigma^0 \mathbb{1}_{t < \tau} + \sigma^1(\tau) \mathbb{1}_{t \geq \tau}, \\ \bar{\sigma}_t = \bar{\sigma}^0 \mathbb{1}_{t < \tau} + \bar{\sigma}^1(\tau) \mathbb{1}_{t \geq \tau}. \end{cases}$$

The aim of this subsection is to provide an explicit price for any bounded  $\mathcal{G}_T$ -measurable European option  $\xi$ , together with a replicating strategy  $\pi = (\pi^0, \pi^1, \pi^2)$  ( $\pi_t^i$  corresponds to the number of share of  $S^i$  held at time  $t$ ). We introduce the following proportionality assumption which ensures the viability of the market (no free lunch), see [6]:

(HP) The following proportionality relation holds true

$$\frac{r^1(\theta) - \mu^1(\theta)}{\sigma^1(\theta)} = \frac{r^1(\theta) - \bar{\mu}^1(\theta)}{\bar{\sigma}^1(\theta)}, \quad \theta \in \mathbb{R}_+, .$$

Let  $\pi = (\pi^0, \pi^1, \pi^2)$  be a  $\mathcal{P}(\mathbb{G})$ -measurable self-financing strategy. The wealth process  $Y$  associated to this strategy satisfies

$$Y_t = \pi_t^0 S_t^0 + \pi_t^1 S_t^1 + \pi_t^2 S_t^2, \quad 0 \leq t \leq T. \quad (3.10)$$

Since  $\pi$  is a self-financing strategy, we have

$$dY_t = \pi_t^0 dS_t^0 + \pi_t^1 dS_t^1 + \pi_t^2 dS_t^2, \quad 0 \leq t \leq T.$$

Combining this last equation with (3.10), we get

$$\begin{aligned} dY_t &= (r_t Y_t + (b_t - r_t) \pi_t^1 S_t^1 + (\bar{b}_t - r_t) \pi_t^2 S_t^2) dt \\ &\quad + (\pi_t^1 \sigma_t S_t^1 + \pi_t^2 \bar{\sigma}_t S_t^2) dW_t + \pi_t^1 \beta S_t^1 dN_t, \quad 0 \leq t \leq T. \end{aligned} \quad (3.11)$$

Define the predictable processes  $Z$  and  $U$  by

$$Z_t = \pi_t^1 \sigma_t S_t^1 + \pi_t^2 \bar{\sigma}_t S_t^2 \quad \text{and} \quad U_t = \pi_t^1 \beta S_t^1, \quad 0 \leq t \leq T. \quad (3.12)$$

Then, (3.11) can be written under the form

$$dY_t = \left[ r_t Y_t - \frac{r_t - \bar{b}_t}{\bar{\sigma}_t} Z_t - \left( \frac{r_t - b_t}{\beta} - \frac{\sigma_t (r_t - \bar{b}_t)}{\beta \bar{\sigma}_t} \right) U_t \right] dt + Z_t dW_t + U_t dN_t, \quad 0 \leq t \leq T.$$

Therefore, the problem of valuing and hedging of the contingent claim  $\xi$  consists in solving the following BSDE

$$\begin{cases} -dY_t = \left[ \frac{r_t - \bar{b}_t}{\bar{\sigma}_t} Z_t + \left( \frac{r_t - b_t}{\beta} - \frac{\sigma_t (r_t - \bar{b}_t)}{\beta \bar{\sigma}_t} \right) U_t - r_t Y_t \right] dt \\ \quad - Z_t dW_t - U_t dN_t, \quad 0 \leq t \leq T, \\ Y_T = \xi. \end{cases} \quad (3.13)$$

The recursive system of Brownian BSDEs associated to (3.13) is then given by

$$\begin{cases} -dY_t^1(\theta) &= \left[ \frac{r^1(\theta) - \bar{b}^1(\theta)}{\bar{\sigma}^1(\theta)} Z_t^1(\theta) - r^1(\theta) Y_t^1(\theta) \right] dt - Z_t^1(\theta) dW_t, \quad \theta \leq t \leq T, \\ Y_T^1(\theta) &= \xi^1(\theta), \end{cases} \quad (3.14)$$

and

$$\begin{cases} -dY_t^0 &= \left[ \frac{r^0 - \bar{b}^0}{\bar{\sigma}^0} Z_t + \left( \frac{r^0 - b^0}{\beta} - \frac{\sigma^0(r^0 - \bar{b}^0)}{\beta \bar{\sigma}^0} \right) (Y_t^1(t) - Y_t^0) - r^0 Y_t^0 \right] dt \\ &\quad - Z_t dW_t, \quad 0 \leq t \leq T, \\ Y_T^0 &= \xi^0. \end{cases} \quad (3.15)$$

**Proposition 3.2.** *Under (HB), BSDE (3.13) admits a solution in  $\mathcal{S}_{\mathbb{G}}^{\infty}[0, T] \times L_{\mathbb{G}}^2[0, T] \times L^2(\mu)$ .*

**Proof.** Using the same argument as in Step 1 of the proof of Proposition 3.1, we can assume w.l.o.g. that the coefficients of BSDEs (3.14) and (3.15) are bounded. Then, BSDE (3.14) is a linear BSDE with bounded coefficients and a bounded terminal condition. From Theorem 2.3 in [20], we get the existence of a solution  $(Y^1(\theta), Z^1(\theta))$  in  $\mathcal{S}_{\mathbb{F}}^{\infty}[\theta, T] \times L_{\mathbb{F}}^2[\theta, T]$  to (3.14) for all  $\theta \in [0, T]$ . Moreover, from Proposition 2.1 in [20], we have

$$\sup_{\theta \in [0, T]} \|Y^1(\theta)\|_{\mathcal{S}^{\infty}[\theta, T]} < \infty. \quad (3.16)$$

Applying Proposition C.1 with  $\mathcal{X} = [0, T]$  and  $d\rho(\theta) = \gamma_0(\theta)d\theta$  we can choose the solution  $(Y^1, Z^1)$  as a  $\mathcal{P}(\mathbb{F}) \otimes \mathcal{B}([0, T])$ -measurable process.

Estimate (3.16) gives that BSDE (3.15) is also a linear BSDE with bounded coefficients. Applying Theorem 2.3 and Proposition 2.1 in [20] as previously, we get the existence of a solution  $(Y^0, Z^0)$  in  $\mathcal{S}_{\mathbb{F}}^{\infty}[0, T] \times L_{\mathbb{F}}^2[0, T]$  to (3.15). Applying Theorem 3.1, we get the result.  $\square$

Since BSDEs (3.14) and (3.15) are linear, we have explicit formulae for the solutions. For  $Y^1(\theta)$ , we get:

$$Y_t^1(\theta) = \frac{1}{\Gamma_t^1(\theta)} \mathbb{E} \left[ \xi^1(\theta) \Gamma_T^1(\theta) \middle| \mathcal{F}_t \right], \quad \theta \leq t \leq T,$$

with  $\Gamma^1(\theta)$  defined by

$$\Gamma_t^1(\theta) = \exp \left( \frac{r^1(\theta) - \bar{b}^1(\theta)}{\bar{\sigma}^1(\theta)} W_t - \frac{1}{2} \left| \frac{r^1(\theta) - \bar{b}^1(\theta)}{\bar{\sigma}^1(\theta)} \right|^2 t - r^1(\theta) t \right), \quad \theta \leq t \leq T.$$

For  $Y^0$ , we get :

$$Y_t^0 = \frac{1}{\Gamma_t^0} \mathbb{E} \left[ \xi^0 \Gamma_T^0 + \int_t^T c_s \Gamma_s^0 ds \middle| \mathcal{F}_t \right], \quad 0 \leq t \leq T,$$

with  $\Gamma^0$  defined by

$$\Gamma_t^0 = \exp \left( \int_0^t d_s dW_s - \frac{1}{2} \int_0^t |d_s|^2 ds + \int_0^t a_s ds \right), \quad 0 \leq t \leq T,$$

where the parameters  $a$ ,  $d$  and  $c$  are given by

$$\begin{cases} a_t = -r^0 - \left( \frac{r^0 - b^0}{\beta} - \frac{\sigma^0(r^0 - \bar{b}^0)}{\beta\bar{\sigma}^0} \right), \\ d_t = \frac{r^0 - \bar{b}^0}{\bar{\sigma}^0}, \\ c_t = \left( \frac{r^0 - b^0}{\beta} - \frac{\sigma^0(r^0 - \bar{b}^0)}{\beta\bar{\sigma}^0} \right) Y_t^1(t). \end{cases}$$

The price at time  $t$  of the European option  $\xi$  is equal to  $Y_t^0$  if  $t < \tau$  and  $Y_t^1(\tau)$  if  $t \geq \tau$ . Once we know the processes  $Y$  and  $Z$ , a hedging strategy  $\pi = (\pi^0, \pi^1, \pi^2)$  is given by (3.10) and (3.12).

## 4 Uniqueness

In this section, we provide a uniqueness result based on a comparison theorem. We first provide a general comparison theorem which allows to compare solutions to the studied BSDEs as soon as we can compare solutions to the associated system of recursive Brownian BSDEs. We then illustrate our general result with a concrete example in a convex framework.

### 4.1 The general comparison Theorem

We consider two BSDEs with coefficients  $(\underline{f}, \underline{\xi})$  and  $(\bar{f}, \bar{\xi})$ . We denote by  $(\underline{Y}, \underline{Z}, \underline{U})$  and  $(\bar{Y}, \bar{Z}, \bar{U})$  their respective solutions in  $\mathcal{S}_{\mathbb{G}}^{\infty}[0, T] \times L_{\mathbb{G}}^2[0, T] \times L^2(\mu)$ . We consider the decomposition  $(\underline{Y}^k)_{0 \leq k \leq n}$  (resp.  $(\bar{Y}^k)_{0 \leq k \leq n}$ ,  $(\underline{Z}^k)_{0 \leq k \leq n}$ ,  $(\bar{Z}^k)_{0 \leq k \leq n}$ ,  $(\underline{U}^k)_{0 \leq k \leq n}$ ,  $(\bar{U}^k)_{0 \leq k \leq n}$ ) of  $\underline{Y}$  (resp.  $\bar{Y}$ ,  $\underline{Z}$ ,  $\bar{Z}$ ,  $\underline{U}$ ,  $\bar{U}$ ). For ease of notation, we shall write  $\underline{F}^k(t, y, z)$  and  $\bar{F}^k(t, y, z)$  instead of  $\underline{f}(t, y, z, \underline{Y}_t^{k+1}(\tau_{(k)}, t, \zeta_{(k)}, \cdot) - y)$  and  $\bar{f}(t, y, z, \bar{Y}_t^{k+1}(\tau_{(k)}, t, \zeta_{(k)}, \cdot) - y)$  for each  $k = 0, \dots, n-1$ , and  $\underline{F}^n(t, y, z)$  and  $\bar{F}^n(t, y, z)$  instead of  $\underline{f}(t, y, z, 0)$  and  $\bar{f}(t, y, z, 0)$ .

We shall make, throughout the sequel, the standing assumption known as **(H)**-hypothesis:

(HC) Any  $\mathbb{F}$ -martingale remains a  $\mathbb{G}$ -martingale.

**Remark 4.1.** Since  $W$  is an  $\mathbb{F}$ -Brownian motion, we get under (HC) that it remains a  $\mathbb{G}$ -Brownian motion. Indeed, using (HC), we have that  $W$  is a  $\mathbb{G}$ -local martingale with quadratic variation  $\langle W, W \rangle_t = t$ . Applying Lévy's characterization of Brownian motion (see e.g. Theorem 39 in [27]), we obtain that  $W$  remains a  $\mathbb{G}$ -Brownian motion.

**Definition 4.1.** We say that a generator  $g : \Omega \times [0, T] \times \mathbb{R} \times \mathbb{R}^d \rightarrow \mathbb{R}$  satisfies a comparison theorem for Brownian BSDEs if for any bounded  $\mathbb{G}$ -stopping times  $\nu_2 \geq \nu_1$ , any generator  $g' : \Omega \times [0, T] \times \mathbb{R} \times \mathbb{R}^d \rightarrow \mathbb{R}$  and any  $\mathcal{G}_{\nu_2}$ -measurable r.v.  $\zeta$  and  $\zeta'$  such that  $g \leq g'$  and  $\zeta \leq \zeta'$  (resp.  $g \geq g'$  and  $\zeta \geq \zeta'$ ), we have  $Y \leq Y'$  (resp.  $Y \geq Y'$ ) on  $[\nu_1, \nu_2]$ . Here,  $(Y, Z)$  and  $(Y', Z')$  are solutions in  $\mathcal{S}_{\mathbb{G}}^{\infty}[0, T] \times L_{\mathbb{G}}^2[0, T]$  to BSDEs with data  $(\zeta, g)$  and  $(\zeta', g')$ :

$$Y_t = \zeta + \int_t^{\nu_2} g(s, Y_s, Z_s) ds - \int_t^{\nu_2} Z_s dW_s, \quad \nu_1 \leq t \leq \nu_2,$$

and

$$Y'_t = \zeta' + \int_t^{\nu_2} g'(s, Y'_s, Z'_s) ds - \int_t^{\nu_2} Z'_s dW_s, \quad \nu_1 \leq t \leq \nu_2.$$

We can state the general comparison theorem.

**Theorem 4.1.** *Suppose that  $\underline{\xi} \leq \bar{\xi}$ ,  $\mathbb{P}$ -a.s. Suppose moreover that for each  $k = 0, \dots, n$*

$$\underline{F}^k(t, y, z) \leq \bar{F}^k(t, y, z), \quad \forall (t, y, z) \in [0, T] \times \mathbb{R} \times \mathbb{R}^d, \quad \mathbb{P} - a.s.,$$

*and the generators  $\bar{F}^k$  or  $\underline{F}^k$  satisfy a comparison theorem for Brownian BSDEs. Then, if  $\bar{U}_t = \underline{U}_t = 0$  for  $t > \tau_n$ , we have under (HC)*

$$\underline{Y}_t \leq \bar{Y}_t, \quad 0 \leq t \leq T, \quad \mathbb{P} - a.s.$$

**Proof.** The proof is performed in four steps. We first identify the BSDEs of which the terms appearing in the decomposition of  $\bar{Y}$  and  $\underline{Y}$  are solutions in the filtration  $\mathbb{G}$ . We then modify  $\bar{Y}^k$  and  $\underline{Y}^k$  outside of  $[\tau_k, \tau_{k+1})$  to get càd-làg processes for each  $k = 0, \dots, n$ . We then compare the modified processes by killing their jumps. Finally, we retrieve a comparison for the initial processes since the modification has happened outside of  $[\tau_k, \tau_{k+1})$  (where they coincide with  $\bar{Y}$  and  $\underline{Y}$ ).

**Step 1.** Since  $(\bar{Y}, \bar{Z}, \bar{U})$  (resp.  $(\underline{Y}, \underline{Z}, \underline{U})$ ) is solution to the BSDE with parameters  $(\bar{\xi}, \bar{f})$  (resp.  $(\underline{\xi}, \underline{f})$ ), we obtain from the decomposition in the filtration  $\mathbb{F}$  that  $(\bar{Y}^n, \bar{Z}^n)$  (resp.  $(\underline{Y}^n, \underline{Z}^n)$ ) is solution to

$$\begin{aligned} \bar{Y}_t^n(\tau(n), \zeta(n)) &= \bar{\xi} + \int_t^T \bar{F}^n(s, \bar{Y}_s^n(\tau(n), \zeta(n)), \bar{Z}_s^n(\tau(n), \zeta(n))) ds \\ &\quad - \int_t^T \bar{Z}_s^n(\tau(n), \zeta(n)) dW_s, \quad \tau_n \wedge T \leq t \leq T, \end{aligned} \quad (4.1)$$

$$\begin{aligned} \text{(resp. } \underline{Y}_t^n(\tau(n), \zeta(n)) &= \underline{\xi} + \int_t^T \underline{F}^n(s, \underline{Y}_s^n(\tau(n), \zeta(n)), \underline{Z}_s^n(\tau(n), \zeta(n))) ds \\ &\quad - \int_t^T \underline{Z}_s^n(\tau(n), \zeta(n)) dW_s, \quad \tau_n \wedge T \leq t \leq T) \end{aligned} \quad (4.2)$$

and  $(\bar{Y}^k, \bar{Z}^k)$  (resp.  $(\underline{Y}^k, \underline{Z}^k)$ ) is solution to

$$\begin{aligned} \bar{Y}_t^k(\tau(k), \zeta(k)) &= [\bar{Y}_{\tau_{k+1}}^{k+1}(\tau(k+1), \zeta(k+1)) - \bar{U}_{\tau_{k+1}}(\zeta(k+1))] \mathbb{1}_{\tau_{k+1} \leq T} + \bar{\xi} \mathbb{1}_{\tau_{k+1} > T} \\ &\quad + \int_t^{\tau_{k+1} \wedge T} \bar{F}^k(s, \bar{Y}_s^k(\tau(k), \zeta(k)), \bar{Z}_s^k(\tau(k), \zeta(k))) ds \\ &\quad - \int_t^{\tau_{k+1} \wedge T} \bar{Z}_s^k(\tau(k), \zeta(k)) dW_s, \quad \tau_k \wedge T \leq t < \tau_{k+1} \wedge T, \end{aligned} \quad (4.3)$$

$$\begin{aligned} \text{(resp. } \underline{Y}_t^k(\tau(k), \zeta(k)) &= [\underline{Y}_{\tau_{k+1}}^{k+1}(\tau(k+1), \zeta(k+1)) - \underline{U}_{\tau_{k+1}}(\zeta(k+1))] \mathbb{1}_{\tau_{k+1} \leq T} + \underline{\xi} \mathbb{1}_{\tau_{k+1} > T} \\ &\quad + \int_t^{\tau_{k+1} \wedge T} \underline{F}^k(s, \underline{Y}_s^k(\tau(k), \zeta(k)), \underline{Z}_s^k(\tau(k), \zeta(k))) ds \\ &\quad - \int_t^{\tau_{k+1} \wedge T} \underline{Z}_s^k(\tau(k), \zeta(k)) dW_s, \quad \tau_k \wedge T \leq t < \tau_{k+1} \wedge T) \end{aligned} \quad (4.4)$$



for each  $k = 0, \dots, n-1$ .

**Step 2.** We introduce a family of processes  $(\tilde{Y}^k)_{0 \leq k \leq n}$  (resp.  $(\underline{\tilde{Y}}^k)_{0 \leq k \leq n}$ ). We define it recursively by

$$\tilde{Y}_t^n = \bar{Y}_t^n(\tau_{(n)}, \zeta_{(n)}) \mathbf{1}_{t \geq \tau_n} \quad (\text{resp. } \underline{\tilde{Y}}_t^n = \underline{Y}_t^n(\tau_{(n)}, \zeta_{(n)}) \mathbf{1}_{t \geq \tau_n}), \quad 0 \leq t \leq T,$$

and for  $k = 0, \dots, n-1$

$$\begin{aligned} \tilde{Y}_t^k &= \bar{Y}_t^k(\tau_{(k)}, \zeta_{(k)}) \mathbf{1}_{\tau_k \leq t < \tau_{k+1}} + \tilde{Y}_t^{k+1} \mathbf{1}_{t \geq \tau_{k+1}} \\ (\text{resp. } \underline{\tilde{Y}}_t^k &= \underline{Y}_t^k(\tau_{(k)}, \zeta_{(k)}) \mathbf{1}_{\tau_k \leq t < \tau_{k+1}} + \underline{\tilde{Y}}_t^{k+1} \mathbf{1}_{t \geq \tau_{k+1}}), \quad 0 \leq t \leq T. \end{aligned}$$

These processes are càd-làg with jumps only at times  $\tau_l$ ,  $l = 1, \dots, n$ . Notice also that  $\tilde{Y}^n$  (resp.  $\underline{\tilde{Y}}^n$ ,  $\tilde{Y}^k$ ,  $\underline{\tilde{Y}}^k$ ) satisfies equation (4.1) (resp. (4.2), (4.3), (4.4)).

**Step 3.** We prove by a backward induction that  $\underline{\tilde{Y}}^n \leq \tilde{Y}^n$  on  $[\tau_n \wedge T, T]$  and  $\underline{\tilde{Y}}^k \leq \tilde{Y}^k$  on  $[\tau_k \wedge T, \tau_{k+1} \wedge T)$ , for each  $k = 0, \dots, n-1$ .

• Since  $\underline{\xi} \leq \bar{\xi}$ ,  $\underline{F}^n \leq \bar{F}^n$  and  $\bar{F}^n$  or  $\underline{F}^n$  satisfy a comparison theorem for Brownian BSDEs, we immediately get from (4.1) and (4.2)

$$\underline{\tilde{Y}}_t^n \leq \tilde{Y}_t^n, \quad \tau_n \wedge T \leq t \leq T.$$

• Fix  $k \leq n-1$  and suppose that  $\underline{\tilde{Y}}_t^{k+1} \leq \tilde{Y}_t^{k+1}$  for  $t \in [\tau_{k+1} \wedge T, \tau_{k+2} \wedge T)$ . Denote by  ${}^p\tilde{Y}^l$  (resp.  ${}^p\underline{\tilde{Y}}^l$ ) the predictable projection of  $\tilde{Y}^l$  (resp.  $\underline{\tilde{Y}}^l$ ) for  $l = 0, \dots, n$ . Since the random measure  $\mu$  admits an intensity absolutely continuous w.r.t. the Lebesgue measure on  $[0, T]$ ,  $\tilde{Y}^l$  (resp.  $\underline{\tilde{Y}}^l$ ) has inaccessible jumps (see Ch. IV of [8]). We then have

$${}^p\tilde{Y}_t^l = \tilde{Y}_{t-}^l \quad (\text{resp. } {}^p\underline{\tilde{Y}}_t^l = \underline{\tilde{Y}}_{t-}^l), \quad 0 \leq t \leq T.$$

From equations (4.3) and (4.4), and the definition of  $\tilde{Y}^l$  (resp.  $\underline{\tilde{Y}}^l$ ), we have for  $l = k$

$$\begin{aligned} {}^p\tilde{Y}_t^k &= {}^p\tilde{Y}_{\tau_{k+1}}^{k+1} \mathbf{1}_{\tau_{k+1} \leq T} + \bar{\xi} \mathbf{1}_{\tau_{k+1} > T} + \int_t^{\tau_{k+1} \wedge T} \bar{F}^k(s, {}^p\tilde{Y}_s^k, \bar{Z}_s^k(\tau_{(k)}, \zeta_{(k)})) ds \\ &\quad - \int_t^{\tau_{k+1} \wedge T} \bar{Z}_s^k(\tau_{(k)}, \zeta_{(k)}) dW_s, \quad \tau_k \wedge T \leq t < \tau_{k+1} \wedge T. \end{aligned} \quad (4.5)$$

$$\begin{aligned} (\text{resp. } {}^p\underline{\tilde{Y}}_t^k &= {}^p\underline{\tilde{Y}}_{\tau_{k+1}}^{k+1} \mathbf{1}_{\tau_{k+1} \leq T} + \underline{\xi} \mathbf{1}_{\tau_{k+1} > T} + \int_t^{\tau_{k+1} \wedge T} \underline{F}^k(s, {}^p\underline{\tilde{Y}}_s^k, \underline{Z}_s^k(\tau_{(k)}, \zeta_{(k)})) ds \\ &\quad - \int_t^{\tau_{k+1} \wedge T} \underline{Z}_s^k(\tau_{(k)}, \zeta_{(k)}) dW_s, \quad \tau_k \wedge T \leq t < \tau_{k+1} \wedge T) \end{aligned} \quad (4.6)$$

Since  $\tilde{Y}_{\tau_{k+1}}^{k+1} \geq \underline{\tilde{Y}}_{\tau_{k+1}}^{k+1}$ , we get  ${}^p\tilde{Y}_{\tau_{k+1}}^{k+1} \geq {}^p\underline{\tilde{Y}}_{\tau_{k+1}}^{k+1}$ . This together with conditions on  $\bar{\xi}$ ,  $\underline{\xi}$ ,  $\bar{F}^k$  and  $\underline{F}^k$  give the result.

**Step 4.** Since  $\tilde{Y}^k$  (resp.  $\underline{\tilde{Y}}^k$ ) coincides with  $\bar{Y}$  (resp.  $\underline{Y}$ ) on  $[\tau_k \wedge T, \tau_{k+1} \wedge T)$ , we get the result.  $\square$

## 4.2 Uniqueness via comparison

In this form, the previous theorem is not usable since the condition on the generators of the Brownian BSDEs is implicit: it involves the solution of the previous Brownian BSDEs at each step. We give, throughout the sequel, an explicit example for which Theorem 4.1 provides uniqueness. This example is based on a comparison theorem for quadratic BSDEs given by Briand and Hu [7]. We first introduce the following assumptions.

(HUQ1) The function  $f(t, y, \cdot, u)$  is concave for all  $(t, y, u) \in [0, T] \times \mathbb{R} \times \mathbb{R}^E$ .

(HUQ2) There exists a constant  $L$  s.t.

$$|f(t, y, z, (u(e) - y)_{e \in E}) - f(t, y', z, (u(e) - y')_{e \in E})| \leq L|y - y'|$$

for all  $(t, y, y', z, u) \in [0, T] \times [\mathbb{R}]^2 \times \mathbb{R}^d \times \mathbb{R}^E$ .

(HUQ3) There exist a constant  $C > 0$  such that

$$|f(t, y, z, u)| \leq C \left( 1 + |y| + |z|^2 + \int_E |u(e)| \lambda_t(e) de \right)$$

for all  $(t, y, z, u) \in [0, T] \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^E$ .

(HUQ4)  $f(t, \cdot, u) = f(t, \cdot, 0)$  for all  $u \in \mathbb{R}^E$  and all  $t \in (\tau_n \wedge T, T]$ .

**Theorem 4.2.** *Under (HBI), (HC), (HUQ1), (HUQ2), (HUQ3) and (HUQ4), BSDE (2.4) admits at most one solution.*

**Proof.** Let  $(Y, Z, U)$  and  $(Y', Z', U')$  be two solutions of (2.4) in  $\mathcal{S}_{\mathbb{G}}^{\infty}[0, T] \times L_{\mathbb{G}}^2[0, T] \times L^2(\mu)$ . Define the process  $\tilde{U}$  (resp.  $\tilde{U}'$ ) by

$$\tilde{U}_t(e) \text{ (resp. } \tilde{U}'_t(e)) = U_t(e) \mathbf{1}_{t \leq \tau_n} \text{ (resp. } U'_t(e) \mathbf{1}_{t \leq \tau_n}), \quad (t, e) \in [0, T] \times E.$$

Then, from (HUQ4), we have that  $(Y, Z, \tilde{U})$  and  $(Y', Z', \tilde{U}')$  are also solutions to (2.4) in  $\mathcal{S}_{\mathbb{G}}^{\infty}[0, T] \times L_{\mathbb{G}}^2[0, T] \times L^2(\mu)$ . Using Remark 4.1 and Theorem 5 in [7], we obtain that the generator  $f$  satisfies a comparison theorem in the sense of Definition 4.1. We can then apply Theorem 4.1 and we get that  $Y \leq Y'$ . Since  $Y$  and  $Y'$  play the same role we obtain that  $Y' \leq Y$  and  $Y = Y'$ . Identifying the finite variation part and the unbounded variation part of  $Y$  we get  $Z = Z'$ . Then, identifying the pure jump part of  $Y$  we get  $\tilde{U} = \tilde{U}'$ . Since  $\tilde{U} = U$  (resp.  $\tilde{U}' = U'$ ) in  $L^2(\mu)$ , we finally get  $(Y, Z, U) = (Y', Z', U')$ .  $\square$

## 5 Exponential utility maximization in a jump market model

We consider a financial market model with a riskless bond assumed for simplicity equal to one, and a risky asset subjects to some counterparty risks. We suppose that the Brownian motion  $W$  is one dimensional ( $d = 1$ ). The dynamics of the risky asset is affected by other firms, the counterparties, which may default at some random times, inducing consequently some jumps in the asset price. However, this asset still exists and can be traded after the

default of the counterparties. We keep the notation of previous sections.

Throughout the sequel, we suppose that (HD) and (HC) are satisfied. We consider that the price process  $S$  evolves according to the equation

$$S_t = S_0 + \int_0^t S_{u-} \left( b_u du + \sigma_u dW_u + \int_E \beta_u(e) \mu(de, du) \right), \quad 0 \leq t \leq T.$$

All processes  $b$ ,  $\sigma$  and  $\beta$  are assumed to be  $\mathbb{G}$ -predictable. We introduce the following assumptions on the coefficients appearing in the dynamic of  $S$ :

(HS1) The processes  $b$ ,  $\sigma$  and  $\beta$  are uniformly bounded: there exists a constant  $C$  s.t.

$$|b_t| + |\sigma_t| + |\beta_t(e)| \leq C, \quad 0 \leq t \leq T, \quad e \in E, \quad \mathbb{P} - a.s.$$

(HS2) There exists a positive constant  $c_\sigma$  such that

$$\sigma_t \geq c_\sigma, \quad 0 \leq t \leq T, \quad \mathbb{P} - a.s.$$

(HS3) The process  $\beta$  satisfies:

$$\beta_t(e) > -1, \quad 0 \leq t \leq T, \quad e \in E, \quad \mathbb{P} - a.s.$$

(HS4) The process  $\vartheta$  defined by  $\vartheta_t = \frac{b_t}{\sigma_t}$ ,  $t \in [0, T]$ , is uniformly bounded: there exists a constant  $C$  such that

$$|\vartheta_t| \leq C, \quad 0 \leq t \leq T, \quad \mathbb{P} - a.s.$$

We notice that (HS1) allows the process  $S$  to be well defined and (HS3) ensures it to be positive.

A self-financing trading strategy is determined by its initial capital  $x \in \mathbb{R}$  and the amount of money  $\pi_t$  invested in the stock, at time  $t \in [0, T]$ . The wealth at time  $t$  associated with a strategy  $(x, \pi)$  is

$$X_t^{x, \pi} = x + \int_0^t \pi_s b_s ds + \int_0^t \pi_s \sigma_s dW_s + \int_0^t \int_E \pi_s \beta_s(e) \mu(de, ds), \quad 0 \leq t \leq T.$$

We consider a contingent claim, that is a random payoff at time  $T$  described by a  $\mathcal{G}_T$ -measurable random variable  $B$ . We suppose that  $B$  is bounded. Then, we define

$$V(x) = \sup_{\pi \in \mathcal{A}} \mathbb{E} \left[ -\exp(-\alpha(X_T^{x, \pi} - B)) \right], \quad (5.1)$$

the maximal expected utility that we can achieve by starting at time 0 with the initial capital  $x$ , using some admissible strategy  $\pi \in \mathcal{A}$  (which is defined throughout the sequel) on  $[0, T]$  and paying  $B$  at time  $T$ .  $\alpha$  is a given positive constant which can be seen as a coefficient of absolute risk aversion.

Finally, we introduce a compact subset  $\mathcal{C}$  of  $\mathbb{R}$  with  $0 \in \mathcal{C}$ , which represents an eventual constraint imposed to the trading strategies, that is,  $\pi_t(\omega) \in \mathcal{C}$ . We then define the space  $\mathcal{A}$  of admissible strategies.

**Definition 5.1.** The set  $\mathcal{A}$  of admissible strategies consists of all  $\mathbb{R}$ -valued  $\mathcal{P}(\mathbb{G})$ -measurable processes  $\pi = (\pi_t)_{0 \leq t \leq T}$  which satisfy  $\mathbb{E} \int_0^T |\pi_t \sigma_t|^2 dt + \mathbb{E} \int_0^T \int_E |\pi_t \beta_t(e)| \lambda_t(e) de dt < \infty$ , and  $\pi_t \in \mathcal{C}$ ,  $dt \otimes d\mathbb{P} - a.e.$ , as well as the uniform integrability of the family

$$\left\{ \exp(-\alpha X_\tau^{x,\pi}) : \tau \text{ stopping time valued in } [0, T] \right\}.$$

We first notice that the compactness of  $\mathcal{C}$  implies the integrability conditions imposed to the admissible strategies.

**Lemma 5.1.** Any  $\mathcal{P}(\mathbb{G})$ -measurable process  $\pi$  valued in  $\mathcal{C}$  satisfy  $\pi \in \mathcal{A}$ .

The proof is exactly the same as in Morlais [22]. We therefore omit it.

In order to characterize the value function  $V(x)$  and an optimal strategy, we construct, as in Hu *et al.* [12] and Morlais [22], a family of stochastic processes  $(R^{(\pi)})_{\pi \in \mathcal{A}}$  with the following properties:

- (i)  $R_T^{(\pi)} = -\exp(-\alpha(X_T^{x,\pi} - B))$  for all  $\pi \in \mathcal{A}$ ,
- (ii)  $R_0^{(\pi)} = R_0$  is constant for all  $\pi \in \mathcal{A}$ ,
- (iii)  $R^{(\pi)}$  is a supermartingale for all  $\pi \in \mathcal{A}$  and there exists  $\hat{\pi} \in \mathcal{A}$  such that  $R^{(\hat{\pi})}$  is a martingale.

Given processes owning these properties we can compare the expected utilities of the strategies  $\pi \in \mathcal{A}$  and  $\hat{\pi} \in \mathcal{A}$  by

$$\mathbb{E}[-\exp(-\alpha(X_T^{x,\pi} - B))] \leq R_0(x) = \mathbb{E}[-\exp(-\alpha(X_T^{x,\hat{\pi}} - B))] = V(x),$$

whence  $\hat{\pi}$  is the desired optimal strategy. To construct this family, we set

$$R_t^{(\pi)} = -\exp(-\alpha(X_t^{x,\pi} - Y_t)), \quad 0 \leq t \leq T, \quad \pi \in \mathcal{A},$$

where  $(Y, Z, U)$  is a solution of the BSDE

$$Y_t = B + \int_t^T f(s, Z_s, U_s) ds - \int_t^T Z_s dW_s - \int_t^T \int_E U_s(e) \mu(de, ds), \quad 0 \leq t \leq T. \quad (5.2)$$

We have to choose a function  $f$  for which  $R^{(\pi)}$  is a supermartingale for all  $\pi \in \mathcal{A}$ , and there exists a  $\hat{\pi} \in \mathcal{A}$  such that  $R^{(\hat{\pi})}$  is a martingale. We assume that there exists a triple  $(Y, Z, U)$  solving a BSDE with jumps of the form (5.2), with terminal condition  $B$  and with a driver  $f$  to be determined. We first apply Itô's formula to  $R^{(\pi)}$  for any strategy  $\pi$ :

$$\begin{aligned} dR_t^{(\pi)} &= R_t^{(\pi)} \left[ \left( -\alpha(f(t, Z_t, U_t) + \pi_t b_t) + \frac{\alpha^2}{2} (\pi_t \sigma_t - Z_t)^2 \right) dt - \alpha(\pi_t \sigma_t - Z_t) dW_t \right. \\ &\quad \left. + \int_E (\exp(-\alpha(\pi_t \beta_t(e) - U_t(e))) - 1) \mu(de, dt) \right]. \end{aligned}$$

Thus, the process  $R^{(\pi)}$  satisfies the following SDE:

$$dR_t^{(\pi)} = R_t^{(\pi)} dM_t^{(\pi)} + R_t^{(\pi)} dA_t^{(\pi)}, \quad 0 < t \leq T,$$

with  $M^{(\pi)}$  a local martingale and  $A^{(\pi)}$  a finite variation continuous process given by

$$\begin{cases} dM_t^{(\pi)} = -\alpha(\pi_t\sigma_t - Z_t)dW_t + \int_E (\exp(-\alpha(\pi_t\beta_t(e) - U_t(e))) - 1)\tilde{\mu}(de, dt), \\ dA_t^{(\pi)} = \left( -\alpha(f(t, Z_t, U_t) + \pi_t b_t) + \frac{\alpha^2}{2}(\pi_t\sigma_t - Z_t)^2 \right. \\ \quad \left. + \int_E (\exp(-\alpha(\pi_t\beta_t(e) - U_t(e))) - 1)\lambda_t(e)de \right) dt. \end{cases}$$

It follows that  $R^{(\pi)}$  has the multiplicative form

$$R_t^{(\pi)} = R_0^{(\pi)} \mathfrak{E}(M^{(\pi)})_t \exp(A_t^{(\pi)}),$$

where  $\mathfrak{E}(M^{(\pi)})$  denotes the Doleans-Dade exponential of the local martingale  $M^{(\pi)}$ . Since  $\exp(-\alpha(\pi_t\beta_t(e) - U_t(e))) - 1 > -1$ ,  $\mathbb{P} - a.s.$ , the Doleans-Dade exponential of the discontinuous part of  $M^{(\pi)}$  is a positive local martingale and hence, a supermartingale. The supermartingale condition in (iii) holds true, provided, for all  $\pi \in \mathcal{A}$ , the process  $\exp(A^{(\pi)})$  is nondecreasing, this entails

$$-\alpha(f(t, Z_t, U_t) + \pi_t b_t) + \frac{\alpha^2}{2}(\pi_t\sigma_t - Z_t)^2 + \int_E (\exp(-\alpha(\pi_t\beta_t(e) - U_t(e))) - 1)\lambda_t(e)de \geq 0.$$

This condition holds true, if we define  $f$  as follows

$$\begin{aligned} f(t, z, u) &= \inf_{\pi \in \mathcal{C}} \left\{ \frac{\alpha}{2} \left| \pi\sigma_t - \left( z + \frac{\vartheta_t}{\alpha} \right) \right|^2 + \int_E \frac{\exp(\alpha(u(e) - \pi\beta_t(e))) - 1}{\alpha} \lambda_t(e)de \right\} \\ &\quad - \vartheta_t z - \frac{|\vartheta_t|^2}{2\alpha}, \end{aligned}$$

recall that  $\vartheta_t = b_t/\sigma_t$  for  $t \in [0, T]$ .

**Theorem 5.1.** *Under (HS1), (HS2), (HS3) and (HS4), the value function of the optimization problem (5.1) is given by*

$$V(x) = -\exp(-\alpha(x - Y_0)), \quad (5.3)$$

where  $Y_0$  is defined as the initial value of the unique solution  $(Y, Z, U) \in \mathcal{S}_{\mathbb{G}}^{\infty}[0, T] \times L_{\mathbb{G}}^2[0, T] \times L^2(\mu)$  of the BSDE

$$Y_t = B + \int_t^T f(s, Z_s, U_s)ds - \int_t^T Z_s dW_s - \int_t^T \int_E U_s(e)\mu(de, ds), \quad 0 \leq t \leq T, \quad (5.4)$$

with

$$\begin{aligned} f(t, z, u) &= \inf_{\pi \in \mathcal{C}} \left\{ \frac{\alpha}{2} \left| \pi\sigma_t - \left( z + \frac{\vartheta_t}{\alpha} \right) \right|^2 + \int_E \frac{\exp(\alpha(u(e) - \pi\beta_t(e))) - 1}{\alpha} \lambda_t(e)de \right\} \\ &\quad - \vartheta_t z - \frac{|\vartheta_t|^2}{2\alpha}, \end{aligned}$$

for all  $(t, z, u) \in [0, T] \times \mathbb{R} \times \mathbb{R}^E$ . There exists an optimal trading strategy  $\hat{\pi} \in \mathcal{A}$  which satisfies

$$\hat{\pi}_t \in \arg \min_{\pi \in \mathcal{C}} \left\{ \frac{\alpha}{2} \left| \pi\sigma_t - \left( z + \frac{\vartheta_t}{\alpha} \right) \right|^2 + \int_E \frac{\exp(\alpha(u(e) - \pi\beta_t(e))) - 1}{\alpha} \lambda_t(e)de \right\}, \quad (5.5)$$

for all  $t \in [0, T]$ .

**Proof. Step 1.** We first prove the existence of a solution to BSDE (5.4). For that we apply Theorem 3.1. Let  $\sigma^k$ ,  $\vartheta^k$  and  $\beta^k$ ,  $k = 0, \dots, n$ , be the respective terms appearing in the decomposition of  $\sigma$ ,  $\vartheta$  and  $\beta$  given by Lemma 2.1. Using (HS1) and (HS4), we can assume w.l.o.g. that these terms are uniformly bounded.

Then, in the decomposition of the generator  $f$ , we can choose the functions  $f^k$ ,  $k = 0, \dots, n$ , as

$$f^n(t, z, u, \theta, e) = \inf_{\pi \in \mathcal{C}} \left\{ \frac{\alpha}{2} \left| \pi \sigma_t^n(\theta, e) - \left( z + \frac{\vartheta_t^n(\theta, e)}{\alpha} \right) \right|^2 \right\} - \vartheta_t^n(\theta, e)z - \frac{|\vartheta_t^n(\theta, e)|^2}{2\alpha},$$

and

$$\begin{aligned} f^k(t, z, u, \theta_{(k)}, e_{(k)}) &= \inf_{\pi \in \mathcal{C}} \left\{ \frac{\alpha}{2} \left| \pi \sigma_t^k(\theta_{(k)}, e_{(k)}) - \left( z + \frac{\vartheta_t^k(\theta_{(k)}, e_{(k)})}{\alpha} \right) \right|^2 \right. \\ &\quad \left. + \int_E \frac{\exp(\alpha(u(e') - \pi \beta_t^k(\theta_{(k)}, e_{(k)}, e'))) - 1}{\alpha} \lambda_t^{k+1}(e', \theta_{(k)}, e_{(k)}) de' \right\} \\ &\quad - \vartheta_t^k(\theta_{(k)}, e_{(k)})z - \frac{|\vartheta_t^k(\theta_{(k)}, e_{(k)})|^2}{2\alpha}, \end{aligned}$$

for  $k = 0, \dots, n-1$  and  $(\theta, e) \in \Delta_n \times E^n$ .

Notice also that since  $B$  is bounded, we can choose  $B^k$ ,  $k = 0, \dots, n$ , uniformly bounded. We now prove by backward induction on  $k$  that the BSDEs (we shall omit the dependence on  $(\theta, e)$ )

$$Y_t^n = B^n + \int_t^T f^n(s, Z_s^n, 0) ds - \int_t^T Z_s^n dW_s, \quad \theta_n \wedge T \leq t \leq T, \quad (k = n) \quad (5.6)$$

and

$$\begin{aligned} Y_t^k &= B^k + \int_t^T f^k(s, Z_s^k, Y_s^{k+1}(s, \cdot) - Y_s^k) ds \\ &\quad - \int_t^T Z_s^k dW_s, \quad \theta_k \wedge T \leq t \leq T, \quad (k = 0, \dots, n-1) \end{aligned} \quad (5.7)$$

admit a solution  $(Y^k, Z^k)$  in  $\mathcal{S}_{\mathbb{F}}^\infty[\theta_k \wedge T, T] \times L_{\mathbb{F}}^2[\theta_k \wedge T, T]$  such that  $Y^k$  (resp.  $Z^k$ ) is  $\mathcal{P}\mathcal{M}(\mathbb{F}) \otimes \mathcal{B}(\Delta_k) \otimes \mathcal{B}(E^k)$  (resp.  $\mathcal{P}(\mathbb{F}) \otimes \mathcal{B}(\Delta_k) \otimes \mathcal{B}(E^k)$ )-measurable with

$$\sup_{(\theta, e) \in \Delta_n \times E^n} \|Y^k(\theta_{(k)}, e_{(k)})\|_{\mathcal{S}^\infty[\theta_k \wedge T, T]} + \|Z^k(\theta_{(k)}, e_{(k)})\|_{L^2[\theta_k \wedge T, T]} < \infty,$$

for all  $k = 0, \dots, n$ .

• Since  $0 \in \mathcal{C}$ , we have

$$-\vartheta_t^n z - \frac{|\vartheta_t^n|^2}{2\alpha} \leq f^n(t, z, 0) \leq \frac{\alpha}{2} |z|^2.$$

Therefore, we can apply Theorem 2.3 of [20], and we get that for any  $(\theta, e) \in \Delta_n \times E^n$ , there exists a solution  $(Y^n(\theta, e), Z^n(\theta, e))$  to BSDE (5.6) in  $\mathcal{S}_{\mathbb{F}}^2[\theta_n \wedge T, T] \times L_{\mathbb{F}}^2[\theta_n \wedge T, T]$ . Moreover, this solution is constructed as a limit of Lipschitz BSDEs (see [20]). Using Proposition C.1,

we get that  $Y^n$  (resp.  $Z^n$ ) is  $\mathcal{P}\mathcal{M}(\mathbb{F}) \otimes \mathcal{B}(\Delta_n) \otimes \mathcal{B}(E^n)$  (resp.  $\mathcal{P}(\mathbb{F}) \otimes \mathcal{B}(\Delta_n) \otimes \mathcal{B}(E^n)$ )-measurable.

Then, using Proposition 2.1 of [20], we get the existence of a constant  $K$  such that

$$\sup_{(\theta, e) \in \Delta_n \times E^n} \|Y^n(\theta, e)\|_{\mathcal{S}^\infty[\theta_n \wedge T, T]} + \|Z^n(\theta, e)\|_{L^2[\theta_n \wedge T, T]} \leq K.$$

- Suppose that BSDE (5.7) admits a solution at rank  $k+1$  ( $k \leq n-1$ ) with

$$\sup_{(\theta, e) \in \Delta_n \times E^n} \left\{ \|Y^{k+1}(\theta_{(k+1)}, e_{(k+1)})\|_{\mathcal{S}^\infty[\theta_{k+1} \wedge T, T]} + \|Z^{k+1}(\theta_{(k+1)}, e_{(k+1)})\|_{L^2[\theta_{k+1} \wedge T, T]} \right\} < \infty. \quad (5.8)$$

We denote  $g^k$  the function defined by

$$g^k(t, y, z, \theta_{(k)}, e_{(k)}) = f^k(t, z, Y_t^{k+1}(\theta_{(k)}, t, e_{(k)}, \cdot) - y, \theta_{(k)}, e_{(k)}),$$

for all  $(t, y, z) \in [0, T] \times \mathbb{R} \times \mathbb{R}$  and  $(\theta, e) \in \Delta_n \times E^n$ . Since  $g^k$  has an exponential growth in the variable  $y$  in the neighborhood of  $-\infty$ , we can not directly apply our previous results. We then prove via a comparison theorem that there exists a solution by introducing another BSDE which admits a solution and whose generator coincides with  $g$  in the domain where the solution lives.

Let  $(\underline{Y}^k(\theta_{(k)}, e_{(k)}), \underline{Z}^k(\theta_{(k)}, e_{(k)}))$  be the solution in  $\mathcal{S}_{\mathbb{F}}^\infty[\theta_k \wedge T, T] \times L_{\mathbb{F}}^2[\theta_k \wedge T, T]$  to the linear BSDE

$$\begin{aligned} \underline{Y}_t^k(\theta_{(k)}, e_{(k)}) &= B^k(\theta_{(k)}, e_{(k)}) + \int_t^T \underline{g}^k(s, \underline{Y}_s^k, \underline{Z}_s^k)(\theta_{(k)}, e_{(k)}) ds \\ &\quad - \int_t^T \underline{Z}_s^k(\theta_{(k)}, e_{(k)}) dW_s, \quad \theta_k \wedge T \leq t \leq T, \end{aligned}$$

where

$$\underline{g}^k(t, y, z, \theta_{(k)}, e_{(k)}) = -\vartheta_t^k(\theta_{(k)}, e_{(k)})z - \frac{\vartheta_t^k(\theta_{(k)}, e_{(k)})}{2\alpha},$$

for all  $(t, y, z) \in [0, T] \times \mathbb{R} \times \mathbb{R}$ . Since  $B^k$  and  $\vartheta^k$  are uniformly bounded, we have

$$\sup_{(\theta_{(k)}, e_{(k)}) \in \Delta_k \times E^k} \|\underline{Y}^k(\theta_{(k)}, e_{(k)})\|_{\mathcal{S}^\infty[\theta_k \wedge T, T]} < \infty. \quad (5.9)$$

Then, define the generator  $\tilde{g}^k$  by

$$\tilde{g}^k(t, y, z, \theta_{(k)}, e_{(k)}) = g^k(t, y \vee \underline{Y}_t^k(\theta_{(k)}, e_{(k)}), z, \theta_{(k)}, e_{(k)}),$$

for all  $(t, y, z) \in [0, T] \times \mathbb{R} \times \mathbb{R}$  and  $(\theta, e) \in \Delta_n \times E^n$ .

Moreover, since  $0 \in \mathcal{C}$ , we get from (5.8) and (5.9) the existence of a constant  $K$  such that

$$|\tilde{g}^k(t, y, z, \theta_{(k)}, e_{(k)})| \leq K(1 + |z|^2),$$

for all  $(t, y, z) \in [0, T] \times \mathbb{R} \times \mathbb{R}$  and  $(\theta, e) \in \Delta_n \times E^n$ . We can then apply Theorem 2.3 of [20], and we obtain that the BSDE

$$\begin{aligned} \tilde{Y}_t^k(\theta_{(k)}, e_{(k)}) &= B^k(\theta_{(k)}, e_{(k)}) + \int_t^T \tilde{g}^k(s, \tilde{Y}_s^k, \tilde{Z}_s^k)(\theta_{(k)}, e_{(k)}) ds \\ &\quad - \int_t^T \tilde{Z}_s^k(\theta_{(k)}, e_{(k)}) dW_s, \quad \theta_k \wedge T \leq t \leq T, \end{aligned}$$

admits a solution  $(\tilde{Y}^k(\theta_{(k)}, e_{(k)}), \tilde{Z}^k(\theta_{(k)}, e_{(k)})) \in \mathcal{S}_{\mathbb{F}}^\infty[\theta_k \wedge T, T] \times L_{\mathbb{F}}^2[\theta_k \wedge T, T]$ . Using Proposition 2.1 of [20], we get

$$\sup_{(\theta_{(k)}, e_{(k)}) \in \Delta_k \times E^k} \|\tilde{Y}^k(\theta_{(k)}, e_{(k)})\|_{\mathcal{S}^\infty[\theta_k \wedge T, T]} < \infty.$$

Then, since  $\tilde{g}^k \geq \underline{g}^k$  and since  $\underline{g}^k$  is Lipschitz continuous, we get from the comparison theorem for BSDEs that  $\tilde{Y}^k \geq \underline{Y}^k$ . Hence,  $(\tilde{Y}^k, \tilde{Z}^k)$  is solution to BSDE (5.7). Notice then that we can choose  $\tilde{Y}^k$  (resp.  $\tilde{Z}^k$ ) as a  $\mathcal{P}\mathcal{M}(\mathbb{F}) \otimes \mathcal{B}(\Delta_k) \otimes \mathcal{B}(E^k)$  (resp.  $\mathcal{P}(\mathbb{F}) \otimes \mathcal{B}(\Delta_k) \otimes \mathcal{B}(E^k)$ )-measurable process. Indeed, these processes are solutions to quadratic BSDEs and hence can be written as the limit of solutions to Lipschitz BSDEs (see [20]). Using Proposition C.1 with  $\mathcal{X} = \Delta_k \times E^k$  and  $d\rho(\theta, e) = \gamma_0(\theta, e) d\theta de$  we get that the solutions to Lipschitz BSDEs are  $\mathcal{P}(\mathbb{F}) \otimes \mathcal{B}(\Delta_k) \otimes \mathcal{B}(E^k)$ -measurable and hence  $\tilde{Y}^k$  (resp.  $\tilde{Z}^k$ ) is  $\mathcal{P}\mathcal{M}(\mathbb{F}) \otimes \mathcal{B}(\Delta_k) \otimes \mathcal{B}(E^k)$  (resp.  $\mathcal{P}(\mathbb{F}) \otimes \mathcal{B}(\Delta_k) \otimes \mathcal{B}(E^k)$ )-measurable.

**Step 2.** We now prove the uniqueness of a solution to BSDE (5.4). Let  $(Y^1, Z^1, U^1)$  and  $(Y^2, Z^2, U^2)$  be two solutions of BSDE (5.4) in  $\mathcal{S}_{\mathbb{G}}^\infty[0, T] \times L_{\mathbb{G}}^2[0, T] \times L^2(\mu)$ .

Applying an exponential change of variable, we obtain that  $(\tilde{Y}^i, \tilde{Z}^i, \tilde{U}^i)$  defined for  $i = 1, 2$  by

$$\begin{aligned} \tilde{Y}_t^i &= \exp(\alpha Y_t^i), \\ \tilde{Z}_t^i &= \alpha \tilde{Y}_t^i Z_t^i, \\ \tilde{U}_t^i(e) &= \tilde{Y}_t^i (\exp(\alpha U_t^i(e)) - 1), \end{aligned}$$

for all  $t \in [0, T]$ , are solution in  $\mathcal{S}_{\mathbb{G}}^\infty[0, T] \times L_{\mathbb{G}}^2[0, T] \times L^2(\mu)$  to the BSDE

$$\tilde{Y}_t = \exp(\alpha B) + \int_t^T \tilde{f}(s, \tilde{Y}_s, \tilde{Z}_s, \tilde{U}_s) ds - \int_t^T \tilde{Z}_s dW_s - \int_t^T \int_E \tilde{U}_s(e) \mu(de, ds),$$

where the generator  $\tilde{f}$  is defined by

$$\tilde{f}(t, y, z, u) = \inf_{\pi \in \mathcal{C}} \left\{ \frac{\alpha^2}{2} |\pi \sigma_t|^2 y - \alpha \pi \sigma_t (z + \vartheta_t y) + \int_E \left[ e^{-\alpha \pi \beta_t(e)} (u(e) + y) - y \right] \lambda_t(e) de \right\}.$$

We then notice that

- $\tilde{f}$  satisfies (HUQ1) since it is an infimum of linear functions in the variable  $z$ ,
- $\tilde{f}$  satisfies (HUQ2). Indeed, from the definition of  $\tilde{f}$  we have

$$\tilde{f}(t, y, z, u(\cdot) - y) - \tilde{f}(t, y', z, u(\cdot) - y') \geq \inf_{\pi \in \mathcal{C}} \left\{ (y - y') (\vartheta_t + \frac{\alpha}{2} \pi \sigma_t) \alpha \pi \sigma_t \right\} - (y - y') \int_E \lambda_t(e) de,$$



for all  $(t, z, u) \in [0, T] \times \mathbb{R} \times \mathbb{R}^E$  and  $y, y' \in \mathbb{R}$ . Since  $\mathcal{C}$  is compact, we get from (HBI) the existence of a constant  $C$  such that

$$\tilde{f}(t, y, z, u - y) - \tilde{f}(t, y', z, u - y') \geq -C|y - y'|.$$

Inverting  $y$  and  $y'$  we get the result.

•  $\tilde{f}$  satisfies (HUQ3). Indeed, since  $0 \in \mathcal{C}$ , we get from (HBI) the existence of a constant  $C$  such that

$$\tilde{f}(t, y, z, u) \leq C\left(|y| + \int_E |u(e)|\lambda_t(e)de\right), \quad (t, y, z, u) \in [0, T] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^E.$$

We get then from (HBI), there exists a constant positive constant  $C$  s.t.

$$\begin{aligned} \tilde{f}(t, y, z, u) &\geq \inf_{\pi \in \mathcal{C}} \left\{ \frac{\alpha^2}{2} |\pi\sigma_t|^2 y - \alpha\pi\sigma_t(z + \vartheta_t y) \right\} \\ &\quad + \inf_{\pi \in \mathcal{C}} \left\{ \int_E e^{-\alpha\pi\beta_t(e)} (u(e) + y)\lambda_t(e)de \right\} - C|y|. \end{aligned}$$

Then, from (HS1), (HS2) and the compactness of  $\mathcal{C}$ , we get

$$\tilde{f}(t, y, z, u) \geq -C\left(1 + |y| + |z| + \int_E |u(e)|\lambda_t(e)de\right), \quad (t, y, z, u) \in [0, T] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^E.$$

•  $\tilde{f}$  satisfies (HUQ4) since at time  $t$  it is an integral of the variable  $u$  w.r.t.  $\lambda_t$ , which vanishes on the interval  $(\tau_n, \infty)$ .

Since  $\tilde{f}$  satisfies (HUQ1), (HUQ2), (HUQ3) and (HUQ4), we get from Theorem 4.2 that  $\tilde{Y}^1 = \tilde{Y}^2$ . Then, identifying the pure jump part and the infinite variation part of  $\tilde{Y}^1$ , we get  $(Y^1, Z^1, U^1) = (Y^2, Z^2, U^2)$  in  $\mathcal{S}_{\mathbb{G}}^{\infty}[0, T] \times L_{\mathbb{G}}^2[0, T] \times L^2(\mu)$ .

**Step 3.** We check that  $M^{(\hat{\pi})}$  is a BMO-martingale. Since  $\mathcal{C}$  is compact, (HS1) holds and  $U$  is bounded as the jump process of the bounded process  $Y$ , it suffices to prove that  $\int_0^\cdot Z_s dW_s$  is a BMO-martingale.

Let  $M$  denote the upper bound of the uniformly bounded process  $Y$ . Applying Itô's formula to  $(Y - M)^2$ , we obtain for any stopping time  $\tau \leq T$

$$\begin{aligned} \mathbb{E}\left[\int_{\tau}^T |Z_s|^2 ds \middle| \mathcal{G}_{\tau}\right] &= \mathbb{E}[(\xi - M)^2 | \mathcal{G}_{\tau}] - |Y_{\tau} - M|^2 \\ &\quad + 2\mathbb{E}\left[\int_{\tau}^T (Y_s - M)f(s, Z_s, U_s) ds \middle| \mathcal{G}_{\tau}\right]. \end{aligned}$$

The definition of  $f$  yields

$$-\vartheta_t Z_t - \frac{|\vartheta_t|^2}{2\alpha} - \frac{1}{\alpha} \int_E \lambda_t(e)de \leq f(t, Z_t, U_t),$$

for all  $t \in [0, T]$ . Therefore, since (HBI) and (HS4) hold, we get

$$\begin{aligned} \mathbb{E}\left[\int_{\tau}^T |Z_s|^2 ds \middle| \mathcal{G}_{\tau}\right] &\leq C\left(1 + \mathbb{E}\left[\int_{\tau}^T |Z_s + 1| ds \middle| \mathcal{G}_{\tau}\right]\right) \\ &\leq C + \frac{1}{2}\mathbb{E}\left[\int_{\tau}^T |Z_s|^2 ds \middle| \mathcal{G}_{\tau}\right]. \end{aligned}$$

Hence,  $\int_0^\cdot Z_s dW_s$  is a BMO-martingale for  $k = 0, \dots, n$ .

**Step 4.** It remains to show that  $R^{(\pi)}$  is a supermartingale for any  $\pi \in \mathcal{A}$ . Since  $\pi \in \mathcal{A}$ , the process  $\mathfrak{E}(M^{(\pi)})$  is a positive local martingale, because it is the Doléans-Dade exponential of a local martingale whose jumps are greater to  $-1$ . Hence, there exists a sequence of stopping times  $(\delta_n)_{n \in \mathbb{N}}$  satisfying  $\lim_{n \rightarrow \infty} \delta_n = T$ ,  $\mathbb{P} - a.s.$ , such that  $\mathfrak{E}(M^{(\pi)})_{\cdot \wedge \delta_n}$  is a positive martingale for each  $n \in \mathbb{N}$ . The process  $A^{(\pi)}$  is nondecreasing. Thus,  $R_{t \wedge \delta_n}^{(\pi)} = R_0 \mathfrak{E}(M^{(\pi)})_{t \wedge \delta_n} \exp(A_{t \wedge \delta_n}^{(\pi)})$  is a supermartingale, i.e. for  $s \leq t$

$$\mathbb{E}[R_{t \wedge \delta_n}^{(\pi)} | \mathcal{G}_s] \leq R_{s \wedge \delta_n}^{(\pi)} .$$

For any set  $A \in \mathcal{G}_s$ , we have

$$\mathbb{E}[R_{t \wedge \delta_n}^{(\pi)} \mathbf{1}_A] \leq \mathbb{E}[R_{s \wedge \delta_n}^{(\pi)} \mathbf{1}_A] . \quad (5.10)$$

On the other hand, since

$$R_t^{(\pi)} = - \exp(-\alpha(X_t^{x, \pi} - Y_t)) ,$$

we use both the uniform integrability of  $(\exp(-\alpha X_\delta^{x, \pi}))$  where  $\delta$  runs over the set of all stopping times and the boundedness of  $Y$  to obtain the uniform integrability of

$$\{R_\tau^{(\pi)} : \tau \text{ stopping time valued in } [0, T]\} .$$

Hence, the passage to the limit as  $n$  goes to  $\infty$  in (5.10) is justified and it implies

$$\mathbb{E}[R_t^{(\pi)} \mathbf{1}_A] \leq \mathbb{E}[R_s^{(\pi)} \mathbf{1}_A] .$$

We obtain the supermartingale property of  $R^{(\pi)}$ .

To complete the proof, we show that the strategy  $\hat{\pi}$  defined by (5.5) is optimal. We first notice that from Lemma 5.1 we have  $\hat{\pi} \in \mathcal{A}$ . By definition of  $\hat{\pi}$ , we have  $A^{(\hat{\pi})} = 0$  and hence,  $R_t^{(\hat{\pi})} = R_0 \mathfrak{E}(M^{(\hat{\pi})})_t$ . Since  $\mathcal{C}$  is compact, (HS1) holds and  $U$  is bounded as jump part of the bounded process  $Y$ , there exists a constant  $\delta > 0$  s.t.

$$\Delta M_t^{(\hat{\pi})} = M_t^{(\hat{\pi})} - M_{t^-}^{(\hat{\pi})} \geq -1 + \delta .$$

Applying Kazamaki criterion to the BMO martingale  $M^{(\hat{\pi})}$  (see [19]) we obtain that  $\mathfrak{E}(M^{(\hat{\pi})})$  is a true martingale. As a result, we get

$$\sup_{\pi \in \mathcal{A}} \mathbb{E}(R_T^{(\pi)}) = R_0 = V(x) .$$

Using that  $(Y, Z, U)$  is the unique solution of the BSDE (5.4), we obtain the expression (5.3) for the value function.  $\square$

## Appendix

## A Proof of decomposition for the progressively measurable processes

We adopt the same approach as in [2]. We first give a decomposition lemma for  $\mathcal{G}_t$ -measurable random variables for any  $t \in \mathbb{R}_+$ . We then apply this lemma to prove the decomposition of progressively measurable processes.

**Lemma A.1.** *Fix  $t \in \mathbb{R}_+$ . Any  $\mathcal{G}_t$ -measurable random variable  $X$  can be written in the form*

$$X = \sum_{i=0}^n \mathbf{1}_{\tau_i \leq t < \tau_{i+1}} X(\tau(i), \zeta(i)),$$

where  $X^i$  is  $\mathcal{F}_t \otimes \mathcal{B}(\Delta_i) \otimes \mathcal{B}(E^i)$ -measurable.

**Proof.** We prove the decomposition result by induction on  $n$ . We denote by  $\mathbb{G}^i := \mathbb{F} \vee \mathbb{D}^1 \vee \dots \vee \mathbb{D}^i$ , for  $i = 1, \dots, n$ .

**Step 1.** Suppose first that  $n = 1$ , so that  $\mathbb{G} = \mathbb{F} \vee \mathbb{D}^1$ . Let us consider generators of the set of  $\mathcal{G}_t$ -measurable random variables, which are random variables in the form

$$\eta f(\xi \mathbf{1}_{\tau \leq t}) g(\tau \wedge t),$$

with  $\eta$  an  $\mathcal{F}_t$ -measurable r.v.,  $f$   $\mathcal{B}(\mathbb{R})$ -measurable and  $g$  measurable defined on  $E \cup \{0\}$ . By taking

$$X^0 = \eta f(0)g(t) \quad \text{and} \quad X^1(\theta_1, e) = \eta f(e \mathbf{1}_{\theta_1 \leq t})g(\theta_1 \wedge t)$$

we see that the decomposition holds for generators of the set of  $\mathcal{G}_t$ -measurable random variables. We then extend this decomposition for any  $\mathcal{G}_t$ -measurable random variable by the monotone class theorem.

**Step 2.** Suppose that the result holds for  $i$  and consider the case with  $i + 1$  ranked jump times, so that  $\mathbb{G} = \mathbb{G}^i \vee \mathbb{D}^{i+1}$ . By the same arguments of enlargement of filtration with one jump time as in Step 1, we derive that any  $\mathcal{G}_t$ -measurable random variable  $X$  is represented as

$$X = X^{0,(i)} \mathbf{1}_{t < \tau_{i+1}} + X^{1,(i)}(\tau_{i+1}, \zeta_{i+1}) \mathbf{1}_{t \geq \tau_{i+1}},$$

where  $X^{0,(i)}$  is a  $\mathcal{G}_t^i$ -measurable random variable, and  $(\omega, \theta_{i+1}, e_{i+1}) \mapsto X^{1,(i)}(\omega, \theta_{i+1}, e_{i+1})$  is  $\mathcal{G}_t^i \otimes \mathcal{B}(\mathbb{R}_+) \otimes \mathcal{B}(E)$ -measurable. Now, from induction hypothesis for  $\mathbb{G}^i$ , we have

$$\begin{aligned} X^{0,(i)} &= X^{0,0,(i)} \mathbf{1}_{t < \tau_1} + \sum_{k=1}^{i-1} X^{k,0,(i)}(\tau_{(k)}, \zeta_{(k)}) \mathbf{1}_{\tau_k \leq t < \tau_{k+1}} \\ &X^{i,0,(i)}(\tau(i), \zeta(i)) \mathbf{1}_{\tau_i \leq t}, \quad t \geq 0, \end{aligned}$$

where  $X^{0,0,(i)}$  is an  $\mathcal{F}_t$ -measurable random variable, and  $X^{k,0,(i)}$  is an  $\mathcal{F}_t \otimes \mathcal{B}(\Delta_k) \otimes \mathcal{B}(E^k)$ -measurable random variable for  $k = 1, \dots, i$ . Similarly, we have

$$\begin{aligned} X^{1,(i)}(\theta_{i+1}, e_{i+1}) &= X^{0,1,(i)}(\theta_{i+1}, e_{i+1}) \mathbf{1}_{t < \tau_1} + \sum_{k=1}^{i-1} X^{k,1,(i)}(\tau_{(k)}, \zeta_{(k)}, \theta_{i+1}, e_{i+1}) \mathbf{1}_{\tau_k \leq t < \tau_{k+1}} \\ &X^{i,1,(i)}(\tau(i), \zeta(i), \theta_{i+1}, e_{i+1}) \mathbf{1}_{\tau_i \leq t}, \quad t \geq 0, \end{aligned}$$

where  $X^{0,1,(i)}$  is an  $\mathcal{F}_t \otimes \mathcal{B}(\mathbb{R}_+) \otimes \mathcal{B}(E)$ -measurable random variable, and  $X^{k,1,(i)}$  is an  $\mathcal{F}_t \otimes \mathcal{B}(\Delta_k \times \mathbb{R}_+) \otimes \mathcal{B}(E^{k+1})$ -measurable random variable for  $k = 1, \dots, i$ . Finally, plugging these two decompositions and recalling that  $\tau_1 \leq \dots \leq \tau_i \leq \tau_{i+1}$ , we get the required decomposition at level  $i + 1$  for  $\mathbb{G}$ .  $\square$

We now prove the decomposition for the progressively measurable processes.

For  $p \geq 0$  we may choose  $X_p^k(\tau_{(k)}, \zeta_{(k)})$  for  $k = 0, \dots, n$  such that

$$X_p = \sum_{i=0}^n \mathbb{1}_{\tau_i \leq p < \tau_{i+1}} X_p^i(\tau_{(i)}, \zeta_{(i)}).$$

Define  $X_t^k(\tau_{(k)}, \zeta_{(k)})$  for  $k = 0, \dots, n$  by setting  $X_t^k(\tau_{(k)}, \zeta_{(k)}) = \liminf_{p \downarrow t} X_p^k(\tau_{(k)}, \zeta_{(k)})$ .  $X^k$ ,  $k = 0, \dots, n$ , are then  $\mathcal{P}\mathcal{M}(\mathbb{F}) \otimes \mathcal{B}(\Delta_k) \otimes \mathcal{B}(E^k)$ -measurable, by [8]. It is easily verified that

$$X_t = \sum_{i=0}^n \mathbb{1}_{\tau_i \leq t < \tau_{i+1}} X_t^i(\tau_{(i)}, \zeta_{(i)}), \quad t \geq 0.$$

## B Proof of Proposition 2.1

We first give a lemma which is a generalization of a proposition in [11]. Throughout the sequel, we denote

$$\mathcal{E}_t^{\mathbb{F},i,k}(G)(\theta_{(i-1)}, e_{(i-1)}) = \int_{\Delta_{k-i+1} \times E^{k-i+1}} \mathbb{1}_{\theta_i > t} \mathbb{E}[G(\theta_{(k)}, e_{(k)}) | \mathcal{F}_t] d\theta_i \dots d\theta_k de_i \dots de_k,$$

for any  $\mathcal{F}_\infty \otimes \mathcal{B}(\Delta_k) \otimes \mathcal{B}(E^k)$ -measurable function  $G$  and any integers  $i$  and  $k$  such that  $1 \leq i \leq k \leq n$ .

**Lemma B.1.** *Fix  $t, s \in \mathbb{R}_+$  such that  $t \leq s$ . Let  $X$  be a positive  $\mathcal{F}_s \otimes \mathcal{B}(\Delta_n) \otimes \mathcal{B}(E^n)$ -measurable function on  $\Omega \times \Delta_n \times E^n$ , then*

$$\mathbb{E}[X(\tau_{(n)}, \zeta_{(n)}) | \mathcal{G}_t] = \sum_{i=0}^n \mathbb{1}_{\tau_i \leq t < \tau_{i+1}} \frac{\mathcal{E}_t^{\mathbb{F},i+1,n}(X\gamma_s)(\tau_{(i)}, \zeta_{(i)})}{\mathcal{E}_t^{\mathbb{F},i+1,n}(\gamma_t)(\tau_{(i)}, \zeta_{(i)})}.$$

**Proof.** Let  $H$  be a positive and  $\mathcal{G}_t$ -measurable test random variable, which can be written

$$H = \sum_{i=0}^n H^i(\tau_{(i)}, \zeta_{(i)}) \mathbb{1}_{\tau_i \leq t < \tau_{i+1}},$$

where  $H^i$  is  $\mathcal{F}_t \otimes \mathcal{B}(\Delta_i) \otimes \mathcal{B}(E^i)$ -measurable for  $i = 0, \dots, n$ . Using the joint density  $\gamma_t(\theta, e)$  of  $(\tau, \zeta)$ , we have on the one hand

$$\mathbb{E}[\mathbb{1}_{\tau_i \leq t < \tau_{i+1}} H X(\tau_{(n)}, \zeta_{(n)})] = \mathbb{E}\left[\int_{(0,t]^i \cap \Delta_i \times E^i} d\theta_{(i)} de_{(i)} H^i(\theta_{(i)}, e_{(i)}) \mathcal{E}_t^{\mathbb{F},i+1,n}(X\gamma_s)(\tau_{(i)}, \zeta_{(i)})\right].$$

On the other hand, we have

$$\begin{aligned}
& \mathbb{E} \left[ \mathbf{1}_{\tau_i \leq t < \tau_{i+1}} H \frac{\mathcal{E}_t^{\mathbb{F}, i+1, n}(X\gamma_s)(\tau(i), \zeta(i))}{\mathcal{E}_t^{\mathbb{F}, i+1, n}(\gamma_t)(\tau(i), \zeta(i))} \right] \\
&= \mathbb{E} \left[ \mathbf{1}_{\tau_i \leq t < \tau_{i+1}} H^i(\tau(i), \zeta(i)) \frac{\mathcal{E}_t^{\mathbb{F}, i+1, n}(X\gamma_s)(\tau(i), \zeta(i))}{\mathcal{E}_t^{\mathbb{F}, i+1, n}(\gamma_t)(\tau(i), \zeta(i))} \right] \\
&= \mathbb{E} \left[ \int_{(0, t]^i \cap \Delta_i \times E^i} d\theta_{(i)} de_{(i)} H_t^i(\theta_{(i)}, e_{(i)}) \frac{\mathcal{E}_t^{\mathbb{F}, i+1, n}(X\gamma_s)(\theta_{(i)}, e_{(i)})}{\mathcal{E}_t^{\mathbb{F}, i+1, n}(\gamma_t)(\theta_{(i)}, e_{(i)})} \mathcal{E}_t^{\mathbb{F}, i+1, n}(\gamma_t)(\theta_{(i)}, e_{(i)}) \right] \\
&= \mathbb{E} [\mathbf{1}_{\tau_i \leq t < \tau_{i+1}} HX(\tau(n), \zeta(n))] .
\end{aligned}$$

□

We now prove Proposition 2.1. To this end, we prove that for any nonnegative  $\mathcal{P}(\mathbb{G}) \otimes \mathcal{B}(E)$ -measurable process  $U$ , any  $T > 0$  and any  $t \in [0, T]$ , we have

$$\mathbb{E} \left[ \int_t^T \int_E U_s(e) \mu(de, ds) \middle| \mathcal{G}_t \right] = \mathbb{E} \left[ \int_t^T \int_E U_s(e) \lambda_s(e) deds \middle| \mathcal{G}_t \right], \quad (\text{B.1})$$

where  $\lambda$  is defined by (2.3).

We first study the left hand side of (B.1). From Lemma 2.1 and Remark 2.1, we can write

$$U_t(e) = \sum_{k=0}^n \mathbf{1}_{\tau_k < t \leq \tau_{k+1}} U_t^k(\tau(k), \zeta(k), e), \quad (t, e) \in [0, T] \times E,$$

where  $U^k$  is a  $\mathcal{P}(\mathbb{G}) \otimes \mathcal{B}(\Delta_k) \otimes \mathcal{B}(E^{k+1})$ -measurable process for  $k = 0, \dots, n$ . Moreover, since  $U$  is nonnegative, we can assume that  $U^k$ ,  $k = 0, \dots, n$ , are nonnegative. Then, from Lemma B.1, we have:

$$\begin{aligned}
& \mathbb{E} \left[ \int_t^T \int_E U_s(e) \mu(de, ds) \middle| \mathcal{G}_t \right] = \sum_{k=1}^n \mathbb{E} [\mathbf{1}_{t < \tau_k \leq T} U_{\tau_k}^{k-1}(\tau(k-1), \zeta(k)) \middle| \mathcal{G}_t] \\
&= \sum_{k=1}^n \sum_{i=0}^n \mathbf{1}_{\tau_i \leq t < \tau_{i+1}} \frac{\mathcal{E}_t^{\mathbb{F}, i+1, n} \left( \mathbf{1}_{t < \theta_k \leq T} U_{\theta_k}^{k-1}(\theta(k-1), e_{(k)}) \gamma_T(\theta, e) \right) (\tau(i), e_{(i)})}{\mathcal{E}_t^{\mathbb{F}, i+1, n}(\gamma_t)(\tau(i), e_{(i)})} \\
&= \sum_{\substack{k, i=0 \\ i \leq k}}^{n-1} \mathbf{1}_{\tau_i \leq t < \tau_{i+1}} \frac{\mathcal{E}_t^{\mathbb{F}, i+1, n} \left( \mathbf{1}_{t < \theta_{k+1} \leq T} U_{\theta_{k+1}}^k(\theta(k), e_{(k+1)}) \gamma_{\theta_{k+1}}(\theta, e) \right) (\tau(i), e_{(i)})}{\mathcal{E}_t^{\mathbb{F}, i+1, n}(\gamma_t)(\tau(i), e_{(i)})} \\
&= \sum_{\substack{k, i=0 \\ i \leq k}}^{n-1} \mathbf{1}_{\tau_i \leq t < \tau_{i+1}} \frac{\mathcal{E}_t^{\mathbb{F}, i+1, k+1} \left( \mathbf{1}_{t < \theta_{k+1} \leq T} U_{\theta_{k+1}}^k(\theta(k), e_{(k+1)}) \gamma_{\theta_{k+1}}^{k+1}(\theta(k+1), e_{(k+1)}) \right) (\tau(i), e_{(i)})}{\mathcal{E}_t^{\mathbb{F}, i+1, n}(\gamma_t)(\tau(i), e_{(i)})}
\end{aligned}$$

We now study the right hand side of (B.1):

$$\begin{aligned}
& \mathbb{E} \left[ \int_t^T \int_E U_s(e) \lambda_s(e) deds \middle| \mathcal{G}_t \right] = \sum_{k=0}^{n-1} \mathbb{E} \left[ \int_t^T \int_E \mathbb{1}_{\tau_k < s \leq \tau_{k+1}} U_s^k(\tau(k), \zeta(k)) \lambda_s^{k+1}(e, \tau(k), \zeta(k)) deds \middle| \mathcal{G}_t \right] \\
&= \sum_{k=0}^{n-1} \sum_{i=0}^n \mathbb{1}_{\tau_i \leq t < \tau_{i+1}} \frac{\mathcal{E}_t^{\mathbb{F}, i+1, n} \left( \int_t^T \int_E \mathbb{1}_{\theta_k < s \leq \theta_{k+1}} U_s^k(\theta(k), e(k)) \lambda_s^{k+1}(e', \theta(k), e(k)) \gamma_s(\theta, e) de' ds \right) (\tau(i), \zeta(i))}{\mathcal{E}_t^{\mathbb{F}, i+1, n}(\gamma_t)(\tau(i), e(i))} \\
&= \sum_{\substack{k, i=0 \\ i \leq k}}^{n-1} \mathbb{1}_{\tau_i \leq t < \tau_{i+1}} \frac{\mathcal{E}_t^{\mathbb{F}, i+1, k} \left( \int_t^T \int_E \mathbb{1}_{\theta_k < s} U_s^k(\theta(k), e(k)) \lambda_s^{k+1}(e, \theta(k), e(k)) \gamma_s^k(\theta(k), e(k)) de' ds \right) (\tau(i), \zeta(i))}{\mathcal{E}_t^{\mathbb{F}, i+1, n}(\gamma_t)(\tau(i), e(i))} \\
&= \sum_{\substack{k, i=0 \\ i \leq k}}^{n-1} \mathbb{1}_{\tau_i \leq t < \tau_{i+1}} \frac{\mathcal{E}_t^{\mathbb{F}, i+1, k} \left( \int_t^T \int_E \mathbb{1}_{\theta_k < s} U_s^k(\theta(k), e(k)) \gamma_s^{k+1}(\theta(k), s, e(k), e') de' ds \right)}{\mathcal{E}_t^{\mathbb{F}, i+1, n}(\gamma_t)(\tau(i), e(i))},
\end{aligned}$$

where the last equality comes from the definition of  $\lambda^k$ . Hence, we get (B.1).

## C Measurability of solutions to BSDEs depending on a parameter

### C.1 Representation for Brownian martingale depending on a parameter

We consider  $\mathcal{X}$  a Borelian subset of  $\mathbb{R}^p$  and  $\rho$  a finite measure on  $\mathcal{B}(\mathcal{X})$ . Let  $\{\xi(x) : x \in \mathcal{X}\}$  be a family of random variables such that the map  $\xi : \Omega \times \mathcal{X} \rightarrow \mathbb{R}$  is  $\mathcal{F}_T \otimes \mathcal{B}(\mathcal{X})$ -measurable and satisfies  $\int_{\mathcal{X}} \mathbb{E}|\xi(x)|^2 \rho(dx) < \infty$ . In the following result, we generalize the representation property as a stochastic integral w.r.t.  $W$  of square-integrable random variables to the family  $\{\xi(x) : x \in \mathcal{X}\}$ . The proof follows the same lines as for the classical Itô representation Theorem which can be found e.g. in [23]. For the sake of completeness we sketch the proof.

**Theorem C.1.** *There exists a  $\mathcal{P}(\mathbb{F}) \otimes \mathcal{B}(\mathcal{X})$ -measurable map  $Z$  such that  $\int_{\mathcal{X}} \int_0^T \mathbb{E}|Z_s(x)|^2 ds \rho(dx) < \infty$  and*

$$\xi(x) = \mathbb{E}[\xi(x)] + \int_0^T Z_s(x) dW_s, \quad \mathbb{P} \otimes \rho - a.e. \quad (\text{C.1})$$

As for the standard representation theorem, we first need a lemma which provides a dense subset of  $L^2(\mathcal{F}_T \otimes \mathcal{B}(\mathcal{X}), \mathbb{P} \otimes \rho)$  generated by easy functions.

**Lemma C.1.** *Random variables of the form*

$$\exp \left( \int_0^T h_t(x) dW_t - \frac{1}{2} \int_0^T |h_t(x)|^2 dt \right), \quad (\text{C.2})$$

where  $h$  is a bounded  $\mathcal{B}([0, T]) \otimes \mathcal{B}(\mathcal{X})$ -measurable map span a dense subset of  $L^2(\mathcal{F}_T \otimes \mathcal{B}(\mathcal{X}), \mathbb{P} \otimes \rho)$ .

**Sketch of the proof.** Let  $\Lambda \in L^2(\mathcal{F}_T \otimes \mathcal{B}(\mathcal{X}), \mathbb{P} \otimes \rho)$  orthogonal to all functions of the form (C.2). Then, in particular, we have

$$G(\alpha_1, \dots, \alpha_n) = \int_{\mathcal{X}} \mathbb{E}[\Lambda \exp(\alpha_1 W_{t_1} + \dots + \alpha_n W_{t_n})] d\rho = 0,$$

for all  $\alpha_1, \dots, \alpha_n \in \mathbb{R}$  and all  $t_1, \dots, t_n \in [0, T]$ . Since  $G$  is identically equal to zero on  $\mathbb{R}^n$  and is analytical it is also identically equal to 0 on  $\mathbb{C}^n$ . We then have for any  $\mathcal{B}(\mathcal{X}) \otimes \mathcal{B}(\mathbb{R}^p)$ -measurable function  $\phi$  such that  $\phi(x, \cdot) \in C^\infty(\mathbb{R}^n)$  with compact support for all  $x \in \mathcal{X}$

$$\begin{aligned} & \int_{\mathcal{X}} \mathbb{E}[Y \phi(x, W_{t_1}, \dots, W_{t_n})] d\rho(x) = \\ & \int_{\mathbb{R}^n \times \mathcal{X}} \hat{\phi}(x, \alpha_1, \dots, \alpha_n) \mathbb{E}[\Lambda \exp(\alpha_1 W_{t_1} + \dots + \alpha_n W_{t_n})] d\rho(x) d\alpha_1 \dots d\alpha_n = 0, \end{aligned}$$

where  $\hat{\phi}(x, \cdot)$  is the Fourier transform of  $\phi(x, \cdot)$ . Hence,  $\Lambda$  is equal to zero since it is orthogonal to a dense subset of  $L^2(\mathcal{F}_T \otimes \mathcal{B}(\mathcal{X}))$ .  $\square$

**Sketch of the proof of Theorem C.1.** First suppose that  $\xi$  has the following form:

$$\xi(x) = \exp\left(\int_0^T h_t(x) dW_t - \frac{1}{2} \int_0^T |h_t(x)|^2 dt\right),$$

with  $h$  a bounded  $\mathcal{B}([0, T]) \otimes \mathcal{B}(\mathcal{X})$ -measurable map. Then, applying Itô's formula to the process  $\exp\left(\int_0^t h_s(x) dW_s - \frac{1}{2} \int_0^t |h_s(x)|^2 ds\right)$ , we get that  $\xi$  satisfies (C.1) where the process  $Z$  is given by

$$Z_t(x) = h_t(x) \exp\left(\int_0^t h_s(x) dW_s - \frac{1}{2} \int_0^t |h_s(x)|^2 ds\right), \quad (t, x) \in [0, T] \times \mathcal{X}.$$

Now for any  $\xi \in L^2(\mathcal{F}_T \otimes \mathcal{B}(\mathcal{X}), \mathbb{P} \otimes \rho)$ , there exists a sequence  $(\xi^n)_{n \in \mathbb{N}}$  such that each  $\xi^n$  satisfies

$$\xi^n(x) = \mathbb{E}[\xi^n(x)] + \int_0^T Z_s^n(x) dW_s, \quad \mathbb{P} \otimes \rho - a.e.$$

and  $(\xi^n)_{n \in \mathbb{N}}$  converges to  $\xi$  in  $L^2(\mathcal{F}_T \otimes \mathcal{B}(\mathcal{X}), \mathbb{P} \otimes dt \otimes \rho)$ . Then, using Itô's Isometry, we get that the sequence  $(Z^n)_{n \in \mathbb{N}}$  is Cauchy and hence converges in  $L^2(\mathcal{P}(\mathbb{F}) \otimes \mathcal{B}(\mathcal{X}), \mathbb{P} \otimes dt \otimes \rho)$  to some  $Z$ . Using again the Itô Isometry, we get that  $(\xi^n)_{n \in \mathbb{N}}$  converges to  $\mathbb{E}[\xi(x)] + \int_0^T Z_s(x) dW_s$  in  $L^2(\mathcal{F}_T \otimes \mathcal{B}(\mathcal{X}), \mathbb{P} \otimes \rho)$ . Identifying the limits, we get the result.  $\square$

**Corollary C.1.** *Let  $M$  be a  $\mathcal{P}(\mathbb{F}) \otimes \mathcal{B}(\mathcal{X})$ -measurable map such that  $(M_t(x))_{0 \leq t \leq T}$  is a martingale for all  $x \in \mathcal{X}$  and  $\int_{\mathcal{X}} \mathbb{E}|M_T(x)|^2 \rho(dx) < \infty$ . Then, there exists a  $\mathcal{P}(\mathbb{F}) \otimes \mathcal{B}(\mathcal{X})$ -measurable map  $Z$  such that  $\int_0^T \int_{\mathcal{X}} \mathbb{E}|Z_s(x)|^2 \rho(dx) ds < \infty$  and*

$$M_t(x) = M_0(x) + \int_0^t Z_s(x) dW_s.$$

The proof is a direct consequence of Theorem C.1 as in [23] so we omit it.

## C.2 BSDEs depending on a parameter

We now study the measurability of solutions to Brownian BSDEs whose data depend on the parameter  $x \in \mathcal{X}$ . We consider

- a family  $\{\xi(x) : x \in \mathcal{X}\}$  of random variables such that the map  $\xi : \Omega \times \mathcal{X} \rightarrow \mathbb{R}$  is  $\mathcal{F}_T \otimes \mathcal{B}(\mathcal{X})$ –measurable and satisfies  $\int_{\mathcal{X}} \mathbb{E}|\xi(x)|^2 \rho(dx) < \infty$ ,
- a family  $\{f(\cdot, x) : x \in \mathcal{X}\}$  of random maps such that the map  $f : \Omega \times [0, T] \times \mathbb{R} \times \mathbb{R}^d \times \mathcal{X} \rightarrow \mathbb{R}$  is  $\mathcal{P}(\mathbb{F}) \otimes \mathcal{B}(\mathbb{R}) \otimes \mathcal{B}(\mathbb{R}^d) \otimes \mathcal{B}(\mathcal{X})$ –measurable and satisfies  $\int_0^T \int_{\mathcal{X}} \mathbb{E}|f(s, 0, 0, x)|^2 \rho(dx) ds < \infty$ .

We then consider the BSDEs depending on the parameter  $x \in \mathcal{X}$ :

$$Y_t(x) = \xi(x) + \int_t^T f(s, Y_s(x), Z_s(x), x) ds - \int_t^T Z_s(x) dW_s, \quad (t, x) \in [0, T] \times \mathcal{X}. \quad (\text{C.3})$$

**Lemma C.2.** *Assume that the generator  $f$  does not depend on  $(y, z)$  i.e.  $f(t, y, z, x) = f(t, x)$ . Then, BSDE (C.3) admits a solution  $(Y, Z)$  such that  $Y$  and  $Z$  are  $\mathcal{P}(\mathbb{F}) \otimes \mathcal{B}(\mathcal{X})$ –measurable.*

**Proof.** Consider the family of martingales  $\{M(x) : x \in \mathcal{X}\}$ , where  $M$  is defined by

$$M_t(x) = \mathbb{E} \left[ \xi(x) + \int_0^T f(s, x) ds \middle| \mathcal{F}_t \right], \quad (t, x) \in [0, T] \times \mathcal{X}.$$

Then, from Corollary C.1, there exists a  $\mathcal{P}(\mathbb{F}) \otimes \mathcal{B}(\mathbb{R}^d)$ –measurable map  $Z$  such that  $\int_0^T \int_{\mathcal{X}} \mathbb{E}|Z_s(x)|^2 \rho(dx) ds < \infty$  and

$$M_t(x) = M_0(x) + \int_0^t Z_s(x) dW_s, \quad (t, x) \in [0, T] \times \mathcal{X}.$$

We then easily check that the process  $Y$  defined by

$$Y_t(x) = M_t(x) - \int_0^t f(s, x) ds, \quad (t, x) \in [0, T] \times \mathcal{X},$$

is  $\mathcal{P}(\mathbb{F}) \otimes \mathcal{B}(\mathcal{X})$ –measurable and that  $(Y, Z)$  satisfies (C.3).  $\square$

We now consider the case where the generator  $f$  is Lipschitz continuous: there exists a constant  $L$  such that

$$|f(t, y, z, x) - f(t, y', z', x)| \leq L(|y - y'| + |z - z'|), \quad (\text{C.4})$$

for all  $(t, y, y', z, z') \in [0, T] \times [\mathbb{R}]^2 \times [\mathbb{R}^d]^2$ .

**Proposition C.1.** *Suppose that  $f$  satisfies (C.4). Then, BSDE (C.3) admits a  $\mathcal{P}(\mathbb{F}) \otimes \mathcal{B}(\mathcal{X})$ –measurable solution  $(Y, Z)$  such that  $\mathbb{E} \int_0^T \int_{\mathcal{X}} (|Y_s(x)|^2 + |Z_s(x)|^2) \rho(dx) ds < \infty$ .*

**Proof.** Consider the sequence  $(Y^n, Z^n)_{n \in \mathbb{N}}$  defined by  $(Y^0, Z^0) = (0, 0)$  and for  $n \geq 1$

$$Y_t^{n+1}(x) = \xi(x) + \int_t^T f(s, Y_s^n(x), Z_s^n(x)) ds - \int_t^T Z_s^{n+1}(x) dW_s, \quad (t, x) \in [0, T] \times \mathcal{X}.$$

From Lemma C.2, we get that  $(Y^n, Z^n)$  is  $\mathcal{P}(\mathbb{F}) \otimes \mathcal{B}(\mathcal{X})$ –measurable for all  $n \in \mathbb{N}$ . Moreover, since  $f$  satisfies (C.4), the sequence  $(Y^n, Z^n)_{n \in \mathbb{N}}$  converges (up to a subsequence) a.e. to  $(Y, Z)$  solution to (C.3) (see [24]). Hence, the solution  $(Y, Z)$  is also  $\mathcal{P}(\mathbb{F}) \otimes \mathcal{B}(\mathcal{X})$ –measurable.  $\square$



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